

Review

String Phenomenology: Past, Present and Future Perspectives [†]

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[†] This article represents the author's light cone in the area of string phenomenology, to which he contributed since the late eighties. The author extends his apologies to those outside this light cone.

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Abstract: The observation of a scalar resonance at the Large Hadron Collider (LHC), compatible with perturbative electroweak symmetry breaking, reinforces the Standard Model (SM) parameterisation of all subatomic data. The logarithmic evolution of the SM gauge and matter parameters suggests that this parameterisation remains viable up to the Planck scale, where gravitational effects are of comparable strength. String theory provides a perturbatively consistent scheme to explore how the parameters of the Standard Model may be determined from a theory of quantum gravity. The free fermionic heterotic string models provide concrete examples of exact string solutions that reproduce the spectrum of the Minimal Supersymmetric Standard Model. Contemporary studies entail the development of methods to classify large classes of models. This led to the discovery of exophobic heterotic-string vacua and the observation of spinor-vector duality, which provides an insight to the global structure of the space of (2,0) heterotic-string vacua. Future directions entail the study of the role of the massive string states in these models and their incorporation in cosmological scenarios. A complementary direction is the formulation of quantum gravity from the principle of manifest phase space duality and the equivalence postulate of quantum mechanics, which suggest that space is compact. The compactness of space, which implies intrinsic regularisation, may be tightly related to the intrinsic finite length scale, implied by string phenomenology.

Keywords: unification; string phenomenology; particle physics; quantum gravity

1. Introduction

The experimental observation of a scalar resonance by the ATLAS (A Toroidal Large Hadron Collider Apparatus) [1] and CMS (Compact Muon Solenoid) [2] experiments of the Large Hadron Collider (LHC) at the European Organization for Nuclear Research (CERN), compatible with the scalar particle of the Standard Electroweak Model [3], is a pivotal moment in the quest for the unification of the fundamental theories of matter and interactions. Indeed, nearly thirty years have elapsed since the experimental discovery of the W^\pm and Z -vector bosons [4,5], and forty years since the demonstration of renormalisability of spontaneously broken non-Abelian gauge symmetries [6], which were the earlier milestones on this journey. The discovery of the Higgs boson solidifies the Standard Model parameterisation of all subatomic experimental observations to date. The observation of a Higgs boson at 125 GeV suggests that the electroweak symmetry breaking mechanism is perturbative, rather than non-perturbative. This reinforces the view that the Standard Model provides a viable perturbative parameterisation of the subatomic interactions up to an energy scale, which is separated by orders of magnitude from the scale within reach of contemporary accelerator experiments. If this is indeed the scenario selected by nature, it entails that alternative experimental tests will be required to establish its validity. These tests will unavoidably look for astrophysical and cosmological imprints that can probe the much higher energy scales.

The possibility that the Standard Model provides a viable effective parameterisation, up to a much higher scale, has been entertained in the context of Grand Unified Theories (GUTs) and string theories [7]. The gauge charges of the Standard Model matter states are strongly suggestive of the embedding of the Standard Model states in representations of larger gauge groups. This is most striking in the context of $SO(10)$ GUT, in which each of the Standard Model chiral generations fits into a 16 spinorial representation of $SO(10)$. The gauge charges of the Standard Model matter states are experimental observables. The Standard Model contains three generations, which are split into six multiplets that are charged under its three gauge sectors. Therefore, in the framework of the Standard Model, one needs fifty-four parameters to account for these gauge charges. Embedding the Standard Model in $SO(10)$ reduces this number of parameters to one parameter, which is the number of spinorial 16 representations of $SO(10)$ needed to accommodate the Standard Model spectrum. Additional evidence for the high scale unification stems from:

- The logarithmic running of the Standard Model parameters, which is compatible with observations in the gauge sectors [8,9] and the heavy generation Yukawa couplings [10]. Logarithmic running in the scalar sector is spoiled by radiative corrections from the Standard Model cut-off scale. Restoration of the logarithmic running mandates the existence of a new symmetry. Supersymmetry is a concrete example that fulfils the task. The observation of a scalar resonance at 125 GeV and the fact that no other particles have been observed up to the multi-TeV energy scale indicate that the resonance is a fundamental scalar rather than a composite state [11]. This outcome agrees with the Higgs states in heterotic-string vacua.
- Further evidence for the validity of the renormalisable Standard Model up to a very high energy scale stems from the suppression of proton decay mediating operators. The Standard Model should be regarded as providing a viable effective parametrisation, but not as a fundamental accounting

of the observable phenomena; the reason being that it does not provide a complete description. Obviously, gravitational effects are not accounted for. Moreover, the Standard Model itself is not mathematically self-consistent. It gives rise to singularities in the ultraviolet limit. For these reasons, the Standard Model can only be regarded as an effective theory below some cut-off. A plausible cut-off is the Planck scale, at which the gravitational coupling is of comparable strength to the gauge couplings. The renormalisability of the Standard Model is not valid beyond its cut-off scale. Non-renormalisable operators are induced by whatever theory extends the Standard Model at and beyond the cut-off scale. We should therefore take into account all the non-renormalisable operators that are allowed by the Standard Model gauge symmetries and that are suppressed by powers of the cut-off scale. Such dimension six operators that are invariant under the Standard Model gauge symmetries lead to proton decay. They indicate that the cut-off scale must be above 10^{16} GeV, unless they are forbidden by some new symmetries. As global symmetries are, in general, expected to be violated by quantum gravity effects, the new symmetries should be either gauge symmetries or local discrete symmetries [12,13].

- Suppression of left-handed neutrino masses is compatible with the generation of heavy mass to the right-handed neutrinos by the seesaw mechanism.

The Standard Model multiplet structure and the additional evidence provided by logarithmic running, proton longevity and neutrino masses indicate that the primary guides in the search of a realistic string vacuum are the existence of three chiral generations and their embedding in $SO(10)$ representations. It should be noted that this embedding does not entail the existence of an $SO(10)$ gauge symmetry in the effective low energy field theory. Rather, the $SO(10)$ symmetry is broken at the string level to a maximal subgroup and, preferably, directly to the Standard Model gauge group.

The Standard Model of particle physics is founded on a causal and renormalisable quantum field theory with local phase invariance under a product of Abelian and non-Abelian gauge symmetries. These symmetry principles encode all the subatomic experimental observations to date. Alas, the effects of the gravitational interactions are not included in this picture. Moreover, there is a fundamental dichotomy between the principles underlying quantum mechanics and gravitational observations; in particular, with regard to the treatment of the vacuum. While quantum field theories give rise to energy sources that contribute to the vacuum energy with a scale of the order of the Quantum chromodynamics (QCD) scale and beyond, observations show that the vacuum energy is smaller by orders of magnitude. Another point of contention is with regard to the nature of space. In general relativity, the contemporary theory of gravity, space is a dynamical field satisfying Einstein's equations of motion. In quantum field theories, on the other hand, space provides background parameters and does not correspond to the fundamental degrees of freedom, which are encoded in the particle wave-functions and their conjugate momenta. Furthermore, gravity as a quantum field theory is not renormalisable, which is therefore plagued with infinities and is inconsistent at a fundamental level.

The conundrum may be seen to arise from the fact that quantum field theories may, in principle, probe space distances that are infinitely small, provided that the corresponding momenta is infinitely large. We may envision that this outcome is fundamentally unphysical, and what we need is a fundamental description of matter and interactions, which excludes the possibility of probing infinitely

small distances. String theory provides such a theory. Moreover, the equivalence postulate formulation of quantum mechanics implies that space is compact and the existence of a fundamental length in quantum mechanics [14]. The fundamental cut-off may therefore be intrinsically built into quantum mechanics, provided that its full set of symmetries are incorporated.

As a finite theory string theory provides a consistent framework for perturbative quantum gravity [15–18], the consistency of string theory at the quantum level dictates that it must accommodate a specific number of worldsheet degrees of freedom to produce an anomaly-free and finite theory. Some of degrees of freedom give rise to the gauge symmetries that we may identify with the subatomic interactions. Moreover, similar consistency constraints at the quantum level in the case of the superstring and heterotic-string give rise to matter states that are charged under the gauge degrees of freedom and may be identified with the Standard Model matter states. Hence, string theory provides a viable framework for the consistent unification of gravity with subatomic matter and interactions. In turn, this feature of string theory allows for the development of a phenomenological approach to quantum gravity.

String theory is therefore a mundane extension of the idealisation of point particles with internal attributes. Furthermore, the rank of the gauge group accounting for the internal attributes is fixed by the consistency conditions of the theory. The string action is parameterised by two worldsheet degrees of freedom, corresponding to the proper time and the string internal dimension. The equation of motion of the worldsheet degrees of freedom is a two-dimensional wave equation. The solutions are separated into left- and right-moving solutions. The physical states of the quantised string give rise to a tachyonic state, which is eliminated from the spectrum if the bosonic worldsheet fields are augmented with fermionic fields. This is achieved provided that the theory possesses $N = 2$ supersymmetry on the worldsheet, which guarantees the existence of $N = 1$ spacetime supersymmetry. Since the tachyonic state does not have a corresponding fermionic superpartner, the existence of spacetime supersymmetry guarantees that the tachyonic state is excluded from the physical spectrum. Additionally, the fermionic string gives rise to spacetime fermions that transforms in representations of the internal gauge symmetry.

String theory is formulated as a perturbative scattering expansion. Using the conformal symmetry on the worldsheet, the lowest order amplitudes can be mapped to the sphere with vertex operator insertions corresponding to the external string states. Higher order amplitudes are mapped to higher genus tori, with the genus one torus being the lowest order quantum correction. The vacuum to vacuum amplitude is the first order correction when there are no external states, and all the physical states can propagate in the closed time-like loop. The conformal worldsheet symmetry is translated to invariance of the torus amplitude under modular transformations of the complex worldsheet parameter, τ . The worldsheet fermionic fields can pick up non-trivial phases when parallel transported around the non-contractible loops of the worldsheet torus. The possible transformations for all the worldsheet fermions are encoded in the so-called spin structures and are mixed non-trivially by the modular transformations. Requiring invariance under modular transformations leads to a set of non-trivial constraints on the allowed spin structures [15–18].

Different string theories may be formulated depending on the existence, or not, of worldsheet fermionic fields in the left- and right-moving sectors of the string. Type IIA and Type IIB superstrings arise if worldsheet fermions are added in both the left- and right-moving sectors. Adding worldsheet fermions only to the left-moving sector produces the heterotic-string with $E_8 \times E_8$, or $SO(32)$ gauge

symmetry in ten dimensions. In the low energy point particle approximation, we expect a string theory to correspond to an effective field theory approximation. That is, when the energy involved is not sufficiently high to reveal the internal structure of the string, we expect that it should be described effectively as some point particle field theory. In the case of the fermionic strings, these are Type IIA or IIB supergravities, or an effective ten-dimensional supergravity with $E_8 \times E_8$ or $SO(32)$ gauge symmetry. Additionally, the non-perturbative effective field theory limits of the ten-dimensional string are related to compactifications of eleven-dimensional supergravity. For example, the Type IIA superstring is related to compactification of eleven-dimensional supergravity on a circle, whereas the ten-dimensional heterotic $E_8 \times E_8$ corresponds to compactification on a circle moded by a Z_2 reflection symmetry. The full set of relations at the quantum level is yet to be unravelled and is traditionally dubbed as M-theory or F-theory [15–18].

The lesson is that our understanding of the synthesis of gravity and the gauge interactions is still very rudimentary. String theory is clearly a step in the right direction. It provides a framework to ask questions about the gauge and gravity unification and to seek consistent answers within that framework. By giving rise to all the basic fields that are used to parameterise the subatomic and gravitational experimental data, it enables the development of a phenomenological approach to quantum gravity. However, it is clear that string theory is not the final answer. The contemporary string theories are believed to be effective limits of a more fundamental theory. From that perspective, each of the string theories can be used to probe some properties of the vacuum of the fundamental theory, but not to fully characterise it. The heterotic $E_8 \times E_8$ string is the effective limit that gives rise to spinorial $SO(10)$ representation in the perturbative spectrum. The heterotic-string, therefore, is the effective limit that should be used if the properties that we would like to preserve are the existence of three chiral generations and their embedding in spinorial $SO(10)$ representations.

2. Past

Realistic string models are obtained by compactifying the heterotic-string from ten to four dimensions. Alternatively, we can construct realistic string models directly in four dimensions by representing the compactified dimensions in terms of internal conformal field theories propagating on the string worldsheet. The simplest such theories are in terms of free worldsheet field theories, *i.e.*, in terms of free bosons [19] or free fermions [20,21], with the main simplification being the implementation of the modular invariance constraints. Nevertheless, constructions using interacting worldsheet conformal field theories exist, as well [22], and can be used to construct phenomenological vacua. It should be remarked that the representations of the four-dimensional string vacua as compactifications on internal manifold or in terms of internal conformal field theories are not necessarily distinct. For example, theories that utilise two-dimensional worldsheet free bosons or free fermions are mathematically equivalent. Similarly, it was demonstrated in some cases that string models with interacting internal conformal field theory (CFT) correspond to string compactification on a Calabi–Yau manifold at specific points in the moduli space [22]. This is an important point for the following reason. While the space of distinct string vacua in the effective field theory limit may seem to be huge, many of these vacua are related by various perturbative and non-perturbative dualities at the string level. The reason is that at the string

level massless and massive physical states can be exchanged. Thus, vacua that are topologically and physically distinct in the effective field theory level are in fact connected at the string level. This feature is particularly important if we envision the existence of a dynamical vacuum selection mechanism in string theory.

The simplest phenomenological string models can therefore be constructed by using a free internal conformal field theory. String theories in which the internal CFT is written in terms of free fermions correspond to compactifications on a flat six-dimensional torus at a special point in the moduli space [23]. Exactly marginal deformations from the free fermionic point are obtained by adding Thirring interactions among the worldsheet fermions [24]. The number of allowed deformations correspond exactly to the number of allowed deformations in compactifications of the corresponding string theory on a flat torus. Compactifications of the heterotic-string on a flat six-dimensional torus produce $N = 4$ spacetime supersymmetry, which is reduced to $N = 1$ by modding out the internal six-dimensional torus by an internal symmetry. This produces the so called orbifold compactifications. The simplest such orbifolds correspond to modding out the internal six-dimensional torus by Z_2 symmetries. Modding out by a single Z_2 reduces the number of spacetime supersymmetries from $N = 4$ to $N = 2$. Therefore, to reduce the number of supersymmetries to $N = 1$ necessitates modding out by two independent Z_2 symmetries, *i.e.*, by $Z_2 \times Z_2$.

2.1. NAHE-Based Models

In the free fermionic formulation [20,21] of toroidal compactifications [25,26], all the internal degrees of freedom needed to cancel the worldsheet conformal anomaly are represented in terms of free fermions propagating on the string worldsheet. In the usual notation, the 64 worldsheet fermions in the light-cone gauge are denoted as:

Left-Movers: $\psi^\mu, \chi_i, y_i, \omega_i \quad (\mu = 1, 2, i = 1, \dots, 6)$

Right-Movers:

$$\bar{\phi}_{A=1, \dots, 44} = \begin{cases} \bar{y}_i, \bar{\omega}_i & i = 1, \dots, 6 \\ \bar{\eta}_i & i = 1, 2, 3 \\ \bar{\psi}_{1, \dots, 5} \\ \bar{\phi}_{1, \dots, 8} \end{cases}$$

In this notation, the $\psi^{1,2}, \chi^{1, \dots, 6}$ are the fermionic superpartners of the left-moving bosonic coordinates. The $\{y, \omega | \bar{y}, \bar{\omega}\}^{1, \dots, 6}$ are the worldsheet real fermions corresponding to the six compactified dimensions of the internal manifold. The remaining sixteen complex fermions generate the Cartan subalgebra of the ten-dimensional gauge group, with $\bar{\psi}^{1, \dots, 5}$ being those that generate the $SO(10)$ symmetry, and $\bar{\phi}^{1, \dots, 8}$ are those that generate the hidden sector gauge group. The $\bar{\eta}^{1,2,3}$ complex worldsheet fermions generate three $U(1)$ symmetries.

Under parallel transport around the noncontractible loops of the torus amplitude, the worldsheet fermionic fields can pick up a phase. The 64 phases are encoded in boundary condition basis vectors, which generate the one loop partition function,

$$Z = \sum_{\substack{\text{all spin} \\ \text{structures}}} c \begin{pmatrix} \vec{\xi} \\ \vec{\beta} \end{pmatrix} Z \begin{pmatrix} \vec{\xi} \\ \vec{\beta} \end{pmatrix}$$

where $\vec{\xi}$ and $\vec{\beta}$ denote all possible combinations of the basis vectors. The requirement of modular invariance leads to a set of constraints on the allowed basis vectors and one loop phase. The basis vectors generate a finite additive group, Ξ , and each sector in the additive group, $\xi \in \Xi$, produces a Fock space by acting on the vacuum with fermionic and bosonic oscillators. Worldsheet fermionic fields that are periodic under parallel transport produce a doubly degenerate vacuum that generate the spinorial charges. The physical states in the Hilbert space are obtained by applying the Generalised Gliozzi, Scherk, Olive (GSO) projections, which arise due to the modular invariance requirement. The cubic level and higher order terms in the superpotential are obtained by calculating scattering amplitudes between vertex operators [27–29]. Finally, string vacua often give rise to a pseudo-anomalous $U(1)$ symmetry, which is cancelled by the Green–Schwarz mechanism [30,31]. The anomalous $U(1)$ gives rise to a Fayet–Iliopoulos D -term [31], which breaks spacetime supersymmetry at the string scale. Restoration of supersymmetry is obtained by assigning non-trivial Vacuum Expectation Value (VEV) to a set of fields in the physical spectrum and imposing that all the supersymmetry breaking F - and D -terms vanish.

In this manner, a large set of string vacua can be obtained. The early quasi-realistic free fermionic models were constructed in the late 1980s to early 1990s, and consist of the so-called NAHE (Nanopoulos, Antoniadis, Hagelin, Ellis. *nahe* means pretty in hebrew)-based models. The NAHE-set is a set of five boundary condition basis vectors, $\{\mathbf{1}, S, b_1, b_2, b_3\}$, which is common to a large class of the early models [32]. The two basis vectors, $\{\mathbf{1}, S\}$, correspond to a toroidally compactified model with $N = 4$ spacetime supersymmetry and an $SO(44)$ gauge group. The sectors, b_1, b_2 and b_3 , correspond to the three twisted sectors of a $Z_2 \times Z_2$ orbifold compactification. They reduce the number of supersymmetries to $N = 1$ and the gauge symmetry to $SO(10) \times SO(6)^3 \times E_8$. Additionally, they produce 48 multiplets in the spinorial $\mathbf{16}$ representation of $SO(10)$. The number of these multiplets is reduced to three by adding three additional basis vectors to the NAHE set, typically denoted by $\{\alpha, \beta, \gamma\}$, which also reduce the gauge symmetry. The $SO(10) \times E_8$ symmetry is reduced to a maximal subgroup, and the flavour $SO(6)^3$ symmetries are reduced to $U(1)^n$, with $n = 3, \dots, 9$. Using this construction three generation models with:

- $SU(5) \times U(1)$ [33];
- $SU(3) \times SU(2) \times U(1)^2$ [34–36];
- $SO(6) \times SO(4)$ [37] and
- $SU(3) \times SU(2)^2 \times U(1)$ [38],

whereas models with $SU(4) \times SU(2) \times U(1)$ [39] did not yield three generations. It is noted that in all these models, the Standard Model weak hypercharge possesses the $SO(10)$ embedding and yields

the canonical GUT normalisation $\sin^2 \theta_W(M_S) = 3/8$, where M_S is the string unification scale. This is an important feature of these models, because it facilitates agreement with the measured gauge coupling parameters at the electroweak scale [40]. It should be also be contrasted with other possible embedding of the weak hypercharge that do not yield the canonical GUT embedding. Such is the case, for instance, in many orientifold models. However, in orientifold models, the string scale may be lowered relative to the gravitational scale. Hence, in orientifold models, smaller values of $\sin^2 \theta_W(M_S)$ may be accommodated. Heterotic-string models may also yield smaller values of $\sin^2 \theta_W(M_S)$, by modifying the identification of the weak hypercharge in the string models. We recall that $\sin^2 \theta_W(M_S)$ arises as a result of the relative normalisation of the weak hypercharge relative to the non-Abelian generators at M_S [41]. In heterotic string models, this normalisation is affected by the number of Cartan generators in the weak hypercharge combination relative to the number of Cartan subgenerators of the non-Abelian group factors. However, in the perturbative heterotic string, the unification scale is fixed, and therefore, lower values of $\sin^2 \theta_W(M_S)$ are disfavoured. This constraint may be relaxed in the non-perturbative heterotic-string [42]. Another point to note in regard to the definition of the weak hypercharge is the existence of string states that carry fractional electric charge. This is a general feature of string models; the reason being the breaking of the non-Abelian gauge symmetries by Wilson lines. A general observation by Wen and Witten [43] and a theorem by Schellekens [44] notes that when a non-Abelian group is broken in string theory by a Wilson line with a left over unbroken $U(1)$ symmetry, it produces states that do not satisfy the $U(1)$ charge quantisation of the unbroken non-Abelian symmetry. This outcome further depends on the identification of the weak-hypercharge. That is, if we relax the canonical GUT embedding of the weak hypercharge, we modify the GUT quantisation of the $U(1)$ charges and may therefore obtain integrally charged states. The important point to note is that these are phenomenological properties of string constructions, and it is yet to be determined how they play out in fully realistic string constructions.

2.2. Phenomenology of String Unification

Subsequent to the construction of the string models and analysis of their spectra, we calculate the cubic level and higher order terms in the superpotential, up to a desired order for a specific phenomenological problem. The next step entails the analysis of supersymmetric F - and D -flat directions. Requiring that the vacuum at the string scale is supersymmetric necessitates the assignment of non-vanishing VEVs to a set of Standard Model singlets in the string models. In this process, some of the higher order terms in the superpotential become effective renormalisable operators, which are suppressed relative to the leading order cubic terms, *i.e.*,

$$V_1^f V_2^f V_3^b \cdots V_N^b \rightarrow V_1^f V_2^f V_3^b \frac{\langle V_4^b \cdots V_N^b \rangle}{M^{N-3}} \quad (1)$$

where $V^{f,b}$ are fermionic and bosonic vertex operators, respectively; N is the order of the non-renormalisable operator; and M is the string cut-off scale. Using this methodology, many of the issues pertaining to the phenomenology of the Standard Model and string unification have been studied in the framework of the quasi-realistic free fermionic heterotic-string models. A partial list includes:

- Top quark mass prediction. The analysis of fermion masses entails the calculation of cubic and higher order terms in the superpotential that are reduced to dimension four terms in Equation (1). The Standard Model fermion mass terms arise from couplings to the electroweak Higgs with an assumed VEV of the order of the electroweak scale. Other fermion mass terms arise from coupling to other scalar fields, and their mass scales may therefore be higher than the electroweak scale. Analysis of Standard Model fermion masses yielded a viable prediction for top quark mass prior to its experimental observation [45]. The calculation proceeds as follows. First the top quark Yukawa coupling is calculated at the cubic level of the superpotential, giving $\lambda_t = \langle Qt_L^c H \rangle = \sqrt{2}g$, where g is the gauge coupling at the unification scale. Subsequently, the Yukawa couplings for the bottom quark and tau lepton are obtained from quartic order terms. The magnitude of the quartic order coefficients are calculated using standard CFT techniques, and the VEV of the Standard Model singlet field in the relevant terms is extracted from analysis of the F - and D -flat directions. This analysis yields effective Yukawa couplings for the bottom quark and tau lepton in terms of the unified gauge coupling given by $\lambda_b = \lambda_\tau = 0.35g^3 \sim 1/8\lambda_t$ [45]. This result for the top quark Yukawa coupling is common in a large class of free fermionic models, whereas those for the bottom quark and tau lepton differ between models. Similarly, the Yukawa coupling for the two lighter generations differ between models and depend on the flat direction VEVs. Subsequent to extracting the Yukawa couplings at the string scale, they are run to the electroweak scale using the Minimal Supersymmetric Standard Model (MSSM) Renormalisation Group Equations (RGEs). It is further assumed that the unified gauge coupling at the string scale is compatible with the value required by the gauge coupling data at the electroweak scale. The bottom Yukawa is further run to the bottom mass scale, which is used to extract a value for $\tan \beta = v_1/v_2$, with v_1 and v_2 being the VEVs of the two MSSMelectroweak Higgs doublets. The top quark mass is then given by:

$$m_t = \lambda_t(m_t) \frac{v_0}{\sqrt{2}} \frac{\tan \beta}{(1 + \tan^2 \beta)^{\frac{1}{2}}}$$

with $v_0 = \sqrt{2(v_1^2 + v_2^2)} = 246$ GeV, yielding $m_t \sim 175$ – 180 GeV. It is noted that, up to the assumptions stated above, the top Yukawa coupling is found near a fixed point. Namely, varying the top Yukawa between 0.5–1.5 at the unification scale yields $\lambda_t(M_Z) \sim 1$ at the electroweak scale. This calculation demonstrates the important advantage of string theory over other attempts of developing a viable framework for quantum gravity. It unifies the gauge and Yukawa couplings and enables the calculation of the Standard Model Yukawa couplings in terms of the unified string coupling. While the calculation of the top Yukawa is robust and shared between a large class of models, the calculation of the corresponding couplings for the lighter quarks and leptons involve a large degree of model dependence. Before investing substantial efforts to calculate the Yukawa couplings of the lighter quarks and leptons in a given model, we should enhance the prospect that a given model is the right model. This line of reasoning underlies the contemporary approach that is outlined below.

- Standard Model fermion masses. The analysis of the effective Yukawa couplings for the lighter two generations proceeds by analysing higher order terms in the superpotential and extracting the effective dimension four operators [27–29]. The analysis should be regarded as demonstrating

in principle the potential of string models to explain the detailed features of the Standard Model flavour parameters. It is still marred by too many uncertainties and built in assumptions to be regarded as a predictive framework. Nevertheless, once an appealing model is constructed the methodology is in place to attempt a more predictive analysis. The explorations to date included, for example, the demonstration of the generation mass hierarchy [46], Cabibbo–Kobayashi–Maskawa (CKM) mixing [47,48], light generation masses [49] and neutrino masses [50,51].

- Gauge coupling unification. An important issue in heterotic-string models is compatibility with the experimental gauge coupling data at the electroweak scale. The perturbative heterotic-string predicts that the gauge couplings unify at the string scale, which is of the order of 5×10^{17} GeV. On the other hand extrapolation of the gauge couplings, assuming MSSM spectra, from the Z -boson mass scale to the GUT scale, shows that the couplings converge at a scale of the order of 2×10^{16} GeV. Thus, the two scales differ by a factor of about 20. This extrapolation should be taken with caution, as the parameters are extrapolated over 14 orders of magnitude, with rather strong assumptions on the physics in the region of extrapolation. Indeed, in view of the more recent results from the LHC, the analysis needs to be revised, as the assumption of MSSM spectrum at the Z -boson scale has been invalidated. Nevertheless, the issue can be studied in detail in perturbative heterotic-string models, and a variety of possible effects have been examined, including heavy string threshold corrections, light supersymmetry (SUSY) thresholds, additional gauge structures and additional intermediate matter states [40]. Within the context of the free fermionic models, only the existence of additional matter states may resolve the discrepancy, and such states indeed exist in the spectrum of concrete string models [52]. This result may be relaxed in the non-perturbative heterotic-string [42] or if the moduli are away from the free fermionic point [53].
- Proton stability. Proton longevity is an important problem in quantum gravity, in general, and in string models in particular. The reason being that we expect only gauge symmetries, or local discrete symmetries that arise as remnants of broken gauge symmetries, to be respected in quantum gravity. Within the Standard Model itself baryon and lepton are accidental global symmetries at the renormalisable level. Thus, we expect, in general, all operators that are compatible with the local gauge and discrete symmetries in given string models to be generated from non-renormalisable terms. Such terms can then give rise to dimension four, five and six baryon and lepton number violating operators that may lead to rapid proton decay. Possible resolutions have been studied in specific free fermionic models and include the existence of an additional light $U(1)$ symmetry [54–59] and local discrete symmetries [12,13].
- Squark degeneracy. String models may, in general, lead to non-degenerate squark masses, depending on the specific SUSY breaking mechanism. For example, SUSY breaking mechanism which is dominated by the moduli F -term will lead to non-degenerate squark masses, because of the moduli dependence of the flavour parameters. Similarly, D -term SUSY breaking depends on the charges of the Standard Model fields under the gauge symmetry in the SUSY breaking sector, and those are in general family non-universal. Free fermionic models can give rise to a family

universal anomalous $U(1)$ [60]. If the SUSY breaking mechanism is dominated by the anomalous $U(1)$ D -term it may produce family universal squark masses of order 1 TeV [61].

- **Minimal Standard Heterotic-String Model (MSHSM).** Three generation semi-realistic string models produce, in general, additional massless vector-like states that are charged under the Standard Model gauge symmetries. Some of these additional vector-like states arise from the Wilson line breaking of the $SO(10)$ GUT symmetry and therefore carry fractional charge with respect to the remnant unbroken $U(1)$ symmetries. In particular, they may carry fractional electric charge, which is highly constrained by observations. These fractionally charged states must therefore be sufficiently massive or diluted to evade the experimental limits. Mass terms for the vector-like states may arise from cubic and higher level terms in the superpotential. In the model of [34] it has been demonstrated in [62] that all the exotic fractionally charged states couple to a set of $SO(10)$ singlets. In [63,64] F - and D -flat solutions that incorporate this set of fields have been found. Additionally, all the extra standard-like fields in the model, beyond the MSSM, receive mass terms by the same set of VEVs. These solutions therefore give rise to the first known string solutions that produce in the low energy effective theory of the observable sector solely the states of the MSSM, and are dubbed Minimal Standard Heterotic-String Model (MSHSM). Three generation Pati–Salam free fermionic models in which fractionally charged exotic states arise only in the massive spectrum were found in [65]. Flat directions that lead to MSHSM with one leading Yukawa coupling were found in an exemplary model in this class [66].
- **Moduli fixing.** An important issue in string models is that of moduli stabilisation. The free fermionic models are formulated near the self-dual point in the moduli space. However, the geometrical moduli that allow deformation from that point exist in the spectrum and can be incorporated in the form of Thirring worldsheet interactions [24]. The correspondence of the free fermionic models with $Z_2 \times Z_2$ orbifold implies that the geometrical moduli correspond to three complex and three Kähler structure moduli. String theory as a theory of quantum geometry, rather than classical geometry, allows for assignment of asymmetric boundary conditions with respect to the worldsheet fermions that correspond to the internal dimensions. These correspond to the asymmetric bosonic identifications under $X_L + X_R \rightarrow X_R - X_L$. In the free fermionic models, and consequently in $Z_2 \times Z_2$ orbifolds, it is possible to assign asymmetric boundary conditions with respect to six circles of the six-dimensional compactified torus. In such a model all the complex and Kähler moduli of the untwisted moduli are projected out [67,68]. Additionally, the breaking of the $N = 2$ worldsheet supersymmetry in the bosonic sector of the heterotic-string results in projection of the would-be twisted moduli [67,68]. Thus, all the fields that are naively identified as moduli in models with $(2, 2)$ worldsheet supersymmetry can be projected out in concrete models. However, the identification of the moduli in models with $(2, 0)$ worldsheet supersymmetry is not well understood and there may exist other fields in the spectrum of such models that may be identified as moduli fields. Furthermore, as long as supersymmetry remains unbroken in the vacuum there exist moduli fields associated with the supersymmetric flat directions. However, it has been proposed that there exist quasi-realistic free fermionic models which do not admit supersymmetric flat directions [69,70]. This is obtained when both symmetric

and asymmetric twistings of the internal dimensions are implemented, resulting in reduction of the number of moduli fields. In the relevant models supersymmetry is broken due to the existence of a Fayet–Iliopoulos term, which is generated by an anomalous $U(1)$ symmetry. It was argued in [69,70] that the relevant models do not admit exact flat directions and therefore supersymmetry is broken at some level. In such models all the moduli are fixed. It should be noted that this possibility arises only in very particular string models, rather than in a generic string vacua [71].

3. Present

Most of the studies discussed so far were done by studying concrete examples of NAHE-based models, *i.e.*, models that contain the common set $\{\mathbf{1}, S, b_1, b_2, b_3\}$ plus the three (or four) additional basis vectors $\{\alpha, \beta, \gamma\}$ that extend the NAHE-set and differ between models, with the most studied models being those of [34,35]. More recent studies involve the exploration of large number of models. This provides an insight into the general properties of the space of vacua, as well as the development of a “fishing algorithm” to fish models with specific phenomenological properties. This method led to discovery of spinor-vector duality [72] and of exophobic vacua [65,66,73,74]. More recently the method has been applied for the classification of flipped $SU(5)$ free fermionic models [75], as well as the classification with respect to the top quark Yukawa coupling [76].

3.1. Classification of Fermionic $Z_2 \times Z_2$ Orbifolds

Over the past decade a systematic method is being developed that allows the explorations of large number of string vacua and analysis of their spectra. In this method the set of basis vectors is fixed. The Pati–Salam class of models is generated by a set of thirteen basis vectors:

$$B = \{v_1, v_2, \dots, v_{13}\}$$

where:

$$\begin{aligned} v_1 = \mathbf{1} &= \{\psi^\mu, \chi^{1,\dots,6}, y^{1,\dots,6}, \omega^{1,\dots,6} | \\ &\quad \bar{y}^{1,\dots,6}, \bar{\omega}^{1,\dots,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,\dots,5}, \bar{\phi}^{1,\dots,8}\}, \\ v_2 = S &= \{\psi^\mu, \chi^{1,\dots,6}\}, \\ v_{2+i} = e_i &= \{y^i, \omega^i | \bar{y}^i, \bar{\omega}^i\}, \quad i = 1, \dots, 6, \\ v_9 = b_1 &= \{\chi^{34}, \chi^{56}, y^{34}, y^{56} | \bar{y}^{34}, \bar{y}^{56}, \bar{\eta}^1, \bar{\psi}^{1,\dots,5}\}, \\ v_{10} = b_2 &= \{\chi^{12}, \chi^{56}, y^{12}, y^{56} | \bar{y}^{12}, \bar{y}^{56}, \bar{\eta}^2, \bar{\psi}^{1,\dots,5}\}, \\ v_{11} = z_1 &= \{\bar{\phi}^{1,\dots,4}\}, \\ v_{12} = z_2 &= \{\bar{\phi}^{5,\dots,8}\}, \\ v_{13} = \alpha &= \{\bar{\psi}^{4,5}, \bar{\phi}^{1,2}\}. \end{aligned} \quad (2)$$

In the notation employed in Equation (2) the worldsheet fields appearing in a given basis vector have periodic boundary conditions, whereas all other fields have anti-periodic boundary conditions. The first twelve vectors in this set are identical to those used in [77,78] for the classification of fermionic $Z_2 \times Z_2$

orbifolds with $SO(10)$ GUT symmetry. The thirteenth basis vector, α , breaks the $SO(10)$ symmetry and generates the Pati–Salam class of models. The set $\{1, S\}$ generate an $N = 4$ supersymmetric model, with $SO(44)$ gauge symmetry. The vectors $e_i, i = 1, \dots, 6$ give rise to all possible symmetric shifts of the six internal fermionised coordinates ($\partial X^i = y^i \omega^i, \bar{\partial} X^i = \bar{y}^i \bar{\omega}^i$). Their addition breaks the $SO(44)$ gauge group, but preserves $N = 4$ supersymmetry. The vectors b_1 and b_2 define the $SO(10)$ gauge symmetry and the $Z_2 \times Z_2$ orbifold twists, which break $N = 4$ to $N = 1$ supersymmetry. The z_1 and z_2 basis vectors reduce the untwisted gauge group generators from $SO(16)$ to $SO(8)_1 \times SO(8)_2$. Finally v_{13} is the additional new vector that breaks the $SO(10)$ GUT symmetry to $SO(6) \times SO(4)$, and the $SO(8)_1$ hidden symmetry to $SO(4)_1 \times SO(4)_2$.

The second ingredient that is needed to define the string vacuum are the Generalised GSO (GGSO) projection coefficients that appear in the one-loop partition function, $c\left(\begin{smallmatrix} v_i \\ v_j \end{smallmatrix}\right) = \exp[i\pi(v_i|v_j)]$, spanning a 13×13 matrix. Only the elements with $i > j$ are independent while the others are fixed by modular invariance. A priori there are therefore 78 independent coefficients corresponding to 2^{78} string vacua. Eleven coefficients are fixed by requiring that the models possess $N = 1$ supersymmetry. Additionally, the phase $c\left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right)$ only affects the overall chirality. Without loss of generality the associated GGSO projection coefficients are fixed, leaving 66 independent coefficients. Each of the 66 independent coefficients can take two discrete values ± 1 and thus a simple counting gives 2^{66} (that is approximately $10^{19.9}$) models in the class of superstring vacua under consideration.

The utility of the classification method is that it provides the means to span all the massless producing sectors in the models. For example, the twisted matter states arise from the sectors:

$$\begin{aligned} B_{\ell_3^1 \ell_4^1 \ell_5^1 \ell_6^1}^1 &= S + b_1 + \ell_3^1 e_3 + \ell_4^1 e_4 + \ell_5^1 e_5 + \ell_6^1 e_6 \\ B_{\ell_1^2 \ell_2^2 \ell_5^2 \ell_6^2}^2 &= S + b_2 + \ell_1^2 e_1 + \ell_2^2 e_2 + \ell_5^2 e_5 + \ell_6^2 e_6 \\ B_{\ell_1^3 \ell_2^3 \ell_3^3 \ell_4^3}^3 &= S + b_3 + \ell_1^3 e_1 + \ell_2^3 e_2 + \ell_3^3 e_3 + \ell_4^3 e_4 \end{aligned}$$

where $\ell_i^j = 0, 1, b_3 = b_1 + b_2 + x = 1 + S + b_1 + b_2 + \sum_{i=1}^6 e_i + \sum_{n=1}^2 z_n$ and x is given by the vector $x = \{\bar{\psi}^{1, \dots, 5}, \bar{\eta}^{1, 2, 3}\}$. These sectors give rise to **16** and $\overline{\mathbf{16}}$ representations of $SO(10)$ decomposed under $SO(6) \times SO(4) \equiv SU(4) \times SU(2)_L \times SU(2)_R$. The important feature of this classification method is that each of the sectors $B_{\ell_1^i \ell_2^i \ell_3^i \ell_4^i}$ for given $\ell_1^i \ell_2^i \ell_3^i \ell_4^i$ gives rise to one spinorial, or one anti-spinorial, or neither, *i.e.* the states arising at each fixed point of the corresponding $Z_2 \times Z_2$ are controlled individually. Similarly, the states from the sectors $B_{\ell_1^i \ell_2^i \ell_3^i \ell_4^i} + x$ produce states in the vectorial **10** representation of $SO(10)$ decomposed under the Pati–Salam gauge group.

The power of the free fermionic classification method is that it enables translation of the GGSO projections into generic algebraic forms. From the general expression for the GSO projections on the states from a given sector $\xi \in \Xi$ [20,21]:

$$e^{i\pi(v_j \cdot F_\xi)} |S\rangle_\xi = \delta_\xi c\left(\begin{smallmatrix} \xi \\ v_j \end{smallmatrix}\right)^* |S\rangle_\xi$$

From this expression we note that, whenever the overlap of periodic fermions between the basis vector v_j and the sector ξ is empty, the operator on the left of this expression is fixed. Hence, depending on the choice of the GGSO phase on the right, the given state is either in or out of the physical spectrum. For any given state from specific sectors there are several basis vectors that act as projectors. Introducing the

notation $c_{a_j}^{(a_i)} = \exp(a_i|a_j)$ with $(a_i|a_j) = 0, 1$, we can collect these projectors into algebraic system of equations of the form: $\Delta^{(i)} U_{16}^{(i)} = Y_{16}^{(i)}$, $i = 1, 2, 3$, where the unknowns are the fixed point labels: $U_{16}^{(i)} = [p_{16}^i, q_{16}^i, r_{16}^i, s_{16}^i]$. The Δ^i and Y_{16}^i are given in terms of the GGSO projection coefficients for each of the three planes. For example, on the first plane for the spinorial $\mathbf{16}$ or $\overline{\mathbf{16}}$ states we have:

$$\Delta^{(1)} = \begin{bmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{bmatrix} \quad (3)$$

and: $Y_{16}^{(1)} = [(e_1|b_1), (e_2|b_1), (z_1|b_1), (z_2|b_2)]$ with similar expressions for the second and third planes. The number of solutions per plane is determined by the relative rank of the matrix Δ^i and the rank of the augmented matrix (Δ^i, Y_{16}^i) . For a given choice of GGSO projection coefficients, the number of states surviving in the spectrum, is therefore readily obtained. Similar, algebraic expressions can be obtained for all the sectors that produce massless states in the given basis, as well as for the chirality of the fermions with periodic boundary conditions.

The methodology outlined above enables the classification of a large number of fermionic $Z_2 \times Z_2$ orbifolds. Compared to the earlier construction it enables a scan of a large number of models and extraction of some of the desired phenomenological properties. We can develop a fishing algorithm to extract models with specific characteristics. For example, a class of Pati–Salam models in which exotic fractionally charged states appear as massive states but not in the massless spectrum was found using these tools. The systematic classification methods were developed to date only for models that admit symmetric boundary conditions with respect to the set of internal worldsheet fermions $\{y, \omega|\bar{y}, \bar{\omega}\}^{1, \dots, 6}$. On the other hand, NAHE-based models were constructed using symmetric and asymmetric boundary conditions, with the assignment of asymmetric boundary conditions having distinct phenomenological implications [79,80].

3.1.1. Spinor-Vector Duality

Another example of the utility of the fermionic classification method is given by the spinor-vector duality, which was discovered by using these methods and elucidates the global structure of the free fermionic models, in particular, and that of the larger string landscape, in general. The spinor-vector duality is a duality in the space of string vacua generated by the basis set v_i with $i = 1, \dots, 12$, and unbroken $SO(10)$ symmetry. The duality entails an invariance under the exchange of the total number of $(\mathbf{16} + \overline{\mathbf{16}})$ representations and the total number of $\mathbf{10}$ representations of $SO(10)$. That is, for a given vacuum with a number of $(\mathbf{16} + \overline{\mathbf{16}})$ and $\mathbf{10}$ representations, there exist another vacuum in which the two numbers are interchanged. The origin of this duality is revealed when the $SO(10)$ symmetry is enhanced to E_6 . Under the decomposition of $E_6 \rightarrow SO(10) \times U(1)$ the $\mathbf{27}$ and $\overline{\mathbf{27}}$ representations decompose as $\mathbf{27} = \mathbf{16} + \mathbf{10} + \mathbf{1}$ and $\overline{\mathbf{27}} = \overline{\mathbf{16}} + \mathbf{10} + \mathbf{1}$. Therefore, in the case of vacua with E_6 symmetry the total number of $(\mathbf{16} + \overline{\mathbf{16}})$ representations is equal to the total of $\mathbf{10}$ representations. Hence, models with enhanced E_6 symmetry are self-dual under the spinor-vector duality map.

The spinor-vector duality therefore arises from the breaking of the E_6 symmetry to $SO(10) \times U(1)$. This breaking is generated in the orbifold language by Wilson-lines, or in the free fermionic construction,

by choices of the GGSO projection coefficients. It is important to recognise that these two descriptions are not distinct, but are mathematically identical. That is we can translate the GGSO projection coefficients to Wilson line and visa versa [23]. Thus, when the E_6 symmetry is broken to $SO(10) \times U(1)$, there exist a choice of GGSO projection coefficients, or of Wilson lines, that keeps a number of spinorial $(\mathbf{16} + \overline{\mathbf{16}})$ and a number of vectorial $\mathbf{10}$ representations of $SO(10)$, and another choice for which the two numbers are interchanged. It is important to note that this is an exact duality symmetry operating in the entire space of string vacua in which the $SO(10)$ symmetry is not enhanced to E_6 [72,81–83]. It is further noted that the spinor-vector duality can be interpreted in terms of a spectral flow operator [83]. In this context the spectral flow operator in the twisted sector may be seen as a deformed version of the operator inducing the Massive Spectral boson-fermion Degeneracy Symmetry (MSDS) [84]. Therefore, the spinor-vector duality extends to the massive sectors [83], albeit in a fashion that still needs to be determined in the general case. Similarly, we note that the generalisation of the spinor-vector duality to the case of interacting internal CFTs can be studied by adopting the following methodology, e.g., in the case of minimal models. The starting point is an heterotic-string compactified to four dimensions with $(2, 2)$ worldsheet supersymmetry and an internal interacting CFT representing the compact space. The next step is to break the worldsheet supersymmetry in the bosonic sector of the heterotic-string. The spectral flow operator then induces a map between distinct $(2, 0)$ vacua [85].

We can also understand the spinor-vector duality operationally in terms of the free phases in the fermionic language [81] or as discrete torsion in the orbifold picture [82,83]. For that purpose we recall the level one $SO(2n)$ characters [86]:

$$\begin{aligned} O_{2n} &= \frac{1}{2} \left(\frac{\theta_3^n}{\eta^n} + \frac{\theta_4^n}{\eta^n} \right), & V_{2n} &= \frac{1}{2} \left(\frac{\theta_3^n}{\eta^n} - \frac{\theta_4^n}{\eta^n} \right) \\ S_{2n} &= \frac{1}{2} \left(\frac{\theta_2^n}{\eta^n} + i^{-n} \frac{\theta_1^n}{\eta^n} \right), & C_{2n} &= \frac{1}{2} \left(\frac{\theta_2^n}{\eta^n} - i^{-n} \frac{\theta_1^n}{\eta^n} \right) \end{aligned}$$

where:

$$\theta_3 \equiv Z_f \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \theta_4 \equiv Z_f \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \theta_2 \equiv Z_f \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \theta_1 \equiv Z_f \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where Z_f is the partition function of a single worldsheet complex fermion, given in terms of theta functions [86]. The partition function of the $E_8 \times E_8$ heterotic-string compactified on a six-dimensional torus is given by:

$$Z_+ = (V_8 - S_8) \left(\sum_{m,n} \Lambda_{m,n} \right)^{\otimes 6} (\overline{O}_{16} + \overline{S}_{16}) (\overline{O}_{16} + \overline{S}_{16}) \quad (4)$$

where as usual, for each circle,

$$p_{L,R}^i = \frac{m_i}{R_i} \pm \frac{n_i R_i}{\alpha'} \quad \text{and} \quad \Lambda_{m,n} = \frac{q^{\frac{\alpha'}{4} p_L^2} \bar{q}^{\frac{\alpha'}{4} p_R^2}}{|\eta|^2}$$

Next, a $Z_2 \times Z'_2 : g \times g'$ projection is applied, where the first Z_2 is a freely acting Scherk–Schwarz-like projection, which couples a fermion number in the observable and hidden sectors with a Z_2 -shift in a compactified coordinate, and is given by $g : (-1)^{(F_1+F_2)} \delta$ where the fermion numbers $F_{1,2}$ act on the spinorial representations of the observable and hidden $SO(16)$ groups as $F_{1,2} : (\overline{O}_{16}^{1,2}, \overline{V}_{16}^{1,2}, \overline{S}_{16}^{1,2}, \overline{C}_{16}^{1,2}) \rightarrow (\overline{O}_{16}^{1,2}, \overline{V}_{16}^{1,2}, -\overline{S}_{16}^{1,2}, -\overline{C}_{16}^{1,2})$ and δ identified points shifted by a Z_2

shift in the X_9 direction, *i.e.*, $\delta X_9 = X_9 + \pi R_9$. The effect of the shift to insert a factor of $(-1)^m$ into the lattice sum in Equation (4), *i.e.*, $\delta : \Lambda_{m,n}^9 \longrightarrow (-1)^m \Lambda_{m,n}^9$. The second Z_2 acts as a twist on the internal coordinates given by $g' : (x_4, x_5, x_6, x_7, x_8, x_9) \longrightarrow (-x_4, -x_5, -x_6, -x_7, +x_8, +x_9)$. The effect of the first Z_2 is to reduce the gauge symmetry from $E_8 \times E_8$ to $SO(16) \times SO(16)$. The Z_2' twist reduces the number of spacetime supersymmetries from $N = 4$ to $N = 2$, and reduces the gauge symmetry arising from $SO(16) \times SO(16)$ to $SO(12) \times SO(4) \times SO(16)$. Additionally, it produces a twisted sector that gives rise to massless states in the spinorial 32 and 32', and vectorial 12, representations of $SO(12)$. In this vacuum the spinor-vector duality operates in terms of the representations of $SO(12) \times SU(2)$ rather than in terms of representations of $SO(10) \times U(1)$, as the enhanced symmetry point possess an E_7 symmetry rather than E_6 . The spinor-vector duality operates identically in the two cases and the case of the single non-freely acting Z_2 twist elucidates more readily the underlying structure of the spinor-vector duality. The orbifold partition function is given by:

$$Z = \left(\frac{Z_+}{Z_g \times Z_{g'}} \right) = \left[\frac{(1+g)(1+g')}{2} \right] Z_+$$

The partition function contains an untwisted sector and three twisted sectors. The winding modes in the sectors twisted by g and gg' are shifted by $1/2$, and therefore these sectors only produce massive states. The sector twisted by g gives rise to the massless twisted matter states. The partition function has two modular orbits and one discrete torsion $\epsilon = \pm 1$. Massless states are obtained for vanishing lattice modes. The terms in the sector g contributing to the massless spectrum take the form:

$$\Lambda_{p,q} \left\{ \frac{1}{2} \left(\left| \frac{2\eta}{\theta_4} \right|^4 + \left| \frac{2\eta}{\theta_3} \right|^4 \right) [P_\epsilon^+ Q_s \bar{V}_{12} \bar{C}_4 \bar{O}_{16} + P_\epsilon^- Q_s \bar{S}_{12} \bar{O}_4 \bar{O}_{16}] + \right. \\ \left. \frac{1}{2} \left(\left| \frac{2\eta}{\theta_4} \right|^4 - \left| \frac{2\eta}{\theta_3} \right|^4 \right) [P_\epsilon^+ Q_s \bar{O}_{12} \bar{S}_4 \bar{O}_{16}] \right\} + \text{massive} \quad (5)$$

where:

$$P_\epsilon^+ = \left(\frac{1 + \epsilon(-1)^m}{2} \right) \Lambda_{m,n} \quad ; \quad P_\epsilon^- = \left(\frac{1 - \epsilon(-1)^m}{2} \right) \Lambda_{m,n} \quad (6)$$

Depending on the sign of the discrete torsion $\epsilon = \pm$ we note from Equation (6) that either the spinorial states, or the vectorial states, are massless. In the case with $\epsilon = +1$ we see from Equation (7) that in this case massless momentum modes from the shifted lattice arise in P_ϵ^+ whereas P_ϵ^- produces only massive modes. Therefore, in his case the vectorial character \bar{V}_{12} in Equation (6) produces massless states, whereas the spinorial character \bar{S}_{12} generates massive states. In the case with $\epsilon = -1$ we note from Equation (8) that exactly the opposite occurs:

$$\epsilon = +1 \quad \Rightarrow \quad P_\epsilon^+ = \Lambda_{2m,n} \quad P_\epsilon^- = \Lambda_{2m+1,n} \quad (7)$$

$$\epsilon = -1 \quad \Rightarrow \quad P_\epsilon^+ = \Lambda_{2m+1,n} \quad P_\epsilon^- = \Lambda_{2m,n} \quad (8)$$

Another observation from the term appearing in Equation (5) is the matching of the number of massless degrees of freedom in the two cases. In the case with $\epsilon = -1$ the number of degrees of freedom in the spinorial representation of $SO(12)$ is 32. In the case with $\epsilon = +1$ the number of degrees of freedom in the vectorial representation of $SO(12)$ is 12. As seen from the first line in Equation (5) the term in the

partition function producing the vectorial states also transforms as a spinor under the $SO(4)$ symmetry. Hence the total number of states is 24, *i.e.*, there is still a mismatch of 8 states between the two cases. However, we note from the second line in Equation (5) that in the case with $\epsilon = +1$ eight additional states are obtained from the first excited states of the internal lattice. We note therefore that the total number of degrees of freedom is preserved under the duality map, *i.e.*, $12 \cdot 2 + 4 \cdot 2 = 32$

Given the relation of free fermionic models to toroidal orbifolds, we can anticipate that the spinor-vector duality can be realised in terms of the moduli of the toroidal lattices. Those are the six-dimensional metric, the antisymmetric tensor field and the Wilson lines [25]. Indeed, the discrete torsion appearing in Equation (5) can be translated to a map between two Wilson lines [83]. We note that in the case of (5) the map between the Wilson lines is continuous. The reason is the fact that we employed a single Z_2 twist on the internal coordinates. The moduli associated with the Wilson line mapping are not projected out in this case and therefore the interpolation between the two Wilson lines is continuous. In the more general case with a $Z_2 \times Z_2$ twist these moduli are projected out and the mapping between the two Wilson lines is discrete.

Additionally, we can understand the spinor-vector duality in terms of a spectral flow operator [83], which may be generalised to other cases. We recall that vacua with E_6 extended gauge symmetry are self-dual under the spinor-vector duality, and that they correspond to vacua with $(2, 2)$ worldsheet supersymmetry. Just like the case of the worldsheet supersymmetry in the supersymmetric sector of the heterotic-string, there is a spectral flow operator that acts as a generator of E_6 in the vacua with enhanced E_6 symmetry. On the supersymmetric side the spectral flow operator mixes states with different spacetime spin, whereas on the non-supersymmetric side it mixes states that differ by their $U(1)$ charge in the decomposition $E_6 \rightarrow SO(10) \times U(1)$, *i.e.*, it mixes the states that transform as spinors and vectors of $SO(10)$. When the E_6 symmetry is broken, *i.e.*, when the worldsheet supersymmetry is broken from $(2, 2)$ to $(2, 0)$, the spectral flow operator induces the spinor-vector duality map between the two distinct vacua [83].

The spinor-vector duality is a novel symmetry that operates in the global space of Z_2 and $Z_2 \times Z_2$ heterotic-string orbifolds and provides valuable insight and interesting questions for future research. First, we note that the spinor-vector duality is a map between vacua that are completely unrelated in the effective field theory limit. For example, we may envision a map between a model with three spinorial 16 representations, and one vectorial 10 representation, to a model with three vectorial 10 representations and one spinorial 16 representation. In terms of the low energy physics the two cases are fundamentally different. On the other hand, from the point of view of string theory they are identical. Namely, there is an exact map from one to the other. The distinction between the string representation versus the effective field theory limit is that the string can access its massive modes, which are not seen in the effective field theory limit. Therefore, vacua that seem distinct in the effective field theory limit are in fact related in the full string theory. We may further envision that at some early stage in the evolution of the universe, when the heavy string modes are excited that the two vacua can in fact mix. This possibility has implications on the counting of distinct string vacua and therefore on the string landscape. It is evident that our contemporary understanding of the string landscape is still very rudimentary and we should proceed with caution before overstating our case. The spinor-vector duality may also have interesting implications from a purely mathematical point of view. Namely, in the effective field theory limit there should exist a

description of the massless degrees of freedom in terms of a smooth effective field theory *i.e.*, in terms of a supergravity theory with a classical geometry (*i.e.*, some Calabi–Yau six-dimensional manifold) with a vector bundle accounting for the gauge degrees of freedom. The existence of the spinor-vector duality map implies that there should be a similar map between the two effective theory limits of the two vacua. This is particularly interesting in terms of the counting of the additional states that are needed to compensate for the mismatch in the number of states between the two vacua. How do they arise in the effective field theory limit? In the very least, the spinor-vector duality provides a valuable tool to study the moduli spaces of $(2, 0)$ heterotic-string compactifications.

4. Other Approaches

The free fermionic models represents one of the approaches to string phenomenology. Several other approaches are being pursued, leading to overlapping and complementary results, in the perturbative and non-perturbative domains. The literature on these subjects is vast and include several monographs, including, for example [7]. A partial and incomplete list of some of these studies include: geometrical studies [87–91]; orbifolds [92–96]; interacting CFTs [97–99]; orientifolds [100–102]. It should be emphasised that the present article does not aim to review these important contributions, but merely those of the author. A comprehensive review is provided in reference [7], as well as in [15–18].

5. Future

With the observation that the agent of electroweak symmetry breaking is compatible with an elementary scalar, particle physics and string phenomenology are set for a bright future. In the particle physics realm the main questions are experimental. Are there additional states associated with the electroweak symmetry breaking mechanism? E.g., is spacetime supersymmetry realised in nature, and within reach of contemporary colliders? Can we improve on the contemporary measurements of the Standard Model parameters and by how much? Can we build accelerators to probe energy scales in the deca-TeV region and above? These are rather general questions and experiments should target more specific questions, e.g., can we cool the muon phase space in a muon storage ring or a muon collider? The construction of a muon based facility will advance the accelerator based technology to a new era, and may be used as a Higgs factory in one of its initial missions [103].

Particle physics and string phenomenology are two sides of the same coin, and should not be regarded as distinct entities. Particle physics shows that experimental data can be parameterised by a model, which is based on the principles of point quantum field theories, *i.e.*, locality, causality and renormalisability. This led to the development of the Standard Model, which is a quantum field theory with internal symmetries. A point quantum gravity theory fails to satisfy these criteria. String theory resolves the problem with the third property by relaxing the first. String models provide consistent approaches to quantum gravity, in which the internal symmetries are dictated by the consistency of the theory. As a common setting for the gauge and gravitational interactions string theory facilitates the calculation of the Standard Particle Model parameters in a reduced framework.

5.1. Toward String Predictions

String theory leads to distinct signatures beyond the Standard Model. In the first instance all the known stable string vacua at the Planck scale are supersymmetric [104]. Whether supersymmetry is manifested within reach of contemporary experiments is a wild speculation. Nevertheless, this hypothesis is motivated on the ground that it facilitates extrapolation of the Standard Model parameters from the unification scale to the electroweak scale. Furthermore, electroweak symmetry breaking at the low scale is generated in the supersymmetric scheme by the interplay of the top quark Yukawa coupling and the gauge coupling of the strong interaction [105,106].

Low scale supersymmetry is therefore not a necessary outcome of string theory, but certainly its observation will provide further evidence that the different structures of string constructions are realised in nature. Specific SUSY breaking scenarios in string models give rise to distinct supersymmetric spectra and that in turn will be used to constrain further the phenomenological string vacua [107]. It is further noted that R -parity is generically broken in string vacua [108] and that the LSP is not expected therefore to provide a viable dark matter candidate.

A generic prediction of string theory is the existence of additional gauge degrees of freedom, beyond those of the Standard Model, and is dictated by the consistency conditions of string theory. However, construction of viable string models that allow for extra gauge symmetries within reach of contemporary experiments is highly non-trivial. On the other hand, an extra $U(1)$ symmetry may be instrumental to understand some phenomenological features of the Supersymmetric Standard Model, like the suppression of proton decay mediating operators and the μ -parameter.

Another generic outcome of string models is the existence of exotic matter states. This feature of string constructions arises as a result of the breaking of the non-Abelian GUT symmetries by Wilson lines, which results in exotic states that do not obey the quantisation rules of the original GUT group. Thus, one can get, for example, states that carry fractional electric charge. The lightest of the fractionally charged states is necessarily stable by electric charge conservations. The experimental restrictions on states that carry fractional electric charge are severe and they must be either sufficiently heavy, and/or sufficiently diluted to evade detection. Nevertheless, given that the bulk of the matter in the universe is dark, *i.e.*, does not interact electromagnetically, stable string relics with a variety of properties can be contemplated [109]. This includes for example the possibility that the string relics come as fractionally charged hadrons and leptons, with charge $\pm 1/2$. Such states will continue to scatter in the early universe until they form a bound hydrogen-like state with another fractionally charged companion. Provided that they are sufficiently heavy and sufficiently rare they could have evaded detection by searches for rare isotopes. Another possibility of exotic stable string relics arises when the $SO(10)$ GUT symmetry is broken to $SU(3) \times SU(2) \times U(1)^2$. This case gives rise to states that carry the regular Standard Model charges, but carry fractional charges with respect to the extra $U(1)_{Z'} \in SO(10)$. This case, depending on the Higgs representations that break the $U(1)_{Z'}$, can result in discrete symmetries that forbid the decay of the exotic states to the Standard Model states. It can therefore give rise to meta-stable heavy string relics that are Standard Model singlets. Depending on the cosmological evolution in the early universe they could have been diluted and reproduced as super-heavy states after reheating [109]. Such states can

produce viable dark matter [109] candidates as well as candidates for Ultra High Energy Cosmic Rays (UHECR) [110].

5.2. Cosmological Evolution

The early studies in string phenomenology, articulated in Section 2, entailed the in depth exploration of exemplary models and the study of phenomenological properties. These studies focussed on the properties of the massless spectra of these exemplary models and led to the construction of the first known Minimal Standard Heterotic String Models (MSHSM) [34,63,64].

The more recent studies, articulated in Section 3, involve the classification of large classes of models and the relations between them. The string vacua in this investigation are fermionic $Z_2 \times Z_2$ orbifolds and are therefore related to the exemplary models in Section 2. More importantly, the contemporary studies involve the analysis of the partition functions associated with this class of string vacua. In that context they aim to explore how the massive string spectrum may play a role in the determination of the phenomenological and mathematical properties of string models. This led to the discovery of spinor-vector duality in heterotic-string models [72].

One direction therefore in future string phenomenology studies will involve the investigation of the associated string partition function, and in particular away from the free fermionic point. The most general form of the partition function affiliated with the $Z_2 \times Z_2$ orbifolds, and hence with the phenomenological free fermionic models is given by:

$$\begin{aligned}
 Z &= \int \frac{d^2\tau}{\tau_2^2} \frac{\tau_2^{-1}}{\eta^{12}\bar{\eta}^{24}} \frac{1}{2^3} \left(\sum (-)^{a+b+ab} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} \vartheta \begin{bmatrix} a+h_1 \\ b+g_1 \end{bmatrix} \vartheta \begin{bmatrix} a+h_2 \\ b+g_2 \end{bmatrix} \vartheta \begin{bmatrix} a+h_3 \\ b+g_3 \end{bmatrix} \right)_{\psi^\mu, \chi} \\
 &\times \left(\frac{1}{2} \sum_{\epsilon, \xi} \bar{\vartheta} \begin{bmatrix} \epsilon \\ \xi \end{bmatrix}^5 \bar{\vartheta} \begin{bmatrix} \epsilon+h_1 \\ \xi+g_1 \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \epsilon+h_2 \\ \xi+g_2 \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \epsilon+h_3 \\ \xi+g_3 \end{bmatrix} \right)_{\bar{\psi}^{1\dots 5}, \bar{\eta}^{1,2,3}} \\
 &\times \left(\frac{1}{2} \sum_{H_1, G_1} \frac{1}{2} \sum_{H_2, G_2} (-)^{H_1 G_1 + H_2 G_2} \bar{\vartheta} \begin{bmatrix} \epsilon+H_1 \\ \xi+G_1 \end{bmatrix}^4 \bar{\vartheta} \begin{bmatrix} \epsilon+H_2 \\ \xi+G_2 \end{bmatrix}^4 \right)_{\bar{\phi}^{1\dots 8}} \\
 &\times \left(\sum_{s_i, t_i} \Gamma_{6,6} \begin{bmatrix} h_i | s_i \\ g_i | t_i \end{bmatrix} \right)_{(y\omega\bar{y}\bar{\omega})^{1\dots 6}} \times e^{i\pi\Phi(\gamma, \delta, s_i, t_i, \epsilon, \xi, h_i, g_i, H_1, G_1, H_2, G_2)}
 \end{aligned}$$

where the internal lattice is for one compact dimension is given by:

$$\Gamma_{1,1}[h] = \frac{R}{\sqrt{\tau_2}} \sum_{\tilde{m}, n} \exp \left[-\frac{\pi R^2}{\tau_2} |(2\tilde{m} + g) + (2n + h) \tau|^2 \right]$$

and Φ is a modular invariant phase. The properties of the string vacua, away from the free fermionic point, can be explored by studying this partition function and the role of the massive states. Furthermore, while the current understanding of string theory is primarily limited to static solutions, exploration of dynamical scenarios can be pursued by compactifying the time coordinate on a circle and using the Scherk–Schwarz mechanism [111] in the compactified time-like coordinate. One then obtains a finite temperature-like partition function that can be used to explore cosmological scenarios. Indeed, this is the string cosmology program pursued by the Paris group over the past few years [112]. In

a similar spirit partition functions of string compactifications to two dimensions have been explored revealing rich mathematical structures and the so-called massive supersymmetry, in which the massive spectrum exhibits Fermi–Bose degeneracy, whereas the massless spectrum does not [84]. One can envision interpolations of the two-dimensional partition functions, associated with the cosmological and massive supersymmetry scenarios to the four-dimensional partition functions associated with the phenomenological free fermionic models. The ultimate aim of this program will be to explore possible mechanisms for dynamical vacuum selection in string theory.

5.3. Dualities and Fundamental Principles

Physics is first and foremost an experimental science. There is no absolute truth. There is only perception of reality as registered in an experimental apparatus (including astrophysical observations). Be that as it may, the language that is used to interpret the experimental signals is mathematics. The scientific methodology then entails: the existence of some initial conditions, which are either preset or set up in experiment; the construction of a mathematical model that predict (or postdict) the outcome of the experiment; the confrontation of the predictions of the mathematical model with the outcome of the experimental observations. A successful mathematical model is the one that is able to account for a wider range of experimental observations. This scientific methodology has been developed over the past five hundred years or so.

To construct a mathematical model one needs to define a set of variables that are to be measured experimentally. The set of variables is key to the interpretation of the experimental outcome. Over the years modern physics has undergone a process of evolution in terms of these basic set of variables. In the Galilean–Newtonian system the basic set of variables are the position and velocities. In modern experiments the relevant measured variables are typically the initial and final energy and momenta. In the Lagrangian formalism the set of variable is generalised to any set of configuration coordinates and their derivatives with respect to time. In the Hamiltonian formalism the set of variables are the generalised configuration coordinates and their conjugate momenta, which constitute the phase space. This represents a nontrivial conceptual evolution from the Galilean–Newtonian system of position and velocities.

String theory provides a consistent framework for the perturbative unification of the gauge and gravitational interactions. The string characterisation of the basic constituents of matter reproduces the picture of elementary particles with internal attributes. String theory unifies the spacetime and internal properties of elementary particles. In the modern description of matter and interactions, the three subatomic interactions are in a sense already unified. They are based on the gauge principle. By giving rise to the mediators of the subatomic interactions that satisfy the gauge principle, and at the same time giving rise to the mediator of the gravitational interactions, that satisfy the gravitational gauge principle, string theory also unifies the principles underlying these theories.

Can string theory be the final chapter in the unification of the gauge and gravitational interactions. Unlike general relativity and quantum mechanics, string theory is not formulated by starting from a fundamental principle and deriving the physical consequences. Ultimately this is what we would like to have.

Perturbative and non-perturbative dualities have played a key role in trying to obtain a rigorous understanding of string theory. T -duality is an important perturbative property of string theory [113]. We may interpret T -duality and phase space duality in compact space. An additional important property of T -duality in string theory is the existence of self-dual states under T -duality.

We may envision promoting phase-space duality to a level of a fundamental principle. This is the program that was undertaken in [114]. The key is the relation between the phase space variables via a generating function S , $p = \partial_q S$. To obtain a dual structure we define a dual generating function T , with $q = \partial_p T$. The two generating functions are related by the dual Legendre transformations:

$$S = p\partial_p T - T \quad (9)$$

and

$$T = q\partial_q S - S \quad (10)$$

Furthermore, one can show that $S(q)$ transforms as a scalar function under the $GL(2, C)$ -transformations:

$$\tilde{q} = \frac{Aq + B}{Cq + D}, \quad \tilde{p} = \rho^{-1}(Cq + D)^2 p, \quad (11)$$

where $\rho = AD - BC \neq 0$. We can associate the two Legendre transformations Equations (9) and (10) with a second order differential equations whose solutions are $\{q\sqrt{p}; \sqrt{p}\}$ and $\{p\sqrt{q}; \sqrt{q}\}$, respectively. A special class of solutions are those which satisfy the two sets of differential equations, *i.e.*, $p = \gamma q$, with $\gamma = \text{constant}$. These are the self-dual solutions under the Legendre duality of Equations (9) and (10).

Given that the Legendre transformations are not defined for linear functions we have that the phase-space duality is not consistent for physical systems with $S = Aq + B$, *i.e.*, precisely for the self-dual states. It is further noted that the second order differential equations are covariant under coordinate transformations, but that their potential functions are only invariant under the Möbius transformations Equation (11). This suggests the fundamental equivalence postulate [114,115]:

Given two physical systems labelled by potential functions $W^a(q^a) \in H$ and $W^b(q^b) \in H$, where H denotes the space of all possible W 's, there always exists a coordinate transformations $q^a \rightarrow q^b = v(q^a)$ such that $W^a(q^a) \rightarrow W^{av}(q^b) = W^b(q^b)$.

This postulate implies that there should always exist a coordinate transformation connecting any state to the state $W^0(q^0) = 0$. Inversely, this means that any nontrivial state $W \in H$ can be obtained from the states $W^0(q^0)$ by a coordinate transformation.

The classical Hamilton–Jacobi (HJ) formalism provides a natural setting to apply this postulate. In the HJ formalism a mechanical problem is solved by using canonical transformations to map the Hamiltonian of a nontrivial physical system, with nonvanishing kinetic and potential energies, to a trivial Hamiltonian. The solution is given by the Classical Hamilton–Jacobi Equation (CHJE) and the functional relation between the phase space variables is extracted by the relation $p = \partial_q S$, with S being the solution of the HJ equation. We can pose a similar question, but imposing the functional relations $p = \partial_q S(q)$ on the trivialising transformation $q \rightarrow q^0(q)$ and $S^0(q^0) = S(q)$. This procedure is not consistent with the CHJE because the state $W^0(q^0)$ is a fixed point under the coordinate transformations [114,115]. Consistency

of the equivalence postulate therefore implies that the CHJE should be deformed. Focussing on the stationary case, the most general deformation is given by:

$$\frac{1}{2m} (\partial_q S_0)^2 + W(q) + Q(q) = 0. \quad (12)$$

The equivalence postulate implies that Equation (12) is covariant under general coordinate transformations. This is obtained provided that the combination $(W + Q)$ transforms as a quadratic differential. On the other hand all nontrivial states should be obtained from the state $W^0(q^0)$ by a coordinate transformation. The basic transformation properties are then:

$$\begin{aligned} W^v(q^v) &= (\partial_{q^v} q^a)^2 W^a(q^a) + (q^a; q^v), \\ Q^v(q^v) &= (\partial_{q^v} q^a)^2 Q^a(q^a) - (q^a; q^v). \end{aligned}$$

Comparing the transformations $q^a \rightarrow q^b \rightarrow q^c$ with $q^a \rightarrow q^c$ fixes the cocycle condition for the inhomogeneous term [116]:

$$(q^a; q^c) = (\partial_{q^c} q^b)^2 [(q^a; q^b) - (q^c; q^b)] \quad (13)$$

The cocycle condition uniquely fixes the transformation properties of the inhomogeneous term, and it is shown to be invariant under Möbius transformations. In the one-dimensional case the Möbius symmetry fixes the functional form of the inhomogeneous term to be given by the Schwarzian derivative $\{q^a, q^b\}$, where the Schwarzian derivative is given by:

$$\{f(q), q\} = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2$$

The Quantum Stationary Hamilton–Jacobi Equation (QSHJE) then takes the form of a Schwarzian identity:

$$(\partial_q S_0)^2 = \beta^2/2 (\{\exp(2iS_0/\beta), q\} - \{S_0, q\}) \quad (14)$$

With $\beta = \hbar$ and the identifications:

$$W(q) = V(q) - E = -\frac{\hbar^2}{4m} \{\exp(2iS_0/\beta), q\} \quad (15)$$

$$Q(q) = \frac{\hbar^2}{4m} \{S_0, q\} \quad (16)$$

the QSHJE takes the form:

$$\frac{1}{2m} \left(\frac{\partial S_0}{\partial q} \right)^2 + V(q) - E + \frac{\hbar^2}{4m} \{S_0, q\} = 0 \quad (17)$$

which can be derived from the Schrödinger equation by taking:

$$\psi(q) = \frac{1}{\sqrt{S_0'}} e^{\pm \frac{iS_0}{\hbar}}$$

It is noted that the QSHJE is a non-linear differential equation, whose solutions are given in terms of the two linearly independent solutions, ψ and ψ_D , of the corresponding Schrödinger equation. Denoting

$w = \psi_D/\psi$, and from the properties of the Schwarzian derivative, it follows that the solution of the QSHJE is given, up to a Möbius transformation, by:

$$e^{\frac{i2}{\hbar}S_0\{\delta\}} = e^{i\alpha} \frac{w + i\bar{\ell}}{w - i\ell} \quad (18)$$

where $\delta = \{\alpha, \ell\}$ with $\alpha \in R$ and $\text{Re}\ell \neq 0$, which is equivalent to the condition $S_0 \neq \text{const}$. We note that the condition that the condition $S_0 \neq \text{const}$ is synonymous with the condition for the definability of all phase space duality for all physical states. Thus, we find that the phase space duality and the equivalence postulate are intimately related. In essence, they are manifestation of the Möbius symmetry that underlies quantum mechanics. It is further noted that the trivialising map to the $W^0(q^0)$ state is given by $q \rightarrow q^0 = w$.

The equivalence postulate formalism reproduces the key phenomenological properties of quantum mechanics, without assuming the probability interpretation of the wave function. It implies that the momentum is real also in the classically forbidden regions, hence implying the tunnelling effect of quantum mechanics [115,117]. It implies quantisation of energy levels for bound states with square integrable wave function. Additionally, it implies that time parameterisation of trajectories is ill defined in quantum mechanics [118]. The last two properties are a direct consequence of the underlying Möbius symmetry. The Möbius symmetry, which includes a symmetry under inversions, entails that space must be compact. In the one-dimensional case this is seen as imposing gluing conditions on the trivialising map at $\pm\infty$ [115,117,119]. If space is compact the energy levels are always quantised.

The compactness of space also explains the inherent probabilistic nature of quantum mechanics, and the inconsistency of a fundamental trajectory parameterisation. There are two primary means to define time parameterisation of trajectories. In Bohmian mechanics [120,121] time parameterisation is obtained by identifying the conjugate momentum with the mechanical momentum, *i.e.*, $p = \partial_q S = m\dot{q}$, where S is the solution of the quantum Hamilton–Jacobi equation. In the classical Hamilton–Jacobi theory time parameterisation is introduced by using Jacobi theorem:

$$t = \frac{\partial \mathcal{S}_0^{\text{cl}}}{\partial E} \quad (19)$$

In classical mechanics this is equivalent to identifying the conjugate momentum with the mechanical momentum. Namely, setting:

$$p = \partial_q \mathcal{S}_0^{\text{cl}} = m\dot{q} \quad (20)$$

yields:

$$t - t_0 = m \int_{q_0}^q \frac{dx}{\partial_x \mathcal{S}_0^{\text{cl}}} = \int_{q_0}^q dx \frac{\partial}{\partial E} \partial_x \mathcal{S}_0^{\text{cl}} = \frac{\partial \mathcal{S}_0^{\text{cl}}}{\partial E} \quad (21)$$

which provides a solution for the equation of motion $q = q(t)$. Therefore, Bohmian mechanics brings back the notion of trajectories for point particles. However, the agreement between the definition of time via the mechanical time $p = m\dot{q}$, and its definition via Jacobi theorem Equation (19) is no longer valid in quantum mechanics. In quantum mechanics we have:

$$t - t_0 = \frac{\partial \mathcal{S}_0^{\text{qm}}}{\partial E} = \frac{\partial}{\partial E} \int_{q_0}^q dx \partial_x \mathcal{S}_0^{\text{qm}} = \left(\frac{m}{2}\right) \int_{q_0}^q dx \frac{1 - \partial_E Q}{(E - V - Q)^{1/2}} \quad (22)$$

The mechanical momentum is then given by:

$$m \frac{dq}{dt} = m \left(\frac{dt}{dq} \right)^{-1} = \frac{\partial_q S_0^{\text{qm}}}{(1 - \partial_E \mathcal{V})} \neq \partial_q S_0^{\text{qm}} \quad (23)$$

where \mathcal{V} denotes the combined potential $\mathcal{V} = V(q) + Q(q)$. Therefore, in quantum mechanics the Bohmian time definition does not coincide with its definition via Jacobi's theorem.

Floyd proposed to define time by using Jacobi's theorem [122], *i.e.*,

$$t - t_0 = \frac{\partial S_0^{\text{qm}}}{\partial E} \quad (24)$$

Floyd's proposal would in principle provide a trajectory representation of quantum mechanics by inverting $t(q) \rightarrow q(t)$, which would seem to be in contradiction with inherently probabilistic nature of quantum mechanics. However, if space is compact then the energy levels are always quantised, albeit with an experimentally indistinguishable splittings [123]. Hence, at a fundamental level one cannot differentiate with respect to time, and time parameterisation of trajectories by Jacobi's theorem is ill defined in quantum mechanics. Hence, time parameterisation of trajectories can only be regarded as an effective semi-classical approximation and in that sense can provide a useful tool in many practical problems [121]. The observation that time cannot be defined as a fundamental variable in quantum mechanics may be extended to space-time. That is the notion of space-time may be a semi-classical approximate notion rather than a fundamental one in quantum gravity. One should emphasise that statements such as "time does not exist" or "space is emergent" are nonsensical. The physical question is "what are the relevant variables to parameterise the outcome of experimental observations?" Thus, the undefinability of time parameterisation of trajectories in quantum mechanics is at the heart of its probabilistic interpretation, which is well documented in experiments.

In this respect it is useful to provide an additional argument that shows that time parameterisation of trajectories is ill defined due to the Möbius symmetry that underlies quantum mechanics, and consequently due to the compactness of space. In Bohmian mechanics the wave function is set as:

$$\psi(q, t) = R(q)e^{iS/\hbar} \quad (25)$$

where $R(q)$ and $S(q)$ are the two real functions of the Quantum Hamilton–Jacobi Equation (QHJE), and $\psi(q)$ is a solution of the Schrödinger equation. The conjugate momentum is then given by:

$$\hbar \text{Im} \frac{\nabla \psi}{\psi}$$

which we may use to define trajectories by identifying it with $m\dot{q}$. The flaw in this argument is in the Bohmian identification of the wave function by Equation (25). The issue is precisely the boundary conditions imposed by the Möbius symmetry that underlies quantum mechanics and the compactness of space. If space is compact then the wave function is necessarily a linear combination of the two solutions of the Schrödinger equation:

$$\psi = R(q) \left(A e^{\frac{i}{\hbar} S} + B e^{-\frac{i}{\hbar} S} \right) \quad (26)$$

albeit one of the coefficients A or B can be very small, but neither can be set identically to zero. In this case the:

$$\nabla S \neq \hbar \text{Im} \frac{\nabla \psi}{\psi}$$

and the Bohmian definition of trajectories is invalid.

The equivalence postulate approach therefore reproduces the main phenomenological characteristics of quantum mechanics. In fact, in retrospect this is not a surprise. It may be regarded as conventional quantum mechanics with the addendum that space is compact, as dictated by the Möbius symmetry that underlies the formalism.

The one-dimensional case reveals the Möbius symmetry that underlies the equivalence postulate and hence underlies quantum mechanics. The equivalence postulate formalism extends to the higher dimensional case both with respect to the Euclidean and Minkowski metrics [124]. For brevity I summarise here only the non-relativistic case. The relativistic extensions as well as the generalisation to the case with gauge coupling are found in [124]. The key to these extensions are the generalisations of the cocycle condition Equation (13), and of the Schwarzian identity Equation (14). Denoting the transformations between two sets of coordinate systems by:

$$q \rightarrow q^v = v(q) \quad (27)$$

and the conjugate momenta by the generating function $S_0(q)$:

$$p_k = \frac{\partial S_0}{\partial q_k} \quad (28)$$

Under the transformations Equation (27) we have $S_0^v(q^v) = S_0(q)$, hence:

$$p_k \rightarrow p_k^v = \sum_{i=1}^D J_{ki} p_i \quad (29)$$

where J is the Jacobian matrix:

$$J_{ki} = \frac{\partial q_i}{\partial q_j^v} \quad (30)$$

Introducing the notation:

$$(p^v|p) = \frac{\sum_k (p_k^v)^2}{\sum_k p_k^2} = \frac{p^t J^t J p}{p^t p} \quad (31)$$

the cocycle condition takes the form:

$$(q^a; q^c) = (p^c|p^b) [(q^a; q^b) - (q^c; q^b)] \quad (32)$$

which captures the symmetries that underlie quantum mechanics. It is shown that the cocycle condition, Equation (32) is invariant under D -dimensional Möbius transformations, which include dilatations, rotations, translations and reflections in the unit sphere [124]. The quadratic identity, Equation (14), is generalised by the basic identity:

$$\alpha^2 (\nabla S_0)^2 = \frac{\Delta(R e^{\alpha S_0})}{R e^{\alpha S_0}} - \frac{\Delta R}{R} - \frac{\alpha}{R^2} \nabla \cdot (R^2 \nabla S_0) \quad (33)$$

which holds for any constant α and any functions R and S_0 . Then, if R satisfies the continuity equation:

$$\nabla \cdot (R^2 \nabla S_0) = 0 \quad (34)$$

and setting $\alpha = i/\hbar$, we have:

$$\frac{1}{2m}(\nabla S_0)^2 = -\frac{\hbar^2}{2m} \frac{\Delta(Re^{\frac{i}{\hbar}S_0})}{Re^{\frac{i}{\hbar}S_0}} + \frac{\hbar^2}{2m} \frac{\Delta R}{R} \quad (35)$$

In complete analogy with the one-dimensional case, we make identifications,

$$W(q) = V(q) - E = \frac{\hbar^2}{2m} \frac{\Delta(Re^{\frac{i}{\hbar}S_0})}{Re^{\frac{i}{\hbar}S_0}} \quad (36)$$

$$Q(q) = -\frac{\hbar^2}{2m} \frac{\Delta R}{R} \quad (37)$$

Equation (36) implies the D -dimensional Schrödinger equation:

$$\left[-\frac{\hbar^2}{2m} \Delta + V(q) \right] \Psi = E\Psi \quad (38)$$

and the general solution:

$$\Psi = R(q) \left(Ae^{\frac{i}{\hbar}S_0} + Be^{-\frac{i}{\hbar}S_0} \right) \quad (39)$$

is mandated by consistency of the equivalence postulate. We note that the key to these generalisations is the symmetry structure that underlies the formalism. Seeking further generalisation of this approach simply entails that this robust symmetry structure is retained.

5.3.1. The Classical Limit

The invariance of the cocycle condition under Möbius transformations may only be implemented if space is compact. The decompactification limit may represent the case when the spectrum of the free quantum particle becomes continuous. In that case time parameterisation of quantum trajectories is consistent with Jacobi's theorem [115,118,122]. However, the decompactification limit can be seen to coincide with the classical limit. For this purpose we examine again the case of the free particle in one dimension. This is sufficient since all physical states can be mapped to this state by a coordinate transformation. The quantum potential associated with the state $W^0(q^0) \equiv 0$ is given by:

$$Q^0 = \frac{\hbar^2}{4m} \{S_0^0, q^0\} = -\frac{\hbar^2(\text{Re } \ell_0)^2}{2m} \frac{1}{|q^0 - i\ell_0|^4} \quad (40)$$

It is noted that the limit $q^0 \rightarrow \infty$ coincides with the limit $Q^0 \rightarrow 0$, *i.e.*, with the classical limit [14]. This observation is consistent with the recent claim that the universe cannot be closed classically [125]. Possible signatures for nontrivial topology in the CMB has been of recent interest [126]. Further experimental support for the equivalence postulate approach to quantum mechanics may arise from modifications of the relativistic-energy momentum relation [127], which affects the propagation of cosmic gamma rays [128].

5.3.2. Where is the Connection with String Theory?

The simple answer to the question may be: in the future. Nevertheless, we may attempt to gather some hints how the connection may exist. String theory is a self-consistent perturbative framework for

quantum gravity. As such it provides an effective approach, but is not formulated from a fundamental principle. An important property of string theory is T-duality, which may be interpreted as phase space duality in compact space. We may conjecture that phase space duality is the fundamental principle and use that as a starting point for formulating quantum gravity. This is what the equivalence postulate approach aims at.

Consistency of the equivalence postulate approach dictates that the CHJE is replaced by the QHJE and that the quantum potential $Q(q)$ is never zero. In the one-dimensional case, the Schwarzian derivative may be interpreted as a curvature term [123,129,130]. In the higher dimensional case it is proportional to the curvature of the function $R(q)$. Thus, we may interpret the quantum potential as an intrinsic curvature term associated with an elementary particle. Point particles do not have curvature. Hence, the interpretation of the quantum potential as a curvature term hints to the connection with internal structure of elementary particles.

The Möbius symmetry underlying quantum mechanics in the equivalence postulate formalism also implies the existence of a finite length scale [131]. For this purpose we can again study the one-dimensional stationary case with $W^0(q^0) = 0$. The Schrödinger equation takes the form:

$$\frac{\partial^2}{\partial q^2} \psi = 0$$

The two linearly independent solutions are $\psi^D = q^0$ and $\psi = \text{const.}$ Consistency of the equivalence postulate mandates that both solutions must be retained. The solution of the corresponding QHJE is given by [114,115]:

$$e^{\frac{2i}{\hbar} S_0^0} = e^{i\alpha \frac{q^0 + i\bar{\ell}_0}{q^0 - i\ell_0}}$$

where ℓ_0 is a constant with the dimension of length [115,131], and the conjugate momentum $p_0 = \partial_{q^0} S_0^0$ takes the form:

$$p_0 = \pm \frac{\hbar(\ell_0 + \bar{\ell}_0)}{2|q^0 - i\ell_0|^2} \quad (41)$$

It is noted that p_0 vanishes only for $q^0 \rightarrow \pm\infty$. The requirement that in the classical limit $\lim_{\hbar \rightarrow 0} p_0 = 0$ suggests that we can set [115,131]:

$$\text{Re } \ell_0 = \lambda_p = \sqrt{\frac{\hbar G}{c^3}} \quad (42)$$

i.e., we identify $\text{Re } \ell_0$ with the Planck length. The invariance under the Möbius transformations mandates the existence of a finite length scale. Additionally, from Equation (41) follows that p_0 is maximal for $q^0 = -\text{Im} \ell_0$, *i.e.*,

$$|p_0(-\text{Im} \ell_0)| = \frac{\hbar}{\text{Re} \ell_0} \quad (43)$$

The equivalence postulate mandates that $\text{Re} \ell_0 \neq 0$. Consequently, p_0 is always finite, and ℓ_0 acts as an ultraviolet cut-off. As we would expect, the existence of an ultraviolet cut-off is tightly linked to the existence of a finite length scale. The fundamental feature is the Möbius symmetry at the core of quantum mechanics.

6. Conclusions

The indication from the LHC of a scalar resonance compatible with perturbative electroweak symmetry breaking reinforces the Standard Model parameterisation of all subatomic experimental data. The logarithmic evolution is the Standard Model gauge, and the matter parameters suggest that the Standard Model provides a viable parameterisation up to the Planck scale. Supersymmetry preserves the logarithmic running also in the scalar sector, which provides reasonable motivation to seek experimental evidence for its validity in the LHC, Very Large Hadron Collider (VLHC) and other future machines. It should be stressed that the viability of the experimental program rests on its ability to deliver a working machine in the first place and to measure the parameters of the Standard Model with better accuracy in the second. Discovering new physics is an added bonus.

The Planck scale is an ultraviolet cut-off, at which gravitational effects are of comparable strength to the gauge interactions. String theory provides a perturbatively consistent framework that incorporates gravity and the gauge interactions and enables the construction of phenomenological models. The state-of-the-art in this regard is string models that reproduce the spectrum of the Minimal Supersymmetric Standard Model. The understanding of string theory, as well as that of the space of string solutions, is still in its infancy. As long as the experimental data does not indicate that this is on the wrong track, its exploration continues to be of interest. Ultimately, in the future, we would like to formulate quantum gravity from a fundamental principle. Phase space duality and the equivalence postulate of quantum mechanics provide a good starting point for that purpose.

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Conflicts of Interest

The author declares no conflicts of interest.

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