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Article

A Note on the Displacement Problem of Elastostatics with Singular Boundary Values

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Abstract: The displacement problem of linear elastostatics in bounded and exterior domains with a non-regular boundary datum $\boldsymbol{a}$ is considered. Precisely, if the elastic body is represented by a domain of class C^k $(k \geq 2)$ of $\mathbb{R}^3$ and $\boldsymbol{a} \in W^{2-k-1/q,q}(\partial\Omega)$, $q \in (1, +\infty)$, then it is proved that there exists a solution which is of class C^∞ in the interior and takes the boundary value in a well-defined sense. Moreover, it is unique in a natural function class.

Keywords: linear elastostatics; simple layer potentials; displacement problem; existence and uniqueness theorems; Fredholm alternative; singular data

MSC: 74B05; 35Q74; 45B05

1. Introduction

The *displacement problem* (classically known as the *Dirichlet problem*) in linear elastostatics consists of finding solutions to the differential system [1]

$$\begin{aligned} \operatorname{div} \mathbb{C}[\nabla \boldsymbol{u}] &= \boldsymbol{0} \quad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{a} \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

In (1) Ω is a bounded domain of $\mathbb{R}^3$, standing for the reference configuration of a linearly elastic body whose unknown displacement field $\boldsymbol{u} = \boldsymbol{u}(x)$ $(x \in \Omega)$ we are looking for, supposing it is assigned on the boundary $\partial\Omega$ through condition $(1)_2$. Concrete examples of displacement problems can be found, for example, in [2], Chapter XIV. Using the components, (1) can be written as

$$\partial_j C_{ijhk} \partial_k u_h = 0,$$

where ∂_i is the derivative with respect to x_i and, hereafter, the summation over repeated indexes is understood. We suppose that the elasticity tensor $\mathbb{C} = (C_{ijhk})$, representing the material properties of the body, is independent of the point (or, in other words, that the body is homogeneous). Recall that $\mathbb{C}$ is a fourth-order tensor, that is, it is a linear map from Lin to Sym, where Lin is the linear space of all second–order tensors and Sym is its subspace of symmetric tensors, such that $\mathbb{C}[\boldsymbol{W}] = \boldsymbol{0}$ for all skew tensors $\boldsymbol{W}$. We require that $\mathbb{C}$ is *symmetric* (or, in other words, that the body is *hyperelastic*), that is,

$$\boldsymbol{E} \cdot \mathbb{C}[\boldsymbol{L}] = \boldsymbol{L} \cdot \mathbb{C}[\boldsymbol{E}], \quad \forall\, \boldsymbol{E}, \boldsymbol{L} \in \text{Lin}. \tag{2}$$

Furthermore, we require that it is *strongly elliptic*, that is,

$$(\boldsymbol{b}\otimes\boldsymbol{c})\cdot\mathbb{C}[\boldsymbol{b}\otimes\boldsymbol{c}]=b_ic_jC_{ijhk}b_hc_k>0,\quad\forall\,\boldsymbol{b},\boldsymbol{c}\neq\boldsymbol{0}. \tag{3}$$

Hereafter, we say that Ω is of class C^k ($k\geq 2$) if for every $\xi\in\partial\Omega$ there is a neighborhood of ξ (on $\partial\Omega$) which is the graph of a function of class C^k. Moreover, $W^{k,q}(\Omega)$, $q\in(1,+\infty)$, is the Sobolev space of all $\varphi\in L^1_{\mathrm{loc}}(\Omega)$ such that $\|\varphi\|_{W^{k,q}(\Omega)}=\|\varphi\|_{L^q(\Omega)}+\|\nabla_k\varphi\|_{L^q(\Omega)}<+\infty$; $W^{k,q}_0(\Omega)$ is the completion of $C^\infty_0(\Omega)$ with respect to $\|\varphi\|_{W^{k,q}(\Omega)}$ and $W^{-k,q'}(\Omega)$, $1/q+1/q'=1$, is its dual space; $W^{k-1/q,q}(\partial\Omega)$ is the trace space of $W^{k,q}(\Omega)$ and $W^{1-k-1/q',q'}(\partial\Omega)$ is its dual space.

If Ω is of class C^k ($k\geq 2$) and $\boldsymbol{a}\in W^{k-1/q,q}(\partial\Omega)$, $q\in(1,+\infty)$, then (1) has a unique solution $\boldsymbol{u}\in W^{k,q}(\Omega)$ and natural estimates hold (see [3–7]). This result also holds when the elastic body is subjected to a body force, that is, if in place of $(1)_1$ we consider the system

$$\operatorname{div}\mathbb{C}[\nabla\boldsymbol{u}]=\boldsymbol{f}\quad\text{in }\Omega \tag{4}$$

with $\boldsymbol{f}\in C^\infty_0(\Omega)$.

As, in applications, the boundary data are often represented by singular fields, it is undoubtly interesting to investigate problem (1) when $\boldsymbol{a}$ satisfies weaker regularity hypotheses.

Using the theory of layer integral equations (see [8], Chapters 2/3 and [2], Chapters IV/V) and the Fredholm alternative (see Section 2), we prove (in Theorem 1) that if $\boldsymbol{a}\in W^{2-k-1/q,q}(\partial\Omega)$, then (1) has a solution, $\boldsymbol{u}$, expressed by a simple layer potential and, thus, taking the boundary value in a well-defined sense. Moreover, it is unique in a reasonable function class. The result also holds for exterior domains (see Theorem 2).

To obtain these results, we recall some established facts about simple layer potentials associated to the system $(1)_1$.

2. The Simple Layer Potentials

For every $\boldsymbol{\psi}\in L^1(\partial\Omega)$, the field

$$\boldsymbol{v}[\boldsymbol{\psi}](x)=\int_{\partial\Omega}\boldsymbol{U}(x-\zeta)\boldsymbol{\psi}(\zeta)\mathrm{d}\sigma_\zeta, \tag{5}$$

where $\boldsymbol{U}(x-y)$ is the fundamental solution to $(1)_1$ (see, e.g., [9], Chapter III), defines the *simple layer potential* with *density* $\boldsymbol{\psi}$. Recall that (see, e.g., [2,8]) $\boldsymbol{v}[\boldsymbol{\psi}]$ is an analytical solution of $(1)_1$ in $\mathbb{R}^3\setminus\partial\Omega$ and inherits from $\boldsymbol{U}$ the following asymptotic behavior

- $\nabla_k\boldsymbol{v}[\boldsymbol{\psi}](x)=O(|x|^{-1-k})$;
- $\int_{\partial\Omega}\boldsymbol{\psi}=\boldsymbol{0}\ \Rightarrow\ \nabla_k\boldsymbol{v}[\boldsymbol{\psi}](x)=O(|x|^{-2-k})$.

If $\boldsymbol{\psi}\in W^{k-1-1/q,q}(\partial\Omega)$, then

$$\|\boldsymbol{v}[\boldsymbol{\psi}]\|_{W^{k,q}(\Omega)}\leq c\|\boldsymbol{\psi}\|_{W^{k-1-1/q,q}(\partial\Omega)} \tag{6}$$

with c independent of $\boldsymbol{\psi}$, and the following limit exists

$$\lim_{\epsilon\to 0^\pm}\boldsymbol{v}[\boldsymbol{\psi}](\xi-\epsilon\boldsymbol{l}(\xi))=\mathcal{S}[\boldsymbol{\psi}](\xi) \tag{7}$$

for almost all $\xi\in\partial\Omega$ and axis $\boldsymbol{l}$ in a ball tangent to $\partial\Omega$ at ξ.

The map

$$\mathcal{S} : W^{k-1-1/q,q}(\partial\Omega) \to W^{k-1/q,q}(\partial\Omega) \tag{8}$$

defined by (7) and representing the trace of the simple layer potential with density $\boldsymbol{\psi}$, is continuous, so that

$$\|\mathcal{S}[\boldsymbol{\psi}]\|_{W^{k-1/q,q}(\partial\Omega)} \leq c\|\boldsymbol{\psi}\|_{W^{k-1-1/q,q}(\partial\Omega)}, \tag{9}$$

for some constant c depending only on k, q, and Ω. Moreover, $\mathcal{S}$ can be extended to a linear and continuous operator

$$\mathcal{S}' : W^{1-k-1/q',q'}(\partial\Omega) \to W^{2-k-1/q',q'}(\partial\Omega),$$

which coincides with the adjoint of $\mathcal{S}$ and defines the trace of the simple layer with density $\boldsymbol{\psi} \in W^{1-k-1/q',q'}(\partial\Omega)$:

$$\boldsymbol{v}[\boldsymbol{\psi}](x) = \int_{\partial\Omega}^{\star} \boldsymbol{U}(x-\zeta)\boldsymbol{\psi}(\zeta)\mathrm{d}\sigma_\zeta. \tag{10}$$

In (10) and hereafter, we use the notation $\int_{\partial\Omega}^{\star} f\varphi$ to denote the duality pairing $<,>$ between f and φ; that is, the value of the functional f belonging to (for instance) $W^{-k,q'}(\partial\Omega)$ at $\varphi \in W_0^{k,q}(\partial\Omega)$.

By (6), one obtains

$$\|\boldsymbol{v}[\boldsymbol{\psi}]\|_{W^{2-k,q'}(\Omega)} \leq c\|\boldsymbol{\psi}\|_{W^{1-k-1/q',q'}(\partial\Omega)}. \tag{11}$$

In the next section, we will prove the existence of a solution to (1) with singular boundary values by making use of the Fredholm alternative—we recall for the sake of completeness—applied to a suitable functional equation translating the boundary value problem (1).

If $\mathcal{B}$ and $\mathcal{D}$ are two Banach spaces and $\mathcal{B}'$, $\mathcal{D}'$ are their dual spaces, a linear and continuous map $\mathcal{T} : \mathcal{B} \to \mathcal{D}$ is said to be *Fredholmian* if its range is closed and $\dim \operatorname{Kern} \mathcal{T} = \dim \operatorname{Kern} \mathcal{T}' \in \mathbb{N}_0$, where $\mathcal{T}' : \mathcal{D}' \to \mathcal{B}'$ is the adjoint of $\mathcal{T}$. The classical *Fredholm alternative* (see [10], Chapter 5) assures us that the equation

$$a = \mathcal{T}[u]$$

has a solution if and only if

$$\langle \phi', a\rangle = 0, \quad \forall\, \phi' \in \operatorname{Kern} \mathcal{T}'.$$

Moreover, the equation

$$a' = \mathcal{T}'[u']$$

has a solution if and only if

$$\langle a', \phi\rangle = 0, \quad \forall\, \phi \in \operatorname{Kern} \mathcal{T}.$$

3. Existence and Uniqueness of Solutions to (1) with Singular Data

We are in a position to prove the following existence and uniqueness theorem for the displacement problem (1) with non-regular boundary data. To this end, we need the following result (Theorem 1 in [11]).

Lemma 1. *Let Ω be a bounded domain of class C^k $(k \geq 2)$. The operator $\mathcal{S}$ is Fredholmian and* $\operatorname{Kern} \mathcal{S} = \operatorname{Kern} \mathcal{S}' = \mathbf{0}$.

Theorem 1. *Let Ω be a bounded domain of class C^k $(k \geq 2)$. If $\boldsymbol{a} \in W^{2-k-1/q,q}(\partial\Omega)$, $q \in (1, +\infty)$, then, (1) has a solution $\boldsymbol{u}$ expressed by a simple layer potential with density $\boldsymbol{\psi} \in W^{1-k-1/q,q}(\partial\Omega)$. It satisfies the estimate*

$$\|\boldsymbol{u}\|_{W^{2-k,q}(\Omega)} \leq c\|\boldsymbol{a}\|_{W^{2-k-1/q,q}(\partial\Omega)}, \tag{12}$$

and is unique in the class of all $\boldsymbol{u} \in W^{2-k,q}(\Omega)$ *such that*

$$\int_\Omega^\star \boldsymbol{u} \cdot \boldsymbol{\phi} = \int_{\partial\Omega}^\star \boldsymbol{a} \cdot \mathbb{C}[\nabla \boldsymbol{z}]\boldsymbol{n}, \tag{13}$$

for all $\boldsymbol{\phi} \in C_0^\infty(\Omega)$, *where* $\boldsymbol{n}$ *denotes the unit normal to* $\partial\Omega$ *(exterior with respect to* Ω*) and* $\boldsymbol{z}$ *is the solution of*

$$\begin{aligned} \operatorname{div} \mathbb{C}[\nabla \boldsymbol{z}] &= \boldsymbol{\phi} \quad \text{in } \Omega, \\ \boldsymbol{z} &= \boldsymbol{0} \quad \text{on } \partial\Omega. \end{aligned} \tag{14}$$

Proof. In order to prove the existence of a solution to (1) in the form of a simple layer potential $\boldsymbol{u} = \boldsymbol{v}[\boldsymbol{\psi}]$, we have to require that the boundary condition $(1)_2$ is met. Thus, in terms of the operator $\mathcal{S}'$, we have to analyse the functional equation

$$\mathcal{S}'[\boldsymbol{\psi}] = \boldsymbol{a}. \tag{15}$$

By virtue of Lemma 1, (15) has a solution $\boldsymbol{\psi} \in W^{1-k-1/q,q}(\partial\Omega)$ and the field $\boldsymbol{u} = \boldsymbol{v}[\boldsymbol{\psi}]$ is a solution to (1) which is C^∞ in Ω and satisfies $(1)_2$ in the sense of (15). Let $\boldsymbol{a}_j$ be a regular sequence on $\partial\Omega$ which converges to $\boldsymbol{a}$ strongly in $W^{2-k-1/q,q}(\partial\Omega)$. Let $\boldsymbol{v}[\boldsymbol{\psi}_j]$ be the solution of (1) with datum $\boldsymbol{a}_j$:

$$\begin{aligned} \operatorname{div} \mathbb{C}[\nabla \boldsymbol{v}[\boldsymbol{\psi}_j]] &= \boldsymbol{0} \quad \text{in } \Omega, \\ \boldsymbol{v}[\boldsymbol{\psi}_j] &= \boldsymbol{a}_j \quad \text{on } \partial\Omega. \end{aligned} \tag{16}$$

By (11) $\boldsymbol{v}[\boldsymbol{\psi}_j]$ converges to $\boldsymbol{v}[\boldsymbol{\psi}]$ strongly in $W^{2-k,q}(\Omega)$. Let consider the scalar product of $(14)_1$ and $\boldsymbol{v}[\boldsymbol{\psi}_j]$ and the scalar product of $(16)_1$ and $\boldsymbol{z}$. Taking into account the boundary conditions $(14)_2$ and $(16)_2$, then integrating by parts twice gives

$$\int_\Omega \boldsymbol{v}[\boldsymbol{\psi}_j] \cdot \boldsymbol{\phi} = \int_\Omega \boldsymbol{v}[\boldsymbol{\psi}_j] \cdot \operatorname{div} \mathbb{C}[\nabla \boldsymbol{z}] = \int_{\partial\Omega} \boldsymbol{a}_j \cdot \mathbb{C}[\nabla \boldsymbol{z}]\boldsymbol{n} - \int_\Omega \nabla \boldsymbol{v}[\boldsymbol{\psi}_j] \cdot \mathbb{C}[\nabla \boldsymbol{z}] \tag{17}$$

and

$$0 = \int_\Omega \boldsymbol{z} \cdot \operatorname{div} \mathbb{C}[\nabla \boldsymbol{v}[\boldsymbol{\psi}_j]] = -\int_\Omega \nabla \boldsymbol{z} \cdot \mathbb{C}[\nabla \boldsymbol{v}[\boldsymbol{\psi}_j]]. \tag{18}$$

As $\mathbb{C}$ is symmetric, from (17) and (18), we obtain

$$\int_\Omega \boldsymbol{v}[\boldsymbol{\psi}_j] \cdot \boldsymbol{\phi} = \int_{\partial\Omega} \boldsymbol{a}_j \cdot \mathbb{C}[\nabla \boldsymbol{z}]\boldsymbol{n}. \tag{19}$$

By the trace theorem and well-known estimates for the solutions of system (14), we obtain

$$\begin{aligned} \left| \int_{\partial\Omega} \boldsymbol{a}_j \cdot \mathbb{C}[\nabla \boldsymbol{z}]\boldsymbol{n} \right| &\le \|\boldsymbol{a}_j\|_{W^{2-k-1/q,q}(\partial\Omega)} \|\mathbb{C}[\nabla \boldsymbol{z}]\boldsymbol{n}\|_{W^{k-1-1/q',q'}(\partial\Omega)} \\ &\le \|\boldsymbol{a}_j\|_{W^{2-k-1/q,q}(\partial\Omega)} \|\boldsymbol{\phi}\|_{W^{k-2,q'}(\Omega)} \end{aligned}. \tag{20}$$

Hence, by letting $j \to +\infty$ in (19) we obtain (13) and (12) by a duality argument. □

We can also consider the problem

$$\begin{aligned} \operatorname{div} \mathbb{C}[\nabla \boldsymbol{u}] &= \boldsymbol{0} \quad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{a} \quad \text{on } \partial\Omega, \\ \lim_{|x| \to +\infty} \boldsymbol{u}(x) &= \boldsymbol{0}, \end{aligned} \tag{21}$$

where Ω in now an exterior domain of $\mathbb{R}^3$, that is, $\Omega = \mathbb{R}^3 \setminus \overline{\Omega'}$, with Ω' a bounded domain (see, e.g., [12–14]). This problem is very intriguing in applications, where one has to consider, for example, the deformations of an elastic body with some holes (defects).

With a proof analogous to the above one for bounded domains, we obtain the following result.

Theorem 2. *Let Ω be an exterior domain of class C^k $(k \geq 2)$. If $\boldsymbol{a} \in W^{2-k-1/q,q}(\partial\Omega)$, with $q \in (1, +\infty)$, then (21) has a solution $\boldsymbol{u}$ expressed by a simple layer potential with density $\boldsymbol{\psi} \in W^{1-k-1/q,q}(\partial\Omega)$. It satisfies the estimate*

$$\|\boldsymbol{u}\|_{W^{2-k,q}(\Omega)} \leq c\|\boldsymbol{a}\|_{W^{2-k-1/q,q}(\partial\Omega)}, \tag{22}$$

and is unique in the class of all $\boldsymbol{u} \in W^{2-k,q}_{\mathrm{loc}}(\Omega)$ such that

$$\int_{\Omega}^{\star} \boldsymbol{u} \cdot \boldsymbol{\phi} = -\int_{\partial\Omega}^{\star} \boldsymbol{a} \cdot \mathbb{C}[\nabla \boldsymbol{z}]\boldsymbol{n}, \tag{23}$$

for all $\boldsymbol{\phi} \in C_0^\infty(\Omega)$, where $\boldsymbol{n}$ denotes the unit normal to $\partial\Omega$ (exterior with respect to Ω') and $\boldsymbol{z}$ is the solution of

$$\begin{aligned} \operatorname{div} \mathbb{C}[\nabla \boldsymbol{z}] &= \boldsymbol{\phi} \quad \text{in } \Omega, \\ \boldsymbol{z} &= \boldsymbol{0} \quad \text{on } \partial\Omega, \\ \lim_{|x|\to+\infty} \boldsymbol{z}(x) &= \boldsymbol{0}. \end{aligned} \tag{24}$$

Proof. First of all, we observe that Lemma 1 also holds for exterior domains (Theorem 1 in [11]). Thus, we can apply the Fredholm alternative again, obtaining a solution $\boldsymbol{\psi}$ to (15) and the corresponding solution $\boldsymbol{u} = \boldsymbol{v}[\boldsymbol{\psi}]$ to (21). Then, with the analogous meaning of $\boldsymbol{a}_j$ and $\boldsymbol{v}[\boldsymbol{\psi}_j]$, in place of (17) and (18), we get

$$\begin{aligned} \int_{\Omega\cap B_R} \boldsymbol{v}[\boldsymbol{\psi}_j] \cdot \boldsymbol{\phi} = -\int_{\partial\Omega} \boldsymbol{a}_j \cdot \mathbb{C}[\nabla \boldsymbol{z}]\boldsymbol{n} + \int_{\partial B_R} \boldsymbol{v}[\boldsymbol{\psi}_j] \cdot \mathbb{C}[\nabla \boldsymbol{z}]\boldsymbol{e}_R \\ - \int_{\Omega\cap B_R} \nabla \boldsymbol{v}[\boldsymbol{\psi}_j] \cdot \mathbb{C}[\nabla \boldsymbol{z}] \end{aligned} \tag{25}$$

and

$$0 = \int_{\partial B_R} \boldsymbol{z} \cdot \mathbb{C}[\nabla \boldsymbol{v}[\boldsymbol{\psi}_j]]\boldsymbol{e}_R - \int_{\Omega\cap B_R} \nabla \boldsymbol{z} \cdot \mathbb{C}[\nabla \boldsymbol{v}[\boldsymbol{\psi}_j]], \tag{26}$$

where B_R is a ball of sufficiently large radius R containing $\partial\Omega$ and $\boldsymbol{e}_R$ is the unit normal to its boundary ∂B_R. By virtue of (2), we obtain

$$\begin{aligned} \int_{\Omega\cap B_R} \boldsymbol{v}[\boldsymbol{\psi}_j] \cdot \boldsymbol{\phi} = -\int_{\partial\Omega} \boldsymbol{a}_j \cdot \mathbb{C}[\nabla \boldsymbol{z}]\boldsymbol{n} + \int_{\partial B_R} \boldsymbol{v}[\boldsymbol{\psi}_j] \cdot \mathbb{C}[\nabla \boldsymbol{z}]\boldsymbol{e}_R \\ - \int_{\partial B_R} \boldsymbol{z} \cdot \mathbb{C}[\nabla \boldsymbol{v}[\boldsymbol{\psi}_j]]\boldsymbol{e}_R. \end{aligned} \tag{27}$$

Taking into account the asymptotic behavior of $\boldsymbol{v}[\boldsymbol{\psi}]$ and $\boldsymbol{z}$, we obtain the thesis by first letting $R \to +\infty$, and then $j \to +\infty$. $\square$

4. Conclusions

In this paper, existence and uniqueness theorems for the displacement problem of linear elastostatics with singular data are proved for three-dimensional bounded and exterior domains of class C^k $(k \geq 2)$. The difficulty of the problem lies in defining the attainability of the boundary datum, which belongs to a space of non-regular fields (namely, $W^{2-k-1/q,q}(\partial\Omega)$, $q \in (1, +\infty)$). The proofs of the theorems make use of the theory of layer integral equations, of the existence and uniqueness results for regular data and of the analysis of the trace operator associated to the simple layer potentials.

As far as the two-dimensional case is concerned, the situation is more involved (also for regular data) because of the behavior of the fundamental solution ($\boldsymbol{U}(x-y) = O(\ln(|x-y|))$. As pointed out in [15] (see also [16]), in this case, the search for a solution in the form of a simple layer potential $v[\boldsymbol{\psi}]$ could not lead to existence and uniqueness for degenerate-scale problems. To overcome this difficulty, one may search for the solution in the form of a sum $v[\boldsymbol{\psi}] + \boldsymbol{c}$, with $\boldsymbol{c}$ constant and $\int_{\partial\Omega} \boldsymbol{\psi} = \boldsymbol{0}$ [15]. This could be the starting point for further research into existence and uniqueness with singular data in two-dimensional domains.

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