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On Almost b-Metric Spaces and Related Fixed Point Results

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Abstract: In this manuscript, we introduce almost b-metric spaces and prove modifications of fixed point theorems for Reich and Hardy–Rogers type contractions. We present an approach generalizing some fixed point theorems to the case of almost b-metric spaces by reducing almost b-metrics to the corresponding b-metrics. Later, we show that this approach can not work for all kinds of contractions. To confirm this, we present a proof in which the contraction condition is such that it cannot be reduced to corresponding b-metrics.

Keywords: fixed point; Reich contraction; Hardy–Rogers contraction; almost b-metric space

MSC: 46T99; 47H10; 54H25

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1. Introduction

In [1] Filipović and Kukić considered some classical contraction principles of Kannan [2], Reich [3] and Hardy–Rogers [4] in b-metric spaces and rectangular b-metric spaces without the assumption of continuity of the corresponding metric. The fact that a b-metric d need not be continuous must remind us to use caution in the proofs.

As possibly more general forms of the theorems proven in [1], here we further try to, as many authors before, generalize metric spaces. Plenty of generalizations in previous two decades were done. Starting from 1989, b-metric spaces were introduced in [5]. After, partial b-metric spaces [6], metric-like spaces [7] and b-dislocated metric spaces [8] have been given. For related contraction principles in the setting of above spaces, the readers can see [9–19].

As an attempt to continue in that spirit, we initiate the concept of almost b-metric spaces. The motivation of this initiation comes from [20] where Mitrović, George and Hussain introduced almost rectangular b-metric spaces.

2. Preliminaries

Bakhtin in [5] and Czerwik in [21] introduced b-metric spaces as a generalization of standard metric spaces.

Definition 1 (Ref. [5,21])**.** *Let X be a nonempty set and $s \geq 1$. The function $d_b : X \times X \to [0, +\infty)$ is a b-metric if and only if, for all $\chi, \zeta, \sigma \in X$, we have*

(bM1) $d_b(\chi, \zeta) = 0$ *if and only if* $\chi = \zeta$,
(bM2) $d_b(\chi, \zeta) = d_b(\zeta, \chi)$,
(bM3) $d_b(\chi, \sigma) \leq s(d_b(\chi, \zeta) + d_b(\zeta, \sigma))$.

(X, d_b, s) *is said a b-metric space and* $s \geq 1$ *is its coefficient.*

In particular, if $s = 1$ then (X, d) is a standard metric space.

Recall that a sequence $\{\chi_n\}$ in X, b-converges to $\chi \in X$ if and only if $d_b(\chi_n, \chi) \to 0$ as $n \to \infty$. $\{\chi_n\}$ is b-Cauchy if and only if $d_b(\chi_n, \chi_m) \to 0$ as $n, m \to \infty$. If each b-Cauchy sequence is b-convergent in X, then (X, d_b, s) is said to be b-complete.

If in previous definition, we assume that only (bM1) and (bM3) hold, then we denote d_b as d_q and we call (X, d_q, s) a quasi-b-metric space.

In next few lines, we make a brief overview of some well known types of contractions. Let (X, d) be a metric space and $T : X \to X$ be such that

- $d(T\chi, T\zeta) \leq \lambda d(\chi, \zeta)$, $\lambda \in [0, 1)$, a Banach type of contraction;
- $d(T\chi, T\zeta) \leq \lambda \left(d(\chi, T\chi) + d(\zeta, T\zeta)\right)$, $\lambda \in [0, \frac{1}{2})$, a Kannan type of contraction;
- $d(T\chi, T\zeta) \leq \lambda \left(d(\chi, T\zeta) + d(\zeta, T\chi)\right)$, $\lambda \in [0, \frac{1}{2})$, a Chatterjea type of contraction;
- $d(T\chi, T\zeta) \leq \alpha d(\chi, \zeta) + \beta d(\chi, T\chi) + \gamma d(\zeta, T\zeta)$ where $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$, a Reich type of contraction;
- $d(T\chi, T\zeta) \leq \alpha d(\chi, \zeta) + \beta d(\chi, T\chi) + \gamma d(\zeta, T\zeta) + \delta d(\chi, T\zeta) + \mu d(\zeta, T\chi)$ where $\alpha, \beta, \gamma, \delta, \mu \geq 0$ with $\alpha + \beta + \gamma + \delta + \mu < 1$, a Hardy–Rogers type of contraction.

In [1] Filipović and Kukić proved new theorems with additional conditions that are necessary to prove the theorems without assumption of continuity of b-metric. Here, we cite only formulations of those theorems and for the proofs, we refer on [1].

Theorem 1. *Ref. [1] let T be a self-mapping on a complete b-metric space* $(X, d_b, s \geq 1)$ *such that*

$$d_b(T\chi, T\zeta) \leq \lambda d_b(\chi, \zeta) + \mu d_b(\chi, T\chi) + \delta d_b(\zeta, T\zeta),$$

for all $\chi, \zeta \in X$, *where* $\lambda, \mu, \delta \geq 0$ *with* $\lambda + \mu + \delta < 1$ *and*

$$\delta < \frac{1}{s}.$$

Then there is a unique fixed point of T.

Theorem 2. *Ref. [1] let* $(X, d_b, s \geq 1)$ *be a complete b-metric space and* $T : X \to X$ *be a mapping satisfying*

$$d_b(T\chi, T\zeta) \leq a_1 d_b(\chi, \zeta) + a_2 d_b(\chi, T\chi) + a_3 d_b(\zeta, T\zeta) + a_4 d_b(\chi, T\zeta) + a_5 d_b(\zeta, T\chi),$$

for all $\chi, \zeta \in X$, *where* a_1, a_2, a_3, a_4, $a_5 \geq 0$ *are such that* $a_1 + a_2 + a_3 + s(a_4 + a_5) < 1$ *and* $a_1 > 1 - \frac{2}{s}$. *Then T has a unique fixed point.*

In the sequel of this paper, we introduce almost-b-metric spaces and present the related previous theorems in this setting. At the end, we also give some results for different type of contractions, where the proofs cannot be reduced to the corresponding b-metrics.

3. Main Results

In this section, let us firstly introduce the concept of almost-b-metric spaces, as a class of quasi-b-metric spaces with the additional requirement that diminishes a lack of symmetry. We set a demand that existence of the left limit of sequence implies the existence of the right limit (bM2l) or that existence of the right limit of sequence implies the existence of the left limit of the same sequence (bM2r). After that, we introduce a couple of examples of almost-b-metrics and also an example of a quasi-b-metric, which is not an almost-b-metric. Finally, we prove Theorems 1 and 2 with the assumption (bM2left) instead of (bM2).

Definition 2. *Let X be a nonempty set and $s \geq 1$. Let $d_{ab} : X \times X \to [0, +\infty)$ be a function such that for all $\chi, \zeta, \sigma, \chi_n \in X$,*

(bM1) $d_{ab}(\chi, \zeta) = 0$ *iff* $\chi = \zeta$,
(bM2l) $d_{ab}(\chi_n, \chi) \to 0$, $n \to \infty$ *implies* $d_{ab}(\chi, \chi_n) \to 0$, $n \to \infty$,
(bM2r) $d_{ab}(\chi, \chi_n) \to 0$, $n \to \infty$ *implies* $d_{ab}(\chi_n, \chi) \to 0$, $n \to \infty$,
(bM3) $d_{ab}(\chi, \zeta) \leq s(d_{ab}(\chi, \sigma) + d_{ab}(\sigma, \zeta))$.

Then (X, d_{ab}, s) is called an

1. *l-almost-b-metric space if (bM1), (bM2l) and (bM3) hold;*
2. *r-almost-b-metric space if (bM1), (bM2r) and (bM3) hold;*
3. *almost-b-metric space if (bM1), (bM2l), (bM2r) and (bM3) hold.*

In the next two examples, we present two quasi-b-metrics, which are also almost-b-metrics.

Example 1. *Let $X = \{0, 1, 2\}$. Choose $\alpha \geq 2$. Consider the b-metric $d_{ab} : X \times X \to [0, +\infty)$ defined by*

$$d_{ab}(0,0) = d_{ab}(1,1) = d_{ab}(2,2) = 0,$$

$$d_{ab}(1,0) = 1, \quad d_{ab}(0,1) = \frac{3}{2},$$

$$d_{ab}(2,1) = 1, \quad d_{ab}(1,2) = \frac{3}{2},$$

$$d_{ab}(2,0) = \alpha, \quad d_{ab}(0,2) = \alpha + 1.$$

Note that d_{ab} satisfies (bM1), (bM3), (bM2l) and (bM2r) (but not (bM2)). For $\alpha > 2$, the ordinary triangle inequality is not verified. Indeed,

$$d_{ab}(0,2) = \alpha + 1 > 3 = \frac{3}{2} + \frac{3}{2} = d_{ab}(0,1) + d_{ab}(1,2).$$

However, the following is satisfied for all $x, y, z \in X$,

$$d_{ab}(x,y) \leq \frac{\alpha+2}{2}(d_{ab}(x,z) + d_{ab}(z,y)).$$

Example 2. *Let $X = [0, +\infty)$ and define $d_{ab} : X \times X \to [0, +\infty)$ as*

$$d_{ab}(x,y) = \begin{cases} (x-y)^3, & x \geq y \\ 4(y-x)^3, & x < y \end{cases}$$

Then $(X, d_{ab}, 4)$ is an almost b-metric space. (bM1), (bM2l) and (bM2r) are obvious. It remains to prove that for all $x, y, z, \in X$,

$$d_{ab}(x,y) \leq 4(d_{ab}(x,z) + d_{ab}(z,y)).$$

Case 1. *$x \geq y$ and $d_{ab}(x,y) = (x-y)^3$. Starting from the inequality $(\alpha + \beta)^3 \leq 4(\alpha^3 + \beta^3)$, we separate the cases:*

$y \leq z \leq x$:

$$\begin{aligned} d_{ab}(x,y) &= (x-y)^3 = (x-z+z-y)^3 \\ &\leq 4((x-z)^3 + (y-z)^3) = 4(d_{ab}(x,z) + d_{ab}(z,y)), \end{aligned}$$

$z \le y \le x$:
$$\begin{aligned} d_{ab}(x,y) &= (x-y)^3 \le 4((x-z)^3 + (y-z)^3) \\ &\le 4((x-z)^3 + 4(y-z)^3) = 4(d_{ab}(x,z) + d_{ab}(z,y)), \end{aligned}$$

$y \le x \le z$:
$$\begin{aligned} d_{ab}(x,y) &= (x-y)^3 \le 4((x-z)^3 + (z-y)^3) \\ &\le 4(4(z-x)^3 + (z-y)^3) = 4(d_{ab}(x,z) + d_{ab}(z,y)). \end{aligned}$$

Case 2. *$x < y$ and $d_{ab}(x,y) = 4(y-x)^3$. Again, we separate the cases:*

$x \le z \le y$:
$$\begin{aligned} d_{ab}(x,y) &= 4(y-x)^3 = 4(y-z+z-x)^3 \\ &\le 4(4(y-z)^3 + 4(z-x)^3) = 4(d_{ab}(x,z) + d_{ab}(z,y)), \end{aligned}$$

$z \le x \le y$:
$$\begin{aligned} d_{ab}(x,y) &= 4(y-x)^3 \le 4 \cdot 4((y-z)^3 + (z-x)^3) \\ &= 4(4(y-z)^3 + 4(z-x)^3) \\ &\le 4(4(y-z)^3 + (x-z)^3) = 4(d_{ab}(x,z) + d_{ab}(z,y)), \end{aligned}$$

$x \le y \le z$:
$$\begin{aligned} d_{ab}(x,y) &= 4(y-x)^3 \le 4 \cdot 4((y-z)^3 + (z-x)^3) \\ &= 4 \cdot (4(y-z)^3 + 4(z-x)^3) \\ &\le 4((z-y)^3 + 4(z-x)^3) = 4(d_{ab}(x,z) + d_{ab}(z,y)). \end{aligned}$$

In the two previous examples, we constructed an almost-*b*-metric, which is also a quasi-*b* metric. The next example shows that there is a quasi-*b*-metric d_q, that it is not an almost-*b*-metric.

Example 3. *Let $X = \mathbb{R}$ and define $d_q : X \times X \to [0,\infty)$ as*

$$d_q(x,y) = \begin{cases} (x-y)^3, & x \ge y \\ 1, & x < y \end{cases}$$

As in the previous example, (bM3) and (bM1) are obvious. Notice that

$$d_q(\frac{1}{n},0) \to 0,\ n \to \infty \quad but \quad d_q(0,\frac{1}{n}) = 1,$$

so (bM2l) does not hold and it is the same for (bM2r). We conclude that $(X, d_q, 4)$ is a quasi-b-metric space, but it is not an almost-b-metric space.

There are many examples of *b*-metrics that are not continuous. Here, we modify one of such examples in sense that we do not demand symmetry.

Example 4. *Let $A = \mathbb{N} \cup \{\infty\}$ and define $d_q : A \times A \to [0,+\infty)$:*

$$d_q(x,y) = \begin{cases} 0, & x = y \\ \dfrac{1}{x} - \dfrac{1}{y}, & \text{if } x < y \text{ and one of } x \text{ and } y \text{ is odd and the other} \\ & \text{is odd or } \infty \\ \dfrac{1}{2}\left(\dfrac{1}{y} - \dfrac{1}{x}\right), & \text{if } y < x \text{ and one of } x \text{ and } y \text{ is odd and the other} \\ & \text{is odd or } \infty \\ 3, & \text{if one of } x \text{ and } y \text{ is even and the other is even or } \infty \\ 2, & \text{otherwise.} \end{cases}$$

Then $(A, d_q, \frac{3}{2})$ is a quasi-b-metric space (it is also an almost-b-metric space). Note that d_q is not continuous. Indeed, $d_q(2n+1, \infty) \to 0$, when $n \to \infty$. But, $d_q(2n+1, 2) = 2$, while $d_q(\infty, 2) = 3$.

Here, we introduce some basic concepts for almost-*b*-metric spaces. The following notions are quite standard and also valid in quasi-*b*- metric spaces.

Definition 3. *Let (X, d_{ab}, s) be an almost-b-metric space. A sequence $\{\chi_n\}$ in X is said to be*

left-Cauchy *if and only if for each $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $d_{ab}(\chi_n, \chi_m) < \varepsilon$ for all $n \geq m > n_0$, which can be written as $\lim\limits_{n \geq m \to \infty} d_{ab}(\chi_n, \chi_m) = 0$,*

right-Cauchy *if and only if for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ so that $d_{ab}(\chi_n, \chi_m) < \varepsilon$ for all $m \geq n > n_0$, which can be written as $\lim\limits_{m \geq n \to \infty} d(\chi_n, \chi_m) = 0$,*

Cauchy *if and only if for each $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ so that $d_{ab}(\chi_n, \chi_m) < \varepsilon$ for all $n, m > n_0$.*

In a quasi-*b*-metric space, a sequence is Cauchy if and only if it is left-Cauchy and right-Cauchy. The same is satisfied in almost-*b*-metric spaces. An almost-*b*-metric space (X, d_{ab}, s) is left-complete if and only if each left-Cauchy sequence $\{\chi_n\}$ in X satisfies $\lim\limits_{n \to \infty} d_{ab}(\chi_n, \chi) = 0$, right-complete if and only if each right-Cauchy sequence $\{\chi_n\}$ in X satisfies $\lim\limits_{n \to \infty} d_{ab}(\chi, \chi_n) = 0$ and is complete if and only if each Cauchy sequence in X is convergent.

In the next lemma, we will associate a *b*-metric to a given quasi-*b*-metric or an almost-*b*-metric. For some kind of contractions, by virtue of this correlation, the proofs from *b*-metric spaces can easily be translated into quasi-*b*-metric spaces and almost-*b*-metric spaces as their subclass.

Lemma 1. *If (X, d_q, s) is a quasi-b-metric space with $s \geq 1$, then (X, l, s) is a b-metric space, where*

$$l(\chi, \zeta) = \frac{d_q(\chi, \zeta) + d_q(\zeta, \chi)}{2}.$$

Proof. $l(x, y)$ is a *b*-metric.

(bM1) Suppose that $l(x, y) = 0$. Then $\frac{d_q(x,y)+d_q(y,x)}{2} = 0$ and since $d_q(x, y) \geq 0$, we obtain that $d_q(x, y) = d_q(y, x) = 0$ and that is, $x = y$, so we conclude that $l(x, y)$ satisfies (bM1).

(bM2) $l(x, y)$ is symmetric by definition:

$$l(x, y) = \frac{d_q(x, y) + d_q(y, x)}{2} = \frac{d_q(y, x) + d_q(x, y)}{2} = l(y, x).$$

(bM3) For all $x, y, z, \in X$, the following is satisfied:

$$d_q(x, z) \leq s(d_q(x, y) + d_q(y, z)).$$

Simply, by adding the following inequality to the previous

$$d_q(z, x) \leq s(d_q(z, y) + d_q(y, x))$$

and dividing the resulted sum by two, we obtain

$$l(x, z) \leq s(l(x, y) + l(y, z)).$$

□

Remark 1. *If (X, d_{ab}, s) is a complete almost-b-metric space, then from (bM2l) and (bM2r), we conclude that (X, l, s) is a complete b-metric space.*

The following theorems are modifications of Theorems 1 and 2 for quasi-b metric spaces and almost-b-metric spaces. Since almost-b-metric spaces are contained in quasi-b-metric spaces, we denote a metric by d_q.

Theorem 3. *Let (X, d_q, s) be a b-complete quasi-b-metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping such that*

$$d_q(Tx, Ty) \le \lambda d_q(x, y) + \mu d_q(x, Tx) + \delta d_q(Ty, y), \tag{1}$$

for all $x, y, z \in X$, where $\lambda, \mu, \delta \ge 0$ and

$$\lambda + 2 \cdot max\{\mu, \delta\} < 1 \quad and \quad max\{\mu, \delta\} < \frac{1}{s}. \tag{2}$$

Then T has a unique fixed point.

Proof. From Lemma 1, we conclude that (X, l, s) is a complete b-metric space. Further, from (1), the b-metric $l(x, y)$ satisfies:

$$\begin{aligned}
l(Tx, Ty) &= \frac{d_q(Tx, Ty) + d_Q(Ty, Tx)}{2} \\
&\le \frac{1}{2}\left(\lambda d_q(x, y) + \mu d_q(x, Tx) + \delta d_q(Ty, y)\right) \\
&+ \frac{1}{2}\left(\lambda d_q(y, x) + \mu d_q(y, Ty) + \delta d_q(Tx, x)\right) \\
&= \lambda l(x, y) + \frac{1}{2}(\mu d_q(x, Tx) + \delta d_q(Tx, x)) + \frac{1}{2}(\mu d_q(y, Ty) + \delta d_q(Ty, y)) \\
&\le \lambda l(x, y) + \frac{1}{2} \cdot max\{\mu, \delta\}(d_q(x, Tx) + d_q(Tx, x)) \\
&+ \frac{1}{2} \cdot max\{\mu, \delta\}(d_q(y, Ty) + d_q(Ty, y)) \\
&= \lambda l(x, y) + max\{\mu, \delta\} l(x, Tx) + max\{\mu, \delta\} l(y, Ty).
\end{aligned}$$

Now, from Theorem 1, we conclude that T has a unique fixed point. □

In the next result, we propose a Hardy–Rogers type contraction for quasi-b metric spaces and almost-b-metric spaces.

Theorem 4. *Let (X, d_q, s) be a complete quasi-b-metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying*

$$d_q(Tx, Ty) \le a_1 d_q(x, y) + a_2 d_q(x, Tx) + a_3 d_q(Ty, y) + a_4 d_q(x, Ty) + a_5 d_q(Tx, y), \tag{3}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, $a_5 \ge 0$ with $a_1 + 2 \cdot max\{a_2, a_3\} + 2s \cdot max\{a_4, a_5\} < 1$ and $a_1 > max\{0, 1 - \frac{2}{s}\}$. Then T has a unique fixed point.

Proof. From Lemma 1, we conclude that (X, l, s) is a complete b-metric space. Starting from (3), we obtain for any $x, y \in X$,

$$\begin{aligned}
2l(Tx, Ty) &= d_q(Tx, Ty) + d_q(Ty, Tx) \\
&\le a_1 d_q(x, y) + a_1 d_q(y, x) + a_2 d_q(x, Tx) + a_2 d_q(y, Ty) + a_3 d_q(Ty, y) \\
&+ a_3 d_q(Tx, x) + a_4 d_q(x, Ty) + a_4 d_q(y, Tx) + a_5 d_q(Tx, y) + a_5 d_q(Ty, x).
\end{aligned}$$

Further, we get

$$\begin{aligned}
l(Tx,Ty) &\leq a_1 l(x,y) + \frac{1}{2}\left(a_2 d_q(x,Tx) + a_3 d_q(Tx,x)\right) + \frac{1}{2}\left(a_2 d_q(y,Ty) + a_3 d_q(Ty,y)\right) \\
&+ \frac{1}{2}\left(a_4 d_q(x,Ty) + a_5 d_q(Ty,x)\right) + \frac{1}{2}\left(a_4 d_q(y,Tx) + a_5 d_q(Tx,y)\right) \\
&\leq a_1 l(x,y) + max\{a_2,a_3\} l(x,Tx) + max\{a_2,a_3\} l(y,Ty) \\
&+ max\{a_4,a_5\} l(x,Ty) + max\{a_4,a_5\} l(y,Tx).
\end{aligned}$$

From Theorem 2 and conditions from Theorem 4, we conclude that self-mapping T on the complete b-metric space (X,l,s) has an unique fixed point, say x^*. Finally, according to Theorem 2, the result follows. □

It is not difficult to see that Theorems 3 and 4 are also satisfied for $s = 1$. To be specific, then $(X,d,1)$ is a quasi-metric space, (X,l) is a metric space, while condition (2) reduces to the well known condition $\lambda + \mu + \delta < 1$ for Reich type contractions, and similar for Hardy–Rogers type contractions.

The following results slightly differ from previous in a sense that we use properties (bM2l) and (bM2r). Before we state our result, we prove an auxiliary lemma that we use it in the proof. Since the lemma is satisfied in the quasi-b-metric spaces, it is also valid in almost-b-metric spaces, so again we denote it by d_q (having in mind that it is also valid for d_{ab}).

Lemma 2. *Let $\{\chi_n\}$ be a sequence in a quasi-b-metric space $(X,d_q,s\geq 1)$ such that*

$$d_q(\chi_n,\chi_{n+1}) \leq \lambda \cdot d_q(\chi_{n-1},\chi_n), \tag{4}$$

for some $\lambda \in [0,\frac{1}{s})$ and each $n \in N$. Then $\{\chi_n\}$ is a right-Cauchy sequence.

Proof. From (4), we get

$$d_q(\chi_n,\chi_{n+1}) \leq \lambda^n d_q(\chi_0,\chi_1). \tag{5}$$

Let $n,m \in \mathbb{N}$ with $n < m$. Then

$$\begin{aligned}
& d_q\left(\chi_n,\chi_m\right) \\
\leq\; & s\left(d_q\left(\chi_n,\chi_{n+1}\right) + d_q\left(\chi_{n+1},\chi_m\right)\right) \\
=\; & sd_q\left(\chi_n,\chi_{n+1}\right) + sd_q\left(\chi_{n+1},\chi_m\right) \\
\leq\; & sd_q\left(\chi_n,\chi_{n+1}\right) + s^2 d_q\left(\chi_{n+1},\chi_{n+2}\right) + s^2 d_q\left(\chi_{n+2},\chi_m\right) \\
\leq\; & sd_q\left(\chi_n,\chi_{n+1}\right) + s^2 d_q\left(\chi_{n+1},\chi_{n+2}\right) + s^3 d_q\left(\chi_{n+2},\chi_{n+3}\right) + ... \\
+\; & s^{m-n-1} d_q\left(\chi_{m-2},\chi_{m-1}\right) + s^{m-n-1} d_q\left(\chi_{m-1},\chi_m\right) \\
\leq\; & \left[s\lambda^n + s^2\lambda^{n+1} + s^3\lambda^{n+2} + ... + s^{m-n-1}\lambda^{m-2}\right] d_q\left(\chi_0,\chi_1\right) \\
+\; & s^{m-n-1}\lambda^{m-1} d_q\left(\chi_0,\chi_1\right) \\
=\; & s\lambda^n\left(1 + (s\lambda) + (s\lambda)^2 + ... + (s\lambda)^{m-n-2}\right) d_q\left(\chi_0,\chi_1\right) + \frac{(s\lambda)^{m-1}}{s^n} d_q\left(\chi_0,\chi_1\right) \\
\leq\; & \left(\frac{s\lambda^n}{1-s\lambda} + \frac{(s\lambda)^{m-1}}{s^n}\right) d_q\left(\chi_0,\chi_1\right) \to 0 \; (m > n \to \infty).
\end{aligned}$$

Since $s\lambda < 1$, we have

$$d_q\left(\chi_n,\chi_m\right) \to 0, m > n, n \to \infty \text{ or equivalently } \lim_{m>n\to\infty} d_q\left(\chi_n,\chi_m\right) = 0,$$

that is, $\{\chi_n\}$ is right-Cauchy. □

The following result is analogue to Lemma 2 for left- Cauchy sequences.

Lemma 3. *Let $\{\chi_n\}$ be a sequence in a quasi-b-metric space $(X, d_q, s \geq 1)$ such that*

$$d_q(\chi_{n+1}, \chi_n) \leq \lambda \cdot d_q(\chi_n, \chi_{n-1}) \tag{6}$$

for some $\lambda \in [0, \frac{1}{s})$ and each $n \in N$. Then $\{\chi_n\}$ is a left-Cauchy sequence.

Proof. The proof follows the same steps as in Lemma 2, where, starting from (6), the condition (5) is replaced by

$$d_q(\chi_{n+1}, \chi_n) \leq \lambda^n d_q(\chi_1, \chi_0). \tag{7}$$

Let $n, m \in \mathbb{N}$ with $n > m$. Then

$$\begin{aligned}
& d_q(\chi_n, \chi_m) \\
\leq\ & s\left(d_q(\chi_n, \chi_{m+1}) + d_q(\chi_{m+1}, \chi_m)\right) \\
=\ & s d_q(\chi_{m+1}, \chi_m) + s d_q(\chi_n, \chi_{m+1}) \\
\leq\ & s d_q(\chi_{m+1}, \chi_m) + s^2 d_q(\chi_n, \chi_{m+2}) + s^2 d_q(\chi_{m+2}, \chi_{m+1}) \\
\leq\ & s d_q(\chi_{m+1}, \chi_m) + s^2 d(\chi_{m+2}, \chi_{m+1}) + \ldots \\
+\ & s^{n-m-1}\left(d_q(\chi_n, \chi_{n-1}) + d_q(\chi_{n-1}, \chi_{n-2})\right) \\
\leq\ & \left[s\lambda^m + s^2\lambda^{m+1} + s^3\lambda^{m+2} + \ldots + s^{n-m-1}\lambda^{n-2}\right] d_q(\chi_1, \chi_0) \\
+\ & s^{n-m-1}\lambda^{n-1} d_q(\chi_1, \chi_0) \\
=\ & s\lambda^m\left(1 + (s\lambda) + (s\lambda)^2 + \ldots + (s\lambda)^{n-m-2}\right) d_q(\chi_1, \chi_0) + \frac{(s\lambda)^{n-1}}{s^m} d_q(\chi_1, \chi_0) \\
\leq\ & \left(\frac{s\lambda^m}{1 - s\lambda} + \frac{(s\lambda)^{n-1}}{s^m}\right) d_q(\chi_1, \chi_0) \to 0 \ (n > m \to \infty).
\end{aligned}$$

Since $s\lambda < 1$, we conclude that

$$d_q(\chi_n, \chi_m) \to 0, n > m, m \to \infty \text{ or equivalently } \lim_{n > m \to \infty} d_q(\chi_n, \chi_m) = 0,$$

that is, $\{\chi_n\}$ is left-Cauchy. □

Remark 2. *It is not hard to see that Lemma 2 and Lemma 3 hold if $\lambda \in [\frac{1}{s}, 1)$. For details, see Lemma 5 in [22].*

In the proof of the next theorem, we use the assumption (bM2r), hence we state it an almost-b-metric, and so denote the metric by d_{ab}.

Theorem 5. *Let (X, d_{ab}, s) be a right-complete r-almost b-metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying*

$$d_{ab}(Tx, Ty) \leq k \cdot max\{d_{ab}(x, y), d_{ab}(x, Tx), d_{ab}(y, Ty)\}, \tag{8}$$

for all $x, y \in X$, where k is such that $0 \leq k < \frac{1}{s}$. Then T has a unique fixed point.

Proof. At the beginning of the proof, let us consider uniqueness of a possible fixed point. To prove that the fixed point is unique, if it exists, suppose that T has two distinct fixed points $x^*, y^* \in X$. Then we get

$$\begin{aligned} d_{ab}(x^*, y^*) &= d_{ab}(Tx^*, Ty^*) \\ &\leq k \cdot max\{d_{ab}(x^*, y^*), d_{ab}(x^*, Tx^*), d_{ab}(y^*, Ty^*)\} \\ &= kd_{ab}(x^*, y^*) < d_{ab}(x^*, y^*), \end{aligned}$$

which is a contradiction.

For an arbitrary $\chi_0 \in X$, consider the sequence $\chi_n = T\chi_{n-1} = T^n\chi_0$, $n \in \mathbb{N}$. If $\chi_n = \chi_{n+1}$ for some n, then χ_n is the unique fixed point of T. We suppose that $d_{ab}(\chi_n, \chi_{n+1}) > 0$ for all $n \in \mathbb{N}$.

We start from (8) for $d_{ab}(\chi_n, \chi_{n+1})$. Then for any $n \in \mathbb{N}$, we get

$$\begin{aligned} d_{ab}(\chi_n, \chi_{n+1}) &= d_{ab}(T\chi_{n-1}, T\chi_n) \\ &\leq k \cdot max\{d_{ab}(\chi_{n-1}, \chi_n), d_{ab}(\chi_{n-1}, T\chi_{n-1}), d_{ab}(\chi_n, T\chi_n)\} \\ &= k \cdot max\{d_{ab}(\chi_{n-1}, \chi_n), d_{ab}(\chi_{n-1}, \chi_n), d_{ab}(\chi_n, \chi_{n+1})\} \\ &= k \cdot max\{d_{ab}(\chi_{n-1}, \chi_n), d_{ab}(\chi_n, \chi_{n+1})\}. \end{aligned} \tag{9}$$

If $d_{ab}(\chi_{m-1}, \chi_m) \leq d_{ab}(\chi_m, \chi_{m+1})$ for some $m \in \mathbb{N}$, then from (9) we get

$$d_{ab}(\chi_m, \chi_{m+1}) \leq k \cdot d_{ab}(\chi_m, \chi_{m+1}) < d_{ab}(\chi_m, \chi_{m+1})$$

which is a contradiction. So, we have

$$d_{ab}(\chi_n, \chi_{n+1}) \leq k \cdot d_{ab}(\chi_{n-1}, \chi_n) \quad \text{for all} \quad n \in \mathbb{N}. \tag{10}$$

From (10) and Lemma 2 we can easily conclude that for some $n_0 \in \mathbb{N}$,

$$d_{ab}(\chi_n, \chi_m) < \varepsilon$$

for all $m \geq n > n_0$, so $\{\chi_n\}$ is a right-Cauchy sequence.

Since $(X, d_{ab}, s > 1)$ is a right-complete r-almost-b-metric space, we get that the sequence $\{\chi_n\}$ right converges to $x^* \in X$, i.e., $d_{ab}(x, \chi_n) \to 0$ as $n \to \infty$. (bM2r) implies that $d_{ab}(\chi_n, x^*) \to 0$ as $n \to \infty$.

The end of the proof is analogue to the standard case. From (bM3) and (8), we obtain

$$\begin{aligned} \frac{1}{s} d_{ab}(x^*, Tx^*) &\leq d_{ab}(x^*, \chi_{n+1}) + d_{ab}(\chi_{n+1}, Tx^*) \\ &= d_{ab}(x^*, \chi_{n+1}) + d_{ab}(T\chi_n, Tx^*) \\ &\leq d_{ab}(x^*, \chi_{n+1}) + k \cdot max\{d_{ab}(\chi_n, x^*), d_{ab}(\chi_n, T\chi_n), d_{ab}(x^*, Tx^*)\} \\ &\to k \cdot d_{ab}(x^*, Tx^*),\ n \to \infty. \end{aligned}$$

Finally, $x^* = Tx^*$. In the last inequality, we used property (bM2r) to obtain that $d_{ab}(\chi_n, x^*) \to 0$ as $n \to \infty$ and also that $d_{ab}(\chi_n, T\chi_n) = d_{ab}(\chi_n, \chi_{n+1}) \to 0$ as $n \to \infty$ since $\{\chi_n\}$ is a right-Cauchy sequence. □

From the previous theorem, we can draw several corollaries that are analogous to Banach, Kannan and Reich type contraction principles, respectively.

Corollary 1. *Let (X, d_{ab}, s) be a right-complete r-almost b-metric space with coefficient $s > 1$ and $T : X \to X$ be such that*

Banach contraction:

$$d_{ab}(Tx, Ty) \leq k \cdot d_{ab}(x, y)$$

for all $x, y \in X$ *where* $0 \leq k < \frac{1}{s}$.

Kannan contraction:

$$d_{ab}(Tx, Ty) \leq k_1 d_{ab}(x, fx) + k_2 d_{ab}(y, fy)$$

for all $x, y \in X$ *where* $k_1, k_2 \geq 0$ *such that* $k_1 + k_2 < \frac{1}{s}$.

Reich contraction:

$$d_{ab}(Tx, Ty) \leq k_1 d_{ab}(x, y) + k_2 d_{ab}(x, fx) + k_3 d_{ab}(y, fy),$$

for all $x, y \in X$ *where* $k_1, k, k_3 \geq 0$ *such that* $k_1 + k_2 + k_3 < \frac{1}{s}$.

Then T has a unique fixed point.

The next result is analogue to Theorem 5 for left-complete l-almost b-metric spaces.

Theorem 6. *Let* (X, d_{ab}, s) *be a left-complete l-almost b-metric space with* $s > 1$ *and* $T : X \to X$ *be such that*

$$d_{ab}(Tx, Ty) \leq k \cdot max\{d_{ab}(x, y), d_{ab}(Tx, x), d_{ab}(Ty, y)\}, \tag{11}$$

for all $x, y \in X$ *where* $0 \leq k < \frac{1}{s}$. *Then T has a unique fixed point.*

Proof. The uniqueness of a possible fixed point is obtained the same way as in proof of Theorem 5.

For arbitrary $\chi_0 \in X$, consider the sequence $\chi_n = T\chi_{n-1} = T^n\chi_0$, $n \in \mathbb{N}$. If $\chi_n = \chi_{n+1}$ for some n, then χ_n is a unique fixed point of T. Hence, we suppose that $d_{ab}(\chi_{n+1}, \chi_n) > 0$ for all $n \in \mathbb{N}$.

We start from (11) for $d_{ab}(\chi_{n+1}, \chi_n)$. Then for any $n \in \mathbb{N}$, using same considerations as in previous proof, we get

$$\begin{aligned} d_{ab}(\chi_{n+1}, \chi_n) &= d_{ab}(T\chi_n, T\chi_{n-1}) \\ &\leq k \cdot max\{d_{ab}(\chi_n, \chi_{n-1}), d_{ab}(T\chi_n, \chi_n), d_{ab}(T\chi_{n-1}, \chi_{n-1})\} \\ &\leq k \cdot d_{ab}(\chi_n, \chi_{n-1}). \end{aligned} \tag{12}$$

From (12) and Lemma 3, we can easily conclude that for some $n_0 \in \mathbb{N}$,

$$d_{ab}(\chi_n, \chi_m) < \varepsilon$$

for all $n \geq m > n_0$, so $\{\chi_n\}$ is a left-Cauchy sequence.

Since $(X, d_{ab}, s > 1)$ is a left-complete l-almost-b-metric space, we get that the sequence $\{\chi_n\}$ left converges to $x^* \in X$, i.e., $d_{ab}(\chi_n, x^*) \to 0$, $n \to \infty$. (bM2l) implies that $d_{ab}(x^*, \chi_n) \to 0$ as $n \to \infty$.

Finally, from (bM3) and (11), we obtain

$$\begin{aligned} \frac{1}{s} d_{ab}(Tx^*, x^*) &\leq d_{ab}(Tx^*, \chi_{n+1}) + d_{ab}(\chi_{n+1}, x^*) \\ &= d_{ab}(Tx^*, T\chi_n) + d_{ab}(\chi_{n+1}, x^*) \\ &\leq k \cdot max\{d_{ab}(x^*, \chi_n), d_{ab}(Tx^*, x^*), d_{ab}(T\chi_n, \chi_n)\} + d_{ab}(\chi_{n+1}, x^*) \\ &\to k \cdot d_{ab}(Tx^*, x^*),\ n \to \infty, \end{aligned}$$

and so $x^* = Tx^*$. In the last inequality, we used property (bM2l) that implies $d_{ab}(x^*, \chi_n) \to 0$, $n \to \infty$ and also that $d_{ab}(T\chi_n, \chi_n) = d_{ab}(\chi_{n+1}, \chi_n) \to 0, n \to \infty$ since $\{\chi_n\}$ is a left-Cauchy sequence. □

The previous considerations should convince the readers that many generalizations of contraction principles may be obtained in almost-b-spaces, which are introduced here, and present a proper subclass of quasi-b-metric spaces. As another benefit of this paper, we point out the principle applied in Theorems 3 and 4 that elegantly proves some contractions in quasi-b-metric spaces.

Finally, we state some open questions in the context of almost-b-metric spaces (respectively quasi-b-metric spaces). If $s = 1$, we have appropriate unresolved questions in the context of quasi-metric spaces. We present formulations for the case of a right-complete r-almost b-metric space, noting that similar issues remain open in left-complete l-almost b-metric spaces.

Problem 1. *(Generalized Ćirić type contraction of first order) Let* $(X, d_{ab}, s \geq 1)$ *be a right-complete r-almost b-metric space and* $T : X \to X$ *be a mapping satisfying*

$$d_{ab}(Tx, Ty) \leq k \max\left\{d_{ab}(x,y), \frac{d_{ab}(x,Tx) + d_{ab}(y,Ty)}{2s}, \frac{d_{ab}(x,Ty) + d_{ab}(y,Tx)}{2s}\right\},$$

for all $x, y \in X$ *where* $0 \leq k < \frac{1}{s}$. *Then* T *has a unique fixed point.*

Problem 2. *(Generalized Ćirić type contraction of second order) Let* $(X, d_{ab}, s \geq 1)$ *be a right-complete r-almost b-metric space and* $T : X \to X$ *be a mapping satisfying*

$$d_{ab}(Tx, Ty) \leq k \max\left\{d_{ab}(x,y), d_{ab}(x,Tx), d_{ab}(y,Ty), \frac{d_{ab}(x,Ty) + d_{ab}(y,Tx)}{2s}\right\},$$

for all $x, y \in X$ *where* $0 \leq k < \frac{1}{s}$. *Then* T *has a unique fixed point.*

Problem 3. *(Quasicontraction of Ćirić type) Let* $(X, d_{ab}, s \geq 1)$ *be a right-complete r-almost b-metric space and* $T : X \to X$ *be such that*

$$d_{ab}(Tx, Ty) \leq k \max\{d_{ab}(x,y), d_{ab}(x,Tx), d_{ab}(y,Ty), d_{ab}(x,Ty), d_{ab}(y,Tx)\},$$

for all $x, y \in X$ *where* $0 \leq k < \frac{1}{s}$. *Then* T *has a unique fixed point.*

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