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Harmonic Starlike Functions with Respect to Symmetric Points

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Abstract: In the paper we define classes of harmonic starlike functions with respect to symmetric points and obtain some analytic conditions for these classes of functions. Some results connected to subordination properties, coefficient estimates, integral representation, and distortion theorems are also obtained.

Keywords: harmonic functions; janowski functions; starlike functions; extreme points; subordination

1. Introduction

We denote by $\mathcal{H}$ the class of complex-valued harmonic functions in the unit disc $\mathbb{U} := \{z : |z| < r\}$. Then $f \in \mathcal{H}$ if $f = h + \overline{g}$, where h, g are functions analytic in $\mathbb{U}$. Let $\mathcal{H}_0$ be the class of function $f \in \mathcal{H}$ with the following normalization:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} \overline{b_n z^n} \quad (z \in \mathbb{U}) \tag{1}$$

and let $\mathcal{S}_{\mathcal{H}}$ denote the class of functions $f \in \mathcal{H}_0$, which are orientation preserving and univalent in $\mathbb{U}$.

For functions $f_1, f_2 \in \mathcal{H}$ of the forms:

$$f_k(z) = \sum_{n=0}^{\infty} a_{k,n} z^n + \sum_{n=1}^{\infty} \overline{b_{k,n} z^n} \quad (z \in \mathbb{U},\ k \in \{1,2\}) \tag{2}$$

by $f_1 * f_2$ we denote the Hadamard product or convolution of f_1 and f_2, defined by:

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{1,n} a_{2,n} z^n + \sum_{n=1}^{\infty} \overline{b_{1,n} b_{2,n} z^n} \quad (z \in \mathbb{U}).$$

We say that a function $f : \mathbb{U} \to \mathbb{C}$ is subordinate to a function $F : \mathbb{U} \to \mathbb{C}$, and write $f(z) \prec F(z)$ (or simply $f \prec F$), if there exists a complex-valued function ω which maps $\mathbb{U}$ into oneself with $\omega(0) = 0$, such that $f = F \circ \omega$. In particular, if F is univalent in $\mathbb{U}$, we have the following equivalence:

$$f(z) \prec F(z) \iff [f(0) = F(0) \text{ and } f(\mathbb{U}) \subset F(\mathbb{U})].$$

In 1956 Sakaguchi [1] introduced the class $\mathcal{S}^{**}$ of analytic univalent functions in $\mathbb{U}$ which are starlike with respect to symmetrical points. An analytic function f is said to be starlike with respect to symmetric points if:

$$\mathrm{Re}\frac{zf'(z)}{f(z) - f(-z)} > 0 \quad (z \in \mathbb{U}). \tag{3}$$

If $f \in \mathcal{S}^{**}$ then the angular velocity of $f(z)$ about the point $f(-z)$ is positive as z traverses the circle $|z| = r$ in a positive direction.

Let A and B be two distinct complex parameters and let $0 \leq \alpha < 1$. In [2] (see also [3]) it is defined the class $\mathcal{S}^{*}_{\mathcal{H}}(A, B)$ of Janowski harmonic starlike functions $f \in \mathcal{S}_{\mathcal{H}}$ such that:

$$\frac{D_{\mathcal{H}} f(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{4}$$

where,

$$D_{\mathcal{H}} f(z) := zh'(z) - \overline{zg'(z)} \quad (z \in \mathbb{U}).$$

The classes $\mathcal{S}^{*}_{\mathcal{H}}(\alpha) := \mathcal{S}^{*}_{\mathcal{H}}(2\alpha - 1, 1)$ and $\mathcal{S}^{c}_{\mathcal{H}}(\alpha) := \mathcal{S}^{c}_{\mathcal{H}}(2\alpha - 1, 1)$ are studied by Jahangiri [4] (see also [5]). In particular, we obtain the classes $\mathcal{S}^{c}_{\mathcal{H}} := \mathcal{S}^{c}_{\mathcal{H}}(0)$ and $\mathcal{S}^{*}_{\mathcal{H}} := \mathcal{S}^{*}_{\mathcal{H}}(0)$ of functions $f \in \mathcal{S}_{\mathcal{H}}$ which are convex in $\mathbb{U}(r)$ or starlike in $\mathbb{U}(r)$, respectively, for any $r \in (0, 1]$.

Motivated by Sakaguchi [1], we define the class $\mathcal{S}^{**}_{\mathcal{H}}(A, B)$ of functions $f \in \mathcal{H}_0$ such that:

$$\frac{2D_{\mathcal{H}} f(z)}{f(z) - f(-z)} \prec \frac{1 + Az}{1 + Bz}. \tag{5}$$

In particular, the class $SH^{*}(\alpha) := \mathcal{S}^{**}_{\mathcal{H}}(2\alpha - 1, 1)$ was introduced by Ahuja and Jahangiri [6] (see also [7,8]). The class $\mathcal{HS}^{*}_{s}(b, \alpha) := \mathcal{S}^{**}_{\mathcal{H}}(2b(\alpha - 1) + 1, 1)$ was investigated by Janteng and Halim [9].

In the present paper we obtain some analytic conditions for defined classes of functions. Some results connected to subordination properties, coefficient estimates, integral representation, and distortion theorems are also obtained. These results generalize the results obtained in [6,9] (see also [7,8]).

2. Analytic Criteria

Theorem 1. *Let* $Tf(z) := f(z) - f(-z)$. *If* $f \in \mathcal{S}^{**}_{\mathcal{H}}(A, B)$, *then* $Tf \in \mathcal{S}^{*}_{\mathcal{H}}(A, B)$.

Proof. Let $f \in \mathcal{S}^{**}_{\mathcal{H}}(A, B)$ and $H(z) := \frac{1+Az}{1+Bz}$. Then:

$$\frac{2D_{\mathcal{H}} f(z)}{f(z) - f(-z)} \prec H(z)$$

and

$$\frac{2D_{\mathcal{H}}(-f)(z)}{f(z) - f(-z)} = \frac{2D_{\mathcal{H}} f(-z)}{f(-z) - f(z)} \prec H(-z) \prec H(z).$$

Thus, we have:

$$\frac{2D_{\mathcal{H}} f(z)}{Tf(z)} \in H(\mathbb{U}) \quad \text{and} \quad \frac{2D_{\mathcal{H}}(-f)(z)}{Tf(z)} \in H(\mathbb{U}) \quad (z \in \mathbb{U}).$$

Since H is the convex function in $\mathbb{U}$, we have:

$$\frac{1}{2}\frac{2D_{\mathcal{H}} f(z)}{Tf(z)} + \frac{1}{2}\frac{2D_{\mathcal{H}}(-f)(z)}{Tf(z)} = \frac{D_{\mathcal{H}}(Tf)(z)}{Tf(z)} \in H(\mathbb{U}) \quad (z \in \mathbb{U}),$$

or equivalently:

$$\frac{D_{\mathcal{H}}(Tf)(z)}{Tf(z)} \prec H(z),$$

which implies that:

$$Tf \in \mathcal{S}^{*}_{\mathcal{H}}(A, B).$$

□

Let $\mathcal{V} \subset \mathcal{H}$, $\mathbb{U}_0 := \mathbb{U} \setminus \{0\}$. Due to Ruscheweyh [10] we define the dual set of $\mathcal{V}$ by:

$$\mathcal{V}^* := \left\{ f \in \mathcal{H}_0 : \bigwedge_{q \in \mathcal{V}} (f * q)(z) \neq 0 \quad (z \in \mathbb{U}_0) \right\}.$$

Theorem 2. *We have:*

$$\mathcal{S}_{\mathcal{H}}^{**}(A,B) = \left\{ \psi_\xi : |\xi| = 1 \right\}^*,$$

where,

$$\begin{aligned} \psi_\xi(z) \quad : \quad &= z \frac{\xi (B-A) + (2 + A\xi + B\xi) z}{(1+z)(1-z)^2} \\ &- \overline{z} \frac{2 + (A+B)\xi - (B\_A)\xi\overline{z}}{(1+\overline{z})(1-\overline{z})^2} \quad (z \in \mathbb{U}). \end{aligned} \tag{6}$$

Proof. Let $f \in \mathcal{H}_0$ be of the form (1). Then $f \in \mathcal{S}_{\mathcal{H}}^{**}(A,B)$ if and only if it satisfies Equation (5) or equivalently:

$$\frac{2D_{\mathcal{H}} f(z)}{f(z) - f(-z)} \neq \frac{1 + A\xi}{1 + B\xi} \quad (z \in \mathbb{U}_0, \ |\xi| = 1). \tag{7}$$

Since,

$$D_{\mathcal{H}} h(z) = h(z) * \frac{z}{(1-z)^2}, \ \frac{h(z) - h(-z)}{2} = h(z) * \frac{z}{1-z^2}$$

the above inequality yields:

$$\begin{aligned} & (1+B\xi) D_{\mathcal{H}} f(z) - (1+A\xi) \frac{f(z) - f(-z)}{2} \\ = \ & (1+B\xi) D_{\mathcal{H}} h(z) - (1+A\xi) \frac{h(z) - h(-z)}{2} \\ & - \left\{ (1+B\xi) \overline{D_{\mathcal{H}} g(z)} + (1+A\xi) \overline{\frac{g(z) - g(-z)}{2}} \right\} \\ = \ & h(z) * \left(\frac{(1+B\xi) z}{(1-z)^2} - \frac{(1+A\xi) z}{1-z^2} \right) \\ & - \overline{g(z)} * \left(\frac{(1+B\xi)\overline{z}}{(1-\overline{z})^2} + \frac{(1+A\xi)\overline{z}}{1-\overline{z}^2} \right) \\ = \ & f(z) * \psi_\xi(z) \neq 0 \quad (z \in \mathbb{U}_0, \ |\xi| = 1). \end{aligned}$$

Thus, $f \in \mathcal{S}_{\mathcal{H}}^{**}(A,B)$ if and only if $f(z) * \psi_\xi(z) \neq 0$ for $z \in \mathbb{U}_0$, $|\xi| = 1$, i.e., $\mathcal{S}_{\mathcal{H}}^{**}(A,B) = \left\{ \psi_\xi : |\xi| = 1 \right\}^*$. □

Theorem 3. *If a function $f \in \mathcal{H}$ of the form (1) satisfies the condition:*

$$\sum_{n=2}^{\infty} \left(|\alpha_n| \, |a_n| + |\beta_n| \, |b_n| \right) \leq B - A, \tag{8}$$

where $-B \leq A < B \leq 1$ and

$$\alpha_n = n(1+B) - (1+A)\left(1 - (-1)^n\right)/2, \ \ \beta_n = n(1+B) + (1+A)\left(1 - (-1)^n\right)/2, \tag{9}$$

then $f \in \mathcal{S}_{\mathcal{H}}^{*}(A,B)$.

Proof. The result of Lewy [11] gives that the f is orientation preserving and locally univalent if:

$$\left|h'(z)\right| > \left|g'(z)\right| \quad (z \in \mathbb{U}). \tag{10}$$

By Equation (9) we have:

$$|\alpha_n| / (B - A) \geq n, \; |\beta_n| / (B - A) \geq n \quad (n = 2, 3, \cdots). \tag{11}$$

Therefore, by Equation (8) we obtain:

$$\sum_{n=2}^{\infty} n \left(|a_n| + |b_n|\right) \leq 1 \tag{12}$$

and

$$\begin{aligned} \left|h'(z)\right| - \left|g'(z)\right| &\geq 1 - \sum_{n=2}^{\infty} n |a_n| |z|^n - \sum_{n=2}^{\infty} n |b_n| |z|^n \geq 1 - |z| \sum_{n=2}^{\infty} (n |a_n| + n |b_n|) \\ &\geq 1 - \frac{|z|}{B - A} \sum_{n=2}^{\infty} (|\alpha_n| |a_n| + |\beta_n| |b_n|) \geq 1 - |z| > 0 \quad (z \in \mathbb{U}). \end{aligned}$$

Therefore, by Equation (10) the function f is locally univalent and sense-preserving in $\mathbb{U}$. Moreover, if $z_1, z_2 \in \mathbb{U}$, $z_1 \neq z_2$, then:

$$\left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| = \left|\sum_{l=1}^{n} z_1^{l-1} z_2^{n-l}\right| \leq \sum_{l=1}^{n} |z_1|^{l-1} |z_2|^{n-l} < n \quad (n = 2, 3, \cdots).$$

Let $f \in \mathcal{H}_0$ be a function of the form (1). Without loss of generality, we can assume that f is not an identity function. Then there exist $n \in \mathbb{N}_2$ such that $a_n \neq 0$ or $b_n \neq 0$. Thus, by Equation (12) we get:

$$\begin{aligned} |f(z_1) - f(z_2)| &\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\ &= \left|z_1 - z_2 - \sum_{n=2}^{\infty} a_n \left(z_1^n - z_2^n\right)\right| - \left|\sum_{n=2}^{\infty} \overline{b_n \left(z_1^n - z_2^n\right)}\right| \\ &\geq |z_1 - z_2| - \sum_{n=2}^{\infty} |a_n| \left|z_1^n - z_2^n\right| - \sum_{n=2}^{\infty} |b_n| \left|z_1^n - z_2^n\right| \\ &= |z_1 - z_2| \left(1 - \sum_{n=2}^{\infty} |a_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| - \sum_{n=2}^{\infty} |b_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right|\right) \\ &> |z_1 - z_2| \left(1 - \sum_{n=2}^{\infty} n |a_n| - \sum_{n=2}^{\infty} n |b_n|\right) \geq 0. \end{aligned}$$

This leads to the univalence of f, i.e., $f \in \mathcal{S}_{\mathcal{H}}$. Therefore, $f \in \mathcal{S}_{\mathcal{H}}^{**}(A, B)$ if and only if there exists a complex-valued function ω, $\omega(0) = 0$, $|\omega(z)| < 1$ $(z \in \mathbb{U})$ such that:

$$\frac{2D_{\mathcal{H}} f(z)}{f(z) - f(-z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (z \in \mathbb{U}),$$

or equivalently:

$$\left|\frac{2D_{\mathcal{H}} f(z) - f(z) + f(-z)}{2BD_{\mathcal{H}} f(z) - A\left(f(z) - f(-z)\right)}\right| < 1 \quad (z \in \mathbb{U}). \tag{13}$$

Thus for $z \in \mathbb{U} \backslash \{0\}$ it suffices to show that:

$$\left| D_{\mathcal{H}} f(z) - \frac{f(z) - f(-z)}{2} \right| - \left| B D_{\mathcal{H}} f(z) - A \frac{f(z) - f(-z)}{2} \right| < 0.$$

Indeed, letting $|z| = r \ (0 < r < 1)$ we have:

$$\begin{aligned}
&\left| D_{\mathcal{H}} f(z) - \frac{f(z) - f(-z)}{2} \right| - \left| B D_{\mathcal{H}} f(z) - A \frac{f(z) - f(-z)}{2} \right| \\
&= \left| \sum_{n=2}^{\infty} \left(n - \frac{1-(-1)^n}{2} \right) a_n z^n - \sum_{n=2}^{\infty} \left(n + \frac{1-(-1)^n}{2} \right) \overline{b_n} \overline{z}^n \right| \\
&- \left| (B - A) z + \sum_{n=2}^{\infty} \left(Bn - A \frac{1-(-1)^n}{2} \right) a_n z^n + \sum_{n=2}^{\infty} \left(Bn + A \frac{1-(-1)^n}{2} \right) \overline{b_n} \overline{z}^n \right| \\
&\leq \sum_{n=2}^{\infty} \left(n - \frac{1-(-1)^n}{2} \right) |a_n| r^n + \sum_{n=2}^{\infty} \left(n + \frac{1-(-1)^n}{2} \right) |b_n| r^n - (B - A) r \\
&+ \sum_{n=2}^{\infty} \left(Bn - A \frac{1-(-1)^n}{2} \right) |a_n| r^n + \sum_{n=2}^{\infty} \left(Bn + A \frac{1-(-1)^n}{2} \right) |b_n| r^n \\
&\leq r \left\{ \sum_{n=2}^{\infty} \left(|\alpha_n| |a_n| + |\beta_n| |b_n| \right) r^{n-1} - (B - A) \right\} < 0.
\end{aligned}$$

Hence $f \in \mathcal{S}_{\mathcal{H}}^{**}(A, B)$. □

Motivated by Silverman [12] we denote by $\mathcal{T}$ the class of functions $f \in \mathcal{H}_0$ of the form (1) such that $a_n = -|a_n|, b_n = |b_n| \ (n = 2, 3, \cdots)$, i.e.,

$$f = h + \overline{g}, \ \ h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \ \ g(z) = \sum_{n=2}^{\infty} |b_n| \overline{z}^n \quad (z \in \mathbb{U}). \tag{14}$$

Moreover, let us define:

$$\mathcal{S}_{\mathcal{T}}^{**}(A, B) := \mathcal{T} \cap \mathcal{S}_{\mathcal{H}}^{**}(A, B), \ \ -B \leq A < B \leq 1.$$

Now, we show that the condition (8) is also the sufficient condition for a function $f \in \mathcal{T}$ to be in the class $\mathcal{S}_{\mathcal{T}}^{**}(A, B)$.

Theorem 4. *Let $f \in \mathcal{T}$ be a function of the form (14). Then $f \in \mathcal{S}_{\mathcal{T}}^{**}(A, B)$ if and only if condition (8) holds true.*

Proof. In view of Theorem 3 we need only show that each function $f \in \mathcal{S}_{\mathcal{T}}^{**}(A, B)$ satisfies the coefficient inequality of Equation (8). If $f \in \mathcal{S}_{\mathcal{T}}^{**}(A, B)$, then it satisfies Equation (13) or equivalently:

$$\left| \frac{\sum_{n=2}^{\infty} \left\{ \left(n - \frac{1-(-1)^n}{2} \right) |a_n| z^n + \left(n + \frac{1-(-1)^n}{2} \right) |b_n| \overline{z}^n \right\}}{(B - A) z - \sum_{n=2}^{\infty} \left\{ \left(Bn - A \frac{1-(-1)^n}{2} \right) |a_n| z^n + \left(Bn + A \frac{1-(-1)^n}{2} \right) |b_n| \overline{z}^n \right\}} \right| < 1 \quad (z \in \mathbb{U}).$$

Therefore, putting $z = r \ (0 \leq r < 1)$ we obtain:

$$\frac{\sum_{n=2}^{\infty} \left\{ \left(n - \frac{1-(-1)^n}{2} \right) |a_n| + \left(n + \frac{1-(-1)^n}{2} \right) |b_n| r^{n-1} \right\}}{(B - A) - \sum_{n=2}^{\infty} \left\{ \left(Bn - A \frac{1-(-1)^n}{2} \right) |a_n| + \left(Bn + A \frac{1-(-1)^n}{2} \right) |b_n| r^{n-1} \right\}} < 1. \tag{15}$$

It is clear that the denominator of the left hand side cannot vanish for $r \in [0,1)$. Moreover, it is positive for $r = 0$, and in consequence for $r \in \langle 0,1)$. Thus, by Equation (25) we have:

$$\sum_{n=2}^{\infty} \left(\alpha_n \left|a_n\right| + \beta_n \left|b_n\right|\right) r^{n-1} < B - A \quad (0 \leq r < 1). \tag{16}$$

The sequence of partial sums $\{S_n\}$ associated with the series $\sum\limits_{n=2}^{\infty} \left(\alpha_n \left|a_n\right| + \beta_n \left|b_n\right|\right)$ is nondecreasing sequence. Moreover, by Equation (16) it is bounded by $B - A$. Hence, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=2}^{\infty} \left(\alpha_n \left|a_n\right| + \beta_n \left|b_n\right|\right) = \lim_{n\to\infty} S_n \leq B - A,$$

which yields the assertion (8). $\square$

Example 1. *For the function:*

$$f(z) = z - \sum_{n=2}^{\infty} \frac{B-A}{2^n \alpha_n} z^n - \sum_{n=2}^{\infty} \frac{B-A}{2^n \beta_n} \overline{z}^n \quad (z \in \mathbb{U})$$

we have,

$$\sum_{n=2}^{\infty} \left(\alpha_n \left|a_n\right| + \beta_n \left|b_n\right|\right) = \sum_{n=2}^{\infty} \frac{B-A}{2^n} + \frac{B-A}{2^n} = (B-A) \sum_{n=1}^{\infty} \frac{1}{2^n} = B - A.$$

Thus, $f \in \mathcal{S}^{**}_{\mathcal{T}}(A,B)$.

3. Topological Properties

Let us consider a metric on $\mathcal{H}$ in which a sequence $\{f_n\}$ in $\mathcal{H}$ converges to f if and only if it converges to f uniformly on each compact subset of $\mathbb{U}$. The metric induces the usual topology on $\mathcal{H}$. It is easy to verify that the obtained topological space is complete. Let $\mathcal{B}$ be a subset of the space $\mathcal{H}$.

We say that a function $f \in \mathcal{B}$ is *the extreme point of* $\mathcal{B}$ if it cannot be presented as nontrivial convex combination of two functions from $\mathcal{B}$. We denote by $E\mathcal{B}$ the set of extreme points of $\mathcal{B}$.

We say that $\mathcal{B}$ is *locally uniformly bounded* if for each r, $0 < r < 1$, there exists $K = K(r) > 0$ such that:

$$|f(z)| \leq K \quad (f \in \mathcal{B},\ |z| \leq r).$$

We say that a set $\mathcal{B}$ is *convex* if it includes all of convex combinations of two functions from $\mathcal{B}$. Let $\overline{co}\mathcal{B}$ denote *the closed convex* hull of $\mathcal{B}$ i.e., the intersection of all closed convex subsets of $\mathcal{H}$ that contain $\mathcal{B}$.

Let $\mathcal{B} \subset \mathcal{H}$ be a convex set and $\mathcal{L}$ be a real-valued functional on $\mathcal{H}$. We say that $\mathcal{L}$ is *convex functional* on $\mathcal{B}$ if:

$$\mathcal{L}(af + (1-a)g) \leq a\mathcal{L}(f) + (1-a)\mathcal{L}(g) \quad (f,g \in \mathcal{B},\ 0 \leq a \leq 1).$$

By using the Krein-Milman theorem (see [13]) we get the following lemma.

Lemma 1. *Let $\mathcal{B}$ be a non-empty compact set on the space $\mathcal{H}$. Then $E\mathcal{B}$ is non-empty and $\overline{co}E\mathcal{B} = \overline{co}\mathcal{B}$.*

Motivated by Hallenbeck and MacGregor ([14], p. 45) we can formulate the following lemma.

Lemma 2. *Let $\mathcal{B}$ be a non-empty convex compact set on the space $\mathcal{H}$ and let $\mathcal{L}$ be a real-valued, convex, and continuous functional on $\mathcal{B}$. Then* $\max\{\mathcal{L}(f) : f \in \mathcal{B}\} = \max\{\mathcal{L}(f) : f \in E\mathcal{B}\}$.

Proof. We observe that there exists $\max\{\mathcal{L}(f) : f \in \mathcal{B}\} =: K$, since $\mathcal{J}$ is the continuous functional on the compact set $\mathcal{B}$. Thus, the set $H := \{f \in \mathcal{B} : \mathcal{L}(f) = K\}$ is non-empty compact subset of $\mathcal{B}$ and, by Lemma 1, we get that H has an extreme point f_0. Let,

$$f_0 = a f_1 + (1-a)\, f_2,$$

where $f_1, f_2 \in \mathcal{B}$ and $0 < a < 1$. Thus,

$$K = \mathcal{L}(f_0) \leq a\mathcal{L}(f_1) + (1-a)\,\mathcal{L}(f_2) = aK + (1-a)\,K = K$$

and, in consequence, $\mathcal{L}(f_1) = \mathcal{L}(f_2) = K$, i.e., $f_1, f_2 \in H$. Since f_0 is an extreme point of H we get $f_1 = f_2 = f_0 \in E\mathcal{B}$. Thus, we obtain that there exists $\max\{\mathcal{L}(f) : f \in E\mathcal{B}\} = K$, and the proof is complete. $\square$

We observe that $\mathcal{H}$ is a complete metric space. Therefore, by Montel's theorem (see [15]) we get the following lemma.

Lemma 3. *A set $\mathcal{B}$ is compact on $\mathcal{H}$ if and only if $\mathcal{B}$ is locally uniformly bounded and closed on $\mathcal{H}$.*

Theorem 5. *The class $\mathcal{S}_{\mathcal{T}}^{**}(A,B)$ is compact and convex subset on $\mathcal{H}$.*

Proof. Let $f_k \in \mathcal{S}_{\mathcal{T}}^{**}(A,B)$ be functions of the form:

$$f_k(z) = z - \sum_{n=2}^{\infty}\left(\left|a_{k,n}\right| z^n - \left|b_{k,n}\right| \overline{z}^n\right) \quad (z \in \mathbb{U},\ k = 1, 2, \ldots) \tag{17}$$

and let $0 \leq \gamma \leq 1$. Since,

$$\gamma f_1(z) + (1-\gamma) f_2(z) = z - \sum_{n=2}^{\infty}\left\{\left(\gamma\left|a_{1,n}\right| + (1-\gamma)\left|a_{2,n}\right|\right) z^n - \left(\gamma\left|b_{1,n}\right| + (1-\gamma)\left|b_{2,n}\right|\right)\overline{z}^n\right\},$$

and by Theorem 4 we have:

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left\{\alpha_n\left(\gamma\left|a_{1,n}\right| + (1-\gamma)\left|a_{2,n}\right|\right) + \beta_n\left(\gamma\left|b_{1,n}\right| + (1-\gamma)\left|b_{2,n}\right|\right)\right\} \\
&= \gamma\sum_{n=2}^{\infty}\left\{\alpha_n\left|a_{1,n}\right| + \beta_n\left|b_{1,n}\right|\right\} + (1-\gamma)\sum_{n=2}^{\infty}\left\{\alpha_n\left|a_{2,n}\right| + \beta_n\left|b_{2,n}\right|\right\} \\
&\leq \gamma\,(B-A) + (1-\gamma)\,(B-A) = B - A,
\end{aligned}$$

the function $\phi = \gamma f_1 + (1-\gamma) f_2$ belongs to the class $\mathcal{S}_{\mathcal{T}}^{**}(A,B)$. Hence, the class is convex. Furthermore, for $f \in \mathcal{S}_{\mathcal{T}}^{**}(A,B)$, $|z| \leq r$, $0 < r < 1$, we have:

$$|f(z)| \leq r + \sum_{n=2}^{\infty}\left(|a_n| + |b_n|\right) r^n \leq r + \sum_{n=2}^{\infty}\left(\alpha_n|a_n| + \beta_n|b_n|\right) \leq r + (B-A)\,. \tag{18}$$

Thus, we conclude that the class $\mathcal{S}_{\mathcal{T}}^{**}(A,B)$ is locally uniformly bounded. By Lemma 3, we only need to show that it is closed, i.e., if $f_k \in \mathcal{S}_{\mathcal{T}}^{**}(A,B)$ $(k \in \mathbb{N})$ and $f_k \to f$, then $f \in \mathcal{S}_{\mathcal{T}}^{**}(A,B)$. Let f_k and f are given by Equations (17) and (14), respectively. Using Theorem 4 we have:

$$\sum_{n=2}^{\infty}\left(\alpha_n\left|a_{k,n}\right| + \beta_n\left|b_{k,n}\right|\right) \leq B - A \quad (k \in \mathbb{N})\,. \tag{19}$$

Since $f_k \to f$, we conclude that $|a_{k,n}| \to |a_n|$ and $|b_{k,n}| \to |b_n|$ as $k \to \infty$ $(n \in \mathbb{N})$. The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=2}^{\infty} (\alpha_n |a_n| + \beta_n |b_n|)$ is nondecreasing sequence. Moreover, by Equation (19) it is bounded by $B - A$. Therefore, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=2}^{\infty} (\alpha_n |a_n| + \beta_n |b_n|) = \lim_{n\to\infty} S_n \leq B - A.$$

This gives the condition (8), and, in consequence, $f \in \mathcal{S}_{\mathcal{T}}^{**}(A,B)$, which completes the proof. □

Theorem 6. *We have:*

$$E\mathcal{S}_{\mathcal{T}}^{**}(A,B) = \{h_n : n \in \mathbb{N}\} \cup \{g_n : n \in \{2,3\ldots\}\},$$

where,

$$h_1(z) = z,\ h_n(z) = z - \frac{B-A}{\alpha_n} z^n,\ g_n(z) = z + \frac{B-A}{\beta_n} \overline{z}^n \qquad (20)$$
$$(n = 2,3,...;\ z \in \mathbb{U}).$$

Proof. Let $0 < a < 1$ and $g_n = af_1 + (1-a) f_2$,where $f_1, f_2 \in \mathcal{S}_{\mathcal{T}}^{**}(A,B)$ are given by Equation (17). Thus, by Equation (8) we get $|b_{1,n}| = |b_{2,n}| = (B-A)/\beta_n$, and consequently $a_{1,k} = a_{2,k} = 0$ $(k \in \{2,3\ldots\})$ and $b_{1,k} = b_{2,k} = 0$ $(k \in \{2,3\ldots\} \setminus \{n\})$. Thus, $g_n = f_1 = f_2$, and, in consequence, $g_n \in E\mathcal{S}_{\mathcal{T}}^{**}(A,B)$. In the same way, we prove that the functions h_n of the form (20) are the extreme points of the class $\mathcal{S}_{\mathcal{T}}^{**}(A,B)$. Suppose that $f \in E\mathcal{S}_{\mathcal{T}}^{**}(A,B)$ and f is not of the form (20). Then there exists $k \in \{2,3,...\}$ such that:

$$0 < |a_k| < (B-A)/\alpha_n \ \text{ or } \ 0 < |b_k| < (B-A)/\beta_n.$$

If $0 < |a_k| < (B-A)/\alpha_n$ and

$$a = \frac{|a_k|\,\alpha_k}{B-A},\ \varphi = \frac{1}{1-a}(f - ah_k),$$

then we obtain $0 < a < 1$, $h_k, \varphi \in \mathcal{S}_{\mathcal{T}}^{**}(A,B)$, $h_k \neq \varphi$, and

$$f = ah_k + (1-a)\varphi.$$

Therefore, $f \notin E\mathcal{S}_{\mathcal{T}}^{**}(A,B)$. Similarly, if $0 < |b_k| < (B-A)/\beta_n$ and

$$a = \frac{|b_k|\,\beta_k}{B-A},\ \phi(z) = \frac{1}{1-a}(f - ag_k),$$

then we obtain $0 < a < 1$, $g_k, \varphi \in \mathcal{S}_{\mathcal{T}}^{**}(A,B)$, $g_k \neq \phi$ and

$$f = ag_k + (1-a)\phi.$$

Thus we get $f \notin E\mathcal{S}_{\mathcal{T}}^{**}(A,B)$, which completes the proof of Theorem 6. □

4. Applications

It is clear that if the class:

$$\mathcal{F} = \{f_n \in \mathcal{H} : n \in \mathbb{N}\}$$

is locally uniformly bounded, then:

$$\overline{co}\mathcal{F} = \left\{ \sum_{n=1}^{\infty} \gamma_n f_n : \quad \sum_{n=1}^{\infty} \gamma_n = 1, \ \gamma_n \geq 0 \ \ (n \in \mathbb{N}) \right\}. \tag{21}$$

Corollary 1.

$$\mathcal{S}_{\mathcal{T}}^{**}(A,B) = \left\{ \sum_{n=1}^{\infty} (\gamma_n h_n + \delta_n g_n) : \ \sum_{n=1}^{\infty} (\gamma_n + \delta_n) = 1, \ \delta_1 = 0, \gamma_n, \delta_n \geq 0 \ \ (n \in \mathbb{N}) \right\}, \tag{22}$$

where h_n, g_n are defined by Equation (20).

Proof. By Theorem 5 and Lemma 1 we have:

$$\mathcal{S}_{\mathcal{T}}^{**}(A,B) = \overline{co}\mathcal{S}_{\mathcal{T}}^{**}(A,B) = \overline{co}E\mathcal{S}_{\mathcal{T}}^{**}(A,B).$$

Thus, by Theorem 6 and Equation (21) we have Equation (22). □

We observe, that the following real-valued functionals are convex and continuous on $\mathcal{H}$:

$$\mathcal{L}(f) = |a_n|, \ \mathcal{L}(f) = |b_n|, \ \mathcal{L}(f) = |f(z)|, \ \mathcal{L}(f) = |D_{\mathcal{H}} f(z)| \quad (f \in \mathcal{H}),$$

and

$$\mathcal{L}(f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^{\gamma} d\theta \right)^{1/\gamma} \quad (f \in \mathcal{H}, \ 0 < r < 1, \gamma \geq 1).$$

Thus, by using Theorem 6 and Lemma 2 we obtain the following two corollaries.

Corollary 2. *If $f \in \mathcal{S}_{\mathcal{T}}^{**}(A,B)$ is a function of the form (14), then:*

$$|a_n| \leq \frac{B-A}{\alpha_n}, \ |b_n| \leq \frac{B-A}{\beta_n} \quad (n = 2, 3, \ldots), \tag{23}$$

with α_n, β_n defined by Equation (9). The result is sharp. The functions h_n, g_n of the form (20) are the extremal functions.

Proof. Since For the extremal functions h_n and g_n we have $|a_n| = \frac{B-A}{\alpha_n}$ and $|b_n| = \frac{B-A}{\beta_n}$. Thus, by Lemma 2 we have Equation (23). □

Example 2. *In particular, since $\frac{B-A+1}{\alpha_3} > \frac{B-A}{\alpha_3}$ the polynomial:*

$$w(z) = z - z^2 - \frac{B-A+1}{\alpha_3} z^3 \quad (z \in \mathbb{U})$$

does not belong to the class $\mathcal{S}_{\mathcal{T}}^{**}(A,B)$.

Corollary 3. *Let $f \in \mathcal{S}_{\mathcal{T}}^{**}(A,B)$, $|z| = r < 1$. Then,*

$$r - \frac{B-A}{2(1+B)} r^2 \leq |f(z)| \leq r + \frac{B-A}{2(1+B)} r^2 \tag{24}$$

and

$$r - \frac{B-A}{1+B} r^2 \leq |D_{\mathcal{H}} f(z)| \leq r + \frac{B-A}{1+B} r^2. \tag{25}$$

The result is sharp. The function h_2 of the form (20) is the extremal function.

Proof. For the extremal functions h_n and g_n of the form (20) we have:

$$\begin{aligned} |h_n(z)| &\leq r + \frac{B-A}{\alpha_n} r^n \leq r + \frac{B-A}{2(1+B)} r^2 \quad (n = 2, 3 \ldots), \\ |g_n(z)| &\leq r + \frac{B-A}{\beta_n} r^n \leq r + \frac{B-A}{2(1+B)} r^2 \quad (n = 2, 3 \ldots), \\ |h_n(z)| &\geq r - \frac{B-A}{\alpha_n} r^n \geq r - \frac{B-A}{2(1+B)} r^2 \quad (n = 2, 3 \ldots), \\ |g_n(z)| &\geq r - \frac{B-A}{\beta_n} r^n \geq r - \frac{B-A}{2(1+B)} r^2 \quad (n = 2, 3 \ldots). \end{aligned}$$

Thus, by Lemma 2 we have Equation (24). Similarly, we prove Equation (25). □

Due to Littlewood [16] we consider the integral means inequalities for functions from the class $\mathcal{S}^{**}_{\mathcal{T}}(A, B)$.

Lemma 4. *[16] Let $f, g \in \mathcal{A}$. If $f \prec g$, then,*

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^\gamma d\theta \leq \int_0^{2\pi} \left| g(re^{i\theta}) \right|^\gamma d\theta \quad (0 < r < 1,\ \gamma > 0).$$

Lemma 5. *Let $0 < r < 1$, $\gamma > 0$. Then,*

$$\frac{1}{2\pi} \int_0^{2\pi} \left| h_n(re^{i\theta}) \right|^\gamma d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \left| h_2(re^{i\theta}) \right|^\gamma d\theta \quad (n = 1, 2, \cdots) \tag{26}$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \left| g_n(re^{i\theta}) \right|^\gamma d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \left| h_2(re^{i\theta}) \right|^\gamma d\theta \quad (n = 2, 3, \cdots), \tag{27}$$

where h_n and g_n are defined by Equation (20).

Proof. Let h_n and g_n are defined by Equation (20) and let $\widetilde{g}_n(z) = z + \frac{B-A}{\beta_n} z^n$ $(n = 2, 3, \cdots)$. Since $\frac{h_n(z)}{z} \prec \frac{h_2(z)}{z}$ and $\frac{\widetilde{g}(z)}{z} \prec \frac{h_2(z)}{z}$, by Lemma 4 we have:

$$\begin{aligned} \int_0^{2\pi} \left| h_n\left(re^{i\theta}\right) \right|^\gamma d\theta &\leq \int_0^{2\pi} \left| h_2\left(re^{i\theta}\right) \right|^\gamma d\theta, \\ \int_0^{2\pi} \left| g_n\left(re^{i\theta}\right) \right|^\gamma d\theta &= \int_0^{2\pi} \left| \widetilde{g}_n\left(re^{i\theta}\right) \right|^\gamma d\theta \leq \int_0^{2\pi} \left| h_2\left(re^{i\theta}\right) \right|^\gamma d\theta, \end{aligned}$$

which complete the proof. □

Corollary 4. *If $f \in \mathcal{S}^{**}_{\mathcal{T}}(A, B)$ then:*

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^\gamma d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \left| h_2(re^{i\theta}) \right|^\gamma d\theta$$

and

$$\frac{1}{2\pi}\int_{0}^{2\pi}\left|D_{\mathcal{H}}f(re^{i\theta})\right|^{\gamma}d\theta \le \frac{1}{2\pi}\int_{0}^{2\pi}\left|D_{\mathcal{H}}h_2(re^{i\theta})\right|^{\gamma}d\theta,$$

where $\gamma \ge 1, 0 < r < 1$ *and* h_2 *is the function defined by Equation* (20).

Remark 1. *Some new and also well-known results can be obtained by choosing the parameters* A, B *in the defined classes of functions* (*see for example* [6–9]). *In particular, for* $A = 2\alpha - 1$, $B = 1$ *we have results obtained by Ahuja and Jahangiri* [6] *(see also* [7,8]*), for* $A = 2b(\alpha - 1) + 1$, $B = 1$ *we have results obtained by Janteng and Halim* [9].

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