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Harrod–Domar Growth Model with Memory and Distributed Lag

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Received: 6 December 2018; Accepted: 11 January 2019; Published: 15 January 2019



Abstract: In this paper, we propose a macroeconomic growth model, in which we take into account memory with power-law fading and gamma distributed lag. This model is a generalization of the standard Harrod–Domar growth model. Fractional differential equations of this generalized model with memory and lag are suggested. For these equations, we obtain solutions, which describe the macroeconomic growth of national income with fading memory and distributed time-delay. The asymptotic behavior of these solutions is described.

Keywords: fractional differential equations; fractional derivative; time delay; distributed lag; gamma distribution; macroeconomics; Harrod–Domar model

MSC: 91B02; 91B55; 26A33

JEL Classification: E00; C02

1. Introduction

Fractional differential equations are equations that contain derivatives of non-integer orders. There are many different types of such operators, among which the most famous are the fractional derivatives that are proposed by Liouville and Riemann, Letnikov and Grünwald, Riesz, Hadamard, Erdelyi and Kober, Caputo [1–6]. The fractional differential equations are a powerful tool to describe power-law fading memory and spatial non-locality. Fractional derivatives of non-integer order have a wide application in mechanics, physics, economics and other sciences. For example, see the eight-volume encyclopedia on fractional calculus and its applications [7].

In this paper, we propose a generalization of the Harrod–Domar growth model [8–12], in which the dynamics of national income is described by fractional differential equations with continuously distributed time delay. The standard Harrod–Domar growth model has been proposed by Roy Harrod [10] and Evsey Domar [11,12] in 1946–1947. A generalization of this model by taking into account the exponentially distributed lag without memory was proposed by William Phillips [13,14] in 1954. The Harrod–Domar growth model with power-law memory was suggested by authors [15,16] in 2016 (see also References [17–19]). The simultaneous consideration of the effects of memory and time delay is important for the description of economic processes. As a starting point of this paper, we take the Harrod–Domar model without memory and lag, which is described in the Allen’s book [8], (pp. 64–65). For simplification, in this paper, we consider one-parameter power-law memory and gamma distributed time delay. We use operators that are the composition of fractional differentiation and continuously distributed translation (shift). We propose the fractional differential equations of generalization of the Harrod–Domar growth model. We obtain solutions of these equations that

describe the macroeconomic growth of national income with power-law fading memory and gamma distributed time-delay. The asymptotic behavior of the solutions, which characterize the technological growth rate of national income for the case of the Erlang distribution of delay time, is suggested.

2. Harrod–Domar Growth Model without Memory and Lag

The Harrod–Domar model with continuous time describes the dynamics of national income $Y(t)$, which is determined by the sum of the non-productive consumption $C(t)$, the induced investment $I(t)$ and the autonomous investment $A(t)$. The balance equation of this model has the form

$$Y(t) = C(t) + I(t) + A(t) \quad (1)$$

In the standard Harrod–Domar model of the growth without memory, the following assumptions are used.

(a) In the Harrod–Domar model, the autonomous investment $A(t)$ is considered as exogenous variables, which is independent of national income $Y(t)$.

(b) In the model without memory, the consumption $C(t)$ is a linear function of national income that is described by the linear multiplier equation

$$C(t) = c Y(t) \quad (2)$$

where c is the marginal propensity to consume ($0 < c < 1$).

(c) In the standard Harrod–Domar model of the growth without memory, it is assumed that induced investment $I(t)$ is determined by the rate of the national income. This assumption is described by the linear accelerator equation

$$I(t) = v Y^{(1)}(t) \quad (3)$$

where v is the positive constant, which is called the investment coefficient indicating the power of accelerator or the capital intensity of the national income, and $Y^{(1)}(t) = dY(t)/dt$ is the first-order derivative of the function $Y(t)$. Equation (3) means that the induced investment is a constant proportion of the current rate of change of income.

Substitution of Equations (2) and (3) into (1) gives

$$Y(t) = c Y(t) + v Y^{(1)}(t) + A(t) \quad (4)$$

This equation can be written as

$$v Y^{(1)}(t) = sY(t) - A(t) \quad (5)$$

where $s = 1 - c$ is the marginal propensity to save ($0 < s < 1$).

The economic dynamics, which is represented by Equation (5), can be qualitatively described in the following form. If independent investments $A(t)$ grow, for example, due to the sudden appearance of large inventions, the multiplier gives rise to a corresponding increase in $A(t)/(1 - c)$ output [8], (p. 65), where c is the marginal propensity to consume ($0 < c < 1$). The expansion of output drives the accelerator and is accompanied by the appearance of other (induced) investments. In turn, these additional investments increase (“multiply”) the products due to the economic multiplier, and a new cycle begins. In general, the result is a progressive increase of national income.

An important characteristic of macroeconomic growth models is the technological growth rate [20], (p. 49), which is also called the Harrod’s warranted rate of growth [8], of the endogenous variable (for example, national income). The technological (warranted) growth rate describes the growth rate in the case of the constant structure of the economy and the absence of external influences. The constant structure means that the parameters of the model are constant (for example, s, v are constants). The absence of external influences means the absence of exogenous variables ($A(t) = 0$).

Mathematically the technological growth rate is described by the asymptotic behavior of the solution of the homogeneous differential equation for the macroeconomic model.

In the standard Harrod–Domar model, the solution of Equation (5) with $A(t) = 0$ has the form $Y(t) = Y(0)exp(\omega t)$. Therefore, the technological growth rate of this standard model is described by the value $\omega = s/v$.

Equation (5) defines the Harrod–Domar model without memory and lag, where the behavior of the national income $Y(t)$ is determined by the dynamics of the autonomous investment $A(t)$. The solution of Equation (5) depends on what is assumed about the change of autonomous expenditure over time. The solution of Equation (5) and its analysis is given in References [8,9].

3. Harrod–Domar Growth Model with Memory

The standard Harrod–Domar model is described by first-order differential Equation (5), which is based on the multiplier Equation (2) and the accelerator Equation (3). Equation (2) assumes that the consumption $C(t)$ changes instantly when income changes. The derivatives of the first order, which are used in Equation (3), imply an instantaneous change of the investment $I(t)$ when changing the growth rate of the national income $Y(t)$. Because of this, accelerator Equation (3) does not take into account memory and lag. As a result, Equation (5) can be used only to describe an economy in which all economic agents have an instantaneous amnesia. This restriction substantially narrows the field of application of macroeconomic models to describe the real economic processes. In many cases, economic agents can remember the history of changes of the national income and investment and this fact influences the decision-making by economic agents. The Harrod–Domar model with one-parameter power-law memory has been proposed by authors of References [10–12] in 2016.

Let us consider the Harrod–Domar model with memory. In the case, the equation of investment accelerator with memory is written in the form

$$I(t) = v \int_0^t M(t - \tau) Y^{(n)}(\tau) d\tau \tag{6}$$

where $M(t - \tau)$ is the memory function or the weighting function (the probability density function) that describes the lag. For $M(t - \tau) = \delta(t - \tau)$ Equation (6) gives Equation (5) of the standard accelerator without memory and lag. Substituting the expression for the investment $I(t)$, which is given by Equation (6), into balance Equation (1), and Expression (2), we obtain the integro-differential equation

$$v \int_0^t M(t - \tau) Y^{(n)}(\tau) d\tau = sY(t) - A(t) \tag{7}$$

For $M(t - \tau) = \delta(t - \tau)$ Equation (7) gives Equation (5) that describes the standard Harrod–Domar model without memory and lag.

Equation (7) determines the dynamics of the national income within the framework of the Harrod–Domar macroeconomic model of growth with memory (and the time delay). If the parameter s, v is given, then the dynamics of national income $Y(t)$ is determined by the behavior of the autonomous investment $A(t)$.

If the function $M(\tau)$ describes memory with power-law fading, i.e., the memory function is described by the expression

$$M(t - \tau) = M_{RL}^{n-\alpha}(t - \tau) = \frac{1}{\Gamma(n - \alpha)} (t - \tau)^{n-\alpha-1} \tag{8}$$

then the equation of the accelerator with memory [18,19] has the form $I(t) = v \left(D_{C,0+}^\alpha Y \right) (t)$. In general, the capital intensity depends on the parameter of memory fading, i.e., $v = v(\alpha)$. For the Equation (8), the Harrod–Domar model with power-law memory is described by the fractional differential equation

$$v \left(D_{C,0+}^\alpha Y \right) (t) = sY(t) - A(t) \tag{9}$$

where $\left(D_{C,0+}^\alpha Y \right) (t)$ is the Caputo fractional derivative of the order $\alpha \geq 0$ that is defined [4], (p. 92), by the equation

$$\left(D_{C,0+}^\alpha Y \right) (t) = \left(M_{RL}^{n-\alpha} * Y^{(n)} \right) (t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} Y^{(n)}(\tau) d\tau \tag{10}$$

where $*$ is the Laplace convolution, $n = [\alpha] + 1$ for non-integer values of α and $n = \alpha$ for integer values of α , $\Gamma(\alpha)$ is the gamma function and $t > 0$. Here we assume that the function $Y(\tau)$ has integer-order derivatives up to $(n - 1)$ -th order, which are absolutely continuous functions on the interval $[0, t]$.

By analogy in physics, in which sub-diffusion ($0 < \alpha < 1$) and super-diffusion ($1 < \alpha < 2$) have been described [21], we can say that the fading parameter $0 < \alpha < 1$ corresponds to sub-growth and the parameter $1 < \alpha < 2$ corresponds to super-growth. This interpretation is based on the results of our works [17,18]. In macroeconomic models, the parameter $0 < \alpha < 1$ leads to a slowdown (inhibition) of economic growth [17,18,22,23]. The fading parameter $1 < \alpha < 2$ leads to an increase in the economic growth and to growth instead of decline [17,18,22,23].

The question arises as to how we can decide the value of fractional order for processes with fading memory. For this purpose, we can use the criteria of the existence of power-law memory for economic processes, which are proposed in Reference [24]. The use of these criteria allows us to apply the fractional calculus to construct dynamic models of economic processes and to define the parameters of memory fading, which are interpreted as the orders of fractional derivatives and integrals [24]. The parameter of memory fading can also be defined by statistical methods [25–31] of analyzing long-range time dependence in time series based on economic data.

Note that the Harrod–Domar model with one-parameter power-law memory has been suggested in References [15,16]. The solutions of the fractional differential Equation (9) and its properties are also described in References [17,18,22]. We proved [17,18,22] that the technological growth rates of macroeconomic models with one-parametric memory do not coincide with the growth rates $\omega = s/v$ of standard Harrod–Domar model. The technological growth rate with memory is equal to the value $\omega_{eff}(\alpha) := \omega^{1/\alpha}$, where $\alpha > 0$ characterizes power-law fading of memory. In References [17,18] we demonstrate that the account of memory effects can significantly change the technological growth rates. The principles of changing of technological growth rates by power-law memory have been suggested in References [17,22]. The technological growth rates may both increase and decrease in comparison with the standard Harrod–Domar model, which does not take into account the memory effects. The accounting of the memory can give a new type of behavior for the same parameters of the macroeconomic model. The memory with the fading parameter $\alpha < 1$ leads to a slowdown in the growth and decline of the economy. In other words, the effect of fading memory with $\alpha < 1$ leads to inhibition of economic growth and decline. The memory with the parameter $\alpha < 1$ leads to stagnation of the economy. The memory with the fading parameter $\alpha > 1$ leads to an improvement in economic dynamics, such as a slowdown in the rate of decline, a replacement of the economic decline by its growth, and an increase in the rate of economic growth.

4. Operators to Take into Account Distributed Delay and Power-Law Memory

The first time in the economics the continuously distributed lag (time delay) has been considered by Phillips [13,14] in 1954. In the growth Phillips models [8,13,14], the continuously distributed lags were proposed in the exponential form. The operators with continuously distributed lag were

described by R.G.D. Allen [8,9] in 1956. The continuously (exponentially) distributed lags are described in Section 1.9 of Reference [8], (pp. 23–29), Section 5.8 of Reference [8], (pp. 166–170), and Reference [9], (pp. 88–94). The exponential distribution is the continuous analogue of the geometric distribution, and its basic property is memoryless. These operators are often used to describe economic processes with time delay [8].

The translation (shift) operator T_τ is defined [1], (pp. 95–96), by the expression

$$(T_\tau Y)(t) = Y(t - \tau) \tag{11}$$

where $\tau > 0$ is the delay time. Using Equation (11), the accelerator with fixed time delay is described by the equation $I(t) = v(T_\tau Y)(t) = v Y^{(1)}(t - \tau)$.

The operator of the continuously distributed delay time can be defined by the equation

$$(T_M Y)(t) = \int_0^\infty M_T(\tau) T_\tau^1 Y(t) d\tau = \int_0^\infty M_T(\tau) Y(t - \tau) d\tau \tag{12}$$

where the function $M(\tau)$ is called weighting function if the condition

$$M_T(\tau) \geq 0, \int_0^\infty M_T(\tau) d\tau = 1 \tag{13}$$

holds of all $t > 0$. Here we assume that $Y(t)$ and $M_T(t)$ are piecewise continuous functions on \mathbb{R} and the integral $\int_0^\infty M_T(\tau) |Y(t - \tau)| d\tau$ converges.

The effects of the time delay (lag) are caused by finite speeds of processes, i.e., the change of one variable does not lead to instant changes of another variable. This allows us to state that the lag (time delay) cannot be interpreted as a memory. As a result, the key property of processes with time delay is memoryless.

For simultaneous and joint consideration of the distributed lag and power-law memory, we can use a composition of a fractional derivative $(D_t^\alpha Y)(t)$ and the translation operator T_M . The fractional derivative with continuously distributed lag can be defined [32] by the expression

$$(D_T^\alpha Y)(t) = (T_M(D_t^\alpha Y))(t) = \int_0^\infty M_T(\tau) (D_t^\alpha Y)(t - \tau) d\tau \tag{14}$$

where $(D_t^\alpha Y)(t)$ is the fractional derivative of the order $\alpha \in \mathbb{R}_+$ of the function $Y(t)$ with respect to time t , and the weighting function $M_T(\tau)$ satisfies the condition of non-negativity and the normalization conditions Equation (13). Here we assume that $(D_t^\alpha Y)(t)$ and $M_T(t)$ are piecewise continuous functions on \mathbb{R} and the integral $\int_0^\infty M_T(\tau) |(D_t^\alpha Y)(t - \tau)| d\tau$ converges. In Equation (14) the Caputo fractional derivative D_{C,t_0}^α can be used. This derivative is defined [4], (p. 92), by the expression

$$(D_{C,t_0}^\alpha Y)(t) := \frac{1}{\Gamma(n - \alpha)} \int_{t_0}^t (t - \tau)^{n - \alpha - 1} Y^{(n)}(\tau) d\tau \tag{15}$$

where $\alpha \geq 0$ is the order of the derivative, $n := [\alpha] + 1$ for non-integer values of α and $n = \alpha$ for integer values of α , $\Gamma(\alpha)$ is the gamma function, $\tau \in [t_0, t]$. Equation (15) assumes that the function $Y(\tau) \in AC[t_0, t]$, i.e., $Y(\tau)$ has integer-order derivatives up to $(n - 1)$ -th order, which are absolutely continuous functions on the interval $[t_0, t]$.

For $t_0 = -\infty$, Equation (15) defines the Caputo fractional derivative $(D_{C,+}^\alpha Y)(t)$ of Liouville type, where $\tau \in (-\infty, t]$. Using this derivative, we can define fractional differentiation with a continuously distributed lag

$$(D_{T,C}^\alpha Y)(t) = (T_M(D_{C,+}^\alpha Y))(t) = \int_0^\infty M_T(\tau) (D_{C,+}^\alpha Y)(t - \tau) d\tau \tag{16}$$

In Equation (16) we assume that $Y(\tau) \in AC(-\infty, t]$, $M_T(t)$ and $(D_{C,+}^\alpha Y)(t)$ are piecewise continuous functions on \mathbb{R} and the integral $\int_0^\infty M_T(\tau) \left| (D_{C,+}^\alpha Y)(t - \tau) \right| d\tau$ converges.

For $t_0 = 0$, we can assume that $(D_{C,0+}^\alpha Y)(t) = 0$ for $t < 0$ since $\tau \in [t_0, t]$ in Equation (15). In this case, we should use upper limit $t > 0$ instead of infinity in Equation (16) such that

$$(D_{T;C;0+}^\alpha Y)(t) = (T_M(D_{C,0+}^\alpha Y))(t) = \int_0^t M_T(\tau) (D_{C,0+}^\alpha Y)(t - \tau) d\tau \tag{17}$$

Let us define the fractional differential operator with the continuously distributed lag, which is distributed by the gamma distribution, in the form

$$(D_{T;C;0+}^{\lambda,a;\alpha} Y)(t) = (M_T^{\lambda,a} * (D_{C,0+}^\alpha Y))(t) = \int_0^t M_T^{\lambda,a}(\tau) (D_{C,0+}^\alpha Y)(t - \tau) d\tau \tag{18}$$

Here the weighting function $M_T(\tau)$ is described by the probability density function of the gamma distribution

$$M_T(\tau) = M_T^{\lambda,a}(\tau) = \begin{cases} \frac{\lambda^a \tau^{a-1} \exp(-\lambda \tau)}{\Gamma(a)} & \tau > 0 \\ 0 & \tau \leq 0 \end{cases} \tag{19}$$

where the parameters $a > 0$ and $\lambda > 0$ describe the shape and rate, respectively. The parameter $\theta = 1/\lambda$ that described the scale is used in econometrics to take into account waiting times. If $a = 1$, the Function (19) describes the exponential distribution. If $a = m \in \mathbb{N}$, then Function (19) describes the Erlang distribution.

In economic models with continuous time, the most popular continuously distributions of time delay are described by the exponential and gamma distributions. Exponential distribution describes the time of receipt of the order for the enterprise, the waiting time for an insurance event, the time between visits by shop, the service life of components of complex products. In economics, the gamma distribution is applied to take into account waiting times. The gamma distributions are used to describe economic processes, in which there is a sharp increase in the average duration of time delays, including delays in payments and delays orders in queues. This distribution is also used to take into account an increase in the likelihood of risk events and insurance events. Since the exponential distribution is a special case of the gamma distribution when the shape parameter is equal to one, we consider the gamma distribution in this paper. In the general case, the delay time $\tau > 0$ can be considered as a random variable, which is distributed by any other probability law (distribution) on positive semiaxis [32], if this law describes a time delay in economic processes.

The Caputo fractional derivative with gamma distributed lag is written by the equation

$$(D_{T;C;0+}^{\lambda,a;\alpha} Y)(t) = \int_0^t M_T^{\lambda,a}(\tau) (D_{C,0+}^\alpha Y)(t - \tau) d\tau = (M_T^{\lambda,a} * (M_{RL}^{n-\alpha} * Y^{(n)}))(t) \tag{20}$$

where $*$ is the Laplace convolution. Using the associativity of the Laplace convolution, we get

$$(M_T^{\lambda,a} * (M_{RL}^{n-\alpha} * Y^{(n)}))(t) = (M_{TRL}^{\lambda,a;n-\alpha} * Y^{(n)})(t) \tag{21}$$

where $M_{TRL}^{\lambda,a;n-\alpha}(t) = (M_T^{\lambda,a} * M_{RL}^{n-\alpha})(t)$. This allows us to represent Operators (20) in the form

$$(D_{T;C;0+}^{\lambda,a;\alpha} Y)(t) = (T_M^{\lambda,a} (D_{C,0+}^\alpha Y))(t) = \int_0^t M_{TRL}^{\lambda,a;n-\alpha}(t - \tau) Y^{(n)}(\tau) d\tau \tag{22}$$

where $n - 1 < \alpha \leq n$ and $M_{TRL}^{\lambda,a;n-\alpha}(t - \tau)$ is defined by the equation $M_{TRL}^{\lambda,a;n-\alpha}(t) = (M_T^{\lambda,a} * M_{RL}^{n-\alpha})(t)$.

To get an explicit expression of the function $M_{TRL}^{\lambda,a;n-\alpha}(t)$, we can use Equation 2.3.6 of Reference [33], (p. 324), in the form

$$\int_0^t (t - \tau)^{\alpha-1} \tau^{\beta-1} \exp(-\lambda \tau) d\tau = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} t^{\alpha+\beta-1} F_{1,1}(\beta; \alpha + \beta; -\lambda t) \tag{23}$$

where $Re(\alpha) > 0, Re(\beta) > 0$. In Equation (23), we use the function $F_{1,1}(a; b; z)$ that is defined [4], (pp. 29–30), by the equation

$$\Phi(a; c; z) = F_{1,1}(a; c; z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(c)}{\Gamma(a)\Gamma(c+k)} \frac{z^k}{k!} \tag{24}$$

where $a, z \in \mathbb{C}$ such that $c \neq 0, -1, -2, \dots$. Series (24) is absolutely convergent for all $z \in \mathbb{C}$. Equation (23) allows us to obtain the representation of the kernel $M_{TRL}^{\lambda,a;n-\alpha}(t)$ in the form

$$M_{TRL}^{\lambda,a;n-\alpha}(t) = \frac{\lambda^a \Gamma(a)}{\Gamma(a + n - \alpha)} t^{a+n-\alpha-1} F_{1,1}(a; a + n - \alpha; -\lambda t) \tag{25}$$

Expression (25) defines the kernel of Operator (22).

It should be noted that the memory kernel, which is given by Equation (25), may also be represented through the three parameter Mittag–Leffler function [34–38] that is defined as

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)\Gamma(\alpha k + \beta)} \frac{z^k}{k!}$$

where $Re(\alpha) > 0, Re(\beta) > 0, \gamma > 0$. This function has been proposed by Prabhakar in Reference [34]. Using Equation 5.1.18 of Reference [35], (p. 99), in the form $F_{1,1}(a; c; z) = \Gamma(c)E_{1,c}^a(z)$, we get the representation of the memory kernel (25) in the form

$$M_{TRL}^{\lambda,a;n-\alpha}(t) = t^{a+n-\alpha-1} E_{1,a+n-\alpha}^a(-\lambda t)$$

Note that the three parameter Mittag–Leffler functions have different applications (for example, see References [38–44]).

As a result, the Caputo fractional derivative with gamma distributed lag is represented [32] by the equation

$$\left(D_{T;C;0+}^{\lambda,a;\alpha} Y \right) (t) = \frac{\lambda^a \Gamma(a)}{\Gamma(a + n - \alpha)} \int_0^t (t - \tau)^{n-\alpha+a-1} F_{1,1}(a; a + n - \alpha; -\lambda(t - \tau)) Y^{(n)}(\tau) d\tau \tag{26}$$

where $n - 1 < \alpha \leq n$.

Expression (26) allows us to consider the proposed integro-differential operator of Equation (26) as a generalized operator with memory kernel given by the confluent hypergeometric function or three parameter Mittag–Leffler function. Operator (26) can be used to describe processes with power-law memory and gamma distributed lag.

5. Fractional Differential Equation for Growth Model with Memory and Lag

Assuming that induced investment $I(t)$ depends on the power-law memory and continuously distributed time delay, which is distributed by the gamma distribution, we can use the accelerator equation with memory and lag in the form

$$I(t) = v \left(D_{T;C;0+}^{\lambda,a;\alpha} Y \right) (t) \tag{27}$$

where $D_{T;C;0+}^{\lambda,a;\alpha}$ is the integro-differential operator that is defined by Equation (26).

Substitution of Equations (2) and (27) into (1) gives the macroeconomic growth model with memory and lag. The Harrod–Domar model with power-law memory and lag, which is distributed by gamma distribution, is described by the fractional differential equation

$$v \left(D_{T;C;0+}^{\lambda,a;\alpha} Y \right) (t) = sY(t) - A(t) \tag{28}$$

where $D_{T;C;0+}^{\lambda,a;\alpha}$ is the fractional derivative of order $0 < \alpha < 2$ ($n = [\alpha] + 1$), where the gamma distributed delay time has the shape parameter $a > 0$ and the rate $\lambda > 0$. For the Erlang distribution, the shape parameter is integer number ($a = m \in \mathbb{N}$).

For simplification, we rewrite Equation (28) in the form

$$\left(D_{T;C;0+}^{\lambda,a;\alpha} Y \right) (t) = \omega Y(t) + F(t) \tag{29}$$

where $\omega = s/v$ and $F(t) = -v^{-1}A(t)$.

The general solution of the nonhomogeneous Equation (29) has the form

$$Y(t) = Y_0(t) + Y_F(t), \tag{30}$$

where $Y_0(t)$ is the solution of the homogeneous equation.

$$\left(D_{T;C;0+}^{\lambda,a;\alpha} Y \right) (t) = \frac{s}{v} Y(t) \tag{31}$$

The function $Y_F(t)$ is a particular solution of the nonhomogeneous equation. This particular solution has the form

$$Y_F(t) = \int_0^t G_\alpha[t - \tau] F(\tau) d\tau = -\frac{1}{v} \int_0^t G_\alpha[t - \tau] A(\tau) d\tau \tag{32}$$

where $G_\alpha[t - \tau]$ is the fractional analog of the Green function [4], (pp. 281, 295). Equation (32) yields a solution $Y_F(t)$ for nonhomogeneous Equation (29) with zero initial conditions, $Y^{(j)}(0) = 0$ for $j = 0, \dots, (n - 1)$.

Let us obtain the solution for the homogeneous fractional differential Equation (31). The Laplace transform of Equation (31) has the form

$$\frac{\lambda^a}{(s + \lambda)^a} \left(s^\alpha (\mathcal{L}Y)(s) - \sum_{j=0}^{n-1} s^{\alpha-j-1} Y^{(j)}(0) \right) = \omega (\mathcal{L}Y)(s) \tag{33}$$

where $\omega = s/v$. Then we get

$$(\mathcal{L}Y)(s) = \sum_{j=0}^{n-1} \frac{\lambda^a s^{\alpha-j-1}}{\lambda^a s^\alpha - \omega (s + \lambda)^a} Y^{(j)}(0) = \sum_{j=0}^{n-1} \frac{s^{\alpha-j-1}}{s^\alpha - \mu (s + \lambda)^a} Y^{(j)}(0) \tag{34}$$

where $\mu = \omega \lambda^{-a}$. Let us define the special function $S_{\alpha,\delta}^\gamma [\mu, \lambda|t]$ in the form

$$S_{\alpha,\delta}^\gamma [\mu, \lambda|t] = - \sum_{k=0}^{\infty} \frac{t^{\delta(k+1)-\alpha k-\gamma-1}}{\mu^{k+1} \Gamma(\delta(k+1) - \alpha k - \gamma)} F_{1,1}(\delta(k+1); \delta(k+1) - \alpha k - \gamma, -\lambda t) \tag{35}$$

where $F_{1,1}(a; b; z)$ is the confluent hypergeometric Kummer Function (24). Using Equation 5.1.18 of Reference [35], (p. 99), in the form $F_{1,1}(a; c; z) = \Gamma(c)E_{1,c}^a(z)$, the Function (35) can be represented as an infinite series with three parameter Mittag–Leffler functions in the form

$$S_{\alpha,\delta}^\gamma [\mu, \lambda|t] = - \sum_{k=0}^{\infty} \mu^{-(k+1)} t^{\delta(k+1)-\alpha k-\gamma-1} E_{1,\delta(k+1)-\alpha k-\gamma}^{\delta(k+1)}(-\lambda t)$$

Note that the infinite series with three parameter Mittag–Leffler functions, and, therefore, the series with the confluent hypergeometric functions in Equation (35), are convergent [45,46].

The S-function $S_{\alpha,\delta}^\gamma [\mu, \lambda|t]$ allows us to represent solutions of the fractional differential equations with derivatives of non-integer order with gamma distributed lag. In the S-function in Equation (35), the parameter $\delta > 0$ is interpreted as the shape of the gamma distribution, the parameter $\lambda > 0$ is interpreted as the rate of the gamma distribution, and the parameter $\alpha > 0$ is interpreted as a parameter of memory fading.

Using the inverse Laplace transform (see Equation 5.4.9 of Reference [47]) in the form

$$\left(\mathcal{L}^{-1} \left(\frac{s^a}{(s+b)^c} \right) \right) (s) = \frac{1}{\Gamma(c-a)} t^{c-a-1} F_{1,1}(c; c-a, -bt) \tag{36}$$

where $Re(c-a) > 0$, we can proof [32] that the Laplace transform of the S-function (35) has the form

$$\mathcal{L} \left(S_{\alpha,\delta}^\gamma [\mu, \lambda|t] \right) (s) = \frac{s^\gamma}{s^\alpha - \mu(s+\lambda)^\delta} \tag{37}$$

Note that we can use Equation 5.1.6 of Reference [35], (p. 98), (or Equation 3 of Reference [42], (p. 316), and Equation 2.5 in Reference [34], (p. 8)), in the form

$$\left(\mathcal{L} \left(t^{\beta-1} E_{\alpha,\beta}^\gamma(\lambda t^\alpha) \right) \right) (s) = \frac{s^{\alpha\gamma-\beta}}{(s^\alpha - \lambda)^\gamma}$$

where $Re(s) > 0$, $Re(\beta) > 0$ and $|s|^\alpha > |\lambda|$, instead of Equation (36) to get the representation of S-function through three parameter Mittag–Leffler functions instead of the confluent hypergeometric functions.

Equation (37) allows us to represent the solution of the homogenous fractional differential equation in the form

$$Y_0(t) = \sum_{j=0}^{n-1} S_{\alpha,a}^{\alpha-j-1}[\omega \lambda^{-a}, \lambda|t] Y^{(j)}(0) \tag{38}$$

where $S_{\alpha,a}^{\alpha-j-1}[\omega \lambda^{-a}, \lambda|t]$ is defined by Equation (35), $a > 0$ and $\lambda > 0$ are the shape and rate parameters of the gamma distribution, respectively. The asymptotic behavior of solution (38) of the homogeneous fractional differential equation is considered in Section 6. For $0 < \alpha < 1$, the solution can be written in the form

$$Y_0(t) = - \sum_{k=1}^{\infty} \omega^{-k} \lambda^{ak} t^{(\alpha+a)k} E_{1,(\alpha+a)k+1}^{ak}(-\lambda t) Y(0)$$

where $E_{\alpha,\beta}^\gamma(\lambda t^\alpha)$ is the three parameter Mittag–Leffler function [35]. Note that the solution of the homogeneous fractional differential equation that describes economic growth with memory in absence of time delay (lag) is expressed through the two parameter Mittag–Leffler function, where the argument depends on ωt^α [17,18,22], instead of the rate parameter $\lambda > 0$ of gamma distribution.

Let us obtain the particular solution of Equation (29). This particular solution has the form Equation (32), where $G_\alpha[t - \tau]$ is the fractional analog of the Green function [4], (pp. 281, 295). Equation (32) yields a solution $Y_F(t)$ for nonhomogeneous Equation (29) with zero initial conditions,

$Y^{(j)}(0) = 0$ for $j = 0, \dots, (n - 1)$. The Laplace transform of Equation (29) with conditions $Y^{(j)}(0) = 0$ has the form

$$\frac{\lambda^a}{(s + \lambda)^a} s^\alpha (\mathcal{L}Y)(s) = \omega (\mathcal{L}Y)(s) + (\mathcal{L}F)(s) \tag{39}$$

Equation (39) gives

$$(\mathcal{L}Y)(s) = \frac{(s + \lambda)^a}{\lambda^a s^\alpha - \omega (s + \lambda)^a} (\mathcal{L}F)(s) \tag{40}$$

Using the transformations

$$\begin{aligned} \frac{(s + \lambda)^a}{\lambda^a s^\alpha - \omega (s + \lambda)^a} &= \frac{1}{\lambda^a s^\alpha / (s + \lambda)^a - \omega} = -\frac{1}{\omega} \frac{1}{1 - \frac{\lambda^a s^\alpha}{\omega (s + \lambda)^a}} = -\frac{1}{\omega} \sum_{k=0}^{\infty} \left(\frac{\lambda^a s^\alpha}{\omega (s + \lambda)^a} \right)^k \\ &= -\frac{1}{\omega} \sum_{k=0}^{\infty} \frac{\lambda^{ak}}{\omega^k} \frac{s^{ak}}{(s + \lambda)^{ak}} \end{aligned} \tag{41}$$

where

$$\left| \frac{\lambda^a s^\alpha}{\omega (s + \lambda)^a} \right| < 1 \tag{42}$$

we get

$$\begin{aligned} (\mathcal{L}Y)(s) &= -\sum_{k=0}^{\infty} \frac{\lambda^{ak}}{\omega^{k+1}} \frac{s^{ak}}{(s + \lambda)^{ak}} (\mathcal{L}F)(s) = -\frac{1}{\omega} (\mathcal{L}F)(s) - \sum_{k=0}^{\infty} \frac{\lambda^{ak}}{\omega^{k+1}} \frac{s^{ak}}{(s + \lambda)^{ak}} (\mathcal{L}F)(s) \\ &= -\frac{1}{\omega} (\mathcal{L}F)(s) - \sum_{k=0}^{\infty} \frac{\lambda^{a(k+1)}}{\omega^{k+2}} \frac{s^{a(k+1)}}{(s + \lambda)^{a(k+1)}} (\mathcal{L}F)(s) \end{aligned} \tag{43}$$

Using Equation 5.4.8 of Reference [47], (p. 238), of the inverse Laplace transform in the form

$$\left(\mathcal{L}^{-1} \left(\frac{s^a}{(s + b)^c} \right) \right) (s) = \frac{1}{\Gamma(c - a)} t^{c-a-1} F_{1,1}(c; c - a, -bt) \tag{44}$$

where $a = \alpha(k + 1)$, $c = a(k + 1)$, $b = \lambda$, we obtain the expression

$$\begin{aligned} G_\alpha[t - \tau] &= -\frac{1}{\omega} \delta(t - \tau) \\ &\quad - \frac{1}{\omega} \sum_{k=0}^{\infty} \frac{1}{\mu^{k+1}} \frac{t^{(k+1)(a-\alpha)-1}}{\Gamma((k+1)(a-\alpha))} F_{1,1}(a(k + 1); (k + 1)(a - \alpha), -\lambda(t - \tau)) \end{aligned} \tag{45}$$

where $-(k + 1)(a - \alpha) \notin \mathbb{N}$.

Using the special function $S_{\alpha,\delta}^\gamma[\mu, \lambda|t]$ with $\delta = a$, $\gamma = \alpha$, we get the representation

$$G_\alpha[t - \tau] = -\frac{1}{\omega} \delta(t - \tau) + \frac{1}{\omega} S_{\alpha,a}^\alpha[\mu, \lambda|t - \tau] \tag{46}$$

As a result, we get the expression

$$Y_F(t) = \frac{1}{v\omega} A(t) - \frac{1}{v\omega} \int_0^t S_{\alpha,a}^\alpha[\mu, \lambda|t - \tau] A(\tau) d\tau$$

that describes the particular solution $Y_F(t)$ of the nonhomogeneous Equation (29) for gamma distribution of delay time.

Remark. For the Erlang distribution, the shape parameter is integer number ($a = m \in \mathbb{N}$) and we have

$$(\mathcal{L}Y)(s) = \sum_{k=0}^m \binom{m}{k} \frac{s^k \lambda^{m-k}}{\lambda^m s^\alpha - \omega (s + \lambda)^m} (\mathcal{L}F)(s) = \sum_{k=0}^m \binom{m}{k} \frac{\lambda^{-k} s^k}{s^\alpha - \mu (s + \lambda)^m} (\mathcal{L}F)(s) \tag{47}$$

where $\mu = \omega \lambda^{-a}$ and we used the binomial expansion

$$(s + \lambda)^m = \sum_{k=0}^m \binom{m}{k} s^k \lambda^{m-k} \tag{48}$$

Using the special function $S_{\alpha,\delta}^\gamma [\mu, \lambda | t]$, we can write

$$(\mathcal{L}Y)(s) = \sum_{k=0}^m \binom{m}{k} \lambda^{-k} \mathcal{L} \left(S_{\alpha,m}^k [\mu, \lambda | t] \right) (s) (\mathcal{L}F)(s) \tag{49}$$

Therefore the fractional Green function for the Erlang distribution of lag has the form

$$\begin{aligned} G_\alpha[t - \tau] &= \sum_{k=0}^m \binom{m}{k} \lambda^{-k} S_{\alpha,m}^k [\mu, \lambda | t - \tau] \\ &= - \sum_{k=0}^m \sum_{j=0}^\infty \binom{m}{j} \frac{\lambda^{-j} t^{m(k+1) - \alpha k - k - 1}}{\mu^{k+1} \Gamma(m(k+1) - \alpha k - k)} F_{1,1}(m(k+1); m(k+1) - \alpha k - k, -\lambda(t - \tau)) \end{aligned} \tag{50}$$

where $\mu = \omega \lambda^{-a} = \lambda^{-a} s / v$. Using the three parameter Mittag-Leffler function, we obtain the particular solution $Y_F(t)$ of nonhomogeneous Equation (29) for the Erlang distribution of delay time in the form

$$Y_F(t) = \frac{1}{v} \int_0^t \sum_{k=0}^m \sum_{j=0}^\infty \binom{m}{j} \frac{\lambda^{-j}}{\mu^{k+1}} t^{m(k+1) - \alpha k - k - 1} E_{1,m(k+1) - \alpha k - k}^{m(k+1)}(-\lambda(t - \tau)) A(\tau) d\tau$$

where $A(t)$ is the exogenous variable that describes autonomous investment.

6. Technological Growth Rate of National Income

Let us consider the technological (warranted) growth rate of national income for the case of the Erlang distribution of the delay time. For this purpose, we consider the asymptotic behavior of the solution of the homogeneous fractional differential equation. This solution is represented through the special function $S_{\alpha,\delta}^\gamma [\mu, \lambda | t]$ or the confluent hypergeometric Kummer function $F_{1,1}(a; c; z) = \Phi(a; c; z)$. The asymptotic behavior of $\Phi(a; c; z) = F_{1,1}(a; c; z)$ at infinity $z \rightarrow -\infty$ has [4], (p. 29), the form

$$\Phi(a; c; z) = \frac{\Gamma(c)}{\Gamma(c - a)} e^{-i\pi a} z^{-a} \left(1 + O\left(\frac{1}{z}\right) \right) \tag{51}$$

Using $z = -\lambda t < 0$, then $t \rightarrow \infty$ means $z \rightarrow -\infty$ and we obtain

$$\begin{aligned} &F_{1,1}(\delta(k+1); \delta(k+1) - \alpha k - \gamma, -\lambda t) \\ &= \frac{\Gamma(\delta(k+1) - \alpha k - \gamma)}{\Gamma(-\alpha k - \gamma)} (\lambda t)^{-\delta(k+1)} e^{-i\pi\delta(k+1)} \left(1 + O\left(\frac{1}{t}\right) \right) \end{aligned} \tag{52}$$

As a result, we get

$$\begin{aligned} S_{\alpha,\delta}^\gamma [\mu, \lambda | t] &= - \sum_{k=0}^\infty \frac{\lambda^{-\delta(k+1)} t^{-\alpha k - \gamma - 1}}{\mu^{k+1} \Gamma(-\alpha k - \gamma)} e^{-i\pi\delta(k+1)} \left(1 + O\left(\frac{1}{t}\right) \right) \\ &= - \frac{\lambda^{-\delta} t^{-\gamma-1}}{\mu^1 \Gamma(-\gamma)} e^{-i\pi\delta} \left(1 + O(t^{-\alpha-\gamma-1}) \right) \end{aligned} \tag{53}$$

at infinity ($t \rightarrow \infty$).

Note that to get an asymptotic expression of the function $S_{\alpha,\delta}^\gamma[\mu, \lambda|t]$, we can use the representation though the three parameter Mittag–Leffler functions $E_{1,\delta(k+1)-\alpha k-\gamma}^{\delta(k+1)}(-\lambda t)$ instead of the representation by the confluent hypergeometric function $F_{1,1}(\delta(k+1); \delta(k+1) - \alpha k - \gamma, -\lambda t)$. To analyze the asymptotic behavior of the obtained solutions, we also can use asymptotic expression the three parameter Mittag–Leffler function (for example, see [39], Theorem 7 of Reference [42], Equation 31 of References [44] and Theorem 2.1 in Reference [45], (pp. 75–76)).

In Solution (38), we use the function $S_{\alpha,a}^{\alpha-j-1}[\mu, \lambda|t]$, where $\mu = \omega \lambda^{-a}$. Then using the asymptotic Expression (53) of the function $S_{\alpha,\delta}^\gamma[\mu, \lambda|t]$, we get the asymptotic behavior of the function $S_{\alpha,a}^{\alpha-j-1}[\mu, \lambda|t]$, at infinity ($t \rightarrow \infty$) in the form

$$S_{\alpha,a}^{\alpha-j-1}[\omega \lambda^{-a}, \lambda|t] = - \sum_{k=0}^{\infty} \frac{\lambda^{-a} t^{-\alpha(k+1)+j}}{\omega^{k+1} \Gamma(-\alpha(k+1) + j + 1)} e^{-i\pi a} \left(1 + O\left(\frac{1}{t}\right)\right) \tag{54}$$

For the Erlang distribution the shape parameter is integer $a = m \in \mathbb{N}$, and we have

$$S_{\alpha,m}^{\alpha-j-1}[\omega \lambda^{-a}, \lambda|t] = - \sum_{k=0}^{\infty} \frac{(-1)^m \lambda^{-m} t^{-\alpha(k+1)+j}}{\omega^{k+1} \Gamma(-\alpha(k+1) + j + 1)} \left(1 + O\left(\frac{1}{t}\right)\right) \tag{55}$$

where we use $e^{-i\pi m} = (-1)^m$. Note that three parameter Mittag–Leffler functions in the long time limit lead to a series, which can be considered as two-parameter Mittag–Leffler function [44]. For our case, this function has a negative first parameter [48] in the form

$$\begin{aligned} - \sum_{k=0}^{\infty} \frac{(-1)^m \lambda^{-m} t^{-\alpha(k+1)+j}}{\omega^{k+1} \Gamma(-\alpha(k+1)+j+1)} &= \frac{(-1)^m \lambda^{-m} t^j}{\Gamma(j+1)} - (-1)^m \lambda^{-m} t^j \sum_{k=0}^{\infty} \frac{(t^{-\alpha}/\omega)^k}{\Gamma(-\alpha k + j + 1)} \\ &= \frac{(-1)^m \lambda^{-m} t^j}{\Gamma(j+1)} - (-1)^m \lambda^{-m} t^j E_{-\alpha, j+1}(t^{-\alpha}/\omega) \end{aligned}$$

which by further asymptotic expansion can give the power-law behavior.

As a result, using Equation (55) we obtain

$$S_{\alpha,a}^{\alpha-j-1}[\omega \lambda^{-a}, \lambda|t] = - \frac{(-1)^m \lambda^{-m} t^{-\alpha+j}}{\omega^1 \Gamma(-\alpha + j + 1)} \left(1 + O\left(t^{-2\alpha+j}\right)\right) \tag{56}$$

Using Equation (56), the asymptotic behavior of Solution (38) can be described by the equation

$$Y(t) = \sum_{j=0}^{n-1} \frac{(-1)^{m+1} \lambda^{-m} t^{-\alpha+j}}{\omega \Gamma(-\alpha + j + 1)} Y^{(j)}(0) \left(1 + O\left(t^{-2\alpha+j}\right)\right) \tag{57}$$

Equation (57) describes the asymptotic behavior national income (for $t \rightarrow \infty$) that is represented by Solution (38) with $a = m \in \mathbb{N}$. We see that this behavior has the power-law form with the power $-\alpha + j$, where j is a smallest values from $\{0, \dots, n - 1\}$ at which $Y^{(j)}(0) \neq 0$. This power defines the technological growth rate in the framework of the Harrod–Domar model with power-law fading memory and lag, which is distributed by the Erlang distribution. We see that the asymptotic behavior of the national income with memory and distributed lag has the power-law type of growth instead of the exponential growth for processes with memory in absence of lag [17,18], where the technological growth rate is defined by the expression $\omega_{eff}(\alpha) := \omega^{1/\alpha}$. We can assume that the distributed lag suppresses the memory effects.

7. Conclusions

The standard Harrod–Domar growth model [8–12] describes the behavior of national income in the absence of distributed lag and memory. A generalization of this model by taking into account the exponentially distributed lag without memory was proposed by Phillips [13,14]. The Harrod–Domar growth model with power-law memory was suggested by authors in References [15–18]. In this paper, we proposed a generalization of Harrod–Domar growth model by taking into account one-parameter power-law memory and gamma distributed time delay (lag). We obtain the fractional differential equation of this generalized model, where we use fractional derivatives with distributed lag to take into account the memory and lag in the economic accelerator. The solution, which describes the behavior of national income, has been proposed. The asymptotic behavior of national income with memory and distributed lag demonstrates the power-law type of growth instead of the exponential growth for processes with memory in the absence of time delay. The technological growth rate with memory [17,18] is equal to the value $\omega_{eff}(\alpha) := \omega^{1/\alpha}$, where $\alpha > 0$ is a memory fading parameter. The memory effects can significantly accelerate the growth rate of the economy [17,18,22,23]. The appearance of a time delay does not accelerate growth due to memory effects. We can assume that the continuously distributed lag suppresses the influence of memory effects. Accounting for the memory effect in processes with distributed lag does not lead to an increase in the growth rate by power-law memory, as it happens in processes without delay [17,18,22,23].

We assume that this model can be used to economic growth modeling by using a generalization of methods applied in References [49–55].

Author Contributions: V.E.T.: Contributed by the ideas, analysis and writing the manuscript in mathematical part. V.V.T.: Contributed by the ideas, analysis and writing the manuscript in economical and mathematical part.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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