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Review

Differential Equations for Classical and Non-Classical Polynomial Sets: A Survey †

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† Dedicated to Prof. Dr. Hari M. Srivastava.

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Abstract: By using the monomiality principle and general results on Sheffer polynomial sets, the differential equation satisfied by several old and new polynomial sets is shown.

Keywords: differential equations; Sheffer polynomial sets; generating functions; monomiality principle

MSC: 33C99; 12E10; 11B83

1. Introduction

In this survey article, a uniform method is presented for constructing the differential equations satisfied by several sets of classical and non classical polynomials. This has been done by starting from the basic elements of the relevant generating functions, using the monomiality principle by G. Dattoli [1] and a general result by Y. Ben Cheikh [2]. Of course, the polynomials considered in this paper are only examples for showing that the method works, but obviously this technique can be theoretically extended to every polynomial set.

This method has been recently applied in several articles (see [3–9]), which include works in collaboration with several authors. The most outstanding of them is Prof. Dr. Hari M. Srivastava, to whom this article is dedicated.

The derived differential equations are generally of infinite order, but they reduce to finite order when applied to polynomials.

It is worth noting that the differential equations for Sheffer polynomial sets have been studied even with different methods (see [10–13]), but here we use only elements directly connected with the theory of polynomials.

We start recalling, in Section 2, the definitions relevant to Sheffer polynomials, the G. Dattoli monomiality principle, and a general result by Y. Ben Cheikh.

The classical polynomial sets, considered in Section 3, are the Bernoulli, Euler, Genocchi and Mittag–Leffler polynomials. In Section 4, we show some new polynomial sets derived from non-classical generating functions.

2. Sheffer Polynomials

The Sheffer polynomials $\{s_n(x)\}$ are introduced [14] by means of the exponential generating function [15] of the type:

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x)\,\frac{t^n}{n!}\,, \tag{1}$$

where

$$\begin{aligned} A(t) &= \sum_{n=0}^{\infty} a_n\,\frac{t^n}{n!}\,, \qquad (a_0 \neq 0)\,, \\ H(t) &= \sum_{n=0}^{\infty} h_n\,\frac{t^n}{n!}\,, \qquad (h_0 = 0)\,. \end{aligned} \tag{2}$$

According to a different characterization (see [16], p. 18), the same polynomial sequence can be defined by means of the pair $(g(t), f(t))$, where $g(t)$ is an invertible series and $f(t)$ is a delta series:

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} g_n\,\frac{t^n}{n!}\,, \qquad (g_0 \neq 0)\,, \\ f(t) &= \sum_{n=0}^{\infty} f_n\,\frac{t^n}{n!}\,, \qquad (f_0 = 0,\ f_1 \neq 0)\,. \end{aligned} \tag{3}$$

Denoting by $f^{-1}(t)$ the compositional inverse of $f(t)$ (i.e., such that $f\left(f^{-1}(t)\right) = f^{-1}\left(f(t)\right) = t$), the exponential generating function of the sequence $\{s_n(x)\}$ is given by

$$\frac{1}{g[f^{-1}(t)]}\,\exp\left(xf^{-1}(t)\right) = \sum_{n=0}^{\infty} s_n(x)\,\frac{t^n}{n!}\,, \tag{4}$$

so that

$$A(t) = \frac{1}{g[f^{-1}(t)]}\,, \qquad H(t) = f^{-1}(t)\,. \tag{5}$$

When $g(t) \equiv 1$, the Sheffer sequence corresponding to the pair $(1, f(t))$ is called the associated Sheffer sequence $\{\sigma_n(x)\}$ for $f(t)$, and its exponential generating function is given by

$$\exp\left(xf^{-1}(t)\right) = \sum_{n=0}^{\infty} \sigma_n(x)\,\frac{t^n}{n!}\,. \tag{6}$$

A list of known Sheffer polynomial sequences and their associated ones can be found in [17].

Shift Operators and Differential Equation

We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators $\hat{P}$ and $\hat{M}$ such that

$$\hat{P}\,(p_n(x)) = np_{n-1}(x)\,, \qquad \hat{M}\,(p_n(x)) = p_{n+1}(x)\,, \qquad (n = 1, 2, \dots). \tag{7}$$

$\hat{P}$ is called the *derivative* operator and $\hat{M}$ the *multiplication* operator, as they act in the same way as classical operators on monomials.

This definition traces back to a paper by J.F. Steffensen [18] recently improved by G. Dattoli and widely used in several applications [19,20].

Y. Ben Cheikh proved that every polynomial set is quasi-monomial under the action of suitable derivative and multiplication operators. In particular, in the same article, the following result is proved, as a particular case of Corollary 3.2 in [2]:

Theorem 1. *Let $(p_n(x))$ denote a Sheffer polynomial set, defined by the generating function*

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} p_n(x)\,\frac{t^n}{n!}\,, \tag{8}$$

where

$$A(t) = \sum_{n=0}^{\infty} \tilde{a}_n t^n\,, \qquad (\tilde{a}_0 \neq 0)\,, \tag{9}$$

and

$$H(t) = \sum_{n=0}^{\infty} \tilde{h}_n\, t^{n+1}\,, \qquad (\tilde{h}_0 \neq 0)\,. \tag{10}$$

Denoting, as before, by $f(t)$ the compositional inverse of $H(t)$, the Sheffer polynomial set $\{p_n(x)\}$ is quasi-monomial under the action of the operators

$$\hat{P} = f(D_x)\,, \qquad \hat{M} = \frac{A'[f(D_x)]}{A[f(D_x)]} + xH'[f(D_x)]\,, \tag{11}$$

where prime denotes the ordinary derivatives with respect to t.

Furthermore, according to the monomiality principle, the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}\,p_n(x) = n\,p_n(x)\,. \tag{12}$$

3. Differential Equations of Classical Polynomials

3.1. Bernoulli Polynomials

The Bernoulli polynomials are defined by the generating function

$$G(t,x) = \frac{t}{e^t - 1}\,e^{x\,t}\,, \tag{13}$$

so that

$$
\begin{aligned}
A(t) &= \frac{t}{e^t - 1} = \sum_{k=0}^{\infty} b_k \frac{t^k}{k!}, \\
G(t,x) &= \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} = \sum_{k=0}^{\infty} \left[\sum_{h=0}^{k} \binom{k}{h} b_{k-h}\, x^h \right] \frac{t^k}{k!}, \\
B_k(x) &= \sum_{h=0}^{k} \binom{k}{h} b_{k-h}\, x^h,
\end{aligned}
\tag{14}
$$

where b_k are the Bernoulli numbers.

Differential Equation of the $B_k(x)$

Note that, recalling that $B_n(1) = (-1)^n\, b_n$, the following expansion holds:

$$
\begin{aligned}
t\,\frac{A'(t)}{A(t)} &= \frac{e^t - te^t - 1}{e^t - 1} = 1 - \frac{te^t}{e^t - 1} = 1 - \sum_{n=0}^{\infty} B_n(1)\, \frac{t^n}{n!} = \\
&= \sum_{n=1}^{\infty} (-1)^{n+1}\, b_n\, \frac{t^n}{n!}.
\end{aligned}
\tag{15}
$$

The shift operators for the Bernoulli polynomials are given by

$$
\begin{aligned}
\hat{P} &= D_x, \\
\hat{M} &= \frac{e^{D_x} - D_x e^{D_x} - 1}{D_x\,(e^{D_x} - 1)}.
\end{aligned}
\tag{16}
$$

Therefore, by using the factorization method, we find

Theorem 2. *The Bernoulli polynomials* $\{B_n(x)\}$ *satisfy the differential equation*

$$
\left(\frac{e^{D_x} - D_x e^{D_x} - 1}{e^{D_x} - 1} + x\, D_x \right) B_n(x) = n\, B_n(x), \tag{17}
$$

that is

$$
\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, b_k}{k!}\, D_x^k + x\, D_x \right) B_n(x) = n\, B_n(x), \tag{18}
$$

or, in equivalent form:

$$
\left(\sum_{k=1}^{n} \frac{(-1)^{k+1}\, b_k}{k!}\, D_x^k + x\, D_x \right) B_n(x) = n\, B_n(x). \tag{19}
$$

Proof. It is sufficient to expand in series the operator (17). Equation (19) follows because, for any fixed n, the series expansion in Equation (18) reduces to a finite sum when applied to a polynomial of degree n. □

3.2. Euler Polynomials

The Euler polynomials are defined by the generating function

$$G(t,x) = \frac{2}{e^t+1}\, e^{x\,t}\,, \tag{20}$$

so that

$$\begin{aligned} A(t) &= \frac{2}{e^t+1} = \sum_{k=0}^{\infty} e_k\, \frac{t^k}{k!}\,, \\ G(t,x) &= \sum_{k=0}^{\infty} E_k(x)\, \frac{t^k}{k!} = \sum_{k=0}^{\infty} \left[\sum_{h=0}^{k} \binom{k}{h} e_{k-h}\, x^h \right] \frac{t^k}{k!}\,, \\ E_k(x) &= \sum_{h=0}^{k} \binom{k}{h} e_{k-h}\, x^h\,, \end{aligned} \tag{21}$$

where e_k are the Euler numbers.

Differential Equation of the $E_k(x)$

Note that the following expansion holds:

$$\begin{aligned} \frac{A'(t)}{A(t)} &= -\frac{e^t}{e^t+1} = -1 + \frac{1}{e^t+1} = -1 + \frac{1}{2}\sum_{n=0}^{\infty} e_n\, \frac{t^n}{n!} = \\ &= -\frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} e_n\, \frac{t^n}{n!} = \sum_{n=0}^{\infty} c_n\, \frac{t^n}{n!}\,, \end{aligned} \tag{22}$$

where $c_0 = -1/2$, and $c_n = e_n/2$.

The shift operators for the Euler polynomials are given by

$$\begin{aligned} \hat{P} &= D_x\,, \\ \hat{M} &= -\frac{e^{D_x}}{e^{D_x}+1}\,. \end{aligned} \tag{23}$$

Therefore, by using the factorization method, we find

Theorem 3. *The Euler polynomials $\{E_n(x)\}$ satisfy the differential equation*

$$\left(-\frac{e^{D_x}\, D_x}{e^{D_x}+1} + x\, D_x \right) E_n(x) = n\, E_n(x)\,. \tag{24}$$

that is

$$\left(\sum_{k=0}^{\infty} \frac{c_k}{k!}\, D_x^{k+1} + x\, D_x \right) E_n(x) = n\, E_n(x)\,, \tag{25}$$

or, in equivalent form:

$$\left(\sum_{k=0}^{n-1} \frac{c_k}{k!}\, D_x^{k+1} + x\, D_x \right) E_n(x) = n\, E_n(x)\,. \tag{26}$$

Proof. It is sufficient to expand in series the operator (24). Equation (26) follows because, for any fixed n, the series expansion in Equation (25) reduces to a finite sum when applied to a polynomial of degree n. □

3.3. Genocchi Polynomials

The Genocchi polynomials are defined by the generating function

$$G(t,x) = \frac{2\,t}{e^t+1}\,e^{x\,t}\,, \tag{27}$$

so that

$$\begin{aligned}
A(t) &= \frac{2\,t}{e^t+1} = \sum_{k=0}^{\infty} g_k\,\frac{t^k}{k!}\,, \\
G(t,x) &= \sum_{k=0}^{\infty} G_k(x)\,\frac{t^k}{k!} = \sum_{k=0}^{\infty}\left[\sum_{h=0}^{k}\binom{k}{h} g_{k-h}\,x^h\right]\frac{t^k}{k!}\,, \\
G_k(x) &= \sum_{h=0}^{k}\binom{k}{h} g_{k-h}\,x^h\,,
\end{aligned} \tag{28}$$

where g_k are the Genocchi numbers.

Differential Equation of the $G_k(x)$

Note that the following expansion holds:

$$\begin{aligned}
t\,\frac{A'(t)}{A(t)} &= \frac{e^t - te^t + 1}{e^t+1} = 1 - \frac{te^t}{e^t+1} = 1 - \frac{1}{2}\,t + \frac{1}{2}\sum_{n=2}^{\infty} e_n\,\frac{t^{n+1}}{n!} = \\
&= \sum_{n=0}^{\infty} d_n\,\frac{t^n}{n!}\,,
\end{aligned} \tag{29}$$

where $d_0 = 1$, $d_1 = -1/2$, and $d_k = e_k/2$, $(k \geq 2)$.

The shift operators for the Genocchi polynomials are given by

$$\begin{aligned}
\hat{P} &= D_x\,, \\
\hat{M} &= \frac{e^{D_x} - D_x e^{D_x} + 1}{D_x\,(e^{D_x}+1)}\,,
\end{aligned} \tag{30}$$

so that the Genocchi polynomials satisfy the differential equation

$$\left(\frac{e^{D_x} - D_x e^{D_x} + 1}{e^{D_x}+1} + x\,D_x\right) G_n(x) = n\,G_n(x)\,. \tag{31}$$

Therefore, by using the factorization method, we find

Theorem 4. *The Genocchi polynomials* $\{G_n(x)\}$ *satisfy the differential equation*

$$\left(\frac{e^{D_x} - D_x e^{D_x} + 1}{e^{D_x}+1} + x\,D_x\right) G_n(x) = n\,G_n(x)\,, \tag{32}$$

that is

$$\left(\sum_{k=0}^{\infty} \frac{d_k}{k!} D_x^k + x\, D_x \right) G_n(x) = n\, G_n(x)\,, \tag{33}$$

or, in equivalent form:

$$\left(\sum_{k=0}^{n} \frac{d_k}{k!} D_x^k + x\, D_x \right) G_n(x) = n\, G_n(x)\,. \tag{34}$$

Proof. It is sufficient to expand in series the operator (32). Equation (34) follows because, for any fixed n, the series expansion in Equation (33) reduces to a finite sum when applied to a polynomial of degree n. □

3.4. The Mittag–Leffler Polynomials

We recall that the Mittag–Leffler polynomials [21] are a special case of associated Sheffer polynomials, defined by the generating function

$$\begin{aligned} &A(t) = 1, \qquad H(t) = \log \frac{1+t}{1-t}\,, \\ &G(t,x) = \left(\frac{1+t}{1-t}\right)^x = \exp\left(x\, \log \frac{1+t}{1-t} \right) = \sum_{n=0}^{\infty} \mathcal{M}_n(x)\, \frac{t^n}{n!}\,. \end{aligned} \tag{35}$$

Therefore, we have

$$\frac{A'(t)}{A(t)} = 0\,, \qquad H'(t) = \frac{2}{1-t^2}\,, \qquad H^{-1}(t) = f(t) = \frac{e^t - 1}{e^t + 1}\,, \tag{36}$$

so that, for the Mittag-Leffer polynomials, we find the shift operators:

$$\begin{aligned} \hat{P} &= \frac{e^{D_x} - 1}{e^{D_x} + 1} = \tanh\left(\frac{D_x}{2}\right)\,, \\ \hat{M} &= x\, \frac{(e^{D_x} + 1)^2}{2\, e^{D_x}} = x\, \left[1 + \cosh(D_x)\right]\,. \end{aligned} \tag{37}$$

3.5. Differential Equation of the $\mathcal{M}_n(x)$

In the present case, according to the identity:

$$[1 + \cosh x]\, \tanh\,(x/2) = \sinh x\,,$$

we can write

$$\hat{M}\, \hat{P} = x\, \frac{e^{2\, D_x} - 1}{2\, e^{D_x}} = x\, \sinh(D_x)\,, \tag{38}$$

so that we have the theorem

Theorem 5. *The Mittag–Leffler polynomials* $\{\mathcal{M}_n(x)\}$ *satisfy the differential equation*

$$x\,\sinh(D_x)\,\mathcal{M}_n(x) = n\,\mathcal{M}_n(x)\,, \tag{39}$$

that is

$$x\sum_{k=0}^{\infty}\frac{D_x^{2k+1}}{(2k+1)!}\,\mathcal{M}_n(x) = n\,\mathcal{M}_n(x)\,, \tag{40}$$

or

$$x\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{D_x^{2k+1}}{(2k+1)!}\,\mathcal{M}_n(x) = n\,\mathcal{M}_n(x)\,, \tag{41}$$

where $\left[\frac{n-1}{2}\right]$ *denotes the* integral part *of* $(n-1)/2$.

Proof. It is sufficient to expand in series the operator (39). Equation (41) follows because, for any fixed n, the series expansion in Equation (40) reduces to a finite sum when applied to a polynomial of degree n. □

4. Differential Equations of Non-Classical Polynomials

4.1. Euler-Type Polynomials

Here, we introduce a Sheffer polynomial set connected with the classical Euler polynomials. Assuming:

$$A(t) = \frac{1}{\cosh t}, \qquad H(t) = \sinh t\,, \tag{42}$$

we consider the Euler-type polynomials $\tilde{E}_n(x)$, defined by the generating function

$$G(t,x) = \frac{1}{\cosh t}\exp\left[x\,\sinh t\right] = \sum_{k=0}^{\infty}\tilde{E}_k(x)\,\frac{t^k}{k!}\,. \tag{43}$$

Note that the Euler numbers are recovered, since we have:

$$G(t,0) = \frac{2}{e^t + e^{-t}} = \sum_{k=0}^{\infty}\tilde{E}_k(0)\,\frac{t^k}{k!}\,, \tag{44}$$

so that $\tilde{E}_n(0) = E_n$.

In what follows, we use the expansions

$$\sinh t = \sum_{k=0}^{\infty}\frac{t^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}\left(\frac{1+(-1)^{k+1}}{2}\right)\frac{t^k}{k!}\,, \tag{45}$$

$$\cosh t = \sum_{k=0}^{\infty}\frac{t^{2k}}{(2k)!} = \sum_{k=0}^{\infty}\left(\frac{1+(-1)^{k}}{2}\right)\frac{t^k}{k!}\,. \tag{46}$$

Note that, in our case, we are dealing with a Sheffer polynomial set, so that, since we have $\psi(t) = e^t$, the operator σ defined by Equation (6) simply reduces to the derivative operator D_x. Furthermore, we have:

$$A(t) = \frac{1}{\cosh t}, \qquad \frac{A'(t)}{A(t)} = -\tanh t\,,$$

$$H(t) = \sinh t = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}\,, \qquad \left(\tilde{h}_k = \left(\frac{1+(-1)^{k+1}}{2}\right)\frac{1}{k!}\right),$$

$$H'(t) = \cosh t\,, \qquad f(t) = H^{-1}(t) = \log(t + \sqrt{t^2+1})\,,$$

so that we have the theorem

Theorem 6. *The Euler-type polynomial set* $\{\tilde{E}_n(x)\}$ *is quasi-monomial under the action of the operators*

$$\hat{P} = \log(D_x + \sqrt{D_x^2+1})\,, \qquad \hat{M} = -\tanh(\operatorname{arcsinh} D_x) + x \operatorname{arcsinh} D_x \tag{47}$$

(by $\operatorname{arcsinh} t = \log(t + \sqrt{t^2+1})$*, we denote the inverse of the function* $\sinh t$*), i.e.,*

$$\hat{P} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{4^k (k!)^2 (2k+1)}\, D_x^{2k+1}\,,$$

$$\hat{M} = -\frac{D_x}{\sqrt{1+D_x^2}} + x\sqrt{1+D_x^2} = (xD_x^2 - D_x + x)(1+D_x^2)^{-1/2}\,, \tag{48}$$

$$\hat{M} = (xD_x^2 - D_x + x) \sum_{k=0}^{\infty} \binom{-1/2}{k} D_x^{2k}\,.$$

There is no problem about the convergence of the above series, since they reduce to finite sums when applied to polynomials.

4.2. Differential Equation of the $\tilde{E}_n(x)$

In the present case, we have

Theorem 7. *The Euler-type polynomials* $\{\tilde{E}_n(x)\}$ *satisfy the differential equation*

$$\begin{aligned}\Bigg\{\Bigg[(xD_x^2 - D_x + x) \sum_{k=0}^{\infty} \binom{-1/2}{k} D_x^{2k}\Bigg] \\ \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{4^k (k!)^2 (2k+1)}\, D_x^{2k+1}\Bigg\}\, \tilde{E}_n(x) = n\, \tilde{E}_n(x)\,,\end{aligned} \tag{49}$$

i.e.,

$$(xD_x^2 - D_x + x) \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \sum_{h=0}^{k} (-1)^h \binom{-1/2}{k-h} \frac{(2h)!}{4^h (h!)^2 (2h+1)} D_x^{2k+1} \tilde{E}_n(x) = n \, \tilde{E}_n(x) \,. \tag{50}$$

Note that, for any fixed n, the Cauchy product of series expansions in Equation (49) reduces to a finite sum, with upper limit $\left[\frac{n-1}{2}\right]$, when it is applied to a polynomial of degree n, because the successive addends vanish.

Remark 1. *The first few Euler-type polynomials are as follows:*

$$\begin{aligned}
&\tilde{E}_0(x) = 1, \\
&\tilde{E}_1(x) = x, \\
&\tilde{E}_2(x) = x^2 - 1, \\
&\tilde{E}_3(x) = x^3 - 2x, \\
&\tilde{E}_4(x) = x^4 - 2x^2 + 5, \\
&\tilde{E}_5(x) = x^5 + 16x, \\
&\tilde{E}_6(x) = x^6 + 5x^4 + 31x^2 - 61, \\
&\tilde{E}_7(x) = x^7 + 14x^5 + 56x^3 - 272x, \\
&\tilde{E}_8(x) = x^8 + 28x^6 + 126x^4 - 692x^2 + 1385, \\
&\tilde{E}_9(x) = x^9 + 48x^7 + 336x^5 - 1280x^3 + 7936x, \\
&\tilde{E}_{10}(x) = x^{10} + 75x^8 + 882x^6 - 1490x^4 + 25,261x^2 - 50,521.
\end{aligned}$$

5. Adjointness for Sheffer Polynomial Sequences

According to the above considerations, Sheffer polynomials are characterized both by the ordered couples $(A(t), H(t))$, or by $(g(t), f(t))$.

Definition 1. *Adjoint Sheffer polynomials are defined by interchanging the ordered couple $(A(t), H(t))$ with $(g(t), f(t))$, when writing the generating function.*

Here and in the following the *tilde* "$\sim$" above the symbol of a polynomial set stands for the adjective "*adjoint*" (see e.g., [4]).

5.1. Adjoint Hahn Polynomials

Assuming:

$$A(t) = \sec t, \qquad H(t) = \tan t \,, \tag{51}$$

we consider the adjoint Hahn $\tilde{R}_n(x)$, defined by the generating function

$$G(t,x) = \sec t \, \exp(x \tan t) = \sum_{n=0}^{\infty} \tilde{R}_n(x) \frac{t^n}{n!} \,. \tag{52}$$

It is a Sheffer set.

We have:

$$\frac{\partial G}{\partial x} = \frac{1}{\cos^3 t} \exp(x \tan t) = \frac{1}{\cos t} G(t, x) \,.$$

Note that, in this case, we have:

$$A(t) = \sec t \,, \qquad H(t) = \tan t \,,$$

$$H'(t) = \sec^2 t \,, \qquad f(t) = H^{-1}(t) = \arctan t \,,$$

$$\frac{A'(t)}{A(t)} = \tan t \,,$$

so that we have the theorem

Theorem 8. *The adjoint Hahn polynomial set* $\{\tilde{R}_n(x)\}$ *is quasi-monomial under the action of the operators*

$$\begin{aligned} \hat{P} &= \arctan D_x \,, \\ \hat{M} &= \tan(\arctan D_x) + x \sec^2(\arctan D_x) \,, \end{aligned} \tag{53}$$

i.e.,

$$\begin{aligned} \hat{P} &= \arctan D_x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} D_x^{2k+1} \,, \\ \hat{M} &= D_x + x\,(1 + D_x^2) = x\,D_x^2 + D_x + x. \end{aligned} \tag{54}$$

5.2. Differential Equation of the $\tilde{R}_n(x)$

In the present case, we have

Theorem 9. *The Sheffer-type adjoint Hahn polynomials* $\{\tilde{R}_n(x)\}$ *satisfy the differential equation*

$$\left(x\,D_x^2 + D_x + x\right) \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^k}{2k+1} D_x^{2k+1}\,\tilde{R}_n(x) = n\,\tilde{R}_n(x) \,. \tag{55}$$

Note that, for any fixed n, in Equation (55), a finite sum appears, with upper limit $\left[\frac{n-1}{2}\right]$, instead of a complete series expansion since, when this series is applied to a polynomial of degree n, the subsequent addends vanish.

Remark 2. *The first few values of the adjoint Hahn polynomials are as follows:*

$$\begin{aligned}
&\tilde{R}_0(x) = 1,\\
&\tilde{R}_1(x) = x,\\
&\tilde{R}_2(x) = x^2 + 1,\\
&\tilde{R}_3(x) = x^3 + 5x,\\
&\tilde{R}_4(x) = x^4 + 14x^2 + 5,\\
&\tilde{R}_5(x) = x^5 + 30x^3 + 61x,\\
&\tilde{R}_6(x) = x^6 + 55x^4 + 331x^2 + 61,\\
&\tilde{R}_7(x) = x^7 + 91x^5 + 1211x^3 + 1385x,\\
&\tilde{R}_8(x) = x^8 + 140x^6 + 3486x^4 + 12,284x^2 + 1385,\\
&\tilde{R}_9(x) = x^9 + 204x^7 + 8526x^5 + 68,060x^3 + 50,521x,\\
&\tilde{R}_{10}(x) = x^{10} + 285x^8 + 18,522x^6 + 281,210x^4 + 663,061x^2 + 50,521.
\end{aligned}$$

Remark 3. *Table of adjoint Hahn numbers*

$$\begin{array}{lll}
\tilde{R}_0(0) = 1 & \tilde{R}_1(0) = 0 & \tilde{R}_2(0) = 1,\\
\tilde{R}_3(0) = 0 & \tilde{R}_4(0) = 5 & \tilde{R}_{2k+1}(0) = 0\,,\ \forall k \geq 2,\\
\tilde{R}_6(0) = 61 & \tilde{R}_8(0) = 1385 & \tilde{R}_{10}(0) = 50,521.
\end{array}$$

Note that the sequence $\{1, 1, 5, 61, 1385, 50,521, \ldots\}$ appears in the Encyclopedia of Integer Sequences [22] under #A000364—Euler (or secant numbers): $a(n)$ = number of downup permutations of $[2n]$.

Example 1. *$a(2)$ = 5 counts 4231, 4132, 3241, 3142, 2143. - David Callan, Nov 21, 2011.*

5.3. Adjoint Bernoulli Polynomials of the Second Kind

Assuming

$$A(t) = \frac{t}{e^t - 1}, \qquad H(t) = e^t - 1\,, \tag{56}$$

we consider the adjoint Bernoulli polynomials of the second kind $\{\tilde{b}_k(x)\}$, defined by the generating function

$$G(t,x) = \frac{t}{e^t - 1} \exp\left[x(e^t - 1)\right] = \sum_{k=0}^{\infty} \tilde{b}_k(x)\, \frac{t^k}{k!}\,. \tag{57}$$

Note that, in our case, we are dealing with a Sheffer polynomial set, so that, since we have $\psi(t) = e^t$, the operator σ defined by Equation (6) simply reduces to the derivative operator D_x. Furthermore, we have

$$A(t) = \frac{t}{e^t - 1}, \qquad H(t) = e^t - 1 = \sum_{k=1}^{\infty} \frac{t^k}{k!}\,, \qquad \left(\tilde{h}_k = 1/(k+1)!\right)\,,$$

$$H'(t) = e^t\,, \qquad f(t) = H^{-1}(t) = \log(t+1)\,,$$

$$\frac{A'(t)}{A(t)} = \frac{e^t - te^t - 1}{t(e^t - 1)} = \frac{1}{t} - \frac{1}{e^t - 1} - 1\,.$$

so that we have the theorem

Theorem 10. *The adjoint Bernoulli polynomials of the second kind $\{\tilde{b}_n(x)\}$ are quasi-monomials under the action of the operators*

$$\begin{aligned}\hat{P} &= \log(D_x + 1)\,,\\ \hat{M} &= \frac{1}{\log(D_x + 1)} - \frac{1}{D_x} - 1 + x\,(D_x + 1)\,,\end{aligned} \tag{58}$$

that is

$$\begin{aligned}\hat{P} &= \log(D_x + 1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, D_x^k\,,\\ \hat{M} &= \frac{1}{\log(D_x + 1)} + \left(x - \frac{1}{D_x}\right)\,(D_x + 1)\,.\end{aligned} \tag{59}$$

5.4. Differential Equation of the $\tilde{b}_n(x)$

In the present case, we have

$$\hat{M}\hat{P} = 1 + \left(x - \frac{1}{D_x}\right)\,(D_x + 1)\,\log(D_x + 1), \tag{60}$$

so that we have the theorem

Theorem 11. *The adjoint Bernoulli polynomials of the second kind $\{\tilde{b}_n(x)\}$ satisfy the differential equation*

$$\left[1 + \left(x - \frac{1}{D_x}\right)\,(D_x + 1)\,\log(D_x + 1)\right]\,\tilde{b}_n(x) = n\,\tilde{b}_n(x)\,, \tag{61}$$

that is

$$\left[1 + (xD_x - 1)\,(D_x + 1)\,\sum_{k=0}^{n} \frac{(-1)^k}{k+1}\, D_x^k\right]\,\tilde{b}_n(x) = n\,\tilde{b}_n(x)\,, \tag{62}$$

because, for any fixed n, the series expansion in Equation (61) reduces to a finite sum when it is applied to a polynomial of degree n.

Note that, in this case, due to the presence of the operator D_x^{-1}, it is necessary to consider derivatives up to the order $n + 1$.

Remark 4. *The first few values of the adjoint Bernoulli polynomials of the second kind are as follows:*

$$
\begin{aligned}
\tilde{b}_0(x) &= 1, \\
\tilde{b}_1(x) &= x - \tfrac{1}{2}, \\
\tilde{b}_2(x) &= x^2 + \tfrac{1}{6}, \\
\tilde{b}_3(x) &= x^3 + \tfrac{3}{2}x^2, \\
\tilde{b}_4(x) &= x^4 + 4x^3 + 2x^2 - \tfrac{1}{30}, \\
\tilde{b}_5(x) &= x^5 + \tfrac{15}{2}x^4 + \tfrac{35}{3}x^3 + \tfrac{5}{2}x^2, \\
\tilde{b}_6(x) &= x^6 + 12x^5 + \tfrac{75}{2}x^4 + 30x^3 + 3x^2 + \tfrac{1}{42}, \\
\tilde{b}_8(x) &= x^8 + 24x^7 + \tfrac{560}{3}x^6 + 560x^5 + 602x^4 + 168x^3 + 4x^2 - \tfrac{1}{30}, \\
\tilde{b}_{10}(x) &= x^{10} + 40x^9 + \tfrac{1155}{2}x^8 + 3780x^7 + 11,585x^6 + 15,540x^5 + \tfrac{15,125}{2}x^4 + 850x^3 + \\
&\quad + 5x^2 + \tfrac{5}{66}, \\
\tilde{b}_{12}(x) &= x^{12} + 60x^{11} + 1386x^{10} + 15,840x^9 + \tfrac{191,961}{2}x^8 + 307,692x^7 + 493,460x^6 + \\
&\quad + 349,800x^5 + 85,503x^4 + 4092x^3 + 6x^2 - \tfrac{691}{2730}.
\end{aligned}
$$

Note that for $x = 0$ the generating function becomes

$$G(t, 0) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \tilde{b}_n(0)\, \frac{t^n}{n!}\,, \tag{63}$$

so that $\tilde{b}_n(0) = B_n$, namely the nth classical Bernoulli number.

6. Conclusions

In this survey article, it has been shown that the common belonging of some polynomial sets to the Sheffer class allows to construct, in a uniform way, the differential equations they verify. This follows from the fact that it is possible to construct their shift operators, on the basis of general results due to G. Dattoli and Y. Ben Cheikh.

The equations derived in such a way are, in general, of infinite order, but they reduce to finite-order equations when they are applied to polynomials of the considered set. This means that the order of the equation increases with the degree of the polynomial, in a similar way to what happens for the order of the recurrence they verify.

Both classic and other polynomials—the so-called associated Sheffer polynomials—have been examined. In fact, it has been shown that, for the polynomials of the Sheffer class, the differential equation follows from the basic elements of their generating function, in a constructive way, using a simple and general method linked to the monomiality principle.

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