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Solutions of the Generalized Abel's Integral Equations of the Second Kind with Variable Coefficients

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Abstract: Applying Babenko's approach, we construct solutions for the generalized Abel's integral equations of the second kind with variable coefficients on R and R^n, and show their convergence and stability in the spaces of Lebesgue integrable functions, with several illustrative examples.

Keywords: Riemann–Liouville fractional integral; Mittag–Leffler function; Babenko's approach; generalized Abel's integral equation

MSC: 45E10; 26A33

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1. Introduction

In 1823, Abel studied a physical problem regarding the relationship between kinetic and potential energies for falling bodies and constructed the integral equation [1–4]

$$g(x) = \int_c^x (x-t)^{-1/2} u(t)dt, \quad c > 0,$$

where $g(x)$ is given and $u(x)$ is unknown. Later on, he worked on a more general integral equation given as

$$g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t)dt, \quad 0 < \alpha < 1, \ a \le x \le b,$$

which is called Abel's integral equation of the first kind. Abel's integral equation of the second kind is generally given as

$$u(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t)dt = g(x), \quad \alpha > 0 \tag{1}$$

where λ is a constant.

Abel's integral equations are related to a wide range of physical problems, such as heat transfer [5], nonlinear diffusion [6], the propagation of nonlinear waves [7], and applications in the theory of neutron transport and traffic theory. There are many studies [8–14] on Abel's integral equations, including their variants and generalizations [15,16]. In 1930, Tamarkin investigated integrable solutions of Abel's integral equations under certain conditions by several integral operators [17]. Sumner [18] studied Abel's integral equations using the convolutional transform. Minerbo and Levy [19] found a numerical solution of Abel's integral equation by orthogonal polynomials. In 1985, Hatcher [20] worked on a nonlinear Hilbert problem of power type, solved in closed form by representing a sectionally holomorphic function by means of an integral with power kernel, and transformed the problem to one of solving a generalized Abel's integral equation. Using a modification of Mikusinski operational calculus, Gorenflo and Luchko [21] obtained an explicit solution of the generalized Abel's integral equation of the second kind, in terms of the Mittag–Leffler function of several variables.

$$u(x) - \sum_{i=1}^{m} \lambda_i (I^{\alpha_i \mu} u)(x) = g(x), \quad \alpha_i > 0, m \geq 1, \mu > 0, x > 0$$

where λ_i is a constant for $i = 1, 2, \cdots, m$, and I^μ is the Riemann–Liouville fractional integral of order $\mu \in R^+$ with initial point zero [22],

$$(I^\mu u)(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} u(t) dt.$$

Lubich [10] constructed the numerical solution for the following Abel's integral equation of the second kind based on fractional powers of linear multistep methods

$$u(x) = g(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, u(t)) dt \quad \text{on} \;\; R^n$$

where $x \in [0, T]$ and $\alpha > 0$. The case $\alpha = 1/2$ is encountered in a variety of problems in physics and chemistry [23]. Pskhu [24] considered the following generalized Abel's integral equation with constant coefficients a_k for $k = 1, 2, \cdots, n$

$$\sum_{k=1}^{n} a_k I^{\alpha_k} u(x) = g(x),$$

where $\alpha_k \geq 0$ and $x \in (0, a)$, and constructed an explicit solution based on the Wright function

$$\phi(\alpha, \beta; z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\alpha n + \beta)}, \quad \alpha > -1, \beta \in \mathbb{C}$$

and convolution. Li et al. [25–27] recently studied Abel's integral Equation (1) for any arbitrary $\alpha \in R$ in the generalized sense based on fractional calculus of distributions, inverse convolutional operators and Babenko's approach [28]. They obtained several new and interesting results that cannot be realized in the classical sense or by the Laplace transform. Many applied problems from physical science lead to integral equations which can be converted to the form of Abel's integral equations for analytic or distributional solutions in the case where classical ones do not exist [15,27].

Letting $\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$ and $a > 0$, we consider the generalized Abel's integral equation of the second kind with variable coefficients

$$u(x) - \sum_{k=1}^{n} a_k(x) I^{\alpha_k} u(x) = g(x), \tag{2}$$

where $x \in (0, a)$, $a_i(x)$ is Lebesgue integrable and bounded on $(0, a)$ for $i = 1, 2, \cdots, n$, $g(x)$ is a given function in $L(0, a)$ and $u(x)$ is the unknown function. Clearly, Equation (2) turns to be

$$u(x) - a_1 I^{\alpha_1} u(x) = g(x) \tag{3}$$

if $n = 1$ and $a_1(x) = a_1$ (constant). Equation (3) is the classical Abel's integral equation of the second kind, with the solution given by Hille and Tamarkin [29]

$$u(x) = g(x) + a_1 \int_0^x (x-t)^{\alpha_1 - 1} E_{\alpha_1, \alpha_1}(a_1 (x-t)^{\alpha_1}) g(t) dt,$$

where

$$E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta > 0$$

is the Mittag–Leffler function.

Following a similar approach, we also establish a convergent and stable solution for the generalized Abel's integral equation on R^n with variable coefficients

$$u(x) - a_1(x) I_1^{\alpha_1} a_2(x) I_2^{\alpha_2} \cdots a_n(x) I_n^{\alpha_n} u(x) = g(x),$$

where $x = (x_1, x_2, \cdots, x_n)$ and I_k^{α} is the partial Riemann–Liouville fractional integral of order $\alpha \in R^+$ with respect to x_k, with initial point 0,

$$(I_k^{\alpha} u)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{x_k} (x_k - t)^{\alpha-1} u(x_1, \cdots, x_{k-1}, t, x_{k+1}, \cdots, x_n) dt$$

where $k = 1, 2, \cdots, n$.

2. The Main Results

Theorem 1. *Let $x \in (0, a)$, $a_i(x)$ be Lebesgue integrable and bounded on $(0, a)$ for $i = 1, 2, \cdots, n$, and $g(x)$ be a given function in $L(0, a)$. Then the generalized Abel's integral equation of the second kind with variable coefficients*

$$u(x) - \sum_{k=1}^{n} a_k(x) I^{\alpha_k} u(x) = g(x)$$

has the following convergent and stable solution in $L(0, a)$

$$u(x) = \sum_{m=0}^{\infty} \left(\sum_{k=1}^{n} a_k(x) I^{\alpha_k} \right)^m g(x),$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$.

Proof. Clearly,

$$u(x) - \sum_{k=1}^{n} a_k(x) I^{\alpha_k} u(x) = \left(1 - \sum_{k=1}^{n} a_k(x) I^{\alpha_k} \right) u(x) = g(x)$$

which implies, by Babenko's approach (treating the operator like a variable), that

$$\begin{aligned} u(x) &= \frac{1}{1 - \sum_{k=1}^{n} a_k(x) I^{\alpha_k}} g(x) = \sum_{m=0}^{\infty} \left(\sum_{k=1}^{n} a_k(x) I^{\alpha_k} \right)^m g(x) \\ &= \sum_{m=0}^{\infty} \sum_{m_1+m_2+\cdots+m_n=m} \binom{m!}{m_1!, m_2!, \cdots, m_n!} (a_1(x) I^{\alpha_1})^{m_1} \cdots (a_n(x) I^{\alpha_n})^{m_n} \; g(x). \end{aligned}$$

Let $\|f\|$ be the usual norm of $f \in L(0, a)$, given by

$$\|f\| = \int_0^a |f(x)| dx < \infty.$$

Then, we have from [30]

$$\|I^{\alpha_i} g\| = \|\Phi_{\alpha_i} * g\| \leq \|\Phi_{\alpha_i}\| \, \|g\|$$

where

$$\Phi_{\alpha_i} = \frac{x_+^{\alpha_i - 1}}{\Gamma(\alpha_i)}.$$

This implies that

$$\|I^{\alpha_i}\| \leq \|\Phi_{\alpha_i}\| = \frac{1}{\Gamma(\alpha_i)} \int_0^a x^{\alpha_i - 1} = \frac{a^{\alpha_i}}{\Gamma(\alpha_i + 1)}.$$

Since $a_i(x)$ is bounded over $(0,a)$, there exists $M > 0$ such that

$$\sup_{x\in(0,a)} |a_i(x)| \leq M$$

for all $i = 1, 2, \cdots, n$. Therefore,

$$\begin{aligned}\|u\| &\leq \sum_{m=0}^{\infty} M^m \sum_{m_1+m_2+\cdots+m_n=m} \binom{m!}{m_1!, m_2!, \cdots, m_n!} \cdot \\ &\quad \|I^{m_1\alpha_1}\| \, \|I^{m_2\alpha_2}\| \cdots \|I^{m_n\alpha_n}\| \, \|g\| \\ &\leq \sum_{m=0}^{\infty} M^m \sum_{m_1+m_2+\cdots+m_n=m} \binom{m!}{m_1!, m_2!, \cdots, m_n!} \cdot \\ &\quad \frac{a^{m_1\alpha_1+\cdots+m_n\alpha_n}}{\Gamma(m_1\alpha_1+1)\cdots\Gamma(m_n\alpha_n+1)} \|g\| .\end{aligned}$$

Let

$$A = \max\{a, 1\}.$$

Then,

$$a^{m_1\alpha_1+\cdots+m_n\alpha_n} \leq A^{m_1\alpha_1+\cdots+m_n\alpha_n} \leq A^{\alpha_1 m}$$

as $\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$. On the other hand,

$$\Gamma(m_1\alpha_1+1)\cdots\Gamma(m_n\alpha_n+1) \geq \Gamma(m_1\alpha_n+1)\cdots\Gamma(m_n\alpha_n+1) \geq \left(\frac{1}{2}\right)^{n-1} \Gamma(\alpha_n \frac{m}{n}+1),$$

since there exists $m_i \geq m/n$ for some i by noting that $m_1 + m_2 + \cdots + m_n = m$, and the factor $\Gamma(m_j\alpha_n + 1) \geq 1/2$ for $j \neq i$. Hence,

$$\frac{1}{\Gamma(m_1\alpha_1+1)\cdots\Gamma(m_n\alpha_n+1)} \leq \frac{2^{n-1}}{\Gamma(\alpha_n \frac{m}{n}+1)},$$

and

$$\begin{aligned}\|u\| &\leq 2^{n-1} \|g\| \sum_{m=0}^{\infty} \frac{M^m n^m A^{\alpha_1 m}}{\Gamma(\alpha_n \frac{m}{n}+1)} = 2^{n-1} \|g\| \sum_{m=0}^{\infty} \frac{(MnA^{\alpha_1})^m}{\Gamma(\alpha_n \frac{m}{n}+1)} \\ &= 2^{n-1} \|g\| \, E_{\alpha_n/n,\,1}(MnA^{\alpha_1}) < \infty\end{aligned}$$

by using

$$\sum_{m_1+m_2+\cdots+m_n=m} \binom{m!}{m_1!, m_2!, \cdots, m_n!} = n^m.$$

Furthermore, the solution

$$u(x) = \sum_{m=0}^{\infty} \left(\sum_{k=1}^{n} a_k(x) I^{\alpha_k} \right)^m g(x)$$

is stable from the last inequality. This completes the proof of Theorem 1. □

3. Illustrative Examples

Let α and β be arbitrary real numbers. Then it follows from [31]

$$\Phi_\alpha * \Phi_\beta = \Phi_{\alpha+\beta}.$$

Example 1. *Assume $\alpha > 0$. Then Abel's integral equation with a variable coefficient*

$$u(x) - x^{\alpha} I^{2.5} u(x) = x, \quad x \in (0, a)$$

has the following stable solution

$$u(x) = x + \sum_{m=1}^{\infty} \frac{\Gamma(\alpha + 4.5)\Gamma(2\alpha + 7) \cdots \Gamma(m\alpha + 4.5 + (m-1)2.5)}{\Gamma(4.5)\Gamma(\alpha + 7) \cdots \Gamma((m-1)\alpha + 4.5 + (m-1)2.5)} \Phi_{m\alpha + 4.5 + (m-1)2.5}(x)$$

in $L(0, a)$.

Indeed,

$$u(x) = x + \sum_{m=1}^{\infty} (x^{\alpha} I^{2.5})^m \cdot x = x + \sum_{m=1}^{\infty} (x^{\alpha} \Phi_{2.5})^m * \Phi_2.$$

Clearly,

$$\begin{aligned}
& x^{\alpha}\Phi_{2.5} * \Phi_2 = x^{\alpha}\Phi_{4.5} = \frac{x^{\alpha+3.5}}{\Gamma(4.5)} = \frac{\Gamma(\alpha+4.5)}{\Gamma(4.5)}\Phi_{\alpha+4.5}, \\
& (x^{\alpha}\Phi_{2.5}) * \frac{\Gamma(\alpha+4.5)}{\Gamma(4.5)}\Phi_{\alpha+4.5} = \frac{\Gamma(\alpha+4.5)}{\Gamma(4.5)} x^{\alpha}\Phi_{\alpha+7} = \frac{\Gamma(\alpha+4.5)}{\Gamma(4.5)} \frac{x^{2\alpha+6}}{\Gamma(\alpha+7)} \\
& = \frac{\Gamma(\alpha+4.5)}{\Gamma(4.5)} \frac{\Gamma(2\alpha+7)}{\Gamma(\alpha+7)} \Phi_{2\alpha+7}, \\
& \cdots, \\
& (x^{\alpha}\Phi_{2.5})^m * \Phi_2 = \frac{\Gamma(\alpha+4.5)\Gamma(2\alpha+7)\cdots\Gamma(m\alpha+4.5+(m-1)2.5)}{\Gamma(4.5)\Gamma(\alpha+7)\cdots\Gamma((m-1)\alpha+4.5+(m-1)2.5)} \\
& \Phi_{m\alpha+4.5+(m-1)2.5}
\end{aligned}$$

where $m \geq 1$.

Example 2. *Let $a > 0$. Then Abel's integral equation*

$$u(x) - x I^{0.5} u(x) - x^{0.5} I u(x) = x^{-0.5}, \quad x \in (0, a)$$

has the following stable solution

$$u(x) = x^{-0.5} + \sqrt{\pi} \sum_{m=1}^{\infty} \sum_{k=0}^{m} C_k B_{m,k} \Phi_{2+1.5(m-1)}(x)$$

in $L(0, a)$, where

$$C_k = \begin{cases} 1 & \text{if } k = 0, \\ \dfrac{\Gamma(2)\Gamma(3.5)\cdots\Gamma(2+1.5(k-1))}{\Gamma(1.5)\Gamma(3)\cdots\Gamma(1.5+1.5(k-1))} & \text{if } k \geq 1 \end{cases}$$

and

$$B_{m,k} = \begin{cases} 1 & \text{if } k = m, \\ \dfrac{\Gamma(2+1.5k)\Gamma(2+1.5(k+1))\cdots\Gamma(2+1.5(m-1))}{\Gamma(1+1.5k)\Gamma(1+1.5(k+1))\cdots\Gamma(1+1.5(m-1))} & \text{if } k < m. \end{cases}$$

Indeed,

$$\begin{aligned}
u(x) &= x^{-0.5} + \sum_{m=1}^{\infty} (x I^{0.5} + x^{0.5} I)^m \cdot x^{-0.5} \\
&= x^{-0.5} + \sqrt{\pi} \sum_{m=1}^{\infty} \sum_{k=0}^{m} \binom{m}{k} (x\,\Phi_{0.5})^{m-k} * (x^{0.5}\,\Phi_1)^k * \Phi_{0.5}.
\end{aligned}$$

Clearly,

$$\begin{aligned}
&(x^{0.5}\,\Phi_1) * \Phi_{0.5} = x^{0.5}\Phi_{1.5} = \frac{x}{\Gamma(1.5)} = \frac{\Gamma(2)}{\Gamma(1.5)}\Phi_2,\\
&(x^{0.5}\,\Phi_1)^2 * \Phi_{0.5} = (x^{0.5}\,\Phi_1) * \frac{\Gamma(2)}{\Gamma(1.5)}\Phi_2 = \frac{\Gamma(2)}{\Gamma(1.5)}x^{0.5}\Phi_3\\
&= \frac{\Gamma(2)}{\Gamma(1.5)}\frac{x^{2.5}}{\Gamma(3)} = \frac{\Gamma(2)\Gamma(3.5)}{\Gamma(1.5)\Gamma(3)}\Phi_{3.5},\\
&\cdots,\\
&(x^{0.5}\,\Phi_1)^k * \Phi_{0.5} = \frac{\Gamma(2)\Gamma(3.5)\cdots\Gamma(2+1.5(k-1))}{\Gamma(1.5)\Gamma(3)\cdots\Gamma(1.5+1.5(k-1))}\Phi_{0.5+1.5k} = C_k\Phi_{0.5+1.5k}
\end{aligned}$$

where C_k is defined as above. Furthermore,

$$\begin{aligned}
&(x\,\Phi_{0.5}) * \Phi_{0.5+1.5k} = x\Phi_{1+1.5k} = \frac{x^{1+1.5k}}{\Gamma(1+1.5k)} = \frac{\Gamma(2+1.5k)}{\Gamma(1+1.5k)}\Phi_{2+1.5k},\\
&(x\,\Phi_{0.5})^2 * \Phi_{0.5+1.5k} = \frac{\Gamma(2+1.5k)}{\Gamma(1+1.5k)}\,x\,\Phi_{2.5+1.5k} = \frac{\Gamma(2+1.5k)}{\Gamma(1+1.5k)}\,x\,\Phi_{1+1.5(k+1)}\\
&= \frac{\Gamma(2+1.5k)\Gamma(2+1.5(k+1))}{\Gamma(1+1.5k)\Gamma(1+1.5(k+1))}\Phi_{2+1.5(k+1)},\\
&\cdots,\\
&(x\,\Phi_{0.5})^{m-k} * \Phi_{0.5+1.5k} = \frac{\Gamma(2+1.5k)\Gamma(2+1.5(k+1))\cdots\Gamma(2+1.5(m-1))}{\Gamma(1+1.5k)\Gamma(1+1.5(k+1))\cdots\Gamma(1+1.5(m-1))}\cdot\\
&\Phi_{2+1.5(m-1)} = B_{m,k}\Phi_{2+1.5(m-1)}
\end{aligned}$$

where $B_{m,k}$ is defined above.

Remark 1. *As far as we know, the solution for the generalized Abel's integral equation with variable coefficients over the interval $(0, a)$ is obtained for the first time. However, this approach seems unworkable if the interval is unbounded, as the Riemann–Liouville fractional integral operator is therefore unbounded. In the proof and computations of the above examples, we should point out that the convolution operations are prior to functional multiplications, according to our approach.*

Assuming that $\omega_i > 0$ for all $i = 1, 2, \cdots, n$, and $\Omega = (0, \omega_1) \times (0, \omega_2) \times \cdots \times (0, \omega_n)$, we can derive the following theorem by a similar procedure.

Theorem 2. *Let $\alpha_k \geq 0$ for $k = 1, 2, \cdots, n$ and there is at least one $\alpha_i > 0$ for some $1 \leq i \leq n$. Then the generalized Abel's integral equation of the second kind with variable coefficients on R^n for a given function $g \in L(\Omega)$*

$$u(x) - a_1(x)I_1^{\alpha_1}a_2(x)I_2^{\alpha_2}\cdots a_n(x)I_n^{\alpha_n}u(x) = g(x)$$

has the following convergent and stable solution in $L(\Omega)$

$$u(x) = \sum_{m=0}^{\infty}\left(a_1(x)I_1^{\alpha_1}a_2(x)I_2^{\alpha_2}\cdots a_n(x)I_n^{\alpha_n}\right)^m g(x), \tag{4}$$

where $a_k(x)$ is Lebesgue integrable and bounded on Ω for $k = 1, 2, \cdots, n$.

Proof. Clearly,

$$u(x) - a_1(x)I_1^{\alpha_1}\cdots a_n(x)I_n^{\alpha_n}u(x) = \left(1 - a_1(x)I_1^{\alpha_1}\cdots a_n(x)I_n^{\alpha_n}\right)u(x) = g(x),$$

and

$$\begin{aligned} u(x) &= \frac{1}{1 - a_1(x) I_1^{\alpha_1} \cdots a_n(x) I_n^{\alpha_n}} \, g(x) \\ &= \sum_{m=0}^{\infty} \left(a_1(x) I_1^{\alpha_1} a_2(x) I_2^{\alpha_2} \cdots a_n(x) I_n^{\alpha_n} \right)^m g(x). \end{aligned}$$

It remains to show that the above is convergent and stable in $L(\Omega)$. Let

$$\begin{aligned} W &= \left(a_1(x) I_1^{\alpha_1} \cdots a_n(x) I_n^{\alpha_n} \right)^m \\ &= \left(a_1(x) I_1^{\alpha_1} \cdots a_n(x) I_n^{\alpha_n} \right) \cdots \left(a_1(x) I_1^{\alpha_1} \cdots a_n(x) I_n^{\alpha_n} \right). \end{aligned}$$

Since $a_k(x)$ is bounded on Ω for $k = 1, 2, \cdots, n$, there exists $M > 0$ such that

$$\sup_{x \in \Omega} |a_k(x)| \leq M.$$

Let $\|f\|$ be the usual norm of $f \in L(\Omega)$, given by

$$\|f\| = \int_{\Omega} |f(x)| dx = \int_{\Omega} |f(x_1, x_2, \cdots, x_n)| dx_1 dx_2 \cdots dx_n < \infty.$$

Then, it follows from [30] for $k = 1, 2, \cdots, n$

$$\left\| I_k^{\alpha_k} g \right\| = \left\| \Phi_{k,\alpha_k} * g \right\| \leq \left\| \Phi_{k,\alpha_k} \right\| \|g\|$$

where

$$\Phi_{k,\alpha_k} = \frac{(x_k)_+^{\alpha_k - 1}}{\Gamma(\alpha_k)}.$$

This implies for $\alpha_k > 0$ that

$$\begin{aligned} &\left\| I_k^{\alpha_k} \right\| \leq \left\| \Phi_{k,\alpha_k} \right\| = \int_{\Omega} \frac{(x_k)_+^{\alpha_k - 1}}{\Gamma(\alpha_k)} dx_1 dx_2 \cdots dx_n \\ &= \omega_1 \cdots \omega_{k-1} \frac{\omega_k^{\alpha_k}}{\Gamma(\alpha_k + 1)} \omega_{k+1} \cdots \omega_n \leq \lambda^{n-1} \frac{\omega_k^{\alpha_k}}{\Gamma(\alpha_k + 1)} \end{aligned}$$

where

$$\lambda = \max\{\omega_1, \omega_2, \cdots, \omega_n\} > 0.$$

In particular for $\alpha_k = 0$,

$$\left\| I_k^0 \right\| \leq \lambda^{n-1}.$$

Therefore,

$$\begin{aligned} \|W\| &\leq M^{nm} \left\| I_1^{\alpha_1 m} \right\| \cdots \left\| I_n^{\alpha_n m} \right\| \\ &\leq M^{nm} \lambda^{n^2 - n} \frac{\omega_1^{\alpha_1 m}}{\Gamma(\alpha_1 m + 1)} \cdots \frac{\omega_n^{\alpha_n m}}{\Gamma(\alpha_n m + 1)} \\ &\leq M^{nm} \lambda^{n^2 - n} S^{nm} \frac{1}{\Gamma(\alpha_1 m + 1)} \cdots \frac{1}{\Gamma(\alpha_n m + 1)}, \end{aligned}$$

where

$$S = \max\{\omega_1^{\alpha_1}, \cdots, \omega_n^{\alpha_n}\}.$$

Without loss of generality, we assume that $\alpha_1 > 0$. Then,

$$\Gamma(\alpha_1 m+1)\cdots\Gamma(\alpha_n m+1) \geq \frac{1}{2^{n-1}}\Gamma(\alpha_1 m+1)$$

since

$$\Gamma(\alpha_k m+1) \geq 1/2$$

for $k = 2, \cdots, n$. This infers that

$$\|u(x)\| \leq \lambda^{n^2-n} 2^{n-1} \|g\| \sum_{m=0}^{\infty} \frac{(M^n S^n)^m}{\Gamma(\alpha_1 m+1)} < +\infty$$

by the Mittag–Leffler function. Furthermore, the solution

$$u(x) = \sum_{m=0}^{\infty} \left(a_1(x) I_1^{\alpha_1} a_2(x) I_2^{\alpha_2} \cdots a_n(x) I_n^{\alpha_n}\right)^m g(x)$$

is stable from the last inequality. This completes the proof of Theorem 2. □

In particular, let $g(x) = \phi_1(x_1)\cdots\phi_n(x_n) \in L(\Omega)$. Then

$$u(x) - a_1(x_1) I_1^{\alpha_1} a_2(x_2) I_2^{\alpha_2} \cdots a_n(x_n) I_n^{\alpha_n} u(x) = \phi_1(x_1)\cdots\phi_n(x_n)$$

has the following convergent and stable solution

$$u(x) = \sum_{m=0}^{\infty} \left(a_1(x_1) I_1^{\alpha_1}\right)^m \phi_1(x_1) \cdots \left(a_n(x_n) I_n^{\alpha_n}\right)^m \phi_n(x_n)$$

in $L(\Omega)$.

4. Conclusions

We establish the convergent and stable solutions for the following generalized Abel's integral equations of the second kind with variable coefficients

$$u(x) - \sum_{k=1}^{n} a_k(x) I^{\alpha_k} u(x) = g(x), \quad x \in (0,a) \subset R$$
$$u(x) - a_1(x) I_1^{\alpha_1} a_2(x) I_2^{\alpha_2} \cdots a_n(x) I_n^{\alpha_n} u(x) = g(x), \quad x \in \Omega \subset R^n$$

in the spaces of Lebesgue integrable functions, and provide applicable examples based on convolutions and gamma functions.

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