Deontic Logics as Axiomatic Extensions of First-Order Predicate Logic: An Approach Inspired by Wolniewicz's Formal Ontology of Situations

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Abstract: The aim of this article is to present a method of creating deontic logics as axiomatic theories built on first-order predicate logic with identity. In the article, these theories are constructed as theories of legal events or as theories of acts. Legal events are understood as sequences (strings) of elementary situations in Wolniewicz's sense. On the other hand, acts are understood as two-element legal events: the first element of a sequence is a choice situation (a situation that will be changed by an act), and the second element of this sequence is a chosen situation (a situation that arises as a result of that act). In this approach, legal rules (i.e., orders, bans, permits) are treated as sets of legal events. The article presents four deontic systems for legal events: AEP, AEPF, AEPOF, AEPOFI. In the first system, all legal events are permitted; in the second, they are permitted or forbidden; in the third, they are permitted, ordered or forbidden; and in the fourth, they are permitted, ordered, forbidden or irrelevant. Then, we present a deontic logic for acts (AAPOF), in which every act is permitted, ordered or forbidden. The theorems of this logic reflect deontic relations between acts as well as between acts and their parts. The direct inspiration to develop the approach presented in the article was the book *Ontology of Situations* by Boguslaw Wolniewicz, and indirectly, Wittgenstein's *Tractatus Logico-Philosophicus*.

Keywords: deontic logic; ontology of situations; semantics of law; formal theory of law; Wittgenstein; Wolniewicz

1. Introduction

Boguslaw Wolniewicz in [1] created a formal ontology of situations. Based on his theory, in [2] I proposed a certain semantics of norms. The approach presented in [2] allows a new understanding of deontic logics, which I would like to present below.

Deontic logics formalize the concepts of obligation, prohibition and permission.

Deontic propositional logics use deontic operators whose arguments are sentences, including compound sentences. Usually, deontic operators are defined similarly to modal (aletic) operators. Following the modal logics, iterations of deontic operators are allowed.

Such an approach seems, at least sometimes, not to be intuitive.

Firstly, it does not seem reasonable to apply deontic operators to any sentences. What is the meaning of the sentence "it is mandatory that Mount Everest is the highest mountain in the world", or "it is forbidden that 2 + 2 = 4"?

Secondly, it is not clear what intuitions regarding obligation, prohibition and permission correspond to compound sentences preceded by deontic operators; for example,

$$O((p \land q) \to q) \to (O(p \land q) \to O q),$$

$$O\ (p \rightarrow (p \lor q)) \rightarrow (O\ p \rightarrow O\ (p \lor q))?$$

Likewise, what would be the meaning of the sentence "it is mandatory (permitted, forbidden) that if Mount Everest is the highest mountain in the world and 2 + 2 = 4 then 2 + 2 = 4"?

Thirdly, it is not clear what intuitions regarding obligation, prohibition and permission correspond to iterated deontic operators; e.g., O P p, O O p. What does the phrase "it is obligatory that it is permitted" or "it is obligatory that it is obligatory" mean?

Fourthly, such an approach is not free of paradoxical consequences; e.g., the widely discussed

$$O\ p \rightarrow O\ (p \lor q)$$

(If it is obligatory to save a drowning person, then it is obligatory to save a drowning person or drink coffee), or

$$F\ p \rightarrow O\ (p \rightarrow q)$$

(If it is forbidden to kill a man, then it is obligatory to rob this man after killing him).

One can find more information on propositional deontic logics and their paradoxical consequences in [3–5].

In turn, deontic logics other than the propositional are often only partially formalized. On the other hand, sometimes non-standard formal means are used. Operators' arguments are people, norms, acts, states of affairs, and sometimes combinations of the aforementioned. Interesting examples of deontic systems built on a modal calculus of names can be found in [4], where sentences such as "x at the moment t can be y", "x at the moment t is obliged to be y", "x at the moment t is allowed to be y", "x at the moment t is forbidden to be y" are considered. The advantage of such deontic systems is that they usually capture more specific properties of deontic modalities than propositional deontic logics allow, although the downside of such deontic systems, in addition to the formalization issues mentioned above, is the lack of a clear concept as to what domain deontic sentences describe.

However, there are also deontic systems based on a previous in-depth analysis of the domain to which deontic modalities relate. For example, in [6], permission, prohibition and obligation are defined in terms of an action system. The author of [6] aptly assumes that deontic modalities do not relate to states of affairs but to actions. It is actions that are prescribed, prohibited or allowed. Deontic logic should therefore be based on an action system. Having a clear concept of the domain of deontic modalities, the author provides his deontic logics with a strong semantic basis. This approach avoids the paradoxes of propositional deontic logics. When assessing this direction of research as the most promising, two points should be noted. First, actions (acts) are not the only events to which deontic modalities relate. A good example is the so-called "consequence crimes": the law does not prohibit an act itself, but prohibits it if it produces certain effects. In this case, a sequence of situations is prohibited, in which an act is only its initial fragment (and "legal causality" occurs between the act and subsequent elements of the sequence). Deontic logic should therefore include a more general concept than the concept of an act, namely the concept of a deontic event (I use the term "legal event" in this sense). Secondly, it should be noted that an act can be a complex act not only as a sequence of simple acts: situations constituting an act may themselves be complex. The act of saving two out of three drowning people consists of rescuing two people and sacrificing the third. It would be good if deontic logic could also describe such relations; that is, the relations between an act and its parts.

Bearing in mind the above, one may be tempted to create deontic logics which achieve the following:

(1)　they shall correspond to the intuitions associated with concepts of obligation, prohibition and permission more accurately than propositional deontic logics do;

(2)　they shall use only standard logical tools and shall be based on a clear concept of the domain of deontic modalities;

(3)　they shall treat acts as a special case of deontic events;

(4)　they shall describe the relations between acts and their parts.

We intend to do this in the following part of this article. At the same time, we want to do this using standard logical means; i.e., means of the first-order predicate logic.

2. Methods

In accordance with Stanislaw Lesniewski's intuitive formalism, formal theories, including logical ones, should have an intuitive interpretation. Of course, the clearer and more intuitive this interpretation is, the better. Therefore, it is worth preceding the selection of axioms and rules of the theory with a careful determination of the domain to be described by this theory.

Such an intuitive interpretation for deontic theories is provided by Boguslaw Wolniewicz's ontology of situations, which is a successful attempt to formally develop the ontology of situations contained in Ludwig Wittgenstein's *Tractatus Logico-Philosophicus*.

In [1], Wolniewicz formally described the logical space from Wittgenstein's *Tractatus*. Wolniewicz creates a mathematical model, namely the structure $< SE, \leq >$, where SE is a non-empty set of objects whose elements he calls "situations" (or "elementary situations"), and $\leq$ is a partial order. Wolniewicz distinguishes in the SE set a set of elementary proper situations (SE") and two inappropriate situations, namely an empty situation (o) and an impossible situation (λ). Wolniewicz assumes that the structure $< SE, \leq >$ is a complete lattice, so each subset of SE has, due to the relation $\leq$, its upper and lower limits. In this lattice, o is the smallest element and λ is the largest; i.e., each elementary situation is contained between the empty situation and the impossible situation. In Wolniewicz's model, every proper elementary situation is an atom or is made of atoms; i.e., elementary situations that cover only the empty situation. The opposites of atoms are possible worlds; i.e., elementary situations that are covered only by the impossible situation. Among the possible worlds, one is singled out as the real world (w_0). A set of elementary situations that are fragments of the real world is the set of real situations, or facts. Other elementary situations are imaginary.

The Wolniewicz's structure above corresponds to a static logical space: reality and alternative worlds at some point of time. Meanwhile, the law orders, bans or permits a situation to be replaced by another; one event shall be followed by another. To reflect these dynamics, in [2], each point of time has a Wolniewicz's structure assigned to it. This way, reality and alternative worlds are represented not by individual possible worlds, but by sequences of possible worlds. Thus, while Wolniewicz's original structure can be compared to a picture of reality and alternative worlds, the elaborate structure resembles a film tape.

This dynamic structure has logical events as its elements. A logical event is a non-empty, finite sequence (string) of proper situations, such that each element of the sequence belongs to a different Wolniewicz's structure. Logical events that meet specific conditions are natural events.

Among natural events, one can distinguish legal events; i.e., natural events subject to legal assessments. In [2], four types of legal events are distinguished:

(1) acts;
(2) multiacts;
(3) indirect acts;
(4) causal events.

The most important of these four types of legal events are acts. They are specific two-element sequences of situations in Wolniewicz's sense: the first element of the sequence is the choice situation (the situation that will be changed by the act), and the second element is the chosen situation (the situation that arises as a result of the act). Multiacts, indirect acts and causal acts are understood in the following way:

$$\textbf{MULTIACT} = \{<x_n, x_{n+1}, x_{n+2}, \dots , x_{n+m} >: \text{for any i from n to } n + m - 1 < x_i, x_{i+1} > \epsilon \textbf{ ACT}\},$$

$$\textbf{INDIRECT\_ACT} = \{<x_n, x_{n+1}, x_{n+2}, \dots , x_{n+m} >: \text{for any i from n to } n + m - 1 < x_i, x_{i+1} > \epsilon \textbf{ ACT}$$
$$\text{or } <x_i, x_{i+1} > \epsilon \textbf{ DET}\},$$

CAUSAL_EVENT = $\{<x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+m} >: < x_n, x_{n+1} > \epsilon$ **ACT** and for any i from n + 1 to
$$n + m - 1 <x_i, x_{i+1} > \epsilon \textbf{ LEG}\},$$

where **ACT** is the set of acts, **DET** is the set of deterministic changes, and **LEG** is the set of changes governed by so called "legal causal relations" (see [7]).

The above can be symbolically represented in the following way:

$$\text{ACTS} \subset \text{LEGAL EVENTS} \subset \text{NATURAL EVENTS} \subset \text{LOGICAL EVENTS}.$$

In this approach, the deontic concepts of obligation, prohibition and permission obtain a clear interpretation in terms of sets: orders, bans and permits are simply sets of legal events. Orders, bans and permits are called "legal rules". To determine any deontic theory, it is sufficient to determine relations between sets of legal events.

3. Results

3.1. First-Order Predicate Logic as the Basis of Deontic Theories

Deontic logics will be constructed below as theories built upon the classical first-order predicate logic with identity.

As a result, the language and grammar of these deontic theories is the language and grammar of classical first-order predicate logic with identity. No additional symbols or grammar rules will be used.

Non-specific axioms and rules of these deontic theories are as follows:

(1)　axioms of classical predicate calculus (including substitutions of axioms of classical propositional calculus);

(2)　axioms for the identity predicate:

    (a)　$\forall x (x = x)$,
    (b)　$\forall x\, y (x = y \leftrightarrow y = x)$,
    (c)　$\forall x\, y\, z (x = y \wedge y = z \rightarrow x = z)$;

(3)　rules of classical predicate calculus (including substitutions of rules of classical propositional calculus).

The deontic theories which we will construct below can be divided into:

(1)　theories of legal events;

(2)　theories of simple acts;

(3)　theories of compound acts.

3.2. Theories of Legal Events

The domain of theories of legal events is the set of events as understood in accordance with Section 2. Thus, all propositions of these theories are propositions about events.

We distinguish five unary predicates:

LEV (x)—read "x is a legal event";
PER (x)—read "x is a permitted event";
FOR (x)—read "x is a forbidden event";
OBL (x)—read "x is an ordered event";
IRR (x)—read "x is an irrelevant event".

The specific axioms of these theories are selected in such a way that they determine the relations between sets of ordered, forbidden and permitted events.

3.2.1. Theory 1: All Legal Events are Permitted (AEP)

Adding one specific axiom to non-specific axioms,
A1. ∀ x (LEV (x) ↔ PER (x)),
We will get a simple deontic theory: AEP.
This corresponds to the following Venn diagram:

PERMITTED EVENTS = LEGAL EVENTS

AEP does not seem interesting from the point of view of logic.

3.2.2. Theory 2: All Legal Events are Either Permitted or Forbidden (AEPF)

By adding two specific axioms to non-specific axioms,
A1. ∀ x (LEV (x) ↔ (PER (x) ∨ FOR (x))),
A2. ⅂∃ x (PER (x) ∧ FOR (x)),
We will get a deontic theory: AEPF.
This corresponds to the following Venn diagram:

PERMITTED EVENTS   FORBIDDEN EVENTS

3.2.3. Theory 3: All Legal Events are Either Permitted or Ordered or Forbidden (AEPOF)

By adding three specific axioms to non-specific axioms,
A1. ∀ x (LEV (x) ↔ (OBL (x) ∨ PER (x) ∨ FOR (x))),
A2. ⅂∃ x (PER (x) ∧ FOR (x)),
A3. ∀ x (OBL (x) → PER (x)),
We will get a deontic theory: AEPOF.
This corresponds to the following Venn diagram:

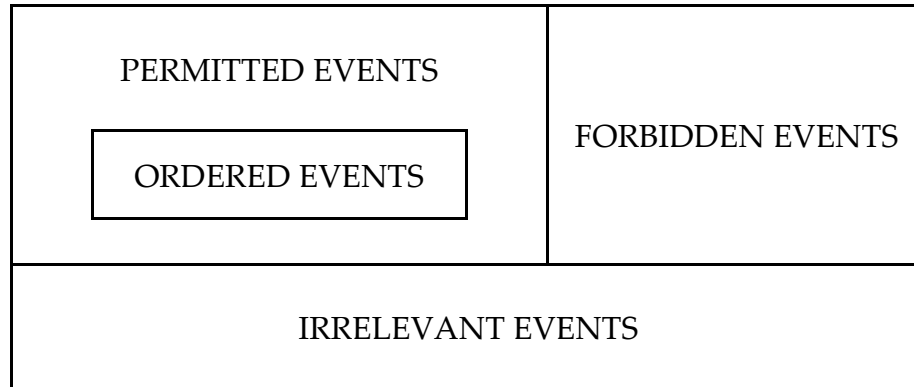
PERMITTED EVENTS

ORDERED EVENTS

FORBIDDEN EVENTS

3.2.4. Theory 4: All Legal Events are Either Permitted or Ordered or Forbidden or Irrelevant (AEPOFI)

By adding five specific axioms to non-specific axioms,
A1. ∀ x (LEV (x) ↔ (OBL (x) ∨ PER (x) ∨ FOR (x) ∨ IRR (x))),
A2. ⅂∃ x (PER (x) ∧ FOR (x)),
A3. ⅂∃ x (IRR (x) ∧ FOR (x)),

A4. ⌐ ∃ x (PER (x) ∧ IRR (x)),
A5. ∀ x (OBL (x) → PER (x)),
We will get a deontic theory: AEPOFI.
This corresponds to the following Venn diagram:

3.2.5. Existence of Legal Events

In the deontic theories set out above, we do not prejudge whether there are legal events. To determine this, a specific axiom should be added to each of these systems:
A0. ∃ x LEV (x).

3.2.6. Selected Theorems of Legal Event Theories

Selected theorems of the theories of legal events are presented below. We omit proofs, because they are quite simple and intuitive.
AEPF, AEPOF, AEPOFI include, in particular, the following theorems:
T1. ∀ x ⌐ (PER (x) ∧ FOR (x));
T2. ∀ x (⌐ PER (x) ∨ ⌐ FOR (x));
T3. ∀ x (PER (x) → ⌐ FOR (x));
T4. ∀ x (FOR (x) → ⌐ PER (x)).
Of course, we also have in AEPOF and AEPOFI the following theorems:
T5. ∀ x (OBL (x) → ⌐ FOR (x));
T6. ∀ x (FOR (x) → ⌐ OBL (x));
T7. ∀ x (⌐ PER (x) → ⌐ OBL (x)).
Theorems T1–T7 have close equivalents in deontic propositional logics.
On the other hand, in AEPF and AEPOF, we have
T8. ∀ x (LEV (x) → (PER (x) ∨ FOR (x)));
T9. ∀ x (LEV (x) → (⌐ PER (x) → FOR (x)));
T10. ∀ x (LEV (x) → (⌐ FOR (x) → PER (x)));
And consequently, we also have
T11. ∀ x (LEV (x) → (PER (x) ↔ ⌐ FOR (x))) which follows from T3, T10;
T12. ∀ x (LEV (x) → (FOR (x) ↔ ⌐ PER (x))) which follows from T4, T9.
Theorems T8–T12 have equivalents in deontic propositional logics. The predecessor of these theorems indicates, however, that the relations described by the successor occur only for legal events and not just for any events.

3.3. Theories of Simple Acts

The domain of the theories of acts is the set of situations as understood in accordance with Section 2 above. Thus, all propositions of these theories are propositions about situations.
We distinguish four binary predicates:

ACT (x, y)—read "replacement x by y is an act";
PER (x, y)—read "replacement x by y is permitted";
FOR (x, y)—read "replacement x by y is forbidden";
OBL (x, y)—read "replacement x by y is ordered".

The specific axioms of these theories are selected in such a way that they determine the relations between sets of ordered, forbidden and permitted acts.

We consider only one such theory below, which is an extension of AEPOF.

3.3.1. Theory: All Acts are Either Permitted or Obligatory or Forbidden (AAPOF)

Every act is a legal event. Thus, the first three AAPOF-specific axioms are the exact counterparts of the AEPOF-specific axioms:

A1. $\forall$ x y (ACT (x, y) $\leftrightarrow$ (OBL (x, y) $\vee$ PER (x, y) $\vee$ FOR (x, y)));

A2. $\neg \exists$ x y (PER (x, y) $\wedge$ FOR (x, y));

A3. $\forall$ x y (OBL (x, y) $\rightarrow$ PER (x, y)).

These three axioms determine the relations between any situations x and y, forming one legal event (i.e., forming a sequence of situations < x, y >).

The next three AAPOF-specific axioms define relations involving three situations, x, y, z, forming two legal events (i.e., forming two sequences of situations: < x, y > and < x, z >).

Axiom A4 states that every act is a choice:

A4. $\forall$ x y (ACT (x, y) $\rightarrow$ $\exists$ z (ACT (x, z) $\wedge$ y $\neq$ z))

(In each choice situation, there are at least two options).

Axiom A5 confirms that the orders are consistent:

A5. $\forall$ x y z (OBL (x, y) $\rightarrow$ (y $\neq$ z $\rightarrow$ FOR (x, z))

(If in a choice situation x, an option y is ordered, then all other options are prohibited in x).

On the other hand, the axiom A6 states that not everything is forbidden:

A6. $\forall$ x y (FOR (x, y) $\rightarrow$ $\exists$ z (ACT (x, z) $\wedge$ y $\neq$ z $\wedge$ $\neg$ FOR (x, z)))

(If in a choice situation x, an option y is forbidden, then some other option is not forbidden in x).

As in the case of the theories of legal events, we do not prejudge whether acts exist. To determine this, it would be necessary to add the specific axiom A0 to AAPOF:

A0. $\exists$ x y ACT (x, y).

(There are choice situations).

3.3.2. Selected Theorems of AAPOF that are Equivalent to Theorems of AEPOF

In AAPOF, we have exact equivalents of theorems T1–T12 of AEPOF:

T1. $\forall$ x y $\neg$ (PER (x, y) $\wedge$ FOR (x, y));

T2. $\forall$ x y ($\neg$ PER (x, y) $\vee$ $\neg$ FOR (x, y));

T3. $\forall$ x y (PER (x, y) $\rightarrow$ $\neg$ FOR (x, y));

T4. $\forall$ x y (FOR (x, y) $\rightarrow$ $\neg$ PER (x, y));

T5. $\forall$ x y (OBL (x, y) $\rightarrow$ $\neg$ FOR (x, y));

T6. $\forall$ x y (FOR (x, y) $\rightarrow$ $\neg$ OBL (x, y));

T7. $\forall$ x y ($\neg$ PER (x, y) $\rightarrow$ $\neg$ OBL (x, y));

T8. $\forall$ x y (ACT (x, y) $\rightarrow$ (PER (x, y) $\vee$ FOR (x, y)));

T9. $\forall$ x y (ACT (x, y) $\rightarrow$ ($\neg$ PER (x, y) $\rightarrow$ FOR (x, y)));

T10. $\forall$ x y (ACT (x, y) $\rightarrow$ ($\neg$ FOR (x, y) $\rightarrow$ PER (x, y)));

T11. $\forall$ x y (ACT (x, y) $\rightarrow$ (PER (x, y) $\leftrightarrow$ $\neg$ FOR (x, y)));

T12. $\forall$ x y (ACT (x, y) $\rightarrow$ (FOR (x, y) $\leftrightarrow$ $\neg$ PER (x, y))).

3.3.3. Selected AAPOF Theorems Specific to Acts

In AAPOF, we also have theorems that do not have their exact counterparts in AEPOF, which are the consequences of adding specific axioms A4–A6 to the system:

T13. ∀ x y z (OBL (x, y) → (y ≠ z → ⌐ PER (x, z)))

(If an option y is ordered in a choice situation x, then no other option is permitted in x);

T14. ∀ x y z (OBL (x, y) → (y ≠ z → ⌐ OBL (x, z)))

(If an option y is ordered in a choice situation x, then no other option is ordered in x);

T15. ∀ x y z (OBL (x, y) ∧ OBL (x, z) → y = z)

(If, in a choice situation, two options are ordered, they are identical);

T16. ∀ x y z (y ≠ z → ⌐ (OBL (x, y) ∧ OBL (x, z)))

(In any choice situation, different options cannot be ordered together);

T17. ∀ x y (FOR (x, y) → ∃ z (y ≠ z ∧ PER (x, z)))

(If an option y is forbidden in a choice situation x, then some other option z is permitted in x);

T18. ∀ x y (OBL (x, y) → ∃ z (y ≠ z ∧ FOR (x, z)))

(If an option y is ordered in a choice situation x, then some other option z is forbidden in x);

T19. ∀ x y z (y ≠ z → (OBL (x, y) → ⌐ PER (x, z)))

(If an option y is ordered in a choice situation x, then no other option is permitted in x);

T20. ∀ x y z (ACT (x, y) ∧ ACT (x, z) ∧ y ≠ z ∧ ∀ w (ACT (x, w) → (w = y ∨ w = z)) → ⌐ (FOR (x, y) ∧ FOR (x, z)))

(If there are exactly two options in a choice situation, both cannot be forbidden);

T21. ∀ x y z (ACT (x, y) ∧ ACT (x, z) ∧ y ≠ z ∧ ∀ w (ACT (x, w) → (w = y ∨ w = z)) → (FOR (x, y) → PER (x, z)))

(If, in a choice situation, there are exactly two options, then if one of them is forbidden, the other is permitted);

T22. ∀ x y z (ACT (x, y) ∧ ACT (x, z) ∧ y ≠ z ∧ ∀ w (ACT (x, w) → (w = y ∨ w = z)) → (PER (x, y) ∨ PER (x, z)))

(If, in a choice situation, there are exactly two options, then at least one of them is permitted);

T23. ∀ x y z (ACT (x, y) ∧ ACT (x, z) ∧ y ≠ z ∧ ∀ w (ACT (x, w) → (w = y ∨ w = z)) → (⌐ PER (x, y) → PER (x, z)))

(If, in a choice situation, there are exactly two options, then if one of them is not permitted, the other is permitted);

T24. ∀ x y z w (FOR (x, y) ∧ ∀ z (FOR (x, z) → y = z) → (ACT (x, w) ∧ w ≠ y → PER (x, w)))

(If, in a choice situation, exactly one option is prohibited, then any other option is permitted).

3.4. Theories of Compound Acts

In deontic propositional logics, deontic operators apply to conjunction or alternative of propositions; for example,

$$O (p \land q) \rightarrow O p \land O q,$$

$$O p \land O q \rightarrow O (p \land q),$$

$$O p \rightarrow O (p \lor q).$$

Such sentences are intended to formalize the intuition that an obligation, prohibition or permission may relate to situations where one is part of the other.

This intuition can be expressed more precisely by developing AAPOF into the theory of compound acts. We do this by adding axioms defining relations between situations, some of which are parts of the others.

To do so, we need to distinguish further one unary predicate "AT (x)", one binary predicate "ε (x, y)" and one ternary predicate "= + (x, y, z)":

AT (x)—read "x is an atomic situation";

ε (x, y)—read "x is a part of y";

= + (x, y, z)—read "x is the sum (composition) of y and z".

Below, we will write "x ε y" instead of "ε (x, y)" and "x = y + z" instead of "= + (x, y, z)".

3.4.1. AAPOF for Compound Acts

First, we will list axioms that will determine when a situation is a part of another situation, when a situation is the sum (composition) of other situations, and when a situation is an atomic situation.

We use Wolniewicz's approach to define the relation of "being a part of":

A7. $\forall\, x\; x\, \varepsilon\, x$;

A8. $\forall\, x\, y\, z\; (x\, \varepsilon\, y \wedge y\, \varepsilon\, z \rightarrow x\, \varepsilon\, z)$;

A9. $\forall\, x\, y\; (x\, \varepsilon\, y \wedge y\, \varepsilon\, x \rightarrow x = y)$.

We also add the A10 axiom for atomic situations:

A10. $\forall\, x\; (\mathrm{AT}\,(x) \leftrightarrow \forall\, y\; (y\, \varepsilon\, x \rightarrow y = x))$

(Every atom is a situation that has no proper parts).

Then, we introduce the sum (composition) of situations:

A11. $x = y + z \leftrightarrow y\, \varepsilon\, x \wedge z\, \varepsilon\, x \wedge \forall\, w\; (\mathrm{AT}\,(w) \rightarrow (w\, \varepsilon\, x \rightarrow (w\, \varepsilon\, y \vee w\, \varepsilon\, z)))$

(A situation x is the sum (composition) of situations y and z, when they are parts of it, and each atom of the situation x is a part of the situation y or a part of the situation z).

Using the concept of a part of situation, we can express the intuition that a part of a situation has the same deontic modality as this situation:

A12. $\forall\, x\, x_1\, y\, y_1\; (x_1\, \varepsilon\, x \wedge y_1\, \varepsilon\, y \rightarrow (\mathrm{OBL}\,(x, y) \rightarrow (\mathrm{ACT}\,(x_1, y_1) \rightarrow \mathrm{OBL}\,(x_1, y_1))))$;

A13. $\forall\, x\, x_1\, y\, y_1\; (x_1\, \varepsilon\, x \wedge y_1\, \varepsilon\, y \rightarrow (\mathrm{PER}\,(x, y) \rightarrow (\mathrm{ACT}\,(x_1, y_1) \rightarrow \mathrm{PER}\,(x_1, y_1))))$;

A14. $\forall\, x\, x_1\, y\, y_1\; (x_1\, \varepsilon\, x \wedge y_1\, \varepsilon\, y \rightarrow (\mathrm{FOR}\,(x, y) \rightarrow (\mathrm{ACT}\,(x_1, y_1) \rightarrow \mathrm{FOR}\,(x_1, y_1))))$.

In turn, using the concept of the sum (composition) of situations, we can express intuition, according to which any situation has the same deontic modality as its parts:

A15. $\forall\, x\, x_1\, x_2\, y\, y_1\, y_2\; (x = x_1 + x_2 \wedge y = y_1 + y_2 \rightarrow (\mathrm{OBL}\,(x_1, y_1) \wedge \mathrm{OBL}\,(x_2, y_2) \rightarrow \mathrm{OBL}\,(x, y)))$;

A16. $\forall\, x\, x_1\, x_2\, y\, y_1\, y_2\; (x = x_1 + x_2 \wedge y = y_1 + y_2 \rightarrow (\mathrm{PER}\,(x_1, y_1) \wedge \mathrm{PER}\,(x_2, y_2) \rightarrow \mathrm{PER}\,(x, y)))$;

A17. $\forall\, x\, x_1\, x_2\, y\, y_1\, y_2\; (x = x_1 + x_2 \wedge y = y_1 + y_2 \rightarrow (\mathrm{FOR}\,(x_1, y_1) \wedge \mathrm{FOR}\,(x_2, y_2) \rightarrow \mathrm{FOR}\,(x, y)))$.

3.4.2. Selected AAPOF Theorems Specific to Compound Acts

The consequences of adopting additional specific axioms A7–A17 include, but are not limited to, the following examples:

T25. $\forall\, x\, x_1\, x_2\, y\, y_1\, y_2\; (x = x_1 + x_2 \wedge y = y_1 + y_2 \rightarrow (\mathrm{OBL}\,(x, y) \rightarrow (\mathrm{ACT}\,(x_1, y_1) \wedge \mathrm{ACT}\,(x_2, y_2) \rightarrow \neg\,(\mathrm{OBL}\,(x_1, y_1) \wedge \mathrm{FOR}\,(x_2, y_2))))$

(If an act is ordered, it is not that one part of it is ordered and the other is forbidden);

T26. $\forall\, x\, x_1\, x_2\, y\, y_1\, y_2\; (x = x_1 + x_2 \wedge y = y_1 + y_2 \rightarrow (\mathrm{OBL}\,(x, y) \rightarrow (\mathrm{ACT}\,(x_1, y_1) \wedge \mathrm{ACT}\,(x_2, y_2) \rightarrow \neg\,(\mathrm{PER}\,(x_1, y_1) \wedge \mathrm{FOR}\,(x_2, y_2))))$

(If an act is ordered, it is not that one part of it is permitted and the other is forbidden);

T27. $\forall\, x\, x_1\, x_2\, y\, y_1\, y_2\; (x = x_1 + x_2 \wedge y = y_1 + y_2 \rightarrow (\mathrm{OBL}\,(x, y) \rightarrow (\mathrm{ACT}\,(x_1, y_1) \wedge \mathrm{ACT}\,(x_2, y_2) \rightarrow \neg\,(\mathrm{FOR}\,(x_1, y_1) \vee \mathrm{FOR}\,(x_2, y_2))))$

(If an act is ordered, it is not that any part of it is forbidden);

T28. $\forall\, x\, x_1\, x_2\, y\, y_1\, y_2\; (x = x_1 + x_2 \wedge y = y_1 + y_2 \rightarrow (\mathrm{OBL}\,(x_1, y_1) \wedge \mathrm{OBL}\,(x_2, y_2) \rightarrow \mathrm{PER}\,(x, y)))$

(If acts are ordered, their composition is permitted);

T29. $\forall\, x\, x_1\, x_2\, y\, y_1\, y_2\; (x = x_1 + x_2 \wedge y = y_1 + y_2 \rightarrow (\mathrm{PER}\,(x_1, y_1) \wedge \mathrm{PER}\,(x_2, y_2) \rightarrow \neg\,\mathrm{FOR}\,(x, y)))$

(If acts are permitted, their composition is not forbidden);

T30. $\forall\, x\, x_1\, x_2\, y\, y_1\, y_2\; (x = x_1 + x_2 \wedge y = y_1 + y_2 \rightarrow (\mathrm{FOR}\,(x_1, y_1) \wedge \mathrm{FOR}\,(x_2, y_2) \rightarrow \neg\,\mathrm{PER}\,(x, y)))$

(If acts are forbidden, their composition is not permitted).

The above relations are useful for reconstructing legal reasoning *a maiori ad minus* and *a minori ad maius*, as well as for reconstructing other similar reasonings.

4. Discussion

A comparison of axioms and theorems of considered deontic theories with axioms and theorems of deontic propositional logics indicates that a number of properties of obligation, prohibition and permission are similarly defined in both approaches.

In particular, the basic theorems of legal event theories, i.e., T1–T7, have close equivalents in deontic propositional logics.

In turn, although theorems T8–T12 have equivalents in deontic propositional logics, their predecessor indicates that successive relations occur only for legal events, not for any events.

For example, T12

$$\forall\, x\, (\text{LEV}\, (x) \rightarrow (\text{FOR}\, (x) \leftrightarrow \neg\, \text{PER}\, (x)))$$

is a counterpart to the definition of prohibition in propositional logics:

$$F\, p \equiv_{\text{def}} \neg\, P\, p.$$

Interestingly, in none of the four theories of legal events under consideration have we a counterpart of the definition of obligation in propositional logics,

$$O\, p \equiv_{\text{def}} \neg\, P\, \neg\, p,$$

which is based on a definition from modal (aletic) logics:

$$\Box\, p \equiv_{\text{def}} \neg\, \Diamond\, \neg\, p.$$

This is because the expression " $P\, \neg\, p$ " has no equivalent in any of these theories.

However, this is not the case in theory of acts, where T19

$$\forall\, x\, y\, z\, (y \neq z \rightarrow (\text{OBL}\, (x, y) \rightarrow \neg\, \text{PER}\, (x, z)))$$

is a counterpart of the aforementioned definition of obligation in propositional logics:

$$O\, p \equiv_{\text{def}} \neg\, P\, \neg\, p.$$

Although, of course, the following proposition is not an AAPOF's theorem:

$$\forall\, x\, y\, z\, (y \neq z \rightarrow (\text{OBL}\, (x, y) \leftrightarrow \neg\, \text{PER}\, (x, z))).$$

Further, T16

$$\forall\, x\, y\, z\, (y \neq z \rightarrow \neg\, (\text{OBL}\, (x, y) \wedge \text{OBL}\, (x, z)))$$

is a counterpart of the theorem

$$\neg\, (O\, p \wedge O\, \neg\, p),$$

while T20

$$\forall\, x\, y\, z\, (\text{ACT}\, (x, y) \wedge \text{ACT}\, (x, z) \wedge y \neq z \wedge \forall\, w\, (\text{ACT}\, (x, w) \rightarrow (w = y \vee w = z)) \rightarrow$$
$$\neg\, (\text{FOR}\, (x, y) \wedge \text{FOR}\, (x, z)))$$

is a counterpart of the theorem

$$\neg\, (F\, p \wedge F\, \neg\, p).$$

In turn, T22

$$\forall\, x\, y\, z\, (\text{ACT}\, (x, y) \wedge \text{ACT}\, (x, z) \wedge y \neq z \wedge \forall\, w\, (\text{ACT}\, (x, w) \rightarrow (w = y \vee w = z)) \rightarrow$$
$$(\text{PER}\, (x, y) \vee \text{PER}\, (x, z)))$$

is a counterpart of the theorem

$$P\,p \lor P\,\urcorner\,p.$$

On the other hand, the A12 axiom

$$\forall\,x\,x_1\,y\,y_1\,(x_1\,\varepsilon\,x \land y_1\,\varepsilon\,y \to (OBL\,(x, y) \to (ACT\,(x_1, y_1) \to OBL\,(x_1, y_1))))$$

is a distant counterpart of the theorem

$$O\,(p \land q) \to O\,p.$$

In turn, axiom A15

$$\forall\,x\,x_1\,x_2\,y\,y_1\,y_2\,(x = x_1 + x_2 \land y = y_1 + y_2 \to (OBL\,(x_1, y_1) \land OBL\,(x_2, y_2) \to OBL\,(x, y)))$$

is a distant counterpart of the theorem

$$O\,p \land O\,q \to O\,(p \land q).$$

Similarly, axiom A13

$$\forall\,x\,x_1\,y\,y_1\,(x_1\,\varepsilon\,x \land y_1\,\varepsilon\,y \to (PER\,(x, y) \to (ACT\,(x_1, y_1) \to PER\,(x_1, y_1))))$$

is a distant counterpart of the theorem

$$P\,(p \land q) \to P\,p.$$

While the axiom A16

$$\forall\,x\,x_1\,x_2\,y\,y_1\,y_2\,(x = x_1 + x_2 \land y = y_1 + y_2 \to (PER\,(x_1, y_1) \land PER\,(x_2, y_2) \to PER\,(x, y)))$$

is a distant counterpart of the theorem

$$P\,p \land P\,q \to P\,(p \land q).$$

Similarly, the A14 axiom

$$\forall\,x\,x_1\,y\,y_1\,(x_1\,\varepsilon\,x \land y_1\,\varepsilon\,y \to (FOR\,(x, y) \to (ACT\,(x_1, y_1) \to FOR\,(x_1, y_1))))$$

is a distant counterpart of the proposition

$$F\,(p \land q) \to F\,p.$$

While the axiom A17

$$\forall\,x\,x_1\,x_2\,y\,y_1\,y_2\,(x = x_1 + x_2 \land y = y_1 + y_2 \to (FOR\,(x_1, y_1) \land FOR\,(x_2, y_2) \to FOR\,(x, y)))$$

is a distant counterpart of the proposition

$$F\,p \land F\,q \to F\,(p \land q).$$

As can be seen, the axioms and theorems of the deontic theories constructed above are usually not the exact equivalents of theorems of deontic propositional logics. They reflect additional restrictions that are necessary for expressing obligation, prohibition and permission in accordance with intuition, but which are inexpressible in propositional logics.

5. Conclusions

Due to the discussed restrictions, the presented systems avoid the non-intuitive properties of propositional deontic logics.

Firstly, deontic sentences do not apply to all domains. They are sentences about legal events, and in particular about acts.

Secondly, in the presented systems we have no equivalents of many non-intuitive sentences of propositional deontic logics, such as those considered in the introduction:

$$O\,((p \wedge q) \to q) \to (O\,(p \wedge q) \to O\,q),$$

$$O\,(p \to (p \vee q)) \to (O\,p \to O\,(p \vee q)).$$

It is a consequence of the accepted limitation that, in the presented systems, deontic sentences are sentences about legal events, and not sentences about any states of affairs.

Thirdly, it is also noteworthy that—for obvious reasons—in the deontic theories presented above, not even far counterparts of propositions that would include iterations of deontic operators exist.

Fourthly, the presented systems have no equivalents to the paradoxical statements of propositional deontic logics such as those considered in the introduction:

$$O\,p \to O\,(p \vee q),$$

$$F\,p \to O\,(p \to q).$$

Once again, it is a consequence of the accepted limitation that, in the presented systems, deontic sentences are sentences about legal events, and not sentences about any states of affairs.

In addition, some axioms and theorems of the deontic theories presented above do not have counterparts in propositional logics at all, and at the same time reflect important intuitions related to deontic modalities. Examples include the A4, A5 and A6 axioms and some theorems obtained with the help of these axioms.

Furthermore, thanks to Wolniewicz's situation ontology, the presented systems are based on a clear concept of deontic modalities: orders, bans and permits are simply sets of legal events.

In the presented approach, a distinction is also made between the deontic properties of any legal events and the deontic properties of acts. The former are described in AEPF, AEPOF, AEPOFI. The latter are expressed, e.g., by axioms A4 to A6 and A7 to A17 of the AAPOF system. Axioms such as A7 to A17 of the AAPOF system also show that it is possible to formally consider the relations between an act and its parts, which is important for the legal applications of deontic logics.

All this leads to the conclusion that deontic theories built on the first-order predicate logic and inspired by Wolniewicz's situation ontology are worthy of attention and development.

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References

1. Wolniewicz, B. *Ontologia Sytuacji (in Polish: Ontology of Situations)*; Państwowe Wydawnictwo Naukowe: Warsaw, Poland, 1985; See also: Wolniewicz, B. *A Formal Ontology of Situations*; Studia Logica 41, 1982; pp. 381–413.
2. Malec, A. Andrzej Malec, Wprowadzenie do semantyki prawa (in Polish: Introduction to semantics of law). Białystok, Poland, 2018.

3. Ziemba, Z. *Deontic Logic, in: Witold Marciszewski (Ed.), Dictionary of Logic as Applied in the Study of Language. Concepts, Methods, Theories*; Martinus Nijhoff Publishers: The Hague, The Netherlands; Martinus Nijhoff Publishers: Boston, Poland; Martinus Nijhoff Publishers: London, UK, 1981; pp. 97–104.

4. Ziemba, Z. *Analityczna Teoria Obowiązku. Studium z Logiki Deontycznej (in Polish: Analytical Theory of Duties. A Study in Deontic Logic)*; Państwowe Wydawnictwo Naukowe: Warsaw, Poland, 1985.

5. Gumański, L. *Istnienie i Logika. Studia z Filozofii (in Polish: Existence and Logic. Studies in Philosophy)*; Wydawnictwo Uniwersytetu Mikolaja Kopernika: Torun, Poland, 2006; pp. 389–446.

6. Czelakowski, J. *Freedom and Enforcement in Action*; Trends in Logic (Studia Logica Library); Springer: Dordrecht, The Netherlands, 2015; Volume 42.

7. Hart, H.L.A. Tony Honore. In *Causation in the Law*, 2nd ed.; Clarendon Press: Oxford, UK, 2002.