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Generalized Hyers–Ulam Stability of the Additive Functional Equation

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Abstract: We will prove the generalized Hyers–Ulam stability and the hyperstability of the additive functional equation $f(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = f(x_1, x_2, \ldots, x_n) + f(y_1, y_2, \ldots, y_n)$. By restricting the domain of a mapping f that satisfies the inequality condition used in the assumption part of the stability theorem, we partially generalize the results of the stability theorems of the additive function equations.

Keywords: additive (Cauchy) equation; additive mapping; Hyers–Ulam stability; generalized Hyers–Ulam stability; hyperstability

MSC: 39B82; 39B5

1. Introduction

In 1940, Ulam [1] gave the question concerning the stability of homomorphisms in a conference of the mathematics club of the University of Wisconsin as follows:

Let $(G, \cdot)$ be a group, and let $(G', \cdot, d)$ be a metric group with the metric d. Given $\delta > 0$, does there exist $\epsilon > 0$ such that if a mapping $h : G \to G'$ satisfies the inequality

$$d(h(xy), h(x)h(y)) \leq \delta$$

for all $x, y \in G$, then there is a homomorphism $H : G \to H$ with

$$d(h(x), H(x)) \leq \epsilon$$

for all $x \in G$?

Next year, the Ulam's conjecture was partially solved by Hyers [2] for the additive functional equation.

Theorem 1. *[2], Let X and Y be Banach spaces. Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon, \quad \forall x, y \in X, \quad \varepsilon : constant.$$

Then, there exists a unique additive mapping

$$A(x+y) = A(x) + A(y),$$

such that $||f(x) - A(x)|| \leq \varepsilon$, where the limit $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$.

Thereafter, this phenomenon has been called the Hyers–Ulam stability.

Theorem 2. *Let X and Y be Banach spaces. Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{1}$$

for all $x, y \in X \backslash \{0\}$, where θ and p are constants with $\theta > 0$ and $p \neq 1$. Then, there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{\theta}{|1 - 2^{p-1}|} \|x\|^p \tag{2}$$

for all $x \in X \backslash \{0\}$.

Theorem 2 is due to Aoki [3] and Rassias [4] for $0 < p < 1$, Gajda [5] for $p > 1$, Hyers [2] for $p = 0$, and Rassias [6] for $p < 0$.

In 1994, Găvruta [7] generalized these results for additive mapping by replacing $\theta(\|x\|^p + \|y\|^p)$ in (1) by a general function $\varphi(x, y)$, which is called the 'generalized Hyers–Ulam stability' in this paper.

In 2001, the term hyperstability was used for the first time probably by G. Maksa and Z. Páles in [8]. However, in 1949, it seems to have created by D. G. Bourgin [9] that the first hyperstability result concerned the ring homomorphisms.

We say that a functional equation $\mathfrak{D}(f) = 0$ is hyperstable if any function f satisfying the equation $\mathfrak{D}(f) = 0$ approximately is a true solution of $\mathfrak{D}(f) = 0$, which is a phenomenon called hyperstability.

The hyperstability results for the additive (Cauchy) equation were investigated by Brzdęk [10,11].

In this paper, let V and W be vector spaces, X be a real normed space, and Y be a real Banach space. We denote the set of natural numbers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$.

For a given mapping $f : V^n \to W$, where V^n denotes $V \times V \times \cdots \times V$, let us consider the additive functional equation

$$f(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = f(x_1, x_2, \ldots, x_n) + f(y_1, y_2, \ldots, y_n), \tag{3}$$

for all $x_i, y_i \in V$ $(i = 1, 2, \ldots, n)$.

Each solution of the additive functional Equation (3) is called an n-variable additive mapping. A typical example for the solutions of Equation (3) is the mapping $f : \mathbb{R}^n \to \mathbb{R}^l$ given by $f(x_1, x_2, \ldots, x_n) = (\sum_{i=1}^n a_{1i}x_i, \sum_{i=1}^n a_{2i}x_i, \ldots, \sum_{i=1}^n a_{li}x_i)$ with real constants a_{ij}.

In this paper, we will prove the generalized Hyers–Ulam stability of the additive functional Equation (3) in the spirit of Găvruta [7], and the hyperstability of the additive functional Equation (3).

2. Main Results

For a given mapping $f : V^n \to W$, we use the following abbreviation:

$$\begin{aligned} Df(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) := {} & f(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ & - f(x_1, x_2, \ldots, x_n) - f(y_1, y_2, \ldots, y_n) \end{aligned}$$

for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$. We need the following lemma to prove main theorems.

Lemma 1. *If a mapping $f : V^n \to W$ satisfies (3) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \backslash \{0\}$, then f satisfies (3) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$.*

Proof. Let $x \in V\backslash\{0\}$ be a fixed element, and let $i \in \{1,2,\ldots,n\}$. For given $x_i, y_i \in V$, let $x_i^{(1)}$, $x_i^{(2)}$, $y_i^{(1)}$, $y_i^{(2)}$ be

$$\begin{aligned}
&x_i^{(1)} = x, x_i^{(2)} = -x, y_i^{(1)} = x, y_i^{(2)} = -x && \text{if } x_i = 0 \text{ and } y_i = 0,\\
&x_i^{(1)} = y_i, x_i^{(2)} = -y_i, y_i^{(1)} = \frac{y_i}{2}, y_i^{(2)} = \frac{y_i}{2} && \text{if } x_i = 0 \text{ and } y_i \neq 0,\\
&x_i^{(1)} = \frac{x_i}{2}, x_i^{(2)} = \frac{x_i}{2}, y_i^{(1)} = x_i, y_i^{(2)} = -x_i && \text{if } x_i \neq 0 \text{ and } y_i = 0,\\
&x_i^{(1)} = \frac{x_i}{2}, x_i^{(2)} = \frac{x_i}{2}, y_i^{(1)} = (k+1)y_i, y_i^{(2)} = -ky_i && \text{if } x_i \neq 0 \text{ and } y_i \neq 0,
\end{aligned}$$

where k is a fixed integer, such that $\frac{x_i}{2} + (k+1)y_i \neq 0$, $\frac{x_i}{2} - ky_i \neq 0$. Then, $x_i^{(1)}, x_i^{(2)}, y_i^{(1)}, y_i^{(2)}, x_i^{(1)} + y_i^{(1)}, x_i^{(2)} + y_i^{(2)} \in V\backslash\{0\}$ and $x_i^{(1)} + y_i^{(1)} + x_i^{(2)} + y_i^{(2)} = x_i + y_i$ for all $i = 1,2,\ldots,n$.

Hence, the equalities $Df(x_1^{(1)}, y_1^{(1)}, \ldots, x_n^{(1)}, y_n^{(1)}) = 0$, $Df(x_1^{(2)}, y_1^{(2)}, \ldots, x_n^{(2)}, y_n^{(2)}) = 0$, $Df(x_1^{(1)}, x_1^{(2)}, x_2^{(1)}, x_2^{(2)}, \ldots, x_n^{(1)}, x_n^{(2)}) = 0$, and $Df(y_1^{(1)}, y_1^{(2)}, y_2^{(1)}, y_2^{(2)}, \ldots, y_n^{(1)}, y_n^{(2)}) = 0$ hold for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$. Since the equality

$$\begin{aligned}
Df(&x_1, y_1, x_2, y_2, \ldots, x_n, y_n)\\
=&Df(x_1^{(1)} + y_1^{(1)}, x_1^{(2)} + y_1^{(2)}, x_2^{(1)} + y_2^{(1)}, x_2^{(2)} + y_2^{(2)}, \ldots, x_n^{(1)} + y_n^{(1)}, x_n^{(2)} + y_n^{(2)})\\
&+ Df(x_1^{(1)}, y_1^{(1)}, x_2^{(1)}, y_2^{(1)}, \ldots, x_n^{(1)}, y_n^{(1)}) + Df(x_1^{(2)}, y_1^{(2)}, x_2^{(2)}, y_2^{(2)}, \ldots, x_n^{(2)}, y_n^{(2)})\\
&- Df(x_1^{(1)}, x_1^{(2)}, x_2^{(1)}, x_2^{(2)}, \ldots, x_n^{(1)}, x_n^{(2)}) - Df(y_1^{(1)}, y_1^{(2)}, y_2^{(1)}, y_2^{(2)}, \ldots, y_n^{(1)}, y_n^{(2)})
\end{aligned}$$

holds for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$, we conclude that f satisfies $Df(x_1, y_1, \ldots, x_n, y_n) = 0$ for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$. □

Thereafter, let $i \in \{1,2,3,\ldots,n\}$. For a given element $(x_1, x_2, \ldots, x_n) \neq (0,0,\ldots,0)$, we can choose a fixed element $x' \neq 0$, such that $x' \in \{x_1, x_2, \ldots, x_n\}$. Moreover, let $x_i^{(1)}$, $x_i^{(2)} \in V\backslash\{0\}$ be the elements defined by

$$\begin{aligned}
&x_i^{(1)} = x_i, x_i^{(2)} = x_i && \text{if } x_i \neq 0,\\
&x_i^{(1)} = x', x_i^{(2)} = -x' && \text{if } x_i = 0.
\end{aligned} \tag{4}$$

By using Lemma 1, we can prove the following set of stability theorems.

Theorem 3. *Suppose that $f : V^n \to Y$ is a mapping for which there exists a function $\varphi : (V\backslash\{0\})^{2n} \to [0,\infty)$, such that*

$$\sum_{m=0}^{\infty} \frac{\varphi(2^m x_1, 2^m y_1, 2^m x_2, 2^m y_2, \ldots, 2^m x_n, 2^m y_n)}{2^m} < \infty \tag{5}$$

and

$$\|Df(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)\| \leq \varphi(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) \tag{6}$$

for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V\backslash\{0\}$. Then, there exists a unique mapping $F : V^n \to Y$ that satisfies

$$DF(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) = 0 \tag{7}$$

for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$ *and*

$$\|f(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)\| \leq \sum_{m=0}^{\infty} \frac{\mu(2^m x_1, 2^m x_2, \ldots, 2^m x_n)}{2^{m+1}} \tag{8}$$

for all $(x_1, x_2, \ldots, x_n) \in V^n \backslash \{(0, 0, \ldots, 0)\}$*, where the function* $\mu : V^n \to \mathbb{R}$ *is defined by*

$$\begin{aligned}
&\mu(x_1, x_2, \ldots, x_n) \\
:=&\varphi\left(x_1^{(1)}, x_1^{(2)}, x_2^{(1)}, x_2^{(2)}, \ldots, x_n^{(1)}, x_n^{(2)}\right) + 2\varphi\left(\frac{x_1^{(1)}}{2}, \frac{x_1^{(2)}}{2}, \frac{x_2^{(1)}}{2}, \frac{x_2^{(2)}}{2}, \ldots, \frac{x_n^{(1)}}{2}, \frac{x_n^{(2)}}{2}\right) \\
&+ \varphi\left(\frac{x_1^{(1)}}{2}, \frac{x_1^{(1)}}{2}, \frac{x_2^{(1)}}{2}, \frac{x_2^{(1)}}{2}, \ldots, \frac{x_n^{(1)}}{2}, \frac{x_n^{(1)}}{2}\right) + \varphi\left(\frac{x_1^{(2)}}{2}, \frac{x_1^{(2)}}{2}, \frac{x_2^{(2)}}{2}, \frac{x_2^{(2)}}{2}, \ldots, \frac{x_n^{(2)}}{2}, \frac{x_n^{(2)}}{2}\right)
\end{aligned}$$

for all $(x_1, x_2, \ldots, x_n) \in V^n \backslash \{(0, 0, \ldots, 0)\}$.

Proof. From the inequality (6) and the equalities

$$\begin{aligned}
&f(2x_1, 2x_2, \ldots, 2x_n) - 2f(x_1, x_2, \ldots, x_n) \\
&\quad= f(2x_1, 2x_2, \ldots, 2x_n) - f\left(x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}\right) - f\left(x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}\right) \\
&\quad\quad - 2f(x_1, x_2, \ldots, x_n) + 2f\left(\frac{x_1^{(1)}}{2}, \frac{x_2^{(1)}}{2}, \ldots, \frac{x_n^{(1)}}{2}\right) + 2f\left(\frac{x_1^{(2)}}{2}, \frac{x_2^{(2)}}{2}, \ldots, \frac{x_n^{(2)}}{2}\right) \\
&\quad\quad + f\left(x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}\right) - 2f\left(\frac{x_1^{(1)}}{2}, \frac{x_2^{(1)}}{2}, \ldots, \frac{x_n^{(1)}}{2}\right) \\
&\quad\quad + f\left(x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}\right) - 2f\left(\frac{x_1^{(2)}}{2}, \frac{x_2^{(2)}}{2}, \ldots, \frac{x_n^{(2)}}{2}\right) \\
&\quad= Df\left(x_1^{(1)}, x_1^{(2)}, x_2^{(1)}, x_2^{(2)}, \ldots, x_n^{(1)}, x_n^{(2)}\right) - 2Df\left(\frac{x_1^{(1)}}{2}, \frac{x_1^{(2)}}{2}, \frac{x_2^{(1)}}{2}, \frac{x_2^{(2)}}{2}, \ldots, \frac{x_n^{(1)}}{2}, \frac{x_n^{(2)}}{2}\right) \\
&\quad\quad + Df\left(\frac{x_1^{(1)}}{2}, \frac{x_1^{(1)}}{2}, \frac{x_2^{(1)}}{2}, \frac{x_2^{(1)}}{2}, \ldots, \frac{x_n^{(1)}}{2}, \frac{x_n^{(1)}}{2}\right) \\
&\quad\quad + Df\left(\frac{x_1^{(2)}}{2}, \frac{x_1^{(2)}}{2}, \frac{x_2^{(2)}}{2}, \frac{x_2^{(2)}}{2}, \ldots, \frac{x_n^{(2)}}{2}, \frac{x_n^{(2)}}{2}\right)
\end{aligned} \tag{9}$$

for all $(x_1, x_2, \ldots, x_n) \in V^n \backslash \{(0, 0, \ldots, 0)\}$, we have

$$\begin{aligned}
&\left\| f(x_1, x_2, \ldots, x_n) - \frac{f(2x_1, 2x_2, \ldots, 2x_n)}{2} \right\| \\
&\leq \left\| Df\left(x_1^{(1)}, x_1^{(2)}, x_2^{(1)}, x_2^{(2)}, \ldots, x_n^{(1)}, x_n^{(2)}\right) \right\| + 2\left\| Df\left(\frac{x_1^{(1)}}{2}, \frac{x_1^{(2)}}{2}, \frac{x_2^{(1)}}{2}, \frac{x_2^{(2)}}{2}, \ldots, \frac{x_n^{(1)}}{2}, \frac{x_n^{(2)}}{2}\right) \right\| \\
&\quad + \left\| Df\left(\frac{x_1^{(1)}}{2}, \frac{x_1^{(1)}}{2}, \frac{x_2^{(1)}}{2}, \frac{x_2^{(1)}}{2}, \ldots, \frac{x_n^{(1)}}{2}, \frac{x_n^{(1)}}{2}\right) \right\| \\
&\quad + \left\| Df\left(\frac{x_1^{(2)}}{2}, \frac{x_1^{(2)}}{2}, \frac{x_2^{(2)}}{2}, \frac{x_2^{(2)}}{2}, \ldots, \frac{x_n^{(2)}}{2}, \frac{x_n^{(2)}}{2}\right) \right\| \\
&\leq \frac{1}{2}\mu(x_1, x_2, \ldots, x_n)
\end{aligned}$$

for all $(x_1, x_2, \ldots, x_n) \in V^n \backslash \{(0,0,\ldots,0)\}$. From the above inequality, we get the (following-4 palces) inequality

$$\begin{aligned}
&\left\| \frac{f\left(2^m x_1, \ldots, 2^m x_n\right)}{2^m} - \frac{f\left(2^{m+m'} x_1, \ldots, 2^{m+m'} x_n\right)}{2^{m+m'}} \right\| \\
&\qquad \leq \sum_{k=m}^{m+m'-1} \left\| \frac{f\left(2^k x_1, \ldots, 2^k x_n\right)}{2^k} - \frac{f\left(2^{k+1} x_1, \ldots, 2^{k+1} x_n\right)}{2^{k+1}} \right\| \\
&\qquad \leq \sum_{k=m}^{m+m'-1} \frac{\mu(2^k x_1, 2^k x_2, \ldots, 2^k x_n)}{2^{k+1}}
\end{aligned} \tag{10}$$

for all $(x_1, x_2, \ldots, x_n) \in V^n \backslash \{(0,0,\ldots,0)\}$ and all positive integers m, m'. Thus, the sequence $\{\frac{f(2^n x_1, \ldots, 2^n x_n)}{2^n}\}_{m \in \mathbb{N}}$ is a Cauchy sequence for all $(x_1, x_2, \ldots, x_n) \in V^n \backslash \{(0,0,\ldots,0)\}$. Since Y is a real Banach space and $\lim_{m\to\infty} \frac{f(2^m 0, 2^m 0, \ldots, 2^m 0)}{2^m} = 0$, we can define a mapping $F : V^n \to Y$ by

$$F\left(x_1, x_2, \ldots, x_n\right) = \lim_{m\to\infty} \frac{f\left(2^m x_1, 2^m x_2, \ldots, 2^m x_n\right)}{2^m}$$

for all $x_1, x_2, \ldots, x_n \in V$. By putting $m = 0$ and by letting $m' \to \infty$ in the inequalities (10), we can obtain the inequalities (8) for all $(x_1, x_2, \ldots, x_n) \in V^n \backslash \{(0,0,\ldots,0)\}$.

From the inequality (6), we can obtain

$$\left\| \frac{Df(2^m x_1, 2^m y_1, 2^m x_2, 2^m y_2, \ldots, 2^m x_n, 2^m y_n)}{2^m} \right\| \leq \frac{\varphi\left(2^m x_1, 2^m y_1, 2^m x_2, \ldots, 2^m x_n, 2^m y_n\right)}{2^m}$$

for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \backslash \{0\}$. Since the right-hand side in the above equality tends to zero as $m \to \infty$, and the equality

$$DF(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) = \lim_{m\to\infty} \frac{Df(2^m x_1, 2^m y_1, 2^m x_2, 2^m y_2, \ldots, 2^m x_n, 2^m y_n)}{2^m}$$

holds, then F satisfies the equality (7) for all $x_1, y_1, \ldots, x_n, y_n \in V \backslash \{0\}$. By Lemma 1, F satisfies the equality (3) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$. If $G : V^n \to Y$ is another n-variable additive mapping that satisfies (8), then we obtain $G(0,0,\ldots,0) = 0 = F(0,0\ldots,0)$ and

$$\begin{aligned}
&\| G\left(x_1, x_2, \ldots, x_n\right) - F\left(x_1, x_2, \ldots, x_n\right) \| \\
&\qquad \leq \left\| \frac{G\left(2^k x_1, 2^k x_2, \ldots, 2^k x_n\right)}{2^k} - \frac{f\left(2^k x_1, 2^k x_2, \ldots, 2^k x_n\right)}{2^k} \right\| \\
&\qquad\quad + \left\| \frac{f\left(2^k x_1, 2^k x_2, \ldots, 2^k x_n\right)}{2^k} - \frac{F\left(2^k x_1, 2^k x_2, \ldots, 2^k x_n\right)}{2^k} \right\| \\
&\qquad \leq \sum_{m=k}^{\infty} \frac{\mu(2^m x_1, 2^m x_2, \ldots, 2^m x_n)}{2^m}
\end{aligned}$$

for all $(x_1, x_2, \ldots, x_n) \in V^n \backslash \{(0,0,\ldots,0)\}$ and all $k \in \mathbb{N}$. Since $\sum_{m=k}^{\infty} \frac{\mu(2^m x_1, 2^m x_2, \ldots, 2^m x_n)}{2^m} \to 0$ as $k \to \infty$, we have $G(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)$ for all $x_1, x_2, \ldots, x_n \in V$. Hence, the mapping F is the unique n-variable additive mapping, as desired. □

The condition $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \backslash \{0\}$ used in the inequality (6) differs from the condition $(x_1, x_2, \ldots, x_n) \neq (0,0,\ldots,0)$ and $(y_1, y_2, \ldots, y_n) \neq (0,0,\ldots,0)$ handled by the other authors. If the function f satisfies the inequality (3.2) for all $(x_1, x_2, \ldots, x_n) \neq (0,0,\ldots,0)$ and $(y_1, y_2, \ldots, y_n) \neq$

$(0,0,\ldots,0)$, then the function f satisfies the inequality (3.2) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \backslash \{0\}$. Therefore, the condition $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \backslash \{0\}$ used in the inequality (3.2) in this paper is a generalization of the conditions used in the inequality (3.2) in the well-known pre-results ([10,11]). This condition will apply until Corollary 1.

Theorem 4. *Suppose that $f : V^n \to Y$ is a mapping for which there exists a function $\varphi : (V \backslash \{0\})^{2n} \to [0, \infty)$ that satisfies*

$$\sum_{i=0}^{\infty} 2^i \varphi \left(\frac{x_1}{2^i}, \frac{y_1}{2^i}, \frac{x_2}{2^i}, \frac{y_2}{2^i}, \ldots, \frac{x_n}{2^i}, \frac{y_n}{2^i} \right) < \infty, \tag{11}$$

and (6) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \backslash \{0\}$. Then, there exists a unique mapping $F : V^n \to Y$ that satisfies (7) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$ and

$$\| f(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n) \| \leq \sum_{m=0}^{\infty} 2^m \mu \left(\frac{x_1}{2^{m+1}}, \frac{x_2}{2^{m+1}}, \ldots, \frac{x_n}{2^{m+1}} \right) \tag{12}$$

for all $(x_1, x_2, \ldots, x_n) \in V^n \backslash \{(0,0,\ldots,0)\}$, where the function $\mu : V^n \to \mathbb{R}$ is defined as Theorem 3.

Proof. By choosing a fixed element $x \in V \backslash \{0\}$, we can obtain

$$\begin{aligned}
\|f(0,0,\ldots,0)\| =& \Bigg\| 2Df \left(\frac{x}{2^m}, \frac{-x}{2^m}, \ldots, \frac{x}{2^m}, \frac{-x}{2^m} \right) - Df \left(\frac{x}{2^{m-1}}, \frac{-x}{2^{m-1}}, \ldots, \frac{x}{2^{m-1}}, \frac{-x}{2^{m-1}} \right) \\
&- Df \left(\frac{x}{2^m}, \frac{x}{2^m}, \ldots, \frac{x}{2^m}, \frac{x}{2^m} \right) - Df \left(\frac{-x}{2^m}, \frac{-x}{2^m}, \ldots, \frac{-x}{2^m}, \frac{-x}{2^m} \right) \Bigg\| \\
\leq& 2\varphi \left(\frac{x}{2^m}, \frac{-x}{2^m}, \ldots, \frac{x}{2^m}, \frac{-x}{2^m} \right) + \varphi \left(\frac{x}{2^{m-1}}, \frac{-x}{2^{m-1}}, \ldots, \frac{x}{2^{m-1}}, \frac{-x}{2^{m-1}} \right) \\
&+ \varphi \left(\frac{x}{2^m}, \frac{x}{2^m}, \ldots, \frac{x}{2^m}, \frac{x}{2^m} \right) + \varphi \left(\frac{-x}{2^m}, \frac{-x}{2^m}, \ldots, \frac{-x}{2^m}, \frac{-x}{2^m} \right) \\
\to& \ 0 \quad \text{as} \quad m \to \infty,
\end{aligned}$$

so $f(0,0,\ldots,0) = 0$. Since the equality (9) holds for all $(x_1, x_2, \ldots, x_n) \in V \backslash \{(0,0,\ldots,0)\}$, the inequality (6) implies the inequality

$$\left\| f(x_1, x_2, \ldots, x_n) - 2f \left(\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_n}{2} \right) \right\| \leq \mu \left(\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_n}{2} \right)$$

for all $(x_1, x_2, \ldots, x_n) \in V^n \backslash \{(0,0,\ldots,0)\}$. From the above inequality, we can also obtain the inequality

$$\begin{aligned}
&\left\| 2^m f \left(\frac{x_1}{2^m}, \frac{x_2}{2^m}, \ldots, \frac{x_n}{2^m} \right) - 2^{m+m'} f \left(\frac{x_1}{2^{m+m'}}, \frac{x_2}{2^{m+m'}}, \ldots, \frac{x_n}{2^{m+m'}} \right) \right\| \\
&\qquad \leq \sum_{k=m}^{m+m'-1} 2^k \mu \left(\frac{x_1}{2^{k+1}}, \frac{x_2}{2^{k+1}}, \ldots, \frac{x_n}{2^{k+1}} \right)
\end{aligned} \tag{13}$$

for all $(x_1, x_2, \ldots, x_n) \in V^n \backslash \{(0,0,\ldots,0)\}$ and all positive integers m, m'. Thus, the sequences $\{2^m f\left(\frac{x_1}{2^m}, \ldots, \frac{x_n}{2^m}\right)\}_{m \in \mathbb{N}}$ is a Cauchy sequence for all $(x_1, \ldots, x_n) \in V^n \backslash \{(0,\ldots,0)\}$. Since $f(0,0,\ldots,0) = 0$ and Y is a real Banach space, we can define a mapping $F : V^n \to Y$ by

$$F(x_1, x_2, \ldots, x_n) = \lim_{m \to \infty} 2^m f \left(\frac{x_1}{2^m}, \frac{x_2}{2^m}, \ldots, \frac{x_n}{2^m} \right)$$

for all $x_1, x_2, \ldots, x_n \in V$. By putting $m = 0$ and by letting $m' \to \infty$ in the inequality (13), we can obtain the inequality (12) for all $(x_1, x_2, \ldots, x_n) \in V^n \backslash \{(0,0,\ldots,0)\}$.

From the inequality (6), we get

$$\left\|2^m Df\left(\frac{x_1}{2^m},\frac{y_1}{2^m},\frac{x_2}{2^m},\frac{y_2}{2^m},\ldots,\frac{x_n}{2^m},\frac{y_n}{2^m}\right)\right\| \le 2^m\varphi\left(\frac{x_1}{2^m},\frac{y_1}{2^m},\frac{x_2}{2^m},\frac{y_2}{2^m},\ldots,\frac{x_n}{2^m},\frac{y_n}{2^m}\right)$$

for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V\backslash\{0\}$. Since the right-hand side in the above equality tends to zero as $m \to \infty$, then F satisfies the equality (7) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V\backslash\{0\}$. By Lemma 1, F satisfies the equality (3) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$. If $G : V^n \to Y$ is another n-variable additive mapping satisfying (12), then we obtain $G(0,0,\ldots,0) = 0 = F(0,0,\ldots,0)$ and

$$\begin{aligned}
&\|G(x_1,x_2,\ldots,x_n) - F(x_1,x_2,\ldots,x_n)\| \\
&\quad\le \left\|2^k G\left(\frac{x_1}{2^k},\frac{x_2}{2^k},\ldots,\frac{x_n}{2^k}\right) - 2^k f\left(\frac{x_1}{2^k},\frac{x_2}{2^k},\ldots,\frac{x_n}{2^k}\right)\right\| \\
&\qquad + \left\|2^k f\left(\frac{x_1}{2^k},\frac{x_2}{2^k},\ldots,\frac{x_n}{2^k}\right) - 2^k F\left(\frac{x_1}{2^k},\frac{x_2}{2^k},\ldots,\frac{x_n}{2^k}\right)\right\| \\
&\quad\le \sum_{m=k}^{\infty} 2^m \mu\left(\frac{x_1}{2^{m+1}},\frac{x_2}{2^{m+1}},\ldots,\frac{x_n}{2^{m+1}}\right) \\
&\quad\to 0 \quad \text{as} \quad k \to \infty
\end{aligned}$$

for all $(x_1, x_2, \ldots, x_n) \in V^n\backslash\{(0,0,\ldots,0)\}$. Hence, the mapping F is the unique n-variable additive mapping, as desired. □

The following corollary follows from Theorems 3 and 4.

Corollary 1. *Let $(X, |||\cdot|||)$ be a normed space, $\theta > 0$, and let p be a real number with $p \neq 1$. Suppose that $f : X^n \to Y$ is a mapping that satisfies*

$$\|Df(x_1,y_1,x_2,y_2,\ldots,x_n,y_n)\| \le \theta(|||x_1|||^p + |||y_1|||^p + |||x_2|||^p + \ldots + |||x_n|||^p + |||y_n|||^p) \tag{14}$$

for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X\backslash\{0\}$. Then, there exists a unique n-variable additive mapping $F : X^n \to Y$, such that

$$\|f(x_1,x_2,\ldots,x_n) - F(x_1,x_2,\ldots,x_n)\| \le \frac{4(2^p+4)n\theta}{2^p|2-2^p|}\max_{x_i\neq 0}\{|||x_i|||^p \ : \ 1 \le i \le n\} \tag{15}$$

for all $(x_1, x_2, \ldots, x_n) \in X^n\backslash\{(0,0,\ldots,0)\}$.

Proof. Put $\varphi(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) := \theta(|||x_1|||^p + |||y_1|||^p + |||x_2|||^p + |||y_2|||^p + \ldots + |||x_n|||^p + |||y_n|||^p)$ for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X\backslash\{0\}$, then $|||x_i^{(1)}|||, |||x_i^{(2)}||| \le \max_{x_i\neq 0}\{|||x_i|||^p \ : \ 1 \le i \le n\}$ for all i from (4). Hence, due to μ of Theorems 3 and 4, we obtain that

$$\begin{aligned}
&\mu(x_1,x_2,\ldots,x_n) \\
&=\varphi\left(x_1^{(1)},x_1^{(2)},x_2^{(1)},x_2^{(2)},\ldots,x_n^{(1)},x_n^{(2)}\right) + 2\varphi\left(\frac{x_1^{(1)}}{2},\frac{x_1^{(2)}}{2},\frac{x_2^{(1)}}{2},\frac{x_2^{(2)}}{2},\ldots,\frac{x_n^{(1)}}{2},\frac{x_n^{(2)}}{2}\right) \\
&\quad+\varphi\left(\frac{x_1^{(1)}}{2},\frac{x_1^{(1)}}{2},\frac{x_2^{(1)}}{2},\frac{x_2^{(1)}}{2},\ldots,\frac{x_n^{(1)}}{2},\frac{x_n^{(1)}}{2}\right) + \varphi\left(\frac{x_1^{(2)}}{2},\frac{x_1^{(2)}}{2},\frac{x_2^{(2)}}{2},\frac{x_2^{(2)}}{2},\ldots,\frac{x_n^{(2)}}{2},\frac{x_n^{(2)}}{2}\right) \\
&\le(2n+\frac{8n}{2^p})\max_{x_i\neq 0}\{|||x_i|||^p \ : \ 1 \le i \le n\}
\end{aligned}$$

for all $(x_1, x_2, \ldots, x_n) \in X^n\backslash\{(0,0,\ldots,0)\}$. Therefore, the inequality (15) can be obtained easily from (8) and (12) in Theorems 3 and 4. □

The following theorem for the hyperstability of n-variable additive functional equation follows from Corollary 1.

Theorem 5. *Let $(X, |||\cdot|||)$ be a normed space and p be a real number with $p < 0$. Suppose that $f : X^n \to Y$ is a mapping that satisfies (14) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X\backslash\{0\}$. Then, f is an n-variable additive mapping itself.*

Proof. By Corollary 1, there exists a unique n-variable additive mapping $F : X^n \to Y$, such that (15) for all $x_1, x_2, \ldots, x_n \in X^n\backslash\{(0,0,\ldots,0)\}$ and $DF(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) = 0$ for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X$.

For a given $(x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)$, let $x' \neq 0$ be a nonzero fixed element in $\{x_1, x_2, \ldots, x_n\}$, and let

$$\begin{aligned} &x_i^{(3)} = (m+1)x_i, x_i^{(4)} = -mx_i && \text{when} \quad x_i \neq 0, \\ &x_i^{(3)} = mx', x_i^{(4)} = -mx' && \text{when} \quad x_i = 0. \end{aligned}$$

Then, we can easily show that $|||x_i^{(3)}|||, |||x_i^{(4)}||| \le m^p \max_{x_i \neq 0}\{|||x_i|||^p \,:\, 1 \le i \le n\}$ for all i from (4). If $(x_1, x_2, \ldots, x_n) \in X\backslash\{(0,0,\ldots,0)\}$, then the equality $f(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)$ follows from the inequalities

$$\begin{aligned} &\| f(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n) \| \\ &\quad = \Big\| Df\left(x_1^{(3)}, x_1^{(4)}, x_2^{(3)}, x_2^{(4)}, \ldots, x_n^{(3)}, x_n^{(4)}\right) - DF\left(x_1^{(3)}, x_1^{(4)}, x_2^{(3)}, x_2^{(4)}, \ldots, x_n^{(3)}, x_n^{(4)}\right) \\ &\qquad + f(x_1^{(3)}, x_2^{(3)}, \ldots, (x_n^{(3)}) + f(x_1^{(4)}, x_2^{(4)}, \ldots, x_n^{(4)}) \\ &\qquad - F(x_1^{(3)}, x_2^{(3)}, \ldots, (x_n^{(3)}) - F(x_1^{(4)}, x_2^{(4)}, \ldots, x_n^{(4)}) \Big\| \\ &\quad \le m^p \cdot 2n\theta \max_{x_i \neq 0}\{|||x_i|||^p \,:\, 1 \le i \le n\} + \left\| f(x_1^{(3)}, x_2^{(3)}, \ldots, x_n^{(3)}) - F(x_1^{(3)}, x_2^{(3)}, \ldots, x_n^{(3)}) \right\| \\ &\qquad + \left\| f(x_1^{(4)}, x_2^{(4)}, \ldots, x_n^{(4)}) - F(x_1^{(4)}, x_2^{(4)}, \ldots, x_n^{(4)}) \right\| \\ &\quad \le m^p \left(1 + \frac{4(2^p+4)}{2^p|2-2^p|}\right) 2n\theta \max_{x_i \neq 0}\{|||x_i|||^p \,:\, 1 \le i \le n\} \end{aligned}$$

as $m \to \infty$. For $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$, if we choose a fixed element of $x \in X\backslash\{0\}$, then the equality $f0, 0, \ldots, 0) = F0, 0, \ldots, 0)$ follows from the inequalities

$$\begin{aligned} &\| f(0, 0, \ldots, 0) - F(0, 0, \ldots, 0) \| \\ &\quad = \| Df(mx, -mx, mx, -mx, \ldots, mx, -mx) - DF(mx, -mx, mx, \ldots, mx, -mx) \\ &\qquad + f(mx, mx, \ldots, mx) + f(-mx, -mx, \ldots, -mx) \\ &\qquad - F(mx, mx, \ldots, mx) - F(-mx, -mx, \ldots, -mx) \| \\ &\quad \le m^p \cdot 2n\theta \|x\|^p + \| f(mx, mx, \ldots, mx) - F(mx, mx, \ldots, mx) \| \\ &\qquad + \| f(-mx, -mx, \ldots, -mx) - F(-mx, -mx, \ldots, -mx) \| \\ &\quad \le m^p \left(1 + \frac{4(2^p+4)}{2^p|2-2^p|}\right) 2n\theta |||x|||^p \end{aligned}$$

as $m \to \infty$. Therefore, f is an n-variable additive mapping itself. □

The following example follows from Theorem 5.

Example 1. *Let* $(\mathbb{R}, |\cdot|)$ *be a normed space with absolute value* $|\cdot|$, $(\mathbb{R}^l, \|\cdot\|)$ *be a Banach space with Euclid norm* $\|\cdot\|$, *and* $p < 0$ *be a real number. Suppose that* $f : \mathbb{R}^n \to \mathbb{R}^l$ *is a continuous mapping such that*

$$\|Df(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)\| \leq \theta(|x_1|^p + |y_1|^p + |x_2|^p + |y_2|^p + \ldots + |x_n|^p + |y_n|^p)$$

for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in \mathbb{R}\backslash\{0\}$. *Then, the mapping* $f : \mathbb{R}^n \to \mathbb{R}^l$ *given by*

$$f(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^{n} a_{1i}x_i, \sum_{i=1}^{n} a_{2i}x_i, \ldots, \sum_{i=1}^{n} a_{li}x_i \right), \tag{16}$$

where $a_{1i}, a_{2i}, \ldots, a_{li}$ *are real constants, indicates that*

$$\begin{aligned} f(1, 0, 0, \ldots, 0) &= (a_{11}, a_{21}, \ldots, a_{l1}), \\ f(0, 1, 0, \ldots, 0) &= (a_{12}, a_{22}, \ldots, a_{l2}), \\ \vdots \quad &\quad \vdots \\ f(0, \ldots, 0, 0, 1) &= (a_{1n}, a_{2n}, \ldots, a_{ln}). \end{aligned}$$

Proof. Since $f : \mathbb{R}^n \to \mathbb{R}^l$ is a continuous n-variable additive mapping by Theorem 5, then the function $f : \mathbb{R}^n \to \mathbb{R}^l$ is given by (16). □

In the following theorems, we replace the domain $(V\backslash\{0\})^{2n}$ of φ and Df in Theorems 3 and 4 with V^{2n}. Then, we can improve the result inequality (8).

Theorem 6. *Suppose that* $f : V^n \to Y$ *is a mapping for which there exists a function* $\varphi : V^{2n} \to [0, \infty)$ *satisfying* (5) *and* (6) *for all* $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$. *Then, there exists a unique mapping* $F : V^n \to Y$, *such that* (7) *for all* $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$ *and*

$$\|f(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)\| \leq \sum_{m=0}^{\infty} \frac{\varphi(2^m x_1, 2^m x_1, 2^m x_2, \ldots, 2^m x_n, 2^m x_n)}{2^{m+1}} \tag{17}$$

for all $x_1, x_2, \ldots, x_n \in V$.

Proof. The equality

$$f(2x_1, 2x_2, \ldots, 2x_n) - 2f(x_1, x_2, \ldots, x_n) = Df(x_1, x_1, x_2, x_2, \ldots, x_n, x_n) \tag{18}$$

for all $x_1, x_2, \ldots, x_n \in V$ and the inequality (6) imply that the inequality

$$\left\| f(x_1, x_2, \ldots, x_n) - \frac{f(2x_1, 2x_2, \ldots, 2x_n)}{2} \right\| \leq \frac{1}{2}\varphi(x_1, x_1, x_2, x_2, \ldots, x_n, x_n)$$

for all $x_1, x_2, \ldots, x_n \in V$. From the above inequality, we can derive the inequalities

$$\begin{aligned} &\left\| \frac{f(2^m x_1, \ldots, 2^m x_n)}{2^m} - \frac{f\left(2^{m+m'} x_1, \ldots, 2^{m+m'} x_n\right)}{2^{m+m'}} \right\| \\ &\qquad \leq \sum_{k=m}^{m+m'-1} \frac{\varphi\left(2^k x_1, 2^k x_1, 2^k x_2, 2^k x_2, \ldots, 2^k x_n, 2^k x_n\right)}{2^{k+1}} \end{aligned} \tag{19}$$

for all $x_1, x_2, \ldots, x_n \in V$ and all positive integers m, m'. The remainder of the proof of this theorem developed after inequality (19) is omitted because it is similar to that of Theorem 3. □

Theorem 7. *Suppose that $f : V^n \to Y$ is a mapping for which there exists a function $\varphi : V^{2n} \to [0,\infty)$ satisfying (11) and (6) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$. Then, there exists a unique mapping $F : V^n \to Y$ that satisfies (7) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$ and*

$$\|f(x_1,\ldots,x_n) - F(x_1,\ldots,x_n)\| \le \sum_{m=0}^{\infty} 2^m \varphi\left(\frac{x_1}{2^{m+1}}, \frac{x_1}{2^{m+1}}, \frac{x_2}{2^{m+1}}, \ldots, \frac{x_n}{2^{m+1}}, \frac{x_n}{2^{m+1}}\right) \tag{20}$$

for all $x_1, x_2, \ldots, x_n \in V$.

Proof. The equality (18) for all $x_1, x_2, \ldots, x_n \in V$ and the inequality (6) imply that the inequality

$$\|f(x_1, x_2, \ldots, x_n) - 2f\left(\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_n}{2}\right)\| \le \varphi\left(\frac{x_1}{2}, \frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_n}{2}, \frac{x_n}{2}\right)$$

for all $x_1, x_2, \ldots, x_n \in V$. From the above inequality, we can derive the inequality

$$\begin{aligned}\left\|2^m f\left(\frac{x_1}{2^m}, \frac{x_2}{2^m}, \ldots, \frac{x_n}{2^m}\right) - 2^{m+m'} f\left(\frac{x_1}{2^{m+m'}}, \frac{x_2}{2^{m+m'}}, \ldots, \frac{x_n}{2^{m+m'}}\right)\right\| \\ \le \sum_{k=m}^{m+m'-1} 2^k \varphi\left(\frac{x_1}{2^{k+1}}, \frac{x_1}{2^{k+1}}, \frac{x_2}{2^{k+1}}, \ldots, \frac{x_n}{2^k}, \frac{x_n}{2^k}\right)\end{aligned} \tag{21}$$

for all $x_1, x_2, \ldots, x_n \in V$ and all positive integers m, m'. The remainder of the proof of this theorem developed after inequality (21) is omitted because it is similar to that of Theorem 4. $\square$

The following corollary follows from Theorems 6 and 7.

Corollary 2. *Let $(X, |||\cdot|||)$ be a normed space and p be a nonnegative real number with $p \neq 1$. Suppose that $f : X^n \to Y$ is a mapping satisfying (14) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X$. Then, there exists a unique n-variable additive mapping $F : X^n \to Y$ such that*

$$\|f(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)\| \le \frac{2\theta}{|2 - 2^p|}\left(|||x_1|||^p + |||x_2|||^p + \ldots + |||x_n|||^p\right) \tag{22}$$

for all $x_1, x_2, \ldots, x_n \in X$.

Proof. By putting $\varphi(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) := \theta(|||x_1|||^p + |||y_1|||^p + |||x_2|||^p + |||y_2|||^p + \cdots + |||x_n|||^p + |||y_n|||^p)$ for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X$, then we easily obtain (22) from (17) and (20) of Theorems 6 and 7. $\square$

3. Conclusions

We obtained two stability results.

Theorems 3 and 4 are the generalized Hyers–Ulam stability for the additive functional Equation (3) on V^n, which is a generalization for the stability of the Cauchy functional equation in papers of Aoki [3], Rassias [4], Gajda [5], Hyers [2], and Găvruta [7].

Theorems 6 and 7 are the hyperstablity of the additive functional Equation (3) on V^n, which is a generalization of the Brzdęk's results [10,11] for the Cauchy functional equation.

If the function f satisfies the inequality (6) for all $(x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)$ and $(y_1, y_2, \ldots, y_n) \neq (0, 0, \ldots, 0)$, then the function f satisfies the inequality (6) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V\backslash\{0\}$. Therefore, the condition $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V\backslash\{0\}$ used in the inequality (3.2) of this paper is a generalization of the conditions used in the inequality (6) in well-known pre-results ([10,11]).

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