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Coincidence Continuation Theory for Multivalued Maps with Selections in a Given Class

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Abstract: This paper considers the topological transversality theorem for general multivalued maps which have selections in a given class of maps.

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1. Introduction

To motivate this study first fix a map Φ (an important case is when Φ is the identity). Many coincidence problems between a map F and Φ (i.e., finding a (coincidence) point x with $F(x) \cap \Phi(x) \neq \emptyset$) arise naturally in applications. For a complicated map F the idea here is to try to relate it to a simpler and solvable coincidence problem between a map G and Φ (i.e., we assume we have a (coincidence) point y with $G(y) \cap \Phi(y) \neq \emptyset$) where the map G is homotopic (in an appropriate way) to F and from this we hope to deduce that there is a coincidence point between F and Φ (i.e., we hope to deduce that there is a (coincidence) point x with $F(x) \cap \Phi(x) \neq \emptyset$). To achieve this we consider general (instead of specific) classes of maps and we present the notion of homotopy for this class of maps which are coincidence free on the boundary of the set considered. In particular, in this paper, we look at multivalued maps F and G with selections in a given class of maps and with $F \cong G$ in this setting. The topological transversality theorem in this setting will state that F is Φ–essential if and only if G is Φ–essential (essential maps were introduced in [1] and extended by many authors in [2–5]). In this paper we discuss the topological transversality theorem in a very general setting using a simple and effective approach. In this paper, we consider a generalization of Φ–essential maps, namely the d–Φ–essential maps.

2. Topological Transversality Theorems

A multivalued map G from a space X to a space Y is a correspondence which associates to every $x \in X$ a subset $G(x) \subseteq Y$. In this paper let E be a completely regular topological space and U an open subset of E.

We will consider classes **A**, **B** and **D** of maps.

Definition 1. *We say $F \in D(\overline{U}, E)$ (respectively $F \in B(\overline{U}, E)$) if $F : \overline{U} \to 2^E$ and $F \in \mathbf{D}(\overline{U}, E)$ (respectively $F \in \mathbf{B}(\overline{U}, E)$); here 2^E denotes the family of nonempty subsets of E and $\overline{U}$ denotes the closure of U in E.*

In this paper we use bold face only to indicate the properties of our maps and usually $D = \mathbf{D}$ etc. Examples of $F \in \mathbf{D}(\overline{U}, E)$ might be that $F : \overline{U} \to K(E)$ is an upper semicontinuous compact map and F has convex values or $F : \overline{U} \to K(E)$ is an upper semicontinuous compact map and F has acyclic values; here $K(E)$ denotes the family of nonempty compact subsets of E.

Definition 2. *We say $F \in A(\overline{U},E)$ if $F : \overline{U} \to 2^E$ and $F \in \mathbf{A}(\overline{U},E)$ and there exists a selection $\Psi \in D(\overline{U},E)$ of F.*

Remark 1. *Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and F a multifunction. We say $F \in PK(Z,W)$ if W is convex and there exists a map $S : Z \to W$ with $Z = \cup\{int\, S^{-1}(w) : w \in W\}$, $co\,(S(x)) \subseteq F(x)$ for $x \in Z$ and $S(x) \neq \emptyset$ for each $x \in Z$; here $S^{-1}(w) = \{z : w \in S(z)\}$, int denotes the interior and co denotes the convex hull. Let E be a Hausdorff topological vector space (note topological vector spaces are completely regular), U an open subset of E and $\overline{U}$ paracompact. In this case we say $F \in \mathbf{A}(\overline{U},E)$ if $F \in PK(\overline{U},E)$ is a compact map, and we say $\Psi \in \mathbf{D}(\overline{U},E)$ if Ψ is a single valued, continuous, compact map. Now [6] guarantees that there exists a continuous, compact selection $f : \overline{U} \to E$ of F.*

In this section we fix a $\Phi \in B(\overline{U},E)$ and now we present the notion of coincidence free on the boundary, Φ–essentiality and homotopy.

Definition 3. *We say $F \in A_{\partial U}(\overline{U},E)$ (respectively $F \in D_{\partial U}(\overline{U},E)$) if $F \in A(\overline{U},E)$ (respectively $F \in D(\overline{U},E)$) with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.*

Definition 4. *We say $F \in A_{\partial U}(\overline{U},E)$ is Φ–essential in $A_{\partial U}(\overline{U},E)$ if for any selection $\Psi \in D(\overline{U},E)$ of F and any map $J \in D_{\partial U}(\overline{U},E)$ with $J|_{\partial U} = \Psi|_{\partial U}$ there exists a $x \in U$ with $J\,(x) \cap \Phi\,(x) \neq \emptyset$.*

Remark 2. *If $F \in A_{\partial U}(\overline{U},E)$ is Φ–essential in $A_{\partial U}(\overline{U},E)$ and if $\Psi \in D(\overline{U},E)$ is any selection of F then there exists an $x \in U$ with $\Psi\,(x) \cap \Phi\,(x) \neq \emptyset$ (take $J = \Psi$ in Definition 4), and $\emptyset \neq \Psi\,(x) \cap \Phi\,(x) \subseteq F\,(x) \cap \Phi\,(x)$.*

Definition 5. *Let E be a completely regular (respectively, normal) topological space and let $\Psi,\ \Lambda \in D_{\partial U}(\overline{U},E)$. We say Ψ is homotopic to Λ in the class $D_{\partial U}(\overline{U},E)$ and we write $\Psi \cong \Lambda$ in $D_{\partial U}(\overline{U},E)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(\,.\,,\eta(\,.\,)) \in \mathbf{D}(\overline{U},E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, $\{x \in \overline{U} : \ \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact (respectively, closed), $H_0 = \Psi$ and $H_1 = \Lambda$ (here $H_t(x) = H(x,t)$).*

Remark 3. *It is of interest to note that in our results below alternatively we could use the following definition for $\cong$ in $D_{\partial U}(\overline{U},E)$: $\Psi \cong \Lambda$ in $D_{\partial U}(\overline{U},E)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H \in \mathbf{D}(\overline{U} \times [0,1],E)$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial\, U$ and $t \in (0,1)$ (here $H_t(x) = H(x,t)$), $\{x \in \overline{U} : \ \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact (respectively, closed), $H_0 = \Psi$ and $H_1 = \Lambda$. Note here if we use this definition then we will also assume for any map $\Theta \in \mathbf{D}(\overline{U} \times [0,1],E)$ and any map $f \in \mathbf{C}(\overline{U}, \overline{U} \times [0,1])$ then $\Theta \circ f \in \mathbf{D}(\overline{U},E)$; here $\mathbf{C}$ denotes the class of single valued continuous functions.*

Now we are in a position to define homotopy ($\cong$) in our class $A_{\partial U}(\overline{U},E)$.

Definition 6. *Let $F,\, G \in A_{\partial U}(\overline{U},E)$. We say F is homotopic to G in the class $A_{\partial U}(\overline{U},E)$ and we write $F \cong G$ in $A_{\partial U}(\overline{U},E)$ if for any selection $\Psi \in D_{\partial U}(\overline{U},E)$ (respectively, $\Lambda \in D_{\partial U}(\overline{U},E)$) of F (respectively, of G) we have $\Psi \cong \Lambda$ in $D_{\partial U}(\overline{U},E)$.*

Next, we present a simple and crucial result that will immediately yield the topological transversality theorem in this setting.

Theorem 1. *Let E be a completely regular (respectively, normal) topological space, U an open subset of E, $F \in A_{\partial U}(\overline{U},E)$ and $G \in A_{\partial U}(\overline{U},E)$ is Φ–essential in $A_{\partial U}(\overline{U},E)$. Suppose also*

$$\begin{cases} \text{for any selection } \ \Psi \in D_{\partial U}(\overline{U},E) \ \ (\text{respectively, } \ \Lambda \in D_{\partial U}(\overline{U},E)) \\ \text{of } \ F \ \ (\text{respectively, of } \ G) \text{ and any map } \ J \in D_{\partial U}(\overline{U},E) \\ \text{with } \ J|_{\partial U} = \Psi|_{\partial U} \ \ \text{we have } \ \Lambda \cong J \ \text{ in } \ D_{\partial U}(\overline{U},E). \end{cases} \tag{1}$$

Then F is Φ–essential in $A_{\partial U}(\overline{U},E)$.

Proof. Let $\Psi \in D_{\partial U}(\overline{U},E)$ be any selection of F and consider any map $J \in D_{\partial U}(\overline{U},E)$ with $J|_{\partial U} = \Psi|_{\partial U}$. It remains to show that there exists an $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$. Let $\Lambda \in D_{\partial U}(\overline{U},E)$ be any selection of G. Now (1) guarantees that there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(\,.\,,\eta(\,.\,)) \in \mathbf{D}(\overline{U},E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, $\{x \in \overline{U} :\ \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact (respectively, closed), $H_0 = \Lambda$, and $H_1 = J$ (here $H_t(x) = H(x,t)$). Let

$$\Omega = \left\{x \in \overline{U} :\ \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

Now since G is Φ–essential in $A_{\partial U}(\overline{U},E)$ then Remark 2 (note $H_0 = \Lambda$) guarantees that $\Omega \neq \emptyset$. Ω is compact (respectively, closed) if E is a completely regular (respectively, normal) topological space. Next note $\Omega \cap \partial U = \emptyset$ and now we can deduce that there exists a continuous map (called a Urysohn map) $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define a map R by $R(x) = H(x,\mu(x))$ for $x \in \overline{U}$. Note $R \in D_{\partial U}(\overline{U},E)$ with $R|_{\partial U} = H_0|_{\partial U} = \Lambda|_{\partial U}$. Now since G is Φ–essential in $A_{\partial U}(\overline{U},E)$ then there exists $x \in U$ with $R(x) \cap \Phi(x) \neq \emptyset$ (i.e., $H_{\mu(x)}(x) \cap \Phi(x) \neq \emptyset$) and so $x \in \Omega$. As a result $\mu(x) = 1$ so $\emptyset \neq H_1(x) \cap \Phi(x) = J(x) \cap \Phi(x)$, and we are finished. $\square$

Now assume

$$\cong \ \text{ in } D_{\partial U}(\overline{U},E) \text{ is an equivalence relation} \tag{2}$$

and

$$\begin{cases} \text{if } F \in A_{\partial U}(\overline{U},E) \text{ and if } \Psi \in D_{\partial U}(\overline{U},E) \text{ is any} \\ \text{selection of } F \text{ and } J \in D_{\partial U}(\overline{U},E) \text{ is any map} \\ \text{with } \Psi|_{\partial U} = J|_{\partial U} \text{ then } \Psi \cong J \text{ in } D_{\partial U}(\overline{U},E). \end{cases} \tag{3}$$

Theorem 2. *Let E be a completely regular (respectively, normal) topological space, U an open subset of E, and assume (2) and (3) hold. Suppose F and G are two maps in $A_{\partial U}(\overline{U},E)$ with $F \cong G$ in $A_{\partial U}(\overline{U},E)$. Now F is Φ–essential in $A_{\partial U}(\overline{U},E)$ if and only if G is Φ–essential in $A_{\partial U}(\overline{U},E)$.*

Proof. Assume G is Φ–essential in $A_{\partial U}(\overline{U},E)$. We use Theorem 1 to show F is Φ–essential in $A_{\partial U}(\overline{U},E)$. Let $\Psi \in D_{\partial U}(\overline{U},E)$ be any selection of F, $\Lambda \in D_{\partial U}(\overline{U},E)$ be any selection of G and consider any map $J \in D_{\partial U}(\overline{U},E)$ with $J|_{\partial U} = \Psi|_{\partial U}$. Now (3) guarantees that $\Psi \cong J$ in $D_{\partial U}(\overline{U},E)$ and this together with $F \cong G$ in $A_{\partial U}(\overline{U},E)$ (so $\Psi \cong \Lambda$ in $D_{\partial U}(\overline{U},E)$) and (2) guarantees that $\Lambda \cong J$ in $D_{\partial U}(\overline{U},E)$. Thus (1) holds so Theorem 1 guarantees that F is Φ–essential in $A_{\partial U}(\overline{U},E)$. A similar argument shows if F is Φ–essential in $A_{\partial U}(\overline{U},E)$ then G is Φ–essential in $A_{\partial U}(\overline{U},E)$. $\square$

Now we consider a generalization of Φ–essential maps, namely the d–Φ–essential maps (these maps were motivated from the notion of the degree of a map). Let E be a completely regular topological space and U an open subset of E. For any map $\Psi \in D(\overline{U},E)$ let $\Psi^\star = I \times \Psi : \overline{U} \to 2^{\overline{U} \times E}$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (\Psi^\star)^{-1}\ (B) \right\} \cup \{\emptyset\} \ \to \ K \tag{4}$$

be any map with values in the nonempty set K; here $B = \{(x,\Phi(x)) :\ x \in \overline{U}\}$.

Next we present the notions of d–Φ–essentiality and homotopy.

Definition 7. *Let $F \in A_{\partial U}(\overline{U},E)$ and write $F^\star = I \times F$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d–Φ–essential if for any selection $\Psi \in D(\overline{U},E)$ of F and any map $J \in D_{\partial U}(\overline{U},E)$ with $J|_{\partial U} = \Psi|_{\partial U}$ we have that $d\left((\Psi^\star)^{-1}\ (B)\right) = d\left((J^\star)^{-1}\ (B)\right) \neq d(\emptyset)$; here $\Psi^\star = I \times \Psi$ and $J^\star = I \times J$.*

Remark 4. *If $F^\star$ is d–Φ–essential then for any selection $\Psi \in D(\overline{U}, E)$ of F (with $\Psi^\star = I \times \Psi$) we have*

$$\emptyset \neq (\Psi^\star)^{-1}(B) = \{x \in \overline{U} : (x, \Psi(x)) \cap (x, \Phi(x)) \neq \emptyset\},$$

so there exists a $x \in U$ with $(x, \Psi(x)) \cap (x, \Phi(x)) \neq \emptyset$ (i.e., $\Phi(x) \cap \Psi(x) \neq \emptyset$ so in particular $\Phi(x) \cap F(x) \neq \emptyset$).

Now we define homotopy in this setting for our class $D_{\partial U}(\overline{U}, E)$.

Definition 8. *Let E be a completely regular (respectively, normal) topological space and let $\Psi, \Lambda \in D_{\partial U}(\overline{U}, E)$. We say Ψ is homotopic to Λ in the class $D_{\partial U}(\overline{U}, E)$ and we write $\Psi \cong \Lambda$ in $D_{\partial U}(\overline{U}, E)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(\,.\,, \eta(\,.\,)) \in \mathbf{D}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, $\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x,t)) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact (respectively, closed), $H_0 = \Psi$ and $H_1 = \Lambda$ (here $H_t(x) = H(x,t)$).*

Remark 5. *There is an analogue Remark 3 in this situation.*

Definition 9. *Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if for any selection $\Psi \in D_{\partial U}(\overline{U}, E)$ (respectively, $\Lambda \in D_{\partial U}(\overline{U}, E)$) of F (respectively, of G) we have $\Psi \cong \Lambda$ in $D_{\partial U}(\overline{U}, E)$ (Definition 8).*

Theorem 3. *Let E be a completely regular (respectively, normal) topological space, U an open subset of E, $B = \{(x, \Phi(x)) : x \in \overline{U}\}$, d is defined in (4), $F \in A_{\partial U}(\overline{U}, E)$, $G \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$ and $G^\star = I \times G$. Suppose $G^\star$ is d–Φ–essential and*

$$\begin{cases} \text{for any selection } \Psi \in D_{\partial U}(\overline{U}, E) \text{ (respectively, } \Lambda \in D_{\partial U}(\overline{U}, E)) \\ \text{of } F \text{ (respectively, of } G) \text{ and any map } J \in D_{\partial U}(\overline{U}, E) \text{ with} \\ J|_{\partial U} = \Psi|_{\partial U} \text{ we have } \Lambda \cong J \text{ in } D_{\partial U}(\overline{U}, E) \text{ (Definition 8) and} \\ d\left((\Psi^\star)^{-1}(B)\right) = d\left((\Lambda^\star)^{-1}(B)\right); \text{ here } \Psi^\star = I \times \Psi \text{ and } \Lambda^\star = I \times \Lambda. \end{cases} \tag{5}$$

Then $F^\star$ is d–Φ–essential.

Proof. Let $\Psi \in D_{\partial U}(\overline{U}, E)$ be any selection of F and consider any map $J \in D_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \Psi|_{\partial U}$. It remains to show $d\left((\Psi^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$; here $\Psi^\star = I \times \Psi$ and $J^\star = I \times J$. Let $\Lambda \in D_{\partial U}(\overline{U}, E)$ be any selection of G and let $\Lambda^\star = I \times \Lambda$. Now (5) guarantees that there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(\,.\,, \eta(\,.\,)) \in \mathbf{D}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, $\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x,t)) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact (respectively, closed), $H_0 = \Lambda$ and $H_1 = J$ (here $H_t(x) = H(x,t)$) and $d\left((\Psi^\star)^{-1}(B)\right) = d\left((\Lambda^\star)^{-1}(B)\right)$. Let

$$\Omega = \{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x,t)) \neq \emptyset \text{ for some } t \in [0,1]\}.$$

Now $\Omega \neq \emptyset$ since $G^\star$ is d–Φ–essential (and $H_0 = \Lambda$). Ω is compact (respectively, closed) if E is a completely regular (respectively, normal) topological space. Next note $\Omega \cap \partial U = \emptyset$ and so there exists a Urysohn map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define a map R by $R(x) = H(x, \mu(x))$ for $x \in \overline{U}$ and write $R^\star = I \times R$. Note $R \in D_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = H_0|_{\partial U} = \Lambda|_{\partial U}$. Since $G^\star$ is d–Φ–essential then

$$d\left((\Lambda^\star)^{-1}(B)\right) = d\left((R^\star)^{-1}(B)\right) \neq d(\emptyset). \tag{6}$$

Now since $\mu(\Omega) = 1$ we have

$$\begin{aligned} (R^\star)^{-1}(B) &= \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, \mu(x))) \neq \emptyset\right\} \\ &= \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, 1)) \neq \emptyset\right\} = (J^\star)^{-1}(B), \end{aligned}$$

so from (6) we have $d\left((\Lambda^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. Now combine with the above and we have $d\left((\Psi^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. $\square$

Now assume

$$\cong \ \text{ in } \ D_{\partial U}(\overline{U}, E) \ \text{(Definition 8) is an equivalence relation} \tag{7}$$

and

$$\begin{cases} \text{if } \ F \in A_{\partial U}(\overline{U}, E) \ \text{ and if } \ \Psi \in D_{\partial U}(\overline{U}, E) \ \text{ is any selection} \\ \text{of } \ F \ \text{and } \ J \in D_{\partial U}(\overline{U}, E) \ \text{ is any map with } \ \Psi|_{\partial U} = J|_{\partial U} \\ \text{then } \ \Psi \cong J \ \text{ in } \ D_{\partial U}(\overline{U}, E) \text{ (Definition 8).} \end{cases} \tag{8}$$

Now we establish the topological transversality theorem in this setting.

Theorem 4. *Let E be a completely regular (respectively, normal) topological space, U an open subset of E, $B = \left\{(x, \Phi(x)) : x \in \overline{U}\right\}$, d is defined in* (4)*, and assume* (7) *and* (8) *hold. Suppose F and G are two maps in $A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$, $G^\star = I \times G$ and $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ (Definition 9). Then $F^\star$ is d–Φ–essential if and only if $G^\star$ is d–Φ–essential.*

Proof. Assume $G^\star$ is d–Φ–essential. Let $\Psi \in D_{\partial U}(\overline{U}, E)$ be any selection of F, $\Lambda \in D_{\partial U}(\overline{U}, E)$ be any selection of G and consider any map $J \in D_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \Psi|_{\partial U}$. If we show (5) then $F^\star$ is d–Φ–essential from Theorem 3. Now (8) guarantees that $\Psi \cong J$ in $D_{\partial U}(\overline{U}, E)$ (Definition 8) and this together with $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ (Definition 9) (so $\Psi \cong \Lambda$ in $D_{\partial U}(\overline{U}, E)$ (Definition 8)) guarantees that $\Lambda \cong J$ in $D_{\partial U}(\overline{U}, E)$ (Definition 8). To complete (5) it remains to show $d\left((\Psi^\star)^{-1}(B)\right) = d\left((\Lambda^\star)^{-1}(B)\right)$; here $\Psi^\star = I \times \Psi$ and $\Lambda^\star = I \times \Lambda$. Note $G \cong F$ in $A_{\partial U}(\overline{U}, E)$ (Definition 9) so let $H : \overline{U} \times [0,1] \to 2^E$ with $H(\,.\,, \eta(\,.\,)) \in \mathbf{D}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, $\left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x,t)) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$ is compact (respectively, closed), $H_0 = \Lambda$ and $H_1 = \Psi$ (here $H_t(x) = H(x,t)$). Let

$$\Omega = \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x,t)) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}.$$

Now $\Omega \neq \emptyset$ and there exists a Urysohn map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define the map R by $R(x) = H(x, \mu(x))$ and write $R^\star = I \times R$. Now $R \in D_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = \Lambda|_{\partial U}$ so since $G^\star$ is d–Φ–essential then $d\left((\Lambda^\star)^{-1}(B)\right) = d\left((R^\star)^{-1}(B)\right) \neq d(\emptyset)$. Now since $\mu(\Omega) = 1$ we have (see the argument in Theorem 3) $(R^\star)^{-1}(B) = (\Psi^\star)^{-1}(B)$ and as a result we have $d\left((\Psi^\star)^{-1}(B)\right) = d\left((\Lambda^\star)^{-1}(B)\right)$. $\square$

Remark 6. *It is also easy to extend the above ideas to other natural situations [3,4]. Let E be a (Hausdorff) topological vector space (so automatically completely regular), Y a topological vector space, and U an open subset of E. Let $L : dom\, L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\, L$ is a vector subspace of E. Finally $T : E \to Y$ will be a linear, continuous single valued map with $L + T : dom\, L \to Y$ an isomorphism (i.e., a linear homeomorphism); for convenience we say $T \in H_L(E, Y)$. We say $F \in A(\overline{U}, Y; L, T)$ if $(L+T)^{-1}(F+T) \in A(\overline{U}, E)$ and we could discuss Φ–essential and d–Φ–essential in this situation.*

Finally, we consider the above in the weak topology situation. Let X be a Hausdorff locally convex topological vector space and U a weakly open subset of C where C is a closed convex subset of X. We will consider classes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ of maps.

Definition 10. *We say $F \in WD(\overline{U^w}, C)$ (respectively $F \in WB(\overline{U^w}, C)$) if $F : \overline{U^w} \to 2^C$ and $F \in \mathbf{D}(\overline{U^w}, C)$ (respectively $F \in \mathbf{B}(\overline{U^w}, C)$); here $\overline{U^w}$ denotes the weak boundary of U in C.*

Definition 11. *We say $F \in WA(\overline{U^w}, C)$ if $F : \overline{U^w} \to 2^C$ and $F \in \mathbf{A}(\overline{U^w}, C)$ and there exists a selection $\Psi \in WD(\overline{U^w}, C)$ of F.*

Now we fix a $\Phi \in WB(\overline{U^w}, C)$ and present the notion of coincidence free on the boundary, Φ–essentiality and homotopy in this setting.

Definition 12. *We say $F \in WA_{\partial U}(\overline{U^w}, C)$ (respectively $F \in WD_{\partial U}(\overline{U^w}, C)$) if $F \in WA(\overline{U^w}, C)$ (respectively $F \in WD(\overline{U^w}, C)$) with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the weak boundary of U in C.*

Definition 13. *We say $F \in WA_{\partial U}(\overline{U^w}, C)$ is Φ–essential in $WA_{\partial U}(\overline{U^w}, C)$ if for any selection $\Psi \in WD(\overline{U^w}, C)$ of F and any map $J \in WD_{\partial U}(\overline{U^w}, C)$ with $J|_{\partial U} = \Psi|_{\partial U}$ there exists a $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$.*

Definition 14. *Let Ψ, $\Lambda \in WD_{\partial U}(\overline{U^w}, C)$. We say $\Psi \cong \Lambda$ in $WD_{\partial U}(\overline{U^w}, C)$ if there exists a map $H : \overline{U^w} \times [0,1] \to 2^C$ with $H(\,.\,, \eta(\,.\,)) \in \mathbf{D}(\overline{U^w}, C)$ for any weakly continuous function $\eta : \overline{U^w} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, $\{x \in \overline{U^w} : \ \Phi(x) \cap H(x,t) \neq \emptyset \ \text{for some} \ t \in [0,1]\}$ is weakly compact, $H_0 = \Psi$ and $H_1 = \Lambda$ (here $H_t(x) = H(x,t)$).*

Definition 15. *Let F, $G \in WA_{\partial U}(\overline{U^w}, C)$. We say $F \cong G$ in $WA_{\partial U}(\overline{U^w}, C)$ if for any selection $\Psi \in WD_{\partial U}(\overline{U^w}, C)$ (respectively, $\Lambda \in WD_{\partial U}(\overline{U^w}, C)$) of F (respectively, of G) we have $\Psi \cong \Lambda$ in $WD_{\partial U}(\overline{U^w}, C)$.*

Theorem 5. *Let X be a Hausdorff locally convex topological vector space and U a weakly open subset of C where C is a closed convex subset of X. Suppose $F \in WA_{\partial U}(\overline{U^w}, C)$ and $G \in WA_{\partial U}(\overline{U^w}, C)$ is Φ–essential in $WA_{\partial U}(\overline{U^w}, C)$ and*

$$\begin{cases} \text{for any selection } \Psi \in WD_{\partial U}(\overline{U^w}, C) \text{ (respectively, } \Lambda \in WD_{\partial U}(\overline{U^w}, C)) \\ \text{of } F \text{ (respectively, of } G) \text{ and any map } J \in WD_{\partial U}(\overline{U^w}, C) \\ \text{with } J|_{\partial U} = \Psi|_{\partial U} \text{ we have } \Lambda \cong J \text{ in } WD_{\partial U}(\overline{U^w}, C). \end{cases} \tag{9}$$

Then F is Φ–essential in $WA_{\partial U}(\overline{U^w}, C)$.

Proof. A slight modification of the argument in Theorem 1 guarantees the result; we just need to note that $X = (X, w)$, the space X endowed with the weak topology, is completely regular. □

Assume

$$\cong \ \text{ in } WD_{\partial U}(\overline{U^w}, C) \text{ is an equivalence relation} \tag{10}$$

and

$$\begin{cases} \text{if } F \in WA_{\partial U}(\overline{U^w}, C) \text{ and if } \Psi \in WD_{\partial U}(\overline{U^w}, C) \text{ is any} \\ \text{selection of } F \text{ and } J \in WD_{\partial U}(\overline{U^w}, C) \text{ is any map} \\ \text{with } \Psi|_{\partial U} = J|_{\partial U} \text{ then } \Psi \cong J \text{ in } WD_{\partial U}(\overline{U^w}, C). \end{cases} \tag{11}$$

A slight modification of the proof of Theorem 2 guarantees the topological transversality theorem in this setting.

Theorem 6. *Let X be a Hausdorff locally convex topological vector space and U a weakly open subset of C where C is a closed convex subset of X and assume* (10) *and* (11) *hold. Suppose F and G are two maps in $WA_{\partial U}(\overline{U^w}, C)$ with $F \cong G$ in $WA_{\partial U}(\overline{U^w}, C)$. Now F is Φ–essential in $WA_{\partial U}(\overline{U^w}, C)$ if and only if G is Φ–essential in $WA_{\partial U}(\overline{U^w}, C)$.*

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