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On the Asymptotic Behavior of a Class of Second-Order Non-Linear Neutral Differential Equations with Multiple Delays

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Abstract: In this work, we present some new sufficient conditions for the oscillation of a class of second-order neutral delay differential equation. Our oscillation results, complement, simplify and improve recent results on oscillation theory of this type of non-linear neutral differential equations that appear in the literature. An example is provided to illustrate the value of the main results.

Keywords: oscillation; nonoscillation; neutral; delay; non-linear; Lebesgue's dominated convergence theorem

1. Introduction

Consider the second-order neutral differential equation with several delays of the form

$$\big(a(y)(w'(y))^{\mu}\big)' + \sum_{i=1}^{m} c_i(y)F_i\big(u(y-q_i)\big) = 0, \quad w(y) = u(y) + b(y)u(y-p) \tag{1}$$

where $p, q_i \in \mathbb{R}_+ = (0, +\infty)$; $b \in PC(\mathbb{R}_+, \mathbb{R})$; $F_i \in C(\mathbb{R}, \mathbb{R})$ is nondecreasing such that $uF_i(u) > 0$ for $u \neq 0$; $c_i, a \in C(\mathbb{R}_+, \mathbb{R}_+)$ for $i = 1, 2, \cdots, m$ and μ is a ratio of two odd positive integers.

Recently there has been an increasing interest in dynamical systems both neutral and involving time delays with applications ranging from Biology and Population Dynamics to Physics and Engineering, and from Economics to Medicine. In particular, it is natural to ask why time-delayed systems are so important. Time delays are intrinsic in many real systems, and therefore must be properly accounted for evolution models. For further details we refer the reader to [1–7]. As a matter of fact, Equation (1) (i.e., half-linear/Emden-Fowler differential equation) arises in a variety of real world problems such as in the study of p-Laplace equations non-Newtonian fluid theory, the turbulent flow of a polytrophic gas in a porous medium, and so forth; see [8–10] for more details.

Now we recall some basic definitions.

A function $u(y) : [y_u, \infty) \to \mathbb{R}$, $y_u \geq y_0$ is said to be a *solution* of (1) if $u(y)$ and $a(y)\big(w'(y)\big)^{\mu}$ are once continuously differentiable and $w(y)$ twice continuously differentiable for all $y \in [y_u, \infty)$ and it satisfies Equation (1) for all $y \in [y_u, \infty)$

We assume that (1) admits a solution in the sense of the above definition.

A solution $w(y)$ of (1) is said to be *non-oscillatory* if it is eventually positive or eventually negative; otherwise, it is said to be *oscillatory*.

The Equation (1) is said to be *oscillatory* if all its solutions are oscillatory.

In this paper, we restrict our attention to study the oscillation and non-oscillation of (1). First of all, it is interesting to make a review in the context of functional differential equations.

In 1978, Brands [11] has proved that the equation

$$u''(y) + c(y)u(y - q(y)) = 0$$

is oscillatory if and only if

$$u''(y) + c(y)u(y) = 0$$

is oscillatory. In [8] Bohner et al. have studied oscillatory behavior of solutions to a class of second-order half-linear dynamic equations with deviating arguments under the assumptions that allow applications to dynamic equations with delayed and advanced arguments. They obtained several Fite–Hille–Wintner-type criteria that do not need some restrictive assumptions required in related results. In [12,13] Chatzarakis et al. have considered a more general equation

$$\big(a(u')^{\mu}\big)'(y) + c(y)u^{\mu}(q(y)) = 0, \tag{2}$$

and established new oscillation criteria for (2) when $A(y) = \int^{y} a(s)^{-\frac{1}{\mu}} ds$, $\lim_{y\to\infty} A(y) = \infty$ and $\lim_{y\to\infty} A(y) < \infty$.

Wong [14] has obtained the oscillation conditions of

$$\big(u(y) + bu(y-p)\big)'' + c(y)F\big(u(y-q)\big) = 0, \quad -1 < b < 0,$$

in which the neutral coefficient and delays are constants. However, we have seen in [15,16] that the authors Baculǐkovǎ and Džurina have studied the equation

$$\Big(a(y)\big(w'(y)\big)^{\mu}\Big)' + c(y)u^{\beta}(q(y)) = 0, \quad w(y) = u(y) + b(y)u(p(y)), \quad y \geq y_0, \tag{3}$$

and established the oscillation of solutions of (3) using comparison techniques when $\mu = \beta = 1$, $0 \leq b(y)$ and $\lim_{y\to\infty} A(y) = \infty$. With the same technique, Baculikova and Džurina [17] have considered (3) and obtained oscillation conditions of (3) considering the assumptions $0 \leq b(y)$ and $\lim_{y\to\infty} A(y) = \infty$. In [18], Tripathy et al. have studied (3) and established several conditions of the solutions of (3) considering the assumptions $\lim_{y\to\infty} A(y) = \infty$ and $\lim_{y\to\infty} A(y) < \infty$ for different ranges of the neutral coefficient b. In [19], Bohner et al. have obtained sufficient conditions for oscillation of solutions of (3) when $\mu = \beta$, $\lim_{y\to\infty} A(y) < \infty$ and $0 \leq b(y) < 1$. Grace et al. [20] studied the oscillation of (3) when $\mu = \beta$ and considering the assumptions $\lim_{y\to\infty} A(y) < \infty$, $\lim_{y\to\infty} A(y) = \infty$ and $0 \leq b(y) < 1$. In [21], Li et al. have established sufficient conditions for the oscillation of the solutions of (3), under the assumptions $\lim_{y\to\infty} A(y) < \infty$ and $b(y) \geq 0$. Karpuz and Santra [22] considered the equation

$$\big(a(y)(u(y) + b(y)u(p(y)))'\big)' + c(y)F\big(u(q(y))\big) = 0,$$

by considering the assumptions $\lim_{y\to\infty} A(y) < \infty$ and $\lim_{y\to\infty} A(y) = \infty$ for different ranges of b.

For further details regarding oscillatory properties of second-order differential equations, we refer the interested reader to [23–51].

2. Necessary and Sufficient Conditions for Oscillation

This section deals with the necessary and sufficient conditions for oscillation of solutions of (1). We introduce the following assumptions for our use in the sequel:

(C1) $-1 < -b_0 \leq b(y) \leq 0$, $b_0 > 0$ for $y \in \mathbb{R}_+$;
(C2) $0 \leq b(y) \leq b_0 < \infty$ for $y \in \mathbb{R}_+$;
(C3) $-\infty < -b_1 \leq b(y) \leq -b_2 < -1$, $b_1, b_2 > 0$ for $y \in \mathbb{R}_+$;
$(A1)$ $F_i(-x) = -F_i(x)$, $x \in \mathbb{R}$;

Theorem 1. *Assume that (C1) and (A1) hold. Furthermore, assume that*

$(A2)$ $\int_Y^{\infty} \sum_{i=1}^{m} c_i(y)F_i(\delta A(y-q_i))\mathrm{d}y < +\infty$ *for every constant* $Y, \delta > 0$ *where* $A(y) = \int_Y^y a(s)^{-\frac{1}{\mu}}\mathrm{d}s$

holds. Then every unbounded solution of (1) *is oscillatory if and only if*

$(A3)$ $\lim_{y\to\infty} A(y) < +\infty$.

Proof. We prove sufficiency by contradiction. Let $u(y)$ be an eventually positive unbounded solution of (1). Then there exists some $y_0 > 0$ such that $u(y) > 0$, $u(y-p) > 0$ and $u(y-q_i) > 0$ for $i = 1, 2, \cdots, m$ and $y \geq y_1 > y_0 + r$ where $r = \max\{p, q_i\}$. We define

$$w(y) = u(y) + b(y)u(y-p). \tag{4}$$

From (1), it follows that

$$\big(a(y)(w'(y))^{\mu}\big)' = -\sum_{i=1}^{m} c_i(y)F_i\big(u(y-q_i)\big) < 0 \tag{5}$$

for $y \geq y_1$. Hence, there exists $y_2 > y_1$ such that $a(y)(w'(y))^{\mu}$ is nonincreasing on $[y_2, \infty)$. Since $w(y)$ is monotonic, then for $y \geq y_3 > y_2$ we have following two possibilities:

Case 1. Let $w(y) < 0$ for $y \geq y_3$. Then $u(y) < u(y-p)$, and hence

$$u(y) < u(y-p) < u(y-2p) < \cdots < u(y_3),$$

that is, $u(y)$ is bounded, which contradicts u being unbounded.

Case 2. Let $w(y) > 0$ for $y \geq y_3$.

Sub-case 2_1**.** Let $a(y)(w'(y))^{\mu} > 0$ for $y \geq y_3$. Since $a(y)(w'(y))^{\mu}$ is nonincreasing on $[y_3, \infty)$, hence there exist a constant $\kappa > 0$ and $y_4 > y_3$ such that $a(y)(w'(y))^{\mu} \leq \kappa$ for $y \geq y_4$. Consequently,

$$w(y) \leq w(y_4) + \kappa^{\frac{1}{\mu}} \int_{y_4}^{y} (a(s))^{-\frac{1}{\mu}}\mathrm{d}s < \infty \tag{6}$$

as $t \to \infty$ due to $(A3)$. On the other hand, $u(y)$ is unbounded, thus there exists $\{\varsigma_n\}$ such that $\varsigma_n \to \infty$ as $n \to \infty$, $u(\varsigma_n) \to \infty$ as $n \to \infty$ and

$$u(\varsigma_n) = \max\{u(\vartheta) : y_3 \leq \vartheta \leq \varsigma_n\}.$$

Therefore,

$$w(\varsigma_n) = u(\varsigma_n) + b(\varsigma_n)u(\varsigma_n - p) \geq (1-b_0)u(\varsigma_n) \to +\infty \quad \text{as} \quad \varsigma_n \to \infty$$

implies that $w(y)$ is unbounded, which leads a contradiction to (6).

Sub-case 2_2**.** Let $a(y)(w'(y))^{\mu} < 0$ for $y \geq y_3$. Since, a is positive and μ is a ratio of two odd positive integers. Therefore, $w'(t) < 0$ and hence w is bounded, which contradicts u being unbounded.

Hence, every positive unbounded solution of (1) oscillates.

If $u(y) < 0$ for $y \geq y_0$, then we set $v(y) = -u(y)$ for $y \geq y_0$ in (1) and we find

$$\big(a(y)(w'(y))^{\mu}\big)' + \sum_{i=1}^{m} c_i(y)F_i\big(v(y-q_i)\big) = 0, \quad w(y) = v(y) + b(y)v(y-p)$$

due to $(A1)$. Hence proceeding as above, we find the same contradiction.

To proof necessary by contradiction, we suppose that $(A3)$ does not hold. Assume that

$$\int_{y_0}^{\infty} (a(s))^{-\frac{1}{\mu}} \mathrm{d}s = +\infty$$

and due to our assumption $(A2)$, let

$$\int_{Y}^{\infty} \sum_{i=1}^{m} c_i(y) F_i(\delta A(y - q_i)) \mathrm{d}y \leq \delta, \quad \text{for all} \quad \delta > 0.$$

In particular, we use a positive ϵ such that $(2\delta)^{\frac{1}{\mu}} = (1 - b_0)\epsilon$ and $0 < \delta^{\frac{1}{\mu}} \leq \frac{(1-b_0)\epsilon}{2^{\frac{1}{\mu}}} < \epsilon$. We define a set of continuous function

$$S = \{u : u \in C([Y - r, +\infty), \mathbb{R}), u(y) = 0 \text{ for } y \in [Y - r, Y) \text{ and } \delta^{\frac{1}{\mu}} A(y) \leq u(y) \leq \epsilon A(y)\}$$

and a mapping $\Phi : S \to C([Y - r, +\infty), \mathbb{R})$ such that

$$(\Phi u)(y) = \begin{cases} 0, & y \in [Y - r, Y) \\ -b(y)u(y - p) + \int_Y^y \frac{1}{(a(v))^{\frac{1}{\mu}}} \Big[\delta + \int_v^{\infty} \sum_{i=1}^{m} c_i(s) F_i\big(u(s - q_i)\big) \mathrm{d}s\Big]^{\frac{1}{\mu}} \mathrm{d}v, & y \geq Y. \end{cases}$$

For every $u \in S$,

$$\begin{aligned} (\Phi u)(y) &\geq \int_Y^y \frac{1}{(a(v))^{\frac{1}{\mu}}} \Big[\delta + \int_v^{\infty} \sum_{i=1}^{m} c_i(s) F_i\big(u(s - q_i)\big) \mathrm{d}s\Big]^{\frac{1}{\mu}} \mathrm{d}v \\ &\geq \delta^{\frac{1}{\mu}} \int_Y^y \frac{\mathrm{d}v}{(a(v))^{\frac{1}{\mu}}} = \delta^{\frac{1}{\mu}} A(y) \end{aligned}$$

and $u(y) \leq \epsilon A(y)$ implies that

$$\begin{aligned} (\Phi u)(y) &\leq -b(y)u(y - p) + (2\delta)^{\frac{1}{\mu}} \int_Y^y \frac{\mathrm{d}v}{(a(v))^{\frac{1}{\mu}}} \\ &\leq b_0 \epsilon A(y) + (2\delta)^{\frac{1}{\mu}} A(y) \\ &= b_0 \epsilon A(y) + (1 - b_0)\epsilon A(y) = \epsilon A(y) \end{aligned}$$

implies that $(\Phi u)(y) \in S$. Define $v_n : [Y - r, +\infty) \to \mathbb{R}$ by the recurrence relation

$$v_n(y) = \big(\Phi v_{n-1}\big)(y), \quad n \geq 1,$$

with the initial condition

$$v_0(y) = \begin{cases} 0, & y \in [Y - r, Y) \\ \delta^{\frac{1}{\mu}} A(y), & y \geq Y. \end{cases}$$

By the mathematical induction, it is easy to prove that

$$\delta^{\frac{1}{\mu}} A(y) \leq v_{n-1}(y) \leq v_n(y) \leq \epsilon A(y).$$

for $y \geq Y$. Therefore for $y \geq Y - r$, $\lim_{n\to\infty} v_n(y)$ exists. Let $\lim_{n\to\infty} v_n(y) = v(y)$ for $y \geq Y - r$. By the Lebesgue's dominated convergence Theorem $v \in S$ and $\big(\Phi v\big)(y) = v(y)$, where $v(y)$ is a solution of Equation (1) such that $v(y) > 0$. Hence, $(A3)$ is necessary. This completes the proof of the Theorem. □

Remark 1. *In the above theorem, the function F_i could be sublinear, superlinear or linear.*

Theorem 2. *Assume that (C1), (A1) and (A3) hold. Furthermore, assume that*

$(A4)$ $\int_Y^\infty \frac{1}{(a(y))^{\frac{1}{\mu}}} \left[\int_Y^y \sum_{i=1}^m c_i(s) F_i(\epsilon A_1(s-q_i)) \mathrm{d}s \right]^{\frac{1}{\mu}} \mathrm{d}y = +\infty$ *for* $Y, \epsilon > 0$ *where* $A_1(y) = \int_y^s \frac{\mathrm{d}\theta}{(a(\theta))^{\frac{1}{\mu}}}$

and

$(A5)$ $\int_Y^\infty \sum_{i=1}^m c_i(s) F_i(\delta) \mathrm{d}s = +\infty$ *for* $\delta > 0$

hold. Then every solution of Equation (1) *either oscillates or converges to zero.*

Proof. On the contrary, let u be an eventually positive solution of (1) that does not converge to zero. By Theorem 1, we have (5) for $y \geq y_1$. Thus, there exists $y_2 > y_1$ such that $w'(y)$ and $w(t)$ are of constant sign on $[y_2, \infty)$. Therefore, we have following two possibilities:

Case 1. Let $w(y) < 0$ for $y \geq y_2$. By Case 1 of Theorem 1, we have that $u(y)$ is bounded. Consequently, w is bounded, and since w is monotonic, $\lim_{y\to\infty} w(y)$ exists. As a result,

$$\begin{aligned} 0 \geq \lim_{y\to\infty} w(y) &= \limsup_{y\to\infty} w(y) \\ &\geq \limsup_{y\to\infty} \big(u(y) - b_0\, u(y-p)\big) \\ &\geq \limsup_{y\to\infty} u(y) + \liminf_{y\to\infty} \big(-b_0\, u(y-p)\big) \\ &= (1-b_0) \limsup_{y\to\infty} u(y) \end{aligned}$$

implies that $\limsup_{y\to\infty} u(y) = 0$, because $1 - b_0 > 0$ and thus $\lim_{y\to\infty} u(y) = 0$.

Case 2. Let $w(y) > 0$ for $y \geq y_2$.

Sub-case 2_1. Let $a(y)(w'(y))^\mu < 0$ for $y \geq y_2$. Therefore, $w(y)$ is bounded and monotonic, hence $\lim_{y\to\infty} w(y)$ exists. Therefore, for $s \geq y > y_2$, $a(s)(w'(s))^\mu \leq a(y)(w'(y))^\mu$ implies that

$$w'(s) \leq \frac{a(y)^{\frac{1}{\mu}} w'(y)}{(a(s))^{\frac{1}{\mu}}}.$$

Since $a(y)(w'(y))^\mu$ is decreasing and μ is the quotient of positive odd integers, $a(y)^{1/\mu}(w'(y))$ is also decreasing. Then

$$w(s) \leq w(y) + (a(y))^{\frac{1}{\mu}} w'(y) \int_y^s \frac{\mathrm{d}\theta}{(a(\theta))^{\frac{1}{\mu}}}.$$

Since, $a(y)(w'(y))^\mu$ is nonincreasing, we can find a constant $\delta > 0$ such that $a(y)(w'(y))^\mu \leq -\delta$ for $y \geq y_2$. As a result, $w(s) \leq w(y) - \delta^{\frac{1}{\mu}} \int_y^s \frac{d\theta}{(a(\theta))^{\frac{1}{\mu}}}$ and hence $0 \leq w(y) - \epsilon A_1(y)$ for $y \geq y_2$, where $\epsilon = \delta^{\frac{1}{\mu}}$, $\epsilon > 0$. From (1), it is not difficult to see that

$$\big(a(y)(w'(y))^\mu\big)' + \sum_{i=1}^m c_i(y) F_i\big(\epsilon A_1(y-q_1)\big) \leq 0.$$

Integrating the last inequality from y_2 to $y (> y_2)$, we obtain

$$\big[a(s)(w'(s))^\mu\big]_{y_2}^y + \int_{y_2}^y \sum_{i=1}^m c_i(s) F_i\big(\epsilon A_1(s-q_i)\big) \mathrm{d}s \leq 0,$$

that is,

$$\int_{y_2}^{y}\sum_{i=1}^{m} c_i(s)F_i\big(\epsilon A_1(s-q_i)\big)\mathrm{d}s \leq -\big[a(s)(w'(s))^{\mu}\big]_{y_2}^{y} \leq -a(y)(w'(y))^{\mu}$$

implies that

$$\frac{1}{(a(y))^{\frac{1}{\mu}}}\Big[\int_{y_2}^{y}\sum_{i=1}^{m} c_i(s)F_i\big(\epsilon A_1(s-q_i)\big)\mathrm{d}s\Big]^{\frac{1}{\mu}} \leq -w'(y)$$

and further integration of the preceeding inequality, we have

$$\int_{Y}^{v}\frac{1}{(a(y))^{\frac{1}{\mu}}}\Big[\int_{Y}^{y}\sum_{i=1}^{m} c_i(s)F_i\big(\epsilon A_1(s-q_i)\big)\mathrm{d}s\Big]^{\frac{1}{\mu}}\mathrm{d}y \leq -\big[w(y)\big]_{Y}^{v} \leq w(Y) < +\infty \quad \text{as} \quad v \to \infty.$$

gives a contradiction to (A4).

Sub-case 2_2. Let $a(y)(w'(y))^{\mu} > 0$ for $y \geq y_2$. Since $w(y)$ is nondecreasing on $[y_2,\infty)$, there exist a constant $\delta > 0$ and $Y > y_2$ such that $w(y) \geq \delta$ for $y \geq Y$. Therefore, (5) becomes

$$\big(a(y)(w'(y))^{\mu}\big)' + \sum_{i=1}^{m} c_i(y)F_i(\delta) \leq 0.$$

We integrate the inequality from Y to $+\infty$ and obtain

$$\int_{Y}^{\infty}\sum_{i=1}^{m} c_i(s)F_i(\delta)\mathrm{d}s < +\infty,$$

which is a contradiction to (A5).

The case where u is eventually negative is very similar and we omit it here. Thus, the theorem is proved. □

Theorem 3. *Assume that (C2), (A1) and (A3) hold. Furthermore, assume that*

(A6) *there exists* $\lambda > 0$ *such that* $\sum_{i=1}^{m} F_i(u) + \sum_{i=1}^{m} F_i(v) \geq \lambda \sum_{i=1}^{m} F_i(u+v)$ *for* $u, v \in \mathbb{R}_+$,
(A7) $\sum_{i=1}^{m} F_i(uv) \leq \sum_{i=1}^{m} F_i(u)\sum_{i=1}^{m} F_i(v)$ *for* $u, v \in \mathbb{R}_+$,
(A8) $\int_{Y}^{\infty}\frac{1}{(a(\eta))^{\frac{1}{\mu}}}\Big[\int_{y_3}^{\eta} c_n(\zeta)\sum_{i=1}^{m} F_i\big(\epsilon A_1(\zeta-q_i)\big)\mathrm{d}\zeta\Big]^{\frac{1}{\mu}}\mathrm{d}\eta = +\infty$ *for* $Y, y_3, \delta, \epsilon > 0$

and

(A9) $\int_{Y}^{\infty} c_n(y)\mathrm{d}y = +\infty$, *where* $c_n(y) = \min\{c_i(y), c_i(y-p)\}$

hold. Then every solution of (1) *oscillates.*

Proof. On the contrary, let $u(y)$ be a eventually positive solution of (1). Proceeding as in Theorem 1, we have that $a(y)(w'(y))^{\mu}$ is nonincreasing and constant sign on $[y_2,\infty)$. Since $w(y)$ is positive. Therefore, we have following two possible cases.

Case 1. Let $a(y)(w'(y))^{\mu} < 0$ for $y \in [y_2,\infty)$. Ultimately, $u(y)$ is bounded. From the system (1) it is not difficult to see that

$$\begin{aligned}\big(a(y)(w'(y))^{\mu}\big)' &+ \sum_{i=1}^{m} c_i(y)F_i\big(u(y-q_i)\big)\\ &+ \sum_{i=1}^{m} F_i(b_0)\Big[\big(a(y-p)(w'(y-p))^{\mu}\big)' + \sum_{i=1}^{m} c_i(y-p)F_i\big(u(y-p-q_i)\big)\Big] = 0.\end{aligned}$$

Using $(A6)$ and $(A7)$ in the above system, it follows that

$$\big(a(y)(w'(y))^{\mu}\big)' + \sum_{i=1}^{m} F_i(b_0)\big(a(y-p)(w'(y-p))^{\mu}\big)' + \lambda c_n(y) \sum_{i=1}^{m} F_i\big(w(y-q_i)\big) \le 0. \tag{7}$$

Since, $u(y)$ is bounded. By Theorem 2 Case 2_1, we have that $w(y) \ge \epsilon A_1(y)$ for $y \ge y_2$. Therefore, (7) becomes

$$\big(a(y)(w'(y))^{\mu}\big)' + \sum_{i=1}^{m} F_i(b_0)\big(a(y-p)(w'(y-p))^{\mu}\big)' + \lambda c_n(y) \sum_{i=1}^{m} F_i\big(\epsilon A_1(y-q_i)\big) \le 0$$

for $y \ge y_3 > y_2$. Integrating the last inequality from y_3 to y $(> y_3)$, we get

$$\big[a(s)(w'(s))^{\mu}\big]_{y_3}^{y} + \sum_{i=1}^{m} F_i(b_0)\big[a(s-p)(w'(s-p))^{\mu}\big]_{y_3}^{y} + \lambda \int_{y_3}^{y} c_n(s) \sum_{i=1}^{m} F_i\big(\epsilon A_1(s-q_i)\big)\mathrm{d}s \le 0,$$

that is,

$$\begin{aligned}
\lambda \int_{y_3}^{y} c_n(s) \sum_{i=1}^{m} F_i\big(\epsilon A_1(s-q_i)\big)\mathrm{d}s &\le -\Big[a(s)(w'(s))^{\mu} + \sum_{i=1}^{m} F_i(b_0)a(s-p)(w'(s-p))^{\mu}\Big]_{y_3}^{y} \\
&\le -\Big[a(y)(w'(y))^{\mu} + \sum_{i=1}^{m} F_i(b_0)a(y-p)(w'(y-p))^{\mu}\Big] \\
&\le -\Big(1 + \sum_{i=1}^{m} F_i(b_0)\Big)a(y)(w'(y))^{\mu}.
\end{aligned}$$

Therefore,

$$\left(\frac{\lambda}{(1+\sum_{i=1}^{m} F_i(b_0))}\right)^{\frac{1}{\mu}} \frac{1}{(a(y))^{\frac{1}{\mu}}} \Big[\int_{y_3}^{y} c_n(s) \sum_{i=1}^{m} F_i\big(\epsilon A_1(s-q_i)\big)\mathrm{d}s\Big]^{\frac{1}{\mu}} \le -w'(y).$$

Integrating the above inequality, we obtain

$$\left(\frac{\lambda}{(1+\sum_{i=1}^{m} F_i(b_0))}\right)^{\frac{1}{\mu}} \int_{Y}^{\infty} \frac{1}{(a(\eta))^{\frac{1}{\mu}}} \Big[\int_{y_3}^{\eta} c_n(\zeta) \sum_{i=1}^{m} F_i\big(\epsilon A_1(\zeta-q_i)\big)\mathrm{d}\zeta\Big]^{\frac{1}{\mu}} \mathrm{d}\eta < \infty$$

which is a contradiction to $(A8)$.

Case 2. Let $a(y)(w'(y))^{\mu} > 0$ for $y \in [y_2, \infty)$. Then there exist a constant $\delta > 0$ and $y_3 > y_2$ such that $w(y) \ge \delta$ for $y \ge y_3$. From (7), it follows that

$$\big(a(y)(w'(y))^{\mu}\big)' + \sum_{i=1}^{m} F_i(b_0)\big(a(y-p)(w'(y-p))^{\mu}\big)' + \lambda c_n(y) \sum_{i=1}^{m} F_i(\delta) \le 0.$$

Integrating the last inequality from Y to $+\infty$, we get a contradiction to $(A9)$.

The case where u is an eventually negative solution is omitted since it can be dealt similarly. □

Theorem 4. *Assume that (C3), (A1) and (A3)–(A5) hold. Furthermore, assume that*

$(A10)$ $\int_{Y}^{\infty} \sum_{i=1}^{m} c_i(s) F_i\big(-b_1^{-1}\beta\big)\mathrm{d}s = \infty$ *for* $\beta < 0$

and

$(A11)$ $\int_{Y}^{\infty} \frac{1}{(a(s))^{\frac{1}{\mu}}} \Big[\int_{Y}^{s} \sum_{i=1}^{m} c_i(\zeta) F_i\big(-b_1^{-1}\beta\big)\mathrm{d}\zeta\Big]^{\frac{1}{\mu}} \mathrm{d}s = +\infty$ *for* $\beta < 0$

hold. Then every bounded solution of the system (1) *either oscillates or converges to zero.*

Proof. Let $u(y)$ be a bounded solution of the system (1). Proceeding as in Theorem 1, it follows that $w(y)$ and $a(y)(w'(y))^{\mu}$ are monotonic on $[y_2, \infty)$. Since $u(y)$ is bounded, then $w(y)$ is also bounded. Hence, $\lim_{y\to\infty} w(y)$ exists. By Theorem 2, we get contradictions to (A4) and (A5) for cases $w(y) > 0, a(y)(w'(y))^{\mu} < 0$ and $w(y) > 0, a(y)(w'(y))^{\mu} > 0$ respectively. Therefore, we have the following two possible cases.

Case 1. Let $w(y) < 0$, $a(y)(w'(y))^{\mu} > 0$ for $y \geq y_2$. We claim that $\lim_{y\to\infty} w(y) = 0$. If not, there exist $\beta < 0$ and $y_3 > y_2$ such that $w(y + p - q_i) < \beta$ for $y \geq y_3$. Hence, $w(y) \geq -b_1 u(y - p)$ implies that $u(y - q_i) \geq -b_1^{-1}\beta$ for $y \geq y_3$. Consequently, Equation (5) reduces to

$$\big(a(y)(w'(y))^{\mu}\big)' + \sum_{i=1}^{m} F_i\big(-b_1^{-1}\beta\big)c_i(y) \leq 0 \tag{8}$$

for $y \geq y_3$. Integrating (8) from Y to $+\infty$, we get

$$\int_Y^{\infty} \sum_{i=1}^{m} c_i(s)F_i\big(-b_1^{-1}\beta\big)\mathrm{d}s < \infty$$

which contradicts to (A10). Therefore, our claim holds and

$$\begin{aligned}
0 = \lim_{y\to\infty} w(y) &= \liminf_{y\to\infty}\big(u(y) + b(y)u(y-p)\big)\\
&\leq \liminf_{y\to\infty}(u(y) - b_2\, u(y-p))\\
&\leq \limsup_{y\to\infty} u(y) + \liminf_{y\to\infty}\big(-b_2\, u(y-p)\big)\\
&= (1 - b_2)\limsup_{y\to\infty} u(y)
\end{aligned}$$

implies that $\limsup_{y\to\infty} u(y) = 0$, because $1 - b_2 < 0$. Thus, $\lim_{y\to\infty} u(y) = 0$.

Case 2. Let $w(y) < 0$, $a(y)(w'(y))^{\mu} < 0$ for $y \geq y_2$. Proceeding as in the previous case, we get (8). Integrating (8) from Y to y, we get

$$\int_Y^{y} \sum_{i=1}^{m} c_i(s)F_i\big(-b_1^{-1}\beta\big)\mathrm{d}s \leq -a(y)(w'(y))^{\mu},$$

that is,

$$\frac{1}{(a(y))^{\frac{1}{\mu}}}\Big[\int_Y^{y} \sum_{i=1}^{m} c_i(s)F_i\big(-b_1^{-1}\beta\big)\mathrm{d}s\Big]^{\frac{1}{\mu}} \leq -w'(y)$$

for $y \geq Y$. Further integration the above inequality from Y to $+\infty$, we get

$$\int_Y^{\infty} \frac{1}{(a(s))^{\frac{1}{\mu}}}\Big[\int_Y^{s} \sum_{i=1}^{m} c_i(\zeta)F_i\big(-b_1^{-1}\beta\big)\mathrm{d}\zeta\Big]^{\frac{1}{\mu}}\mathrm{d}s < \infty$$

which is a contradiction (A11). Thus, $\lim_{y\to\infty} w(y) = 0$. Rest of the case follows from Case 1. Hence, the proof of the Theorem is complete. □

We now present the following example:

Example

Consider the differential equation

$$\left(y^6\left(\left(u(y)-\frac{1}{y}u(y-2)\right)'\right)^3\right)'+\frac{1}{y^3}F_1(u(y))+\frac{1}{(y-1)^5}F_2(u(y-1))=0, y\geq 2, \tag{9}$$

where $a(y) = y^6$, $b(y) = -\frac{1}{y}$, $p = 2$, $c_i(y) = (y-i+1)^{-(2i+1)}$, $F_i(y) = y^{2i+1}$, $\mu = 3$, $q_i = i-1$, with the index $i = 1, 2$. Here $F_i(y)$ is an odd function. Since, $\int_{y_0}^{\infty}\frac{1}{(a(s))^{\frac{1}{\mu}}} = \frac{1}{y_0} < \infty$ and in order to verify (A3), we have

$$\begin{aligned}&\int_{y_0}^{\infty}\Big(c_1(y)F_1(\delta A(y-q_1))+c_2(y)F_2(\delta A(y-q_2))\Big)dy\\&=\int_{y_0}^{\infty}\frac{1}{y^3}\left(\frac{1}{Y}-\frac{1}{y}\right)^3dy+\int_{y_0}^{\infty}\frac{1}{(y-1)^5}\left(\frac{1}{(Y-1)}-\frac{1}{(y-1)}\right)^5dy<\infty.\end{aligned}$$

Therefore, all the assumptions of Theorem 1 are verified. Hence, due to Theorem 1 every solution of (9) is oscillatory.

3. Conclusions

In this work, we obtained necessary and sufficient conditions for the oscillation of a second-order non-linear neutral differential equation with multiple delays (1) under the assumption (A3) when $-1 < b(y) < 0$. However, we failed to obtain the necessary and sufficient conditions for the oscillation of the solution of (1) for the other ranges of the neutral coefficient $b(y)$.

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References

1. Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. Oscillation of second-order differential equations with a sublinear neutral term. *Carpath. J. Math.* **2014**, 30, 1–6.
2. Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. Oscillation of second-order Emden-Fowler neutral delay differential equations. *Ann. Mat. Pura Appl.* **2014**, *193*, 1861–1875. [CrossRef]
3. Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. Even-order half-linear advanced differential equations: improved criteria in oscillatory and asymptotic properties. *Appl. Math. Comput.* **2015**, *266*, 481–490. [CrossRef]
4. Agarwal, R.P.; Zhang, C.; Li, T. Some remarks on oscillation of second order neutral differential equations. *Appl. Math. Comput.* **2016**, *274*, 178–181. [CrossRef]
5. Liu, M.; Dassios, I.; Milano, F. Delay margin comparisons for power systems with constant and time-varying delays. *Electr. Power Syst. Res.* **2021**, *190*, 106627. [CrossRef]
6. Dassios, I.; Zimbidis, A.; Kontzalis, C. The delay effect in classical Samuelson's model. *J. Econ. Struct.* **2014**, *3*, 7. [CrossRef]
7. Liu, M.; Dassios, I.; Milano, F. On the Stability Analysis of Systems of Neutral Delay Differential Equations. *Circuits Syst. Signal Process.* **2019**, *38*, 1639–1653. [CrossRef]
8. Bhoner, M.; Hassan, T.S. Fite-Hille-Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments. *Indag. Math.* **2018**, *29*, 548–560. [CrossRef]
9. Li, T.; Pintus, N.; Viglialoro, G. Properties of solutions to porous medium problems with different sources and boundary conditions. *Z. Angew. Math. Phys.* **2019**, *70*, 86. [CrossRef]

10. Viglialoro, G.; Woolley, T.E. Boundedness in a parabolic-elliptic chemotaxis system with nonlinear diffusion and sensitivity and logistic source. *Math. Methods Appl. Sci.* **2018**, *41*, 1809–1824. [CrossRef]
11. Brands, J.J.M.S. Oscillation Theorems for second-order functional-differential equations. *J. Math. Anal. Appl.* **1978**, *63*, 54–64. [CrossRef]
12. Chatzarakis, G.E.; Dzurina, J.; Jadlovska, I. New oscillation criteria for second-order half-linear advanced differential equations. *Appl. Math. Comput.* **2019**, *347*, 404–416. [CrossRef]
13. Chatzarakis, G.E.; Jadlovska, I. Improved oscillation results for second-order half-linear delay differential equations. *Hacet. J. Math. Stat.* **2019**, *48*, 170–179. [CrossRef]
14. Wong, J.S.W. Necessary and suffcient conditions for oscillation of second-order neutral differential equations. *J. Math. Anal. Appl.* **2000**, *252*, 342–352. [CrossRef]
15. Baculikova, B.; Dzurina, J. Oscillation Theorems for second-order neutral differential equations. *Comput. Math. Appl.* **2011**, *61*, 94–99. [CrossRef]
16. Džurina, J. Oscillation Theorems for second-order advanced neutral differential equations. *Tatra Mt. Math. Publ.* **2011**, *48*, 61–71. [CrossRef]
17. Baculikova, B.; Dzurina, J. Oscillation Theorems for second-order nonlinear neutral differential equations. *Comput. Math. Appl.* **2011**, *62*, 4472–4478. [CrossRef]
18. Tripathy, A.K.; Panda, B.; Sethi, A.K. On oscillatory nonlinear second-order neutral delay differential equations. *Differ. Equ. Appl.* **2016**, *8*, 247–258. [CrossRef]
19. Bohner, M.; Grace, S.R.; Jadlovska, I. Oscillation criteria for second-order neutral delay differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2010**, *2010*, 763278. [CrossRef]
20. Grace, S.R.; Džurina, J.; Jadlovska, I.; Li, T. An improved approach for studying oscillation of second-order neutral delay differential equations. *J. Inequ. Appl.* **2018**, *2018*, 1–13. [CrossRef]
21. Li, H.; Zhao, Y.; Han, Z. New oscillation criterion for Emden-Fowler type nonlinear neutral delay differential equations. *J. Appl. Math. Comput.* **2019**, *60*, 191–200. [CrossRef]
22. Karpuz, B.; Santra, S.S. Oscillation Theorems for second-order nonlinear delay differential equations of neutral type. *Hacet. J. Math. Stat.* **2019**, *48*, 633–643. [CrossRef]
23. Baculikova, B.; Li, T.; Dzurina, J. Oscillation Theorems for second order neutral differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2011**, *74*, 1–13. [CrossRef]
24. Bazighifan, O. Improved Approach for Studying Oscillatory Properties of Fourth-Order Advanced Differential Equations with p-Laplacian Like Operator. *Mathematics* **2020**, *8*, 656. [CrossRef]
25. Bazighifan, O. Kamenev and Philos-types oscillation criteria for fourth-order neutral differential equations. *Adv. Differ. Equ.* **2020**, *201*, 1–12. [CrossRef]
26. Bazighifan, O.; Dassios, I. On the Asymptotic Behavior of Advanced Differential Equations with a Non-Canonical Operator. *Appl. Sci.* **2020**, *10*, 3130. [CrossRef]
27. Bazighifan, O.; Dassios, I. Riccati Technique and Asymptotic Behavior of Fourth-Order Advanced Differential Equations Mathematics. *Mathematics* **2020**, *8*, 590. [CrossRef]
28. Bazighifan, O.; Ruggieri, M.; Santra, S.S.; Scapellato, A. Qualitative Properties of Solutions of Second-Order 20 Neutral Differential Equations. *Symmetry* **2020**, *12*, 1520. [CrossRef]
29. Dassios, I.; Bazighifan, O. Oscillation Conditions for Certain Fourth-Order Non-Linear Neutral Differential Equation. *Symmetry* **2020**, *12*, 1096. [CrossRef]
30. Moaaz, O.; Dassios, I.; Bazighifan, O.; Muhib, A. Oscillation Theorems for Nonlinear Differential Equations of Fourth-Order. *Mathematics* **2020**, *8*, 520. [CrossRef]
31. Moaaz, O.; Dassios, I.; MuhsiN, W.; Muhib, A. Oscillation Theory for Non-Linear Neutral Delay Differential Equations of Third Order. *Appl. Sci.* **2020**, *10*, 4855. [CrossRef]
32. Moaaz, O.; Dassios, I.; Bazighifan, O. Oscillation Criteria of Higher-order Neutral Differential Equations with Several Deviating Arguments. *Mathematics* **2020**, *8*, 402. [CrossRef]
33. Chatzarakis, G.E.; Elabbasy, E.M.; Bazighifan, O. An oscillation criterion in 4th-order neutral differential equations with a continuously distributed delay. *Adv. Differ. Equ.* **2019**, *336*, 1–9.
34. Džurina, J.; Grace, S.R.; Jadlovska, I.; Li, T. Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term. *Math. Nachr.* **2020**, *293*, 910–922. [CrossRef]
35. Li, T.; Rogovchenko, Y.V. Oscillation Theorems for second-order nonlinear neutral delay differential eqquations. *Abstr. Appl. Anal.* **2014**, *2014*, 594190.

36. Li, T.; Rogovchenko, Y.V. Oscillation of second-order neutral differential equations. *Math. Nachr.* **2015**, *288*, 1150–1162. [CrossRef]
37. Li, Q.; Wang, R.; Chen, F.; Li, T. Oscillation of second-order nonlinear delay differential equations with nonpositive neutral coefficients. *Adv. Differ. Equ.* **2015**, *2015*, 35. [CrossRef]
38. Li, T.; Rogovchenko, Y.V. Oscillation criteria for second-order superlinear Emden-Fowler neutral differential equations. *Monatsh. Math.* **2017**, *184*, 489–500. [CrossRef]
39. Pinelas, S.; Santra, S.S. Necessary and sufficient condition for oscillation of nonlinear neutral first-order differential equations with several delays. *J. Fixed Point Theory Appl.* **2018**, *20*, 1–13. [CrossRef]
40. Pinelas, S.; Santra, S.S. Necessary and sufficient conditions for oscillation of nonlinear first order forced differential equations with several delays of neutral type. *Analysis* **2019**, *39*, 97–105. [CrossRef]
41. Qian, Y.; Xu, R. Some new oscillation criteria for higher order quasi-linear neutral delay differential equations. *Differ. Equ. Appl.* **2011**, *3*, 323–335. [CrossRef]
42. Santra, S.S. Existence of positive solution and new oscillation criteria for nonlinear first-order neutral delay differential equations. *Differ. Equ. Appl.* **2016**, *8*, 33–51. [CrossRef]
43. Santra, S.S. Oscillation analysis for nonlinear neutral differential equations of second-order with several delays. *Mathematica* **2017**, *59*, 111–123.
44. Santra, S.S. Necessary and sufficient conditions for oscillation to second-order half-linear delay differential equations. *J. Fixed Point Theory Appl.* **2019**, *21*, 85. [CrossRef]
45. Santra, S.S. Oscillation analysis for nonlinear neutral differential equations of second-order with several delays and forcing term. *Mathematica* **2019**, *61*, 63–78. [CrossRef]
46. Santra, S.S.; Dix, J.G. Necessary and sufficient conditions for the oscillation of solutions to a second-order neutral differential equation with impulses. *Nonlinear Stud.* **2020**, *27*, 375–387. [CrossRef]
47. Santra, S.S. Necessary and sufficient condition for oscillatory and asymptotic behavior of second-order functional differential equations. *Krag. J. Math.* **2020**, *44*, 459–473.
48. Santra, S.S. Necessary and sufficient conditions for oscillatory and asymptotic behavior of solutions to second-order nonlinear neutral differential equations with several delays. *Tatra Mt. Math. Publ.* **2020**, *75*, 121–134.
49. Zhang, C.; Agarwal, R.P.; Bohner, M.; Li, T. Oscillation of second-order nonlinear neutral dynamic equations with noncanonical operators. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 761–778. [CrossRef]
50. Moaaz, O.; Elabbasy, E.M.; Bazighifan, O. On the asymptotic behavior of fourth-order functional differential equations. *Adv. Differ. Equ.* **2017**, *261*, 1–13. [CrossRef]
51. Tripathy, A.K.; Santra, S.S. On oscillatory nonlinear forced neutral impulsive systems of second order. *Nonlinear Ocillations* **2020**, *23*, 274–288.