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Integral Representation of Coherent Lower Previsions by Super-Additive Integrals

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Abstract: Coherent lower previsions generalize the expected values and they are defined on the class of all real random variables on a finite non-empty set. Well known construction of coherent lower previsions by means of lower probabilities, or by means of super-modular capacities-based Choquet integrals, do not cover this important class of functionals on real random variables. In this paper, a new approach to the construction of coherent lower previsions acting on a finite space is proposed, exemplified and studied. It is based on special decomposition integrals recently introduced by Even and Lehrer, in our case the considered decomposition systems being single collections and thus called collection integrals. In special case when these integrals, defined for non-negative random variables only, are shift-invariant, we extend them to the class of all real random variables, thus obtaining so called super-additive integrals. Our proposed construction can be seen then as a normalized super-additive integral. We discuss and exemplify several particular cases, for example, when collections determine a coherent lower prevision for any monotone set function. For some particular collections, only particular set functions can be considered for our construction. Conjugated coherent upper previsions are also considered.

Keywords: coherent lower prevision; collection integral; shift-invariance; super-additive integral

1. Introduction

One of the major tools supporting the standard probability theory is the notion of expectation of a random variable. In generalizations of the probability theory, for example, in the theory of imprecise probabilities, one needs to have a reasonable counterpart of the expected value. This challenging task is realized by means of coherent lower (upper) previsions. Recall that coherent lower previsions are functionals defined on the linear space of all random variables satisfying the axioms of coherence [1] and they are related to coherent lower probabilities. However, they need be not determined by coherent lower probabilities since in some cases, when restricted to events, different coherent lower previsions may yield the same lower probability ([1], Section 2.7.3). Another tool to construct coherent lower previsions is the Choquet integral. Again, they need not be defined always by Choquet integral with respect to a coherent lower probability, because the Choquet integral satisfies the super-additivity, which is one of the defining property of a coherent lower previsions [1], if and only if it is defined with respect to a super-modular lower probability. So, there is a need to introduce some new approaches to the construction of coherent lower (upper) previsions. This need has motivated our study of some new types of super-additive integrals in our preliminary work [2]. In this paper, a deeper study of the problem how to construct coherent lower (upper) previsions by means of super-additive integrals with respect to monotone measures (capacities) is presented, illustrated by numerous examples. Several new types of monotone set functions, such as plausibility and belief measures, possibility

and necessity measures, coherent upper and lower probability measures, monotone measures, and so forth., have been introduced in the past decades to enrich the framework of classical σ-additive measures to represent several types of uncertainty. These measures were axiomatically characterized and relations among them were deeply studied in numerous works, such as Ref. [1], for example.

Classical Lebesgue integral is heavily based on the additivity (and continuity in the case of infinite spaces) of the considered measures. Therefore, new types of set functions generalizing the classical measures require some modified approach to integration, which, typically, leads to the non-linearity of the introduced integrals. Maybe the most famous among these integrals is Choquet integral [3,4], which extends Lebesgue integral (i.e., if the considered measure is a classical one, then both these integrals coincide). As another non-linear integral recall Shilkret integral [5], which is well defined for any of the above mentioned monotone set functions, but it coincides with the Lebesgue integral only in the case when Dirac measures are considered. Both Choquet and Shilkret integrals are based on the standard real operations of addition and multiplication (what is not the case of Sugeno integral [6], for example). Recently, an interesting class of non-linear integrals, called decomposition integrals and considering the standard arithmetic operations $+$ and $\cdot$, was introduced by Even and Lehrer [7], see also Ref. [8]. This class of integrals contains both Choquet and Shilkret integrals, but also the PAN integral [9,10] and the concave integral [11].

The main aim of this paper is a new look on constructions of generalized expectations of random variables, namely of coherent lower previsions and coherent upper previsions [12,13]. Recall that one of basic characterizations of the coherent lower previsions is the super-additivity of this functional. Observe that an important class of coherent lower previsions is identified with the (asymmetric) Choquet integral [3] with respect to a supermodular capacity (ensuring the super-additivity of the related Choquet integral), [4,13]. Inspired by this fact, also some other super-additive integral-based functionals could be considered. Recently, we have introduced collection integrals [2], that is, decomposition integrals [7] with respect to singleton decomposition systems, that is, decomposition systems consisting from a single collection. Some first results relating coherent lower (upper) previsions and collection integrals were already introduced in Ref. [2]. In this paper, a significant extension of the results from Ref. [2] is given, yielding, among others, several interesting by-product results.

Given a non-empty set Ω, a super-additive integral, with respect to a collection $\mathcal{C}$ and a monotone set function μ, is introduced in Ref. [2] to give an integral representation of a coherent lower prevision defined on the linear space of all random variables on Ω. Note that due to the finiteness of Ω, any considered random variable is bounded. Different integral representation of coherent upper conditional previsions in terms of Choquet integral, pan integral and concave integral has been analysed in Ref. [14] where it has been proven that the pan integral and the concave integral with respect to a sub-additive capacity coincide and they represent a coherent upper probability if and only if the underlying capacity is additive.

In this paper the construction method of coherent lower prevision based on the super-additive integral is complemented with motivations and examples. Given a finite non-empty set Ω, a collection $\mathcal{C}$ and a monotone set function μ, in Ref. [15] a collection integral is defined for non-negative functions as a particular case of decomposition integrals; super-additive integrals are extension of shift-invariant collection integrals to the class of all functions. If collections $\mathcal{C}$ are classes of disjoint subsets of Ω, the super-additive integral can be defined for any monotone set function μ since the underlying collection integral is shift-invariant and so it can be used to construct a coherent lower prevision. Moreover for any collection $\mathcal{C}$ there exists a monotone set function μ, which asses the value 1 to the biggest set from $\mathcal{C}$ and the value 0 to all other sets from 2^Ω, so that the corresponding collection integral is shift-invariant and it can be used to define a super-additive integral and a coherent lower prevision.

Collections $\mathcal{C}$ and monotone set functions μ are given such that the corresponding collection integrals are not shift-invariant. In Example 8 it is shown that the collection integral with respect to a collection $\mathcal{C}$ consisting of two non-disjoint sets that do not form a chain, is not shift-invariant for any

(non-trivial) capacity; so its extension cannot be used to define coherent lower previsions. Examples of coherent lower previsions constructed by super-additive integrals are given in the following cases: (a) the collection $\mathcal{C}$ is a chain and the capacity μ is such that the capacity of the biggest set of the chain is positive; (b) the collection $\mathcal{C}$ consists of disjoint sets and μ is any monotone set function, (c) the collection $\mathcal{C}$ is the power set of Ω and μ is a super-modular monotone set function; in this case the super-additive integral is the asymmetric Choquet integral with real values. If the monotone set function μ is not super-modular the corresponding collection integral may be not shift-invariant; an example of capacity is given such that the corresponding collection integral is shift-invariant and the super-additive integral defines a coherent lower prevision and an example of monotone set function, which does not define a coherent lower prevision, is given. Related coherent upper previsions are also defined.

The paper is organized as follows. In the next section, some preliminaries important for the rest of the paper are given. Section 3 extends the collection integrals which are shift-invariant (and acting on non-negative random variables only) to a functional acting on the space of all (bounded) random variables, preserving the original super-additivity of collection integrals. The main core of this paper is contained in Section 4, where, based on our results from Section 3, a novel construction method for coherent lower (upper) previsions is proposed and exemplified. In Section 5, a further generalization of the proposed construction method for coherent lower (upper) previsions are introduced and exemplified. Finally, some concluding remarks are added.

2. Preliminaries

Throughout this paper, we will consider a finite universe Ω equipped with σ-algebra 2^{Ω}. We denote by $\mathcal{L}$ the set of all random variables on $(\Omega, 2^{\Omega})$, that is, the set of all functions $X: \Omega \to \mathbb{R}$. Similarly, $\mathcal{L}_+$ denotes the set of all non-negative random variables from $\mathcal{L}$, and $\mathcal{L}_0$ the set of all random variables from $\mathcal{L}_+$ attaining the value 0 (i.e., $\min X = 0$). Obviously, $\mathcal{L}_0 \subset \mathcal{L}_+ \subset \mathcal{L}$ and $\mathcal{L}$ is a linear space over the field $\mathbb{R}$.

A *monotone set function* or *capacity* is any set function $\mu: 2^{\Omega} \to [0, \infty[$ such that $\mu(\varnothing) = 0$ and $A \subseteq B$ implies $\mu(A) \leq \mu(B)$.

A *coherent lower prevision* $\underline{P}: \mathcal{L} \to \mathbb{R}$ is a functional such that

1. $\underline{P}(X) \geq \inf X$;
2. $\underline{P}(\lambda X) = \lambda \underline{P}(X)$; and
3. $\underline{P}(X + Y) \geq \underline{P}(X) + \underline{P}(Y)$;

for all $X, Y \in \mathcal{L}$ and $\lambda \geq 0$ [1,13]. A *coherent upper prevision* $\overline{P}: \mathcal{L} \to \mathbb{R}$ [1] is defined by the conjugacy property $\overline{P}(X) = -\underline{P}(-X)$. From axioms 1-3 we obtain $\inf X \leq \underline{P}(X) \leq \overline{P}(X) \leq \sup X$ and in particular $\underline{P}(I_{\Omega}) = \overline{P}(I_{\Omega}) = 1$.

A coherent lower prevision, defined on $\mathcal{L}$ and such that $\underline{P}(X) = \overline{P}(X)$ is called a *linear prevision*, see, for example, Ref. [16–20], and it is a linear, positive and positively homogenous functional on $\mathcal{L}$, see Ref. ([1], Corollary 2.8.5).

For $E \in 2^{\Omega}$ the indicator function 1_E is defined by

$$1_E(\omega) = \begin{cases} 1, & \text{if } \omega \in E, \\ 0, & \text{otherwise,} \end{cases}$$

for all $\omega \in \Omega$. A *coherent lower probability* is the restriction of a linear prevision $\underline{P}$ to the class of all indicator functions 1_E with $E \in 2^{\Omega}$.

A functional $\underline{P}$ is coherent if and only if there is a class $M(\underline{P})$ of linear previsions defined on the class of all random variables defined on Ω such that $\underline{P}$ is dominated by every $P \in M(\overline{P})$, that is, $\underline{P} = \inf\{P : P \in M\}$, see, for example, Ref. ([1], Section 3.3.3); in this case $\underline{P}$ is called the *lower envelope* of linear previsions.

Knowing the lower and upper probabilities for events does not determine lower and upper previsions for other random variables. Levi ([21], pp. 407, 416–417) in his example for Case 5 involves two different (closed, convex) sets of probabilities that have the same lower and upper expectations for indicator functions, but where these two sets induce different judgments of E-admissibility for a given decision problem.

The following example shows that coherent lower probabilities may not determine coherent lower previsions since different lower previsions yield the same lower probability when they are restricted to events (for a different example, see, for example, Walley ([1], Section 2.7.3).

Example 1. *Let* $\Omega = \{a, b, c, d\}$ *and let* $P_i = (P_i(a), P_i(b), P_i(c), P_i(d))$ *for* $1 \leq i \leq 4$ *be the probabilities defined on the atoms of* Ω *by*

$$P_1 = \left(\frac{3}{4}, \frac{1}{4}, 0, 0\right); \quad P_2 = \left(\frac{1}{4}, 0, 0, \frac{3}{4}\right); \quad P_3 = \left(\frac{3}{4}, 0, \frac{1}{4}, 0\right); \quad and \quad P_4 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

Each P_i *has a unique extension to a linear prevision defined on* $\mathcal{L}$ *by*

$$P_i(X) = \sum_{\omega \in \Omega} P_i(\omega) X(\omega).$$

Several different coherent lower previsions can be constructed as lower envelopes of these linear previsions such that the lower previsions yield the same lower probabilities when they are restricted to the indicator functions. Let

$$\underline{P}_1 = \min\{P_1, P_2\}; \quad \underline{P}_2 = \min\{P_1, P_2, P_3\}; \quad \underline{P}_3 = \min\{P_1, P_2, P_4\}; \quad \underline{P}_4 = \min\{P_1, P_2, P_3, P_4\}.$$

Let $\underline{P}_i = (\underline{P}_i(a), \underline{P}_i(b), \underline{P}_i(c), \underline{P}_i(d))$ *so we obtain that* $\underline{P}_1, \underline{P}_2, \underline{P}_3, \underline{P}_4$ *are coherent lower previsions which coincide on the indicator functions since they coincide on the atoms of* Ω, *that is,*

$$\underline{P}_1 = \underline{P}_2 = \underline{P}_3 = \underline{P}_4 = \left(\frac{1}{4}, 0, 0, 0\right).$$

Nevertheless, they are different coherent lower previsions because, if we consider the random variable given by $X = 1_{\{a\}} - 1_{\{b\}}$, *we obtain that*

$$\underline{P}_1(X) = \underline{P}_2(X) = \frac{1}{4}; \quad and \quad \underline{P}_3(X) = \underline{P}_4(X) = 0.$$

Moreover coherent lower previsions can be defined by Choquet integral with respect to a coherent lower probability if and only if the underlying coherent lower prevision is super-modular, that is, if and only if

$$\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B).$$

In this case, Choquet integral is super-additive by the super-additivity theorem, see, for example, Ref. [3].

As Ω is finite and μ is defined on 2^Ω, denote by $A_1, ..., A_n$ the atoms of Ω, which are the minimal elements of Ω. If the atoms A_i are enumerated so that $x_i = X(A_i)$ are in descending order, that is, $x_1 \geq x_2 \geq \cdots \geq x_n$ and $x_{n+1} = 0$, Choquet integral with respect to μ is given by

$$\int^{Ch} X \, d\mu = \sum_{i=1}^{n} (x_i - x_{i+1}) \mu(S_i),$$

where $S_i = A_1 \cup A_2 ... \cup A_i$, and $x_{n+1} = 0$.

Example 2. *Let $\Omega = \{a, b, c, d\}$ and let $P_i = (P_i(\{a\}), P_i(\{b\}), P_i(\{c\}), P_i(\{d\}))$ for $i \in \{1, 2\}$ be probabilities of atoms of Ω given by*

$$P_1 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right); \quad and \quad P_2 = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right);$$

and let $\underline{\mu}$ be a coherent lower probability defined by $\underline{\mu}(A) = \min\{P_1(A), P_2(A)\}$. Let $A = \{a, b\}$ and $B = \{a, c\}$; and note that $\underline{\mu}$ is not super-modular since

$$\underline{\mu}(A \cup B) + \underline{\mu}(A \cap B) = \frac{3}{4} + 0 < \frac{1}{2} + \frac{1}{2} = \underline{\mu}(A) + \underline{\mu}(B)$$

and thus

$$\int^{\mathrm{Ch}} \left(1_A + 1_B\right) \mathrm{d}\underline{\mu} = (2-1)\underline{\mu}(\{a\}) + (1-0)\underline{\mu}(\{a, b, c\}) = \frac{3}{4}$$

$$< \frac{1}{2} + \frac{1}{2} = \int^{\mathrm{Ch}} 1_A \, \mathrm{d}\underline{\mu} + \int^{\mathrm{Ch}} 1_B \, \mathrm{d}\underline{\mu},$$

that is, Choquet integral with respect to $\underline{\mu}$ is not super-additive.

In the next sections we introduce a new integral, called a *super-additive integral* , to construct coherent lower previsions and to give their integral representation.

3. Super-Additive Integral Defined by Shift-Invariant Collection Integral

In this section a construction method for coherent lower previsions is proposed; to do that we start by proving a super-additivity property of the collection integral and we need to restrict to a special class of collection integrals that allow one to extend their domain to all functions while preserving super-additivity.

A collection integral was introduced in Ref. [15] as a special case of decomposition integrals, see, for example, Ref. [7,8]. A collection $\mathcal{C}$ is a non-empty subset of $2^\Omega \setminus \{\emptyset\}$.

Definition 1. *A collection integral with respect to a collection $\mathcal{C}$ and a monotone set function μ is a functional*

$$I_{\mathcal{C}}^{\mu}: \mathcal{L}_+ \to [0, \infty[: X \mapsto \bigvee \left\{\sum_{A \in \mathcal{C}} \alpha_A \mu(A): \sum_{A \in \mathcal{C}} \alpha_A 1_A \leq X, \alpha_A \geq 0\right\}.$$

where $\bigvee$ denotes the supremum.

Proposition 1. *$I_{\mathcal{C}}^{\mu}$ is a super-additive functional, that is, $I_{\mathcal{C}}^{\mu}(X + Y) \geq I_{\mathcal{C}}^{\mu}(X) + I_{\mathcal{C}}^{\mu}(Y)$ for all $X, Y \in \mathcal{L}_+$.*

Proof. For $A \in \mathcal{C}$ let $\alpha_A, \beta_A \geq 0$ be non-negative real numbers such that

$$\sum_{A \in \mathcal{C}} \alpha_A 1_A \leq X \quad and \quad \sum_{A \in \mathcal{C}} \beta_A 1_A \leq Y.$$

Then

$$\sum_{A \in \mathcal{C}} (\alpha_A + \beta_A) 1_A \leq X + Y,$$

that is, by summing the previous two we obtain a sub-decomposition of $X + Y$. This implies the super-additivity of $I_{\mathcal{D}}^{\mu}$, that is,

$$I_{\mathcal{C}}^{\mu}(X) + I_{\mathcal{C}}^{\mu}(Y) \leq I_{\mathcal{C}}^{\mu}(X + Y)$$

as needed.   □

The collection integral is defined only on non-negative functions and the aim of this section to introduce an integral defined on the linear space $\mathcal{L}$ and preserving super-additivity. We propose a new definition of integral that is based only on shift-invariant collection integrals.

Definition 2. *Let $\mathcal{C}$ be a collection and μ be a capacity. Then $I_{\mathcal{C}}^{\mu}$ is* shift-invariant *if and only if*

$$I_{\mathcal{C}}^{\mu}(X + c1_{\Omega}) = I_{\mathcal{C}}^{\mu}(X) + cI_{\mathcal{C}}^{\mu}(1_{\Omega})$$

for all $X \in \mathcal{L}_{+}$ and $c \geq 0$.

Remark 1. *For any collection $\mathcal{C}$ there is a monotone set function μ so that $I_{\mathcal{C}}^{\mu}$ is shift-invariant. Indeed, it is enough to consider any maximal set $A \in \mathcal{C}$ and a monotone set function μ such that $\mu(A) = 1$ and $\mu(B) = 0$ for all $B \in \mathcal{C} \setminus \{A\}$. Note that then $I_{\mathcal{C}}^{\mu}(X) = \min X(A)$.*

Proposition 2. *If $\mathcal{C}$ is a disjoint system, that is, $\mathcal{C} = \{A_i\}_{i=1}^{k}$ and $A_i \cap A_j = \varnothing$ if $i \neq j$, then the collection integral $I_{\mathcal{C}}^{\mu}$ is shift-invariant for any capacity μ.*

Proof. Let $\mathcal{C}$ consist of disjoint sets, and let μ be any monotone set function. Then one obtains that

$$I_{\mathcal{C}}^{\mu}(X) = \sum_{i=1}^{k} \mu(A_i) \min X(A_i),$$

from which it follows that $I_{\mathcal{C}}^{\mu}$ is shift-invariant for any $X \in \mathcal{L}_{+}$　□

Definition 3. *Let $\mathcal{C}$ be a collection and μ be a monotone set function such that $I_{\mathcal{C}}^{\mu}$ is a shift-invariant collection integral. A functional*

$$\underline{I}_{\mathcal{C}}^{\mu} : \mathcal{L} \to \mathbb{R} \quad \text{such that} \quad X \mapsto I_{\mathcal{C}}^{\mu}\big(X - (\inf X)1_{\Omega}\big) + (\inf X)I_{\mathcal{C}}^{\mu}(1_{\Omega})$$

is called a super-additive integral *with respect to a collection $\mathcal{C}$ and a monotone set function μ.*

Remark 2. *Note that the definition of $I_{\mathcal{C}}^{\mu}$ involves non-negative functions $X - (\inf X)1_{\Omega}$ and 1_{Ω}, a and thus $\underline{I}_{\mathcal{D}}^{\mu}$ is well-defined.*

In the following theorem some basic properties of the super-additive integral are proven.

Theorem 1. *Let $\underline{I}_{\mathcal{C}}^{\mu}$ be a super-additive integral. Then*

$\underline{I}_{\mathcal{C}}^{\mu}(X) = I_{\mathcal{C}}^{\mu}(X)$ *for all $X \in \mathcal{L}_{+}$, that is, $\underline{I}_{\mathcal{C}}^{\mu}$ extends $I_{\mathcal{C}}^{\mu}$;*
$\underline{I}_{\mathcal{C}}^{\mu}(cX) = c\underline{I}_{\mathcal{C}}^{\mu}(X)$ *for all $X \in \mathcal{L}$ and $c \geq 0$, that is, $\underline{I}_{\mathcal{C}}^{\mu}$ is positively homogeneous;*
$\underline{I}_{\mathcal{C}}^{\mu}(X + c1_{\Omega}) = \underline{I}_{\mathcal{C}}^{\mu}(X) + c\underline{I}_{\mathcal{C}}^{\mu}(1_{\Omega})$ *for all $X \in \mathcal{L}$ and $c \in \mathbb{R}$, that is, $\underline{I}_{\mathcal{C}}^{\mu}$ is shift-invariant;*
$\underline{I}_{\mathcal{C}}^{\mu}(X) \geq (\inf X)\underline{I}_{\mathcal{C}}^{\mu}(1_{\Omega})$ *for all $X \in \mathcal{L}$, that is, $\underline{I}_{\mathcal{C}}^{\mu}$ is bounded below by a (normed) infimum; and*
$\underline{I}_{\mathcal{C}}^{\mu}(X + Y) \geq \underline{I}_{\mathcal{C}}^{\mu}(X) + \underline{I}_{\mathcal{C}}^{\mu}(Y)$ *for all $X, Y \in \mathcal{L}$, that is, $\underline{I}_{\mathcal{C}}^{\mu}$ is super-additive.*

Proof. Let $\underline{I}_{\mathcal{C}}^{\mu}$ be the super-additive integral. Then $I_{\mathcal{C}}^{\mu}$ is shift-invariant. Based on these facts and the fact that $X \in \mathcal{L}_{+}$ one easily obtains

$$\underline{I}_{\mathcal{C}}^{\mu}(X) = I_{\mathcal{C}}^{\mu}(X - (\inf X)1_{\Omega}) + (\inf X)I_{\mathcal{C}}^{\mu}(1_{\Omega}) = I_{\mathcal{C}}^{\mu}(X),$$

because $\inf X \geq 0$. This implies the first property of the theorem.

Now, if $X \in \mathcal{L}$ and $c \geq 0$, and thanks to the positive homogeneity of $I_{\mathcal{C}}^{\mu}$, we obtain that

$$\underline{I}_{\mathcal{C}}^{\mu}(cX) = I_{\mathcal{C}}^{\mu}(cX - (\inf cX)1_{\Omega}) + (\inf cX)I_{\mathcal{C}}^{\mu}(1_{\Omega})$$
$$= c\left(I_{\mathcal{C}}^{\mu}(X - (\inf X)1_{\Omega}) + (\inf X)I_{\mathcal{C}}^{\mu}(1_{\Omega}) \right) = c\underline{I}_{\mathcal{C}}^{\mu}(X),$$

proving the second property of the theorem. To see that the third statement of the theorem is true, it is enough to notice that

$$\underline{I}_{\mathcal{C}}^{\mu}(X + c1_{\Omega}) = I_{\mathcal{C}}^{\mu}(X + c1_{\Omega} - (\inf(X + c1_{\Omega}))1_{\Omega}) + (\inf(X + c1_{\Omega}))I_{\mathcal{C}}^{\mu}(1_{\Omega})$$
$$= I_{\mathcal{C}}^{\mu}(X - (\inf X)1_{\Omega}) + (\inf X)I_{\mathcal{C}}^{\mu}(1_{\Omega}) + cI_{\mathcal{C}}^{\mu}(1_{\Omega}) = \underline{I}_{\mathcal{C}}^{\mu}(X) + c I_{\mathcal{C}}^{\mu}(1_{\Omega})$$

as needed. The fourth property follows directly from the definition, because $I_{\mathcal{C}}^{\mu}$ is non-negative, that is,

$$\underline{I}_{\mathcal{C}}^{\mu}(X) = I_{\mathcal{C}}^{\mu}(X - (\inf X)1_{\Omega}) + (\inf X)I_{\mathcal{C}}^{\mu}(1_{\Omega}) \geq (\inf X)I_{\mathcal{C}}^{\mu}(1_{\Omega}) = (\inf X)\underline{I}_{\mathcal{C}}^{\mu}(1_{\Omega}),$$

and, lastly, to see that $\underline{I}_{\mathcal{C}}^{\mu}$ is super-additive, let $X, Y \in \mathcal{L}$ be two functions and let $r = \inf(X + Y) - \inf X - \inf Y$ and note that $r \geq 0$. Then

$$\underline{I}_{\mathcal{C}}^{\mu}(X) + \underline{I}_{\mathcal{C}}^{\mu}(Y) = I_{\mathcal{C}}^{\mu}(x - (\inf X)1_{\Omega}) + (\inf X)I_{\mathcal{C}}^{\mu}(1_{\Omega}) + I_{\mathcal{C}}^{\mu}(Y - (\inf Y)1_{\Omega}) + (\inf Y)I_{\mathcal{C}}^{\mu}(1_{\Omega})$$
$$\leq I_{\mathcal{C}}^{\mu}(X + Y - (\inf X + \inf Y)1_{\Omega}) + (\inf X + \inf Y)I_{\mathcal{C}}^{\mu}(1_{\Omega})$$
$$= I_{\mathcal{C}}^{\mu}(X + Y - (\inf X + \inf Y + r)1_{\Omega} + r1_{\Omega}) + (\inf X + \inf Y)I_{\mathcal{C}}^{\mu}(1_{\Omega})$$
$$= I_{\mathcal{C}}^{\mu}(X + Y - (\inf(X + y))1_{\Omega}) + (\inf X + \inf Y + r)I_{\mathcal{C}}^{\mu}(1_{\Omega})$$
$$= I_{\mathcal{C}}^{\mu}(X + Y - (\inf(X + Y))1_{\Omega}) + (\inf(X + Y))I_{\mathcal{C}}^{\mu}(1_{\Omega}) = \underline{I}_{\mathcal{C}}^{\mu}(X + Y),$$

that is, $\underline{I}_{\mathcal{C}}^{\mu}$ is a super-additive operator; which proves the theorem. $\square$

On the other hand, there are collections $\mathcal{C}$ admitting capacity μ such that $I_{\mathcal{C}}^{\mu}$ is not shift-invariant.

Example 3. *Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and let $\mathcal{C} = \{A \in 2^{\Omega} : 0 \leq |A| \leq 2\}$. Then*

- *if $\mu(A) = 1$ for all $A \neq \varnothing$, it holds*

$$I_{\mathcal{C}}^{\mu}(X) = X(\omega_1) + X(\omega_2) + X(\omega_3)$$

  and hence $I_{\mathcal{C}}^{\mu}$ is shift-invariant;
- *if*

$$\mu(A) = \begin{cases} 1, & \text{if } |A| \geq 2, \\ 0, & \text{if } |A| \leq 1, \end{cases}$$

  and $X \in \mathcal{L}$ is such that $X(\omega_1) = 2$, $X(\omega_2) = 1$, and $X(\omega_3) = 0$, then $I_{\mathcal{C}}^{\mu}(1_{\Omega}) = 3/2$, $I_{\mathcal{C}}^{\mu}(X) = 1$, and

$$I_{\mathcal{C}}^{\mu}(X + 1_{\Omega}) = 3 > \frac{5}{2} = I_{\mathcal{C}}^{\mu}(X) + I_{\mathcal{C}}^{\mu}(1_{\Omega}),$$

  violating the shift-invariance of $I_{\mathcal{C}}^{\mu}$.

4. Construction of Coherent Lower and Upper Previsions And Examples

In this section we construct coherent lower previsions by means super-additive integrals when $\underline{I}_{\mathcal{C}}^{\mu}(1_{\Omega}) > 0$.

Theorem 2. *Let $\mathcal{C}$ be a collection and μ be a monotone set function such that $\underline{I}^{\mu}_{\mathcal{C}}$ is a super-additive integral and let $\underline{I}^{\mu}_{\mathcal{C}}(1_{\Omega}) > 0$. Then, an operator $\mathrm{clp}^{\mu}_{\mathcal{C}} \colon \mathcal{L} \to \mathbb{R}$ given by*

$$\mathrm{clp}^{\mu}_{\mathcal{C}}(X) = \frac{\underline{I}^{\mu}_{\mathcal{C}}(X)}{\underline{I}^{\mu}_{\mathcal{C}}(1_{\Omega})}$$

is a coherent lower prevision based on the super-additive integral $\underline{I}^{\mu}_{\mathcal{C}}$. A coherent upper prevision $\mathrm{cup}^{\mu}_{\mathcal{C}}$ corresponding to the coherent lower prevision $\mathrm{clp}^{\mu}_{\mathcal{C}}$ is given by

$$\mathrm{cup}^{\mu}_{\mathcal{C}}(X) = -\mathrm{clp}^{\mu}_{\mathcal{C}}(-X).$$

Proof. The properties of coherent lower prevision for $\mathrm{clp}^{\mu}_{\mathcal{C}}$ follow directly from Theorem 1. □

By Theorem 2 we have

Theorem 3. *Let Ω be a finite set, let $\mathcal{C}$ be a collection of subsets of Ω and let μ be a monotone set function such that $\underline{I}^{\mu}_{\mathcal{C}}$ is a super-additive integral and $\underline{I}^{\mu}_{\mathcal{C}}(1_{\Omega}) > 0$. Let $\mathcal{L}^*$ be the class of all indicator functions 1_E with $E \in 2^{\Omega}$.*

The operator $\mathrm{cup}^{\mu}_{\mathcal{C}} \colon \mathcal{L}^* \to \mathbb{R}$ given by

$$\mathrm{clp}^{\mu}_{\mathcal{C}}(I_E) = \frac{\underline{I}^{\mu}_{\mathcal{C}}(I_E)}{\underline{I}^{\mu}_{\mathcal{C}}(1_{\Omega})} = \frac{I^{\mu}_{\mathcal{C}}(I_E)}{I^{\mu}_{\mathcal{C}}(1_{\Omega})}$$

is a coherent lower probability.

Example 4. *Let $\mathcal{C}$ consist of disjoint sets, and let μ be any monotone set function. Then the shift-invariant collection integral $I^{\mu}_{\mathcal{C}}$ is*

$$I^{\mu}_{\mathcal{C}}(X) = \sum_{i=1}^{k} \mu(A_i) \min X(A_i)$$

and

$$I^{\mu}_{\mathcal{C}}(1_{\Omega}) = \sum_{i=1}^{k} \mu(A_i).$$

Consider $\mu(A_i) > 0$ for some $i \in \{1, 2, \ldots, k\}$. Then, for this choice of $\mathcal{C}$ we get the coherent lower prevision

$$\mathrm{clp}^{\mu}_{\mathcal{C}}(X) = \frac{\sum_{i=1}^{k} \mu(A_i) \min X(A_i)}{\sum_{i=1}^{k} \mu(A_i)},$$

that is, the $\mathrm{clp}^{\mu}_{\mathcal{C}}$ is a weighted average of minimal values obtained on A_i; and the corresponding coherent upper prevision is given by

$$\mathrm{cup}^{\mu}_{\mathcal{C}}(X) = \frac{\sum_{i=1}^{k} \mu(A_i) \max X(A_i)}{\sum_{i=1}^{k} \mu(A_i)},$$

that is, this coherent upper prevision is a weighted average of maximal values obtained on A_i.

Remark 3. *If in Example 4 the collection $\mathcal{C}$ is such that all random variables X are constant on the sets $A_i \in \mathcal{C}$ and $\mu(A_i) > 0$ for some $i \in \{1, 2, \ldots, k\}$ then the coherent lower prevision defined in Theorem 2 is linear.*

In particular if $\mathcal{C}$ is the partition of singletons of Ω we have $\min X(A_i) = \max X(A_i) = X(A_i)$; consider $\mu(A_i) > 0$ for some $i \in \{1, 2, \ldots, k\}$, then

$$\operatorname{clp}_{\mathcal{C}}^{\mu}(X) = \operatorname{cup}_{\mathcal{C}}^{\mu}(X) = \frac{\sum_{i=1}^{k} \mu(A_i) X(A_i)}{\sum_{i=1}^{k} \mu(A_i)}$$

and the corresponding linear probability is

$$\operatorname{clp}_{\mathcal{C}}^{\mu}(1_E) = \operatorname{cup}_{\mathcal{C}}^{\mu}(1_E) = \frac{\sum_{A \subseteq E} \mu(A_i)}{\sum_{i=1}^{k} \mu(A_i)}.$$

Example 5. *Let $\mathcal{C} = \{\Omega\}$ then the coherent lower and upper previsions defined in Theorem 2 are the lower and the upper vacuous previsions since*

$$\operatorname{clp}_{\mathcal{C}}^{\mu}(X) = \min X(\Omega)$$

and the corresponding coherent upper prevision is given by

$$\operatorname{cup}_{\mathcal{C}}^{\mu}(X) = \max X(\Omega)$$

Example 6. *Let $\mathcal{C}$ be a chain, that is, $\mathcal{C} = \{A_i\}_{i=1}^{k}$ such that $\Omega \supset A_1 \supset A_2 \supset \cdots \supset A_k \neq \emptyset$, and let μ be a capacity such that $\mu(A_1) > 0$. In Ref. [15] it can be found that*

$$I_{\mathcal{C}}^{\mu}(X) = \mu(A_1) \min X(A_1) + \sum_{i=2}^{k} \mu(A_i) \Big(\min X(A_i) - \min X(A_{i-1}) \Big)$$

for any $X \in \mathcal{L}_{+}$. Then $I_{\mathcal{C}}^{\mu}(1_{\Omega}) = \mu(A_1)$. It can also easily be seen that this integral is indeed shift-invariant. From this one easily obtains the coherent lower prevision

$$\operatorname{clp}_{\mathcal{C}}^{\mu}(X) = \min X(A_1) + \sum_{i=2}^{k} \Big(\min X(A_i) - \min X(A_{i-1}) \Big) \frac{\mu(A_i)}{\mu(A_1)}$$

and the dual coherent upper prevision

$$\operatorname{cup}_{\mathcal{C}}^{\mu}(X) = \max X(A_1) - \sum_{i=2}^{k} \Big(\max X(A_{i-1}) - \max X(A_i) \Big) \frac{\mu(A_i)}{\mu(A_1)}.$$

Moreover, if $\mathcal{C}$ is a chain, then for every normalized capacity $\mu \in \mathcal{M}$ there exists a monotone set function $\tau \in \mathcal{M}$ such that

$$I_{\mathcal{C}}^{\mu}(X) = \int^{\operatorname{Ch}} X \, d\tau,$$

and $\tau(A) = I_{\mathcal{C}}^{\mu}(1_A)$. Due to properties of Choquet integral, $I_{\mathcal{C}}^{\mu}$ is shift-invariant and thus the related normed super-additive integral is a coherent lower prevision and it coincides with the asymmetric Choquet integral with respect to τ. On the other hand, based on Denneberg's result [3] concerning the super-additivity of Choquet integrals, necessarily τ is super-modular, yielding an interesting result: for any monotone measure μ and any chain collection $\mathcal{C}$, the measure τ given by $\tau(A) = I_{\mathcal{C}}^{\mu}(1_A)$ is super-modular. Take, for example, the greatest normed monotone set function μ^* for which $\mu^*(A) = 1$ for any non-empty subset A of Ω. Obviously, μ^* is sub-modular (and not super-modular). For any chain $\mathcal{C} = \{A_1, A_2, \ldots, A_k\}$ with maximal element A_1,

the corresponding measure τ given by $\tau(A) = I_{\mathcal{C}}^{\mu^*}(1_A) = 1$ if A_1 is a subset of A, and 0 otherwise. Clearly, τ is super-modular, and it is called a unanimity game in the game theory [22].

Example 7. *Let $\mathcal{C} = \{A, B\}$ be a collection such that $A, B \subseteq \Omega$ are two sets such that $A \cap B \neq \varnothing$ and $\{A, B\}$ is not a chain. Let μ a monotone set function such that $\mu(A) \geq \mu(B) > 0$. Then, for any $c > 0$ consider the random variable $X \in \mathcal{L}_+$ given by $X = 2 \cdot 1_A + 1_B$. Since $\mu(A) \geq \mu(B)$ we obtain that*

$$I_{\mathcal{C}}^{\mu}(X + c1_{\Omega}) = (c+2)\mu(A) + (c+1)\mu(B) \neq 2\mu(A) + \mu(B) + c\mu(A) = I_{\mathcal{C}}^{\mu}(X) + cI_{\mathcal{C}}^{\mu}(1_{\Omega}),$$

and thus the collection integral $I_{\mathcal{C}}^{\mu}$ is not shift-invariant for any monotone set function μ such that $\mu(A), \mu(B) > 0$. It can not be used for the construction method of coherent lower previsions based on a super-additive integral.

Example 8. *Consider the collection $\mathcal{C} = 2^{\Omega} \setminus \{\varnothing\}$. The corresponding collection integral $I_{\mathcal{C}}^{\mu}$ is the concave integral introduced by Lehrer in Ref. [11]. If μ is a super-modular measure, then $I_{\mathcal{C}}^{\mu} = \mathrm{Ch}_{\mu}$ is Choquet integral [4], and hence $I_{\mathcal{C}}^{\mu}$ is shift-invariant, and $I_{\mathcal{C}}^{\mu}(1_{\Omega}) = \mu(\Omega)$. Our introduced super-additive integral $\underline{I}_{\mathcal{C}}^{\mu}$ is then just the asymmetric Choquet integral acting on $\mathbb{R}$, see Reference [3]. If μ is a normed monotone set function, $\mu(\Omega) = 1$, then $\underline{I}_{\mathcal{D}}^{\mu}$ is a coherent lower prevision.*

If μ is not super-modular, the collection integral $I_{\mathcal{C}}^{\mu}$ need not be shift-invariant and thus cannot be considered to define the super-additive integral.

Example 9. *Let $\Omega = \{1, 2, 3\}$ and $\mathcal{C} = 2^{\Omega}$ and ,*

$$\mu(A) = \begin{cases} 1, & \text{if } |A| \geq 2, \\ 0, & \text{if } |A| \leq 1. \end{cases}$$

Then $I_{\mathcal{C}}^{\mu}(1_{\Omega}) = 3/2$, and for $X \in \mathcal{L}$ such that $X(1) = 1$, $X(2) = 2$, $X(3) = 0$, we have

$$I_{\mathcal{C}}^{\mu}(X + 1_{\Omega}) = 3 \neq 5/2 = I_{\mathcal{C}}^{\mu}(X) + I_{\mathcal{C}}^{\mu}(1_{\Omega}).$$

The following example shows that there exist a collection $\mathcal{C}$ and a capacity μ, which is not super-modular, such that a coherent lower prevision can defined by the super-additive integral with respect to $\mathcal{C}$ and μ.

Example 10. *Let $\Omega = \{1, 2, \ldots, n\}$, $\mathcal{C} = 2^{\Omega}$ and $\mu(A) = 1$ whenever $A \neq \varnothing$, then*

$$I_{\mathcal{C}}^{\mu}(X) = \sum_{i=1}^{n} X(i)$$

and the related $\mathrm{clp}_{\mathcal{C}}^{\mu}(X) = \dfrac{1}{n} \sum_{i=1}^{n} X(i)$ is a coherent lower prevision, though μ is not super-modular.

5. Generalization of the Approach to Construct Coherent Lower and Upper Previsions

Let Γ be an arbitrary non-empty index set. Having a system of coherent lower previsions $\{\underline{P}_{\gamma} : \gamma \in \Gamma\}$, the lower envelope theorem ([1], Section 2.6.3) allows a further generalization of the approach to construct coherent lower (upper) previsions.

Definition 4. *Let Γ be an arbitrary non-empty index set. Given a class of functionals $\{\underline{P}_{\gamma}(X) : \gamma \in \Gamma\}$ defined on $\mathcal{L}$, $\underline{P}$ is the lower envelope of $\{\underline{P}_{\gamma}(X) : \gamma \in \Gamma\}$ if*

$$\underline{P}(X) = \inf \{\underline{P}_{\gamma}(X) : \gamma \in \Gamma\}$$

for all $X \in \mathcal{L}$

$\underline{P}$ is the lower envelope of $\{\underline{P}_\gamma : \gamma \in \Gamma\}$ if and only if the conjugate $\overline{P}$ is the upper envelope of $\{\overline{P}_\gamma : \gamma \in \Gamma\}$, that is $\overline{P}(X) = \sup\{\overline{P}_\gamma(X) : \gamma \in \Gamma\}$.

Theorem 4. *Let* Γ *be an arbitrary non-empty index set. If* $\underline{P}_\gamma$ *is a coherent lower prevision with domain* $\mathcal{F}$ *for every* $\gamma \in \Gamma$, *and* $\underline{P}$ *is the lower envelope of* $\{\underline{P}_\gamma : \gamma \in \Gamma\}$ *then* $\underline{P}$ *is a coherent lower prevision.*

Example 11. *Let* $\Gamma = \{1,2\}$ *and consider two collections* $\mathcal{C}_\gamma$, $\gamma \in \Gamma$, *given by* $\mathcal{C}_1 = \{\{1\},\{1,2\}\}$ *and* $\mathcal{C}_2 = \{\{2\},\{1,2\}\}$. *Note that for these two collections, the corresponding collection integrals*

$$I^\mu_{\mathcal{C}_1}(X) = \min\{X_1, X_2\} + a\max\{0, X_1 - X_2\} \quad \text{and} \quad I^\mu_{\mathcal{C}_2}(X) = \min\{X_1, X_2\} + b\max\{0, X_1 - X_2\},$$

where $X \in \mathcal{L}$ *is such that* $X(1) = X_1$ *and* $X(2) = X_2$, *and* $\mu \in \mathcal{M}$ *is a capacity such that* $\mu(\{1\}) = a$, $\mu(\{2\}) = b$, *and* $\mu(\{1,2\}) = 1$. *Note that if we consider two super-modular capacities* $\mu_1, \mu_2 \in \mathcal{M}$ *given by*

$$\mu_1(\{1\}) = a, \mu_1(\{2\}) = 0, \mu_1(\{1,2\}) = 1, \quad \text{and} \quad \mu_2(\{1\}) = 0, \mu_2(\{2\}) = b, \mu_2(\{1,2\}) = 1,$$

then we obtain that

$$I^\mu_{\mathcal{C}_1}(X) = \int^{\text{Ch}} X \, d\mu_1 \quad \text{and} \quad I^\mu_{\mathcal{C}_2}(X) = \int^{\text{Ch}} X \, d\mu_2.$$

Now, from these super-additive integrals we can obtain coherent lower previsions clp^μ_1 *and* clp^μ_2 *that coincide with* $I^\mu_{\mathcal{D}_1}$ *and* $I^\mu_{\mathcal{D}_2}$, *respectively. Then we can construct the lower envelope and obtain a coherent lower prevision*

$$\text{clp}^\mu(X) = \min\{\text{clp}^\mu_1(X), \text{clp}^\mu_2(X)\} = \min\{X_1, X_2\}.$$

Similarly, from these coherent lower previsions, we can obtain coherent upper previsions by the property of conjugacy. Let us denote them by cup^μ_1 *and* cup^μ_2, *respectively. Then the coherent upper prevision constructed as the upper envelope is equal to*

$$\text{cup}^\mu(X) = \max\{\text{cup}^\mu_1(X), \text{cup}^\mu_2(X)\} = \max\{X_1, X_2\}.$$

Now, if we construct the upper envelope of these coherent lower previsions we obtain that

$$P^\mu(X) = \max\{\text{clp}^\mu_1(X), \text{clp}^\mu_2(X)\} = \int^{\text{Ch}} X \, d\mu$$

and this is a coherent lower prevision if and only if the underlying monotone set function μ *is super-modular, that is, if and only if* $a + b \geq 1$ *holds.*

Moreover the convexity theorem ([1], Section 2.6.4) and the convergence theorem ([1], Section 2.6.5) assure the following results:

Theorem 5. *Let* $\underline{P}_1$ *and* $\underline{P}_2$ *be two coherent lower previsions then their convex combination* $\underline{P}$ *given by*

$$\underline{P}(X) = \lambda \underline{P}_1(X) + (1 - \lambda)\underline{P}_2(X) \quad \text{with} \quad 0 < \lambda < 1$$

is a coherent lower prevision.

Theorem 6. *Let* $\{\underline{P}_i : i \geq 1\}$ *be a sequence of coherent lower previsions, each defined on* $\mathcal{L}$, *which converges point-wise to* $\underline{P}$ *that is the sequence of real numbers* $\{\underline{P}_i(X)\}$ *converges to* $\underline{P}(X)$ *for every* $X \in \mathcal{L}$ *then* $\underline{P}(X)$ *is a coherent lower prevision.*

6. Conclusions and Discussion

In this paper we have introduced, discussed and exemplified a new construction method for coherent lower previsions acting on finite universe Ω, and, by duality, for coherent upper previsions. Our approach was based on a so called collection integral with respect to a monotone measure, a super-additive functional acting on non-negative random variables. In the case when this integral was shift-invariant, we have extended it to a super-additive functional acting on all real random variables on Ω, and called a super-additive integral. Its normalized form was shown to be a coherent lower prevision. We have discussed and exemplified several particular cases, showing the potential of our construction method for obtaining new types of coherent lower (upper) previsions. As a by product of our studies, several new interesting results were obtained. So, for example, we have shown that for any monotone set function μ, which represents the belief of the subject about a random phenomenon, there is a collection $\mathcal{C}$, consisting of disjoint subsets of Ω, such that a coherent lower prevision can be defined by the corresponding super-additive integral. Also, we have shown that for any collection $\mathcal{C}$ there is a unanimity measure μ (i.e., it asses value 1 to some set in $\mathcal{C}$ and each its superset, and value 0 to any other sets) such that a coherent lower prevision can be defined by the super-additive integral. In monotone measure theory, another our result could be of interest, namely, considering any chain collection $\mathcal{C}$ and any monotone measure μ. Then the related coherent lower prevision applied to characteristic functions yields a super-modular measure. We expect applications of our results in all domains where the imprecise probabilities are applied, in particular in decision problems. As a possible further research expanding our results, we think on the study of simple decomposition systems (consisting, e.g., of two collections) and related decomposition integrals, where our approach considered for single collections could be successfully applied and result into another type of coherent lower (upper) previsions. Another branch of research can deal with super-decomposition integrals introduced in Ref. [23] and their use for constructing of coherent upper previsions. Note also that all our work was done on a finite universe Ω. Then, for sure, a challenging task would be the general case, dealing with monotone measure space (Ω, Σ, μ), in which case the coherent lower (upper) previsions are functionals on bounded real random variables.

Results proposed in this paper show that any monotone set function which represents uncertainty can be used to define a coherent lower prevision by the super-additive integral and this non-linear integral is a mathematical tool to aggregate in a coherent way information represented by different uncertainty measures in the finite case.

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