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Static and Dynamic Characteristics of Rough Porous Rayleigh Step Bearing Lubricated with Couple Stress Fluid

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Abstract: In tribology, the Rayleigh step bearing has the maximum load capacity of any feasible bearing geometry. Traditional tribology resources have demonstrated that the Rayleigh step has an ideal geometry which maximizes load capacity. Both in nature and technology, rough and textured surfaces are essential for lubrication. While surface roughness enhances the performance of the bearings as an efficiency measure, it still has a significant impact on the load-carrying capacity of the bearing. In the present study, we investigate the dynamic characteristics of the Rayleigh step bearing with the impact of surface roughness and a porous medium by considering a squeezing action. Couple stress fluid is considered a lubricant with additives in both the film as well as the porous region. Based on Stokes constitutive equations for couple stress fluids, Darcy's law for porous medium, and stochastic theory for rough surfaces, the averaged Reynolds-type equation is derived. Expressions are obtained for the volume flow rate, steady-state characteristics, and dynamic characteristics. The influence of surface roughness and the porous medium on the Rayleigh step bearing is analyzed. We investigated the static and dynamic characteristics of the Rayleigh step bearing. As a result, the couple stress fluid increases (decreases) the steady load-carrying capacity, dynamic stiffness, and dynamic damping coefficients, and decreases (increases) the volume flow rate negatively (positively) skewed roughness in comparison with that of the Newtonian case. The results are compared with those of the smooth case.



Citation: Naduvinamani, N.; Angadi, A. Static and Dynamic Characteristics of Rough Porous Rayleigh Step Bearing Lubricated with Couple Stress Fluid. *Lubricants* **2022**, *10*, 257. <https://doi.org/10.3390/lubricants10100257>

Received: 6 September 2022

Accepted: 10 October 2022

Published: 13 October 2022

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Keywords: micropolar fluid; Rayleigh step bearing; surface roughness; stiffness coefficient; damping coefficient

1. Introduction

In this paper, we analyzed the roughness influence on the dynamic behavior of porous Rayleigh Step Bearings (RSB) lubricated with couple stress fluids. The RSB is often used in industries because of its characteristics of higher Load Carrying Capacity (LCC). To improve LCC, much research has been conducted using an analytic method by solving the Reynolds equation. The concept of a step bearing was initially discussed by Lord Rayleigh [1] in 1918, who identified the ideal design with the highest load capacity per unit width for a specific film thickness and bearing length. This configuration is now referred to as the RSB. Since then, the characteristics of the RSB have been investigated by several researchers. Due to their high load capacity and cheap manufacturing costs, RSBs have been widely used in industries such as thrust bearings and pad bearings. In recent times, researchers have started studying non-Newtonian fluids, especially fluids containing additives or small particles, to analyze their effects on bearing performance, which the classical Newtonian theory failed to extend. The Stokes [2] Couple Stress Fluid is one such non-Newtonian fluid. Later, the use of porous bearings in the industry became widespread. A porous material is used in bearing systems to simplify production techniques, reduce costs, and extend service life. Porous bearings have a simple structure and are low cost. The application of porous bearings in mounting horsepower motors includes water pumps, vacuum cleaners, sewing machines, tape recorders, record players, shaving machines,

coffee grinders, hair dryers, generators, and distributors. Bujurke et al. [3] considered a porous RSB lubricated by a second-order fluid and calculated the load capacity, frictional force, and frictional coefficients. Naduvinamani [4] theoretically presented double-layered porous RSBs with second-order fluid as a lubricant. Naduvinamani and Siddangouda [5] studied the optimum lubrication characteristics of porous RSBs lubricated with couple stress fluids. Later, they theoretically analyzed the lubrication characteristics of a porous inclined stepped composite bearing lubricated with micropolar fluid [6]. Rahmani et al. [7] described the effect of variations in pressure at the boundaries on the optimum properties of the RSB. Vakilian et al. [8] showed that inertia has a considerable effect on the thermohydrodynamic (THD) characteristics of step bearings having high-velocity runner surfaces. Naduvinamani et al. [9] obtained the optimum bearing parameters for the RSB lubricated with non-Newtonian Robinowitsch fluid. Shen et al. [10] investigated the flow characteristics in the Rayleigh step slider bearing with infinite width, both analytically and numerically. Alazwari et al. [11] analyzed the entropy optimization of first-grade viscoelastic nanofluid flow over a stretching sheet by using a classical Keller-box scheme. An analysis was carried out by Patel et al. [12] to enhance the performance of the ferrofluid lubricated porous step bearing by considering different flow models. Jamshed et al. [13] described the thermal efficiency enhancement of solar aircraft by utilizing unsteady hybrid nanofluid using a single-phase optimized entropy analysis. Muhammad Haq et al. [14] analyzed the energy transport of the magnetized forced flow of power-law nanofluid over a horizontal wall.

Surface roughness has a significant impact on bearing performance. Microscopic surface roughness is imposed on the surface during finishing processes such as grinding, lapping, and grit blasting. Christensen and Tonder [15] analyzed a stochastic method to study surface roughness. Andharia et al. [16] studied the impact of roughness on the performance of hydrodynamic slider bearings. Shiralashetti and Mounesha [17] used the wavelet-based decoupled method to investigate the effect of couple stress fluid and surface roughness on the elastohydrodynamic problem. Bijani et al. [18] studied the impact of surface texturing on the coefficient of friction in parallel sliding lubricated surfaces. Andharia and Pandya [19] considered the longitudinal roughness of the RSB. By adopting different porous structure models, Rao and Agarwal [20] analyzed the surface roughness effects on the hydrodynamic lubrication of step slider bearings. Paggi et al. [21] discussed the smoothed particle hydrodynamics (SPH) modeling of hydrodynamic lubrication along rough surfaces. Badescu [22] performed shape optimization of slider bearings operating with couple stress lubricants using a novel direct optimal control approach. Vashi et al. [23] scrutinized longitudinally rough, porous circular stepped plates with the impact of ferrofluid in the presence of couple stress. The deterministic mixed lubrication model was studied by Wang et al. [24] to understand the mechanism of LCC between parallel rough surfaces. Various researchers have recently begun investigating the bearings' static and dynamic performance. Theoretically, Lin [25,26] presented the steady and dynamic performance of hydrostatic circular step thrust bearings with the effects of couple stresses, fluid inertia, and recess volume fluid compressibility and also derived a general Reynolds equation of sliding-squeezing surfaces with non-Newtonian fluids, which is necessary for assessment of the dynamic characteristics of a lubricating system. Lin et al. [27] studied the dynamic characteristics of an infinite-width tapered-land slider bearing by considering the squeezing action. The effect of couple stress fluid on the dynamic characteristics of wide exponential-shaped slider bearings and wide Rayleigh step slider bearings was presented by Lin et al. [28,29]. Naduvinamani and Patil [30] derived the dynamic Reynolds equation for micropolar fluid to study the dynamic characteristics of a finite exponential-shaped slider bearing. Singh and Gupta [31] theoretically investigated the effect of ferrofluid using the shliomis model on the dynamic characteristics of curved slider bearings. A theoretical study of the impact of roughness is analyzed by Siddangouda et al. [32] on the static characteristics of an inclined plane slider-bearing lubricated with Rabinowitsch fluid. Rajeevkumar et al. [33] described the dynamic behavior of non-Newtonian power-law,

micropolar, and couple stress fluids and their influence on the efficiency of journal bearings. Under the influence of micropolar fluid and roughness, the dynamic characteristics of inclined porous slider bearings are analyzed by Naduvinamani and Angadi [34]. Yandong Gu et al. [35] determined the static characteristics of aerostatic porous journal bearings theoretically and numerically. Fang et al. [36] discussed line contact stiffness and damping behaviors under transient elastohydrodynamic lubrication.

In this present paper, we studied the effect of porosity and roughness on the static and dynamic behavior of RSB lubricated with couple stress fluid along with squeezing action, on which a study has not been conducted so far as per the relevant literature known to the authors. An averaged modified Reynolds-type equation for rough porous RSBs has been derived and the numerical computations were carried out to get the required results. The probability density function (PDF) for the random variable is assumed to be asymmetrical with a non-zero mean, which characterizes the surface roughness.

2. Mathematical Formulation

Figure 1 represents the physical geometry of a rough porous RSB of length L lubricated with Stokes non-Newtonian couple stress fluid [2]. The step bearing has a squeezing velocity $\frac{\partial h}{\partial t}$ and a sliding velocity U . The film thickness can be described as

$$H = h(x, t) + h_r \quad (1)$$

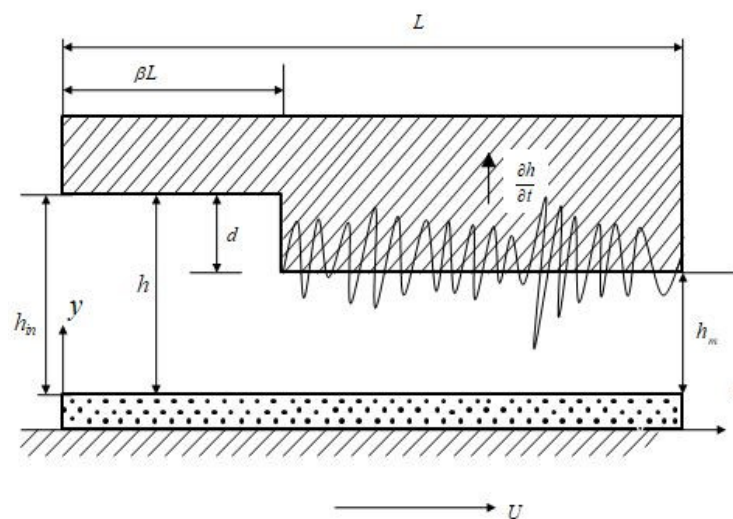


Figure 1. Physical geometry of the considered problem.

Here

$$h(x, t) = \begin{cases} h_{in} = d + h_m(t), & 0 \leq x \leq \beta L \\ h_m(t), & \beta L \leq x \leq L \end{cases} \quad (2)$$

where $h_m(t)$, h_{in} is the outlet, inlet film thickness, d is the difference between steady outlet-inlet film thickness and β denotes the riser location parameter. This analysis assumes that the thin film theory of lubrication is applicable, and the body forces and body couples are absent. The Stokes [2] momentum and continuity equations for the couple stress fluid take the form,

$$\mu \frac{\partial^2 u}{\partial y^2} - \eta \frac{\partial^4 u}{\partial y^4} = \frac{\partial p}{\partial x} \quad (3)$$

$$\frac{\partial p}{\partial y} = 0 \quad (4)$$

$$\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = 0 \quad (5)$$

where p pressure and u and v are the fluid velocity components along x and y directions, respectively, in the film region, μ is the conventional shear viscosity, η is a new material constant accounting for couple stresses and is of the dimension of momentum.

The relevant boundary conditions for the velocity components are

At $y = H$

$$u = 0, v = \frac{\partial h}{\partial t}, \frac{\partial^2 u}{\partial y^2} = 0 \quad (6)$$

At $y = 0$

$$u = U, v = v^*, \frac{\partial^2 u}{\partial y^2} = 0 \quad (7)$$

The velocity component u in the x direction and the dynamic Reynolds-type equation for non-Newtonian couple stress fluid can be obtained by solving Equations (3) and (5) subject to the relevant boundary conditions (6) and (7), and yields,

$$u = U \left(1 - \frac{y}{H}\right) + \frac{1}{2\mu} \frac{\partial p}{\partial x} \left\{ y^2 - Hy + 2l^2 \left[1 - \frac{\cosh[(2y - H)/2l]}{\cosh[H/2l]} \right] \right\} \quad (8)$$

$$\frac{\partial}{\partial x} \left[f(H, l) \frac{\partial p}{\partial x} \right] = 6\mu U \frac{\partial H}{\partial x} + 12\mu \frac{\partial H}{\partial t} - 12v^*|_{y=0} \quad (9)$$

where l is the molecular length of polar additives in a Newtonian fluid, which is given by $l = (\eta/\mu)^{1/2}$ and

$$f(H, l) = H^3 - 12l^2 H + 24l^3 \tanh(H/2l) \quad (10)$$

The flow of couple stress fluid in the porous matrix is ruled by the modified Darcy's law for porous material and is given by

$$\vec{q}^* = \frac{-\phi}{\mu(1 - \zeta)} \nabla \bar{p}^* \quad (11)$$

where $\vec{q}^* = (u^*, v^*)$ is the Darcy's modified velocity vector, ϕ —permeability, and $\zeta = \frac{(\eta/\mu)}{\phi}$ represents the ratio of the microstructure size to the pore size. If $(\eta/\mu)^{1/2} \approx \sqrt{\phi}$, i.e., $\zeta \approx 1$, then the microstructure additives present in the lubricant block the pores in the porous layer and thus reduce the Darcy flow through the porous matrix. When the microstructure size is very small compared to the pore size, i.e., $\zeta \ll 1$, the additives percolate into the porous matrix. Due to the continuity of the fluid in the porous matrix, the pressure $\bar{p}^*$ satisfies the Laplace equation

$$\frac{\partial^2 \bar{p}^*}{\partial x^2} + \frac{\partial^2 \bar{p}^*}{\partial y^2} = 0 \quad (12)$$

Integration regarding y over δ —the thickness of the porous layer, and utilizing the boundary condition $\left(\frac{\partial \bar{p}^*}{\partial y} = 0\right)$ at solid backing $y = -\delta$, we obtain

$$\left(\frac{\partial \bar{p}^*}{\partial y}\right)_{y=0} = \int_{-\delta}^0 \frac{\partial^2 \bar{p}^*}{\partial x^2} dy \quad (13)$$

The assumption of small porous layer thickness δ and the utility of the pressure continuity condition ($p = \bar{p}^*$) at the porous interface ($y = 0$) deduce Equation (13) to

$$\left(\frac{\partial \bar{p}^*}{\partial y}\right)_{y=0} = -\delta \frac{\partial^2 p}{\partial x^2} \quad (14)$$

Then, at the interface ($y = 0$), the velocity component v^* is given by

$$(v^*)_{y=0} = \frac{\phi\delta}{1-\zeta} \left(\frac{\partial^2 p}{\partial x^2} \right) \quad (15)$$

Substituting Equation (15) in Equation (9), the dynamic Reynolds equation for couple stress fluid is acquired in the form

$$\frac{\partial}{\partial x} \left\{ \left[f(N, l, H) + \frac{12\phi\delta}{1-\zeta} \right] \frac{\partial p}{\partial x} \right\} = 6U \frac{\partial H}{\partial x} + 12 \frac{\partial H}{\partial t} \quad (16)$$

$$f(H, l) = H^3 - 12l^2 H + 24l^3 \tanh(H/2l) \quad (17)$$

The Volume Flow Rate (VFR) in the x direction can be evaluated by integrating the velocity component u across the film thickness.

$$Q = \int_0^h u \cdot D \, dy \quad (18)$$

We can determine the VFR by performing the integration with the expression of u .

$$Q = \frac{1}{2} U h \cdot D - \frac{1}{12\mu} \cdot \frac{\partial p}{\partial x} \cdot f(h, l) \cdot D \quad (19)$$

Multiplying both sides of Equations (16) and (19) by $f(h_r)$ and integrating with respect to h_r over the interval $(-C, C)$ and using $\alpha^* = E(h_r)$ as the mean, $\sigma^{*2} = E[(h_r - \alpha^*)^2]$ the standard deviation and $\varepsilon^* = E[(h_r - \alpha^*)^3]$ the measure of symmetry of the random variable h_r , the skewness parameter, where E is the expectation operator defined by

$$E(\bullet) = \int_{-\infty}^{\infty} (\bullet) f(h_r) dh_r \quad (20)$$

The average Reynolds type equation is obtained in the form

$$\frac{\partial}{\partial x} \left[\left[f(h, l, \alpha^*, \sigma^*, \varepsilon^*) + \frac{12\phi\delta}{1-\zeta} \right] \frac{\partial \bar{p}}{\partial x} \right] = 6\mu U \frac{\partial h}{\partial x} + 12\mu \frac{\partial h}{\partial t} \quad (21)$$

$$\bar{Q} = \frac{1}{2} (h + \alpha^*) - \frac{1}{12} \cdot \frac{\partial \bar{p}}{\partial x} \cdot f(h, l, \alpha^*, \sigma^*, \varepsilon^*) \quad (22)$$

where $\bar{p} = E(p)$ and $\bar{Q} = E(Q)$ is the expected value of p and Q

$$f(h, l, \alpha, \sigma, \varepsilon) = h^3 + 3h^2\alpha^* + 3h(\alpha^{*2} + \sigma^{*2}) + \varepsilon^* + 3\alpha^*\sigma^{*2} + \alpha^{*3} - 12l^2(h + \alpha^*) + 24l^3 \tanh\left(\frac{h}{2l}\right) + \left(1 - \tanh^2\left(\frac{h}{2l}\right)\right)(12l^2\alpha^* - \varepsilon^* - \alpha^{*3} - 3\sigma^{*2}\alpha^*) \quad (23)$$

Adding the non-dimensional quantities

$$l^* = \frac{l}{h_{ms}}, \delta = \frac{d}{h_{ms}}, \psi = \frac{\phi\delta^*}{h_{ms}^3}, Q^* = \frac{\bar{Q}}{U h_{ms} D}, p^* = \frac{\bar{p} h_{ms}^2}{\mu U L}, x^* = \frac{x}{L}, t^* = \frac{U t}{L}, h^*(x^*, t^*) = \frac{h(x, t)}{h_{ms}}, \alpha = \frac{\alpha^*}{h_{ms}}, \sigma = \frac{\sigma^*}{h_{ms}}, \varepsilon = \frac{\varepsilon^*}{h_{ms}^3} \quad (24)$$

into Equations (21) and (22) gives the non-dimensional dynamic Reynolds equation and the volume flow rate in the form

$$\frac{\partial}{\partial x^*} \left[\left(f(h^*, l^*, \alpha, \sigma, \varepsilon) + \frac{12\psi}{1-\zeta} \right) \frac{\partial p^*}{\partial x^*} \right] = 6 \frac{\partial h^*}{\partial x^*} + 12 \frac{\partial h^*}{\partial t^*} \quad (25)$$

$$Q^* = \frac{1}{2}(h^* + \alpha) - \frac{1}{12} \cdot \frac{\partial p^*}{\partial x^*} \cdot f(h^*, l^*, \alpha, \sigma, \varepsilon) \quad (26)$$

where

$$f(h^*, l^*, \alpha, \sigma, \varepsilon) = h^{*3} + 3h^{*2}\alpha + 3h^*(\alpha^2 + \sigma^2) + \varepsilon + 3\alpha\sigma^2 + \alpha^3 - 12l^{*2}(h^* + \alpha) + 24l^{*3}\tanh\left(\frac{h^*}{2l^*}\right) + \left(1 - \tanh^2\left(\frac{h^*}{2l^*}\right)\right)(12l^{*2}\alpha - \varepsilon - \alpha^3 - 3\sigma^2\alpha) \quad (27)$$

and

$$h(x^*, t^*) = \begin{cases} h_{in} = \delta + h_m^*(t^*), & 0 \leq x \leq \beta \\ h_m^*(t^*), & \beta \leq x \leq 1 \end{cases} \quad (28)$$

The non-dimensional non-Newtonian couple stress dynamic Reynolds-type equation has two regions based on the geometry of the bearing.

For $0 \leq x^* \leq \beta$: Region 1

$$\frac{\partial}{\partial x^*} \left[\left(f(h^*, l^*, \alpha, \sigma, \varepsilon) + \frac{12\psi}{1-\zeta} \right) \frac{\partial p_1^*}{\partial x^*} \right] = 12V^* \quad (29)$$

Here

$$h^*(x^*, t^*) = \delta + h_m^*(t^*) \quad (30)$$

For $\beta \leq x^* \leq 1$: Region 2

$$\frac{\partial}{\partial x^*} \left[\left(f(h^*, l^*, \alpha, \sigma, \varepsilon) + \frac{12\psi}{1-\zeta} \right) \frac{\partial p_2^*}{\partial x^*} \right] = 12V^* \quad (31)$$

Here

$$h^*(x^*, t^*) = h_m^*(t^*) \quad (32)$$

and $V^* = dh_m^*/dt^*$ is the squeezing velocity in the non-dimensional form. The complementary boundary conditions within Region 1 are:

$$\text{At } x^* = \beta : Q_1^* = Q_{1\beta}^* \quad (33)$$

$$\text{At } x^* = 0 : p_1^* = 0 \quad (34)$$

$$\text{At } x^* = \beta : p_1^* = p_{1\beta}^* \quad (35)$$

Evaluating the volume flow rate (26) with the respective flow boundary condition (33) and solving the non-Newtonian dynamic Reynolds-type Equation (31) with the respective pressure boundary conditions (34) and (35), we obtain

$$Q_{1\beta}^* = \frac{1}{2} \cdot (\delta + h_m^*) - V^* \cdot \alpha - \frac{1}{12} \cdot G_1(h_m^*, V^*) \quad (36)$$

$$p_{1\beta}^* = 12V^* f_A(\beta, h_m^*) + G_1(h_m^*, V^*) \cdot f_B(\beta, h_m^*) \quad (37)$$

where G_1 denotes the integration functions, and

$$f_A(\beta, h_m^*) = \int_0^{\beta} \frac{x^*}{\left(f(h^*, l^*, \alpha, \sigma, \varepsilon) + \frac{12\psi}{1-\zeta} \right)} dx^* \quad (38)$$

$$f_B(\beta, h_m^*) = \int_{\beta}^1 \frac{1}{\left(f(h^*, l^*, \alpha, \sigma, \varepsilon) + \frac{12\psi}{1-\zeta} \right)} dx^* \quad (39)$$

The complementary boundary conditions for Region 2 are:

$$\text{At } x^* = \beta : Q_2^* = Q_{2\beta}^* \quad (40)$$

$$\text{At } x^* = \beta : p_2^* = p_{2\beta}^* \quad (41)$$

$$\text{At } x^* = 1 : p_2^* = 0 \quad (42)$$

Evaluation of the volume flow rate (26) with the corresponding flow boundary condition (40) and solution of the non-Newtonian dynamic Reynolds-type Equation (31) with the corresponding pressure boundary conditions (41) and (42) yields

$$Q_{2\beta}^* = \frac{1}{2} \cdot h_m^* - V^* \cdot \alpha - \frac{1}{12} \cdot G_2(h_m^*, V^*) \quad (43)$$

$$p_{2\beta}^* = 12V^* f_C(\beta, h_m^*) + G_2(h_m^*, V^*) \cdot f_D(\beta, h_m^*) \quad (44)$$

where G_2 denotes the integration functions, and

$$f_C(\beta, h_m^*) = \int_1^{x^*} \frac{x^*}{\left(f(h^*, l^*, \alpha, \sigma, \varepsilon) + \frac{12\psi}{1-\zeta}\right)} dx^* \quad (45)$$

$$f_D(\beta, h_m^*) = \int_1^{x^*} \frac{1}{\left(f(h^*, l^*, \alpha, \sigma, \varepsilon) + \frac{12\psi}{1-\zeta}\right)} dx^* \quad (46)$$

Now at the position $x^* = \beta$, equating the values of the VFR requires

$$Q_{1\beta}^* = Q_{2\beta}^* \quad (47)$$

Similarly, at the position $x^* = \beta$, equating the values of the film pressure requires

$$p_{1\beta}^* = p_{2\beta}^* \quad (48)$$

Applying the conditions (38) and (39), we obtain the expressions of the integration functions.

$$G_1 = \frac{12V^* f_C(\beta, h_m^*) - 12V^* f_A(\beta, h_m^*) - 6\delta f_D(\beta, h_m^*)}{f_B(\beta, h_m^*) - f_D(\beta, h_m^*)} \quad (49)$$

$$G_2 = G_1 - 6\delta \quad (50)$$

The non-Newtonian dynamic film force can be evaluated by integrating film pressure over the fluid-film region. Expression of film force in a non-dimensional form is

$$F^* = \int_0^1 p^* dx^* = \int_0^\beta p_1^* dx^* + \int_\beta^1 p_2^* dx^* \quad (51)$$

By the use of the expression of the film pressure and performing the integrations, one can obtain the non-dimensional dynamic film force as

$$F^*(h_m^*, V^*) = 12V^* F_A(h_m^*) + 12V^* F_C(h_m^*) + G_1(h_m^*, V^*) F_B(h_m^*) + G_1(h_m^*, V^*) F_D(h_m^*) - 6\delta F_D(h_m^*) \quad (52)$$

where the corresponding functions are defined in Appendix A. Following the similar procedures of the study of dynamic characteristics of wide RSB by Lin et al. [26], we can obtain the non-Newtonian steady VFR Q_s^* , non-Newtonian steady LCC W_s^* , non-Newtonian Dynamic Stiffness Coefficient (DSC) S_d^* and non-Newtonian Dynamic Damping Coefficient (DDC) D_d^* .

$$Q_s^* = Q_{1\beta}^*(h_{ms}, 0) = \frac{1}{2} \cdot (\delta + h_{ms}^*) - \frac{1}{12} \cdot G_{1s} \quad (53)$$

$$W_s^* = F^*(h_{ms}^*, 0) = G_{1s}F_B(h_{ms}^*) + G_{1s}F_D(h_{ms}^*) - 6\delta F_D(h_{ms}^*) \quad (54)$$

$$S_d^* = -\left(\frac{\partial F^*}{\partial h_m^*}\right)_s = 6\delta\left(\frac{\partial F_D}{\partial h_m^*}\right)_s - G_{10}\left[\left(\frac{\partial F_B}{\partial h_m^*}\right)_s + \left(\frac{\partial F_D}{\partial h_m^*}\right)_s\right] - \left(\frac{\partial G_1}{\partial h_m^*}\right)_s \cdot [(F_B)_s + (F_D)_s] \quad (55)$$

$$D_d^* = -\left(\frac{\partial F^*}{\partial V^*}\right)_s = -12 \cdot [(F_A)_s + (F_C)_s] - \left(\frac{\partial G_1}{\partial V^*}\right)_s \cdot [(F_B)_s + (F_D)_s] \quad (56)$$

In the above equations, the subscript “s” denotes the bearing operating in a steady state. Appendix A defines various functions and quantities that are associated with various bearings. Using various bearing parameters, Q_s^* , W_s^* , S_d^* and D_d^* are numerically computed and represented graphically. The simplified flowchart of the model is illustrated in Figure 2.

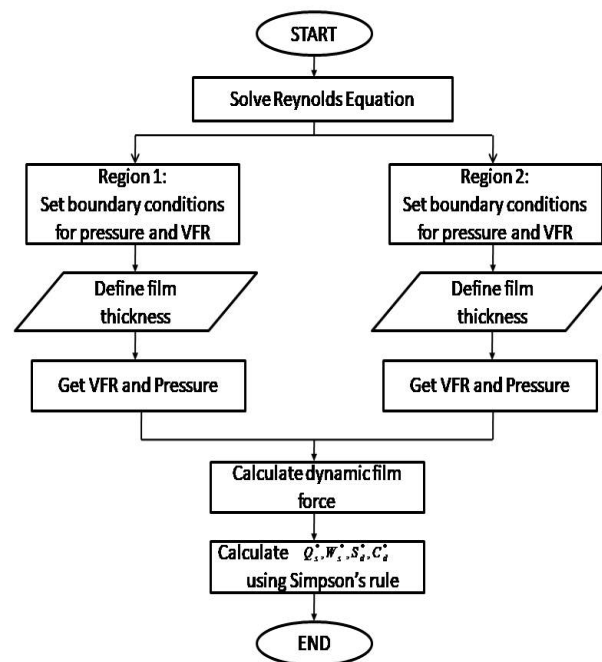


Figure 2. Flow chart of the numerical model.

3. Results

The effect of non-Newtonian couple stress fluid and roughness on the dynamic characteristics of the porous RSB is analyzed. Based on the above analysis, the non-Newtonian couple stress parameter l^* , the riser location parameter β , the shoulder parameter δ , and roughness parameters $\alpha, \sigma, \varepsilon$ are dominant in the RSB characteristics. As the permeability parameter $\psi \rightarrow 0$ and roughness parameters $\alpha, \sigma, \varepsilon \rightarrow 0$, the results obtained in the present article reduce to the case discussed by Lin et al. [26]. The values of the non-dimensional steady volume flow rate, steady LCC W_s^* , DSC S_d^* and DDC C_d^* are calculated by using numerical computations.

3.1. Volume Flow Rate

The variation in the steady VFR Q_s^* as a function of the riser location parameter β for different couple stress parameters l^* and two distinct permeability parameters ψ for $\alpha = -0.05, \sigma = 0.1, \varepsilon = -0.05, \delta = 1$ is represented in Figure 3. We observed that the required VFR increases with the increasing value of β . Compared with the Newtonian case, the effect of couple stress lubricant provides further reductions in VFR. As compared with the smooth case and solid case, higher values of Q_s^* are noticed for the negatively skewed rough case. The effect of the porous layer increases the VFR with an increasing value of the permeability parameter ψ . Figure 4 describes the varying values of Q_s^* as a function of β for distinct values of the roughness parameter α . For fixed values of $l^* = 0.4$,

$\sigma = 0.1$, $\varepsilon = -0.05$, Q_s^* decreases for the negatively increasing value of α and increases for the positively increasing value of α when compared to the smooth case. The varying value of Q_s^* presented in Figure 5 is a function of β for different values of σ with $l^* = 0.4$, $\delta = 1$, $\alpha = -0.05$, $\varepsilon = -0.05$. As σ increases, the VFR increases with the increasing value of the location parameter. The variation of Q_s^* with the location parameter β for different values of ε with $l^* = 0.4$, $\alpha = -0.05$, $\sigma = 0.1$ and $\delta = 1$ is presented in Figure 6. We observed that the VFR increases for the increasing value of ε .

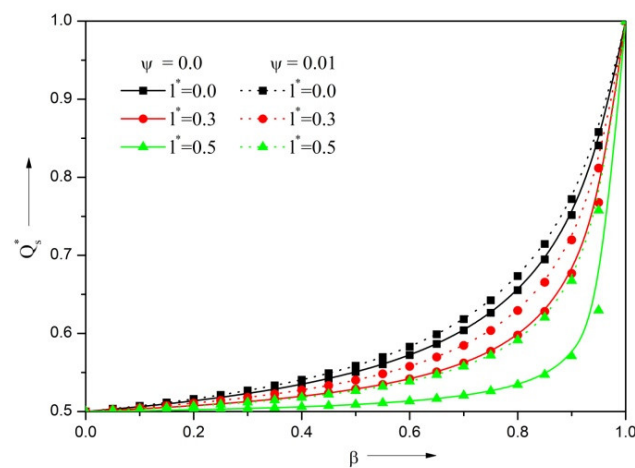


Figure 3. Steady volume flow rate Q_s^* as a function of β for different l and ψ under $\delta = 1$.

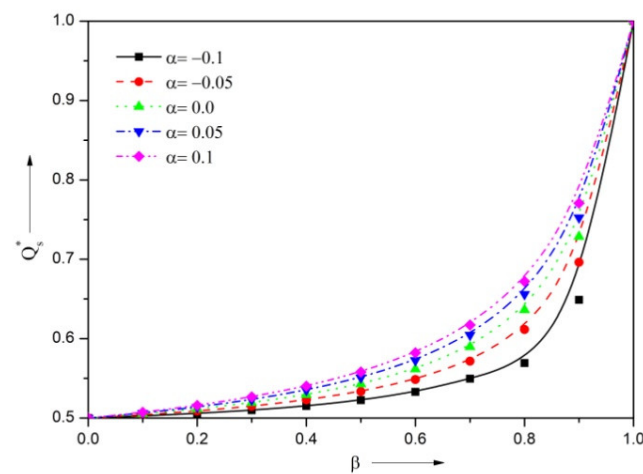


Figure 4. Steady volume flow rate Q_s^* as a function of β for different α under $\delta = 1$.

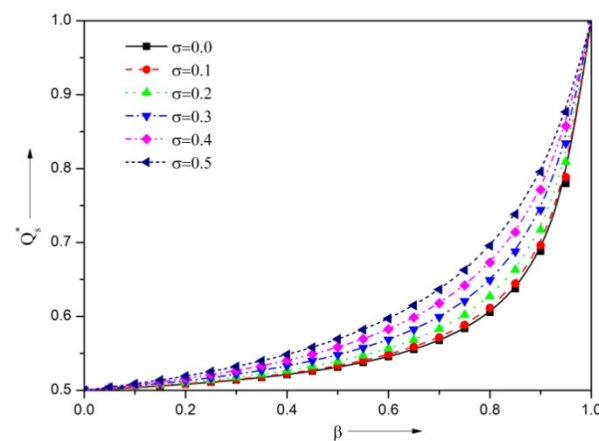


Figure 5. Steady volume flow rate Q_s^* as a function of β for different σ under $\delta = 1$.

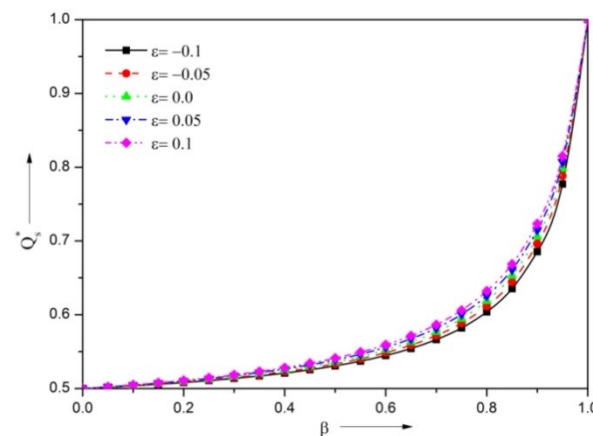


Figure 6. Steady volume flow rate Q_s^* as a function of β for different ε under $\delta = 1$.

3.2. Steady Load-Carrying Capacity

Figure 7 depicts the variation of a non-dimensional steady LCC W_s^* with the riser location parameter β for distinct values of the couple stress parameters l^* and ψ with $\alpha = -0.05$, $\sigma = 0.1$, $\varepsilon = -0.05$, $\delta = 1$. A higher steady LCC is obtained in the presence of non-Newtonian couple stresses for negatively skewed roughness as compared with the smooth case. The rough porous RSB lubricated with couple stress fluids has more LCC as compared with the corresponding Newtonian case. It is observed that the effect of the porous layer is to diminish the load capacity of the bearing as compared with the non-porous case; and also as the value of the permeability parameter increases, LCC decreases. Observing the Newtonian lubricant case, the steady LCC increases with the value of β until a critical riser location β_c is achieved, and thereafter falls as the value of β continues to increase. For $l^* = 0.2$, the critical value $\beta_c = 0.75$. Figures 8–10 describe the varying values of steady LCC with riser location parameter β for various values of the roughness parameter α ($\sigma = 0.1$, $\varepsilon = -0.05$), σ ($\alpha = -0.05$, $\varepsilon = 0.1$) and ε ($\sigma = 0.1$, $\alpha = -0.05$), respectively. In Figure 8, the LCC W_s^* increases for the negatively increasing value of α and decreases for the positively increasing value α . From Figure 9, it is evident that W_s^* decreases for the increasing value of σ . Figure 10 clearly shows how the negatively increasing value of ε increases W_s^* whereas the positively increasing value of ε decreases LCC W_s^* .

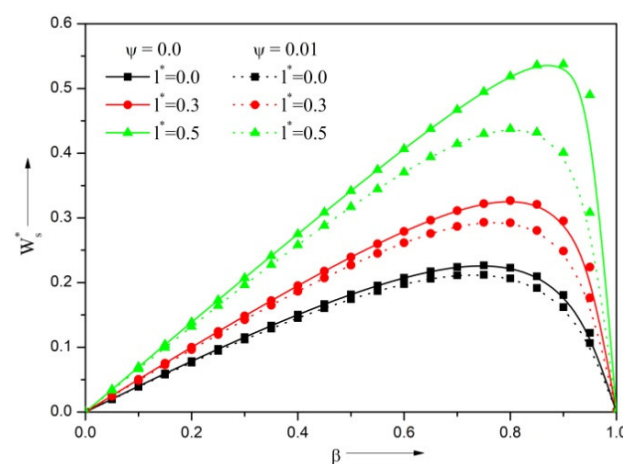


Figure 7. Steady load-carrying capacity W_s^* as a function of β for different l and ψ under $\delta = 1$.

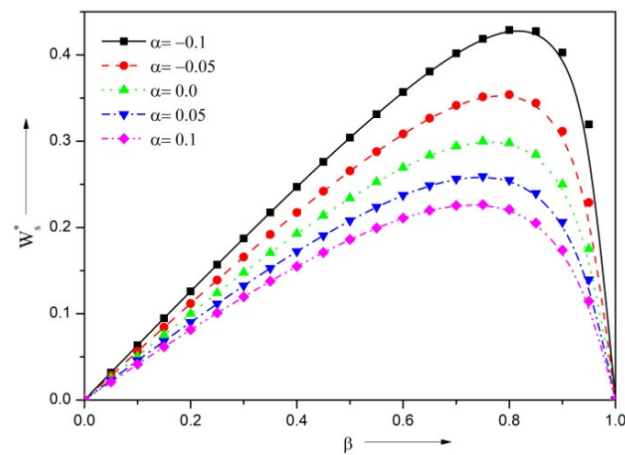


Figure 8. Steady load-carrying capacity W_s^* as a function of β for different α under $\delta = 1$.

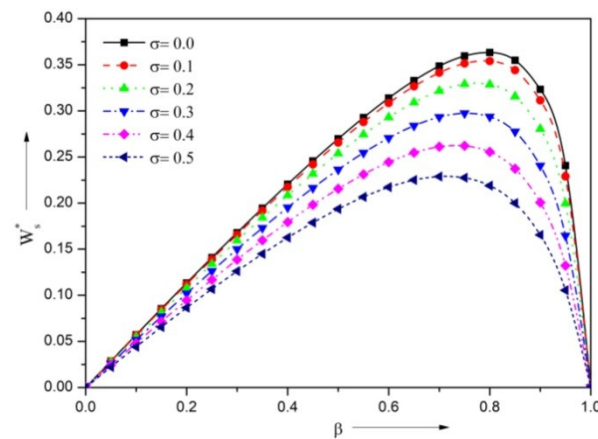


Figure 9. Steady load-carrying capacity W_s^* as a function of β for different σ under $\delta = 1$.

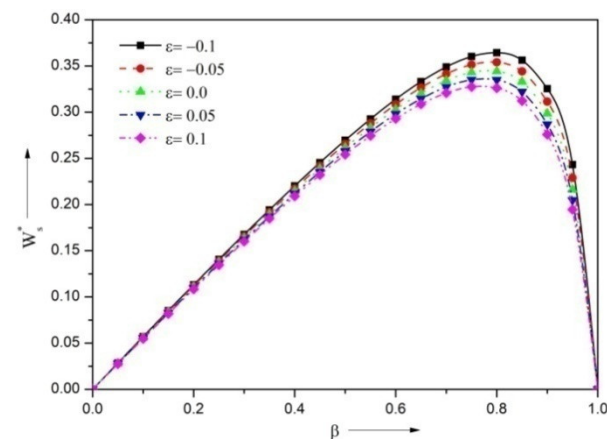


Figure 10. Steady load-carrying capacity W_s^* as a function of β for different ϵ under $\delta = 1$.

3.3. Dynamic Stiffness Coefficient

Figure 11 represents the varying values of non-dimensional DSC S_d^* with the riser location parameter β and ψ for distinct values of the couple stress parameter l^* with $\alpha = -0.05$, $\sigma = 0.1$, $\epsilon = -0.05$, $\zeta = 0.3$, $\delta = 1$. We observed higher DSC S_d^* for increasing values of l^* for negatively skewed roughness as compared with the smooth case. Compared with the Newtonian case, the effects of non-Newtonian couple stress fluid provide a higher bearing stiffness. The effect of permeability is to reduce the DSC as compared with the non-porous case. As the value of the permeability parameter increases, DSC decreases. We

also observed the presence of the critical riser location parameter β_c for the riser location parameter β at which the DSC S_d^* attains its maximum value. For $l^* = 0.2$, the critical value is $\beta_c = 0.85$. Figures 12–14 denote the variation of non-dimensional DSC S_d^* with riser location parameter β for different values of the roughness parameters α ($\sigma = 0.1$, $\varepsilon = -0.05$), σ ($\alpha = -0.05$, $\varepsilon = 0.1$) and ε ($\sigma = 0.1$, $\alpha = -0.05$), respectively. Figure 12 shows that DSC S_d^* increases for the negatively increasing value of α whereas it decreases for the positively increasing values α . From Figure 13, it is evident that S_d^* decreases for the increasing value of σ . Figure 14 shows that the negatively increasing value of ε increases DSC S_d^* whereas the positively increasing value of ε decreases DSC S_d^* .

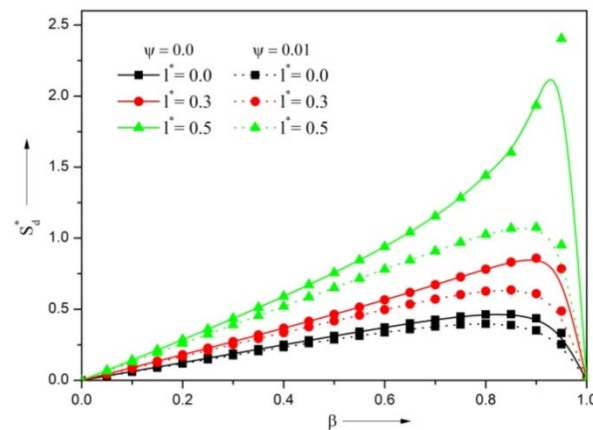


Figure 11. Dynamic stiffness coefficient S_d^* as a function of β for different l and ψ under $\delta = 1$.

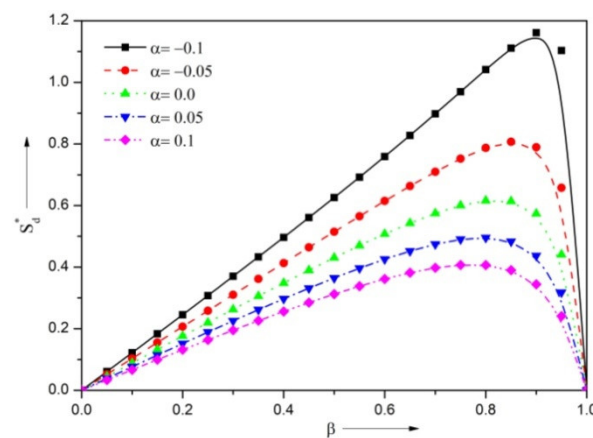


Figure 12. Dynamic stiffness coefficient S_d^* as a function of β for different α under $\delta = 1$.

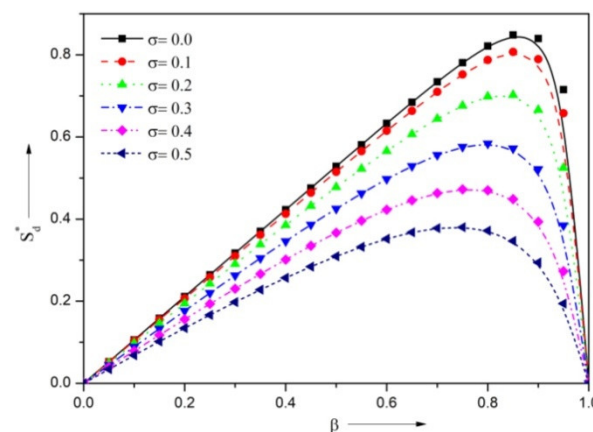


Figure 13. Dynamic stiffness coefficient S_d^* as a function of β for different σ under $\delta = 1$.

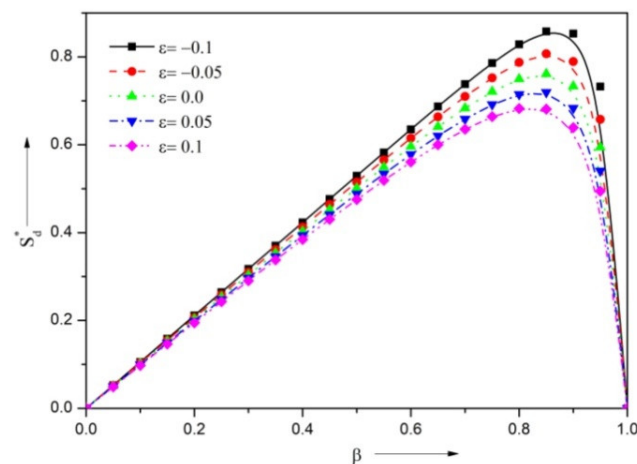


Figure 14. Dynamic stiffness coefficient S_d^* as a function of β for different ε under $\delta = 1$.

3.4. Dynamic Damping Coefficient

Figure 15 describes the variation in non-dimensional DDC C_d^* as a function of the riser location parameter β for different values of l^* and ψ under the shoulder parameter $\delta = 1$, roughness parameters $\alpha = -0.05, \sigma = 0.1, \varepsilon = -0.05, \zeta = 0.3$. The damping coefficient is found to decrease with the increasing value of the riser location parameter and increase with the increasing value of the couple stress parameter l^* . The RSB lubricated with couple stress fluids is observed to provide higher damping coefficients than Newtonian lubricants. As the permeability parameter decreases, DDC increases, and we also observed that DDC decreases with the increasing value of β . The variation of non-dimensional DDC C_d^* with β for different values of $\alpha(\sigma = 0.1, \varepsilon = -0.05)$, $\sigma(\alpha = -0.05, \varepsilon = 0.1)$ and $\varepsilon(\sigma = 0.1, \alpha = -0.05)$ are represented by Figures 16–18, respectively. Figure 16 shows that DDC decreases with the increasing value of the location parameter β , increases for the negatively increasing value of α , and decreases for the positively increasing value of α . From Figure 17, it is observed that C_d^* decreases for the increasing value of σ . The negatively skewed roughness ε decreases C_d^* whereas the positively skewed roughness ε decreases the DDC, as shown in Figure 18. Finally, on the whole, a small riser location parameter and a larger couple stress parameter improve the dynamic damping characteristics of the RSB.

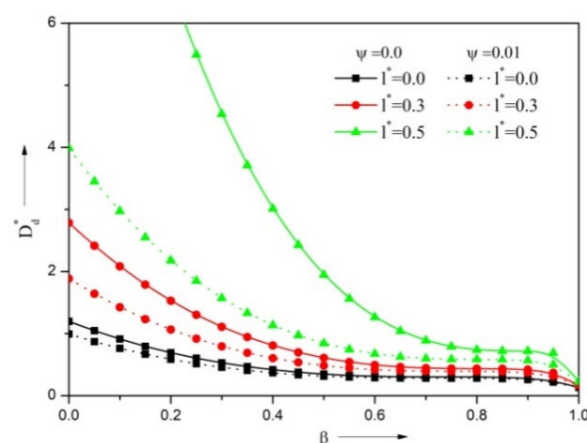


Figure 15. Dynamic damping coefficient D_d^* as a function of β for different l and ψ under $\delta = 1$.

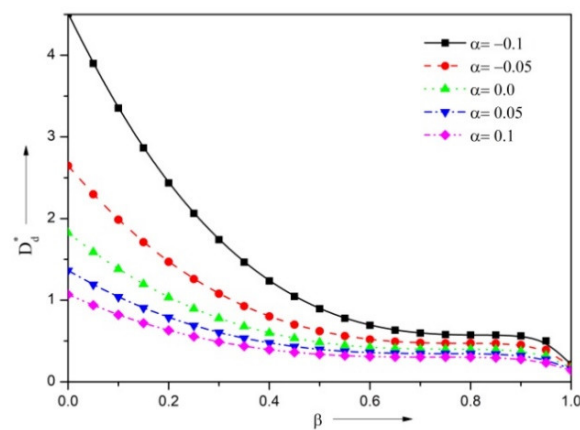


Figure 16. Dynamic damping coefficient D_d^* as a function of β for different α under $\delta = 1$.

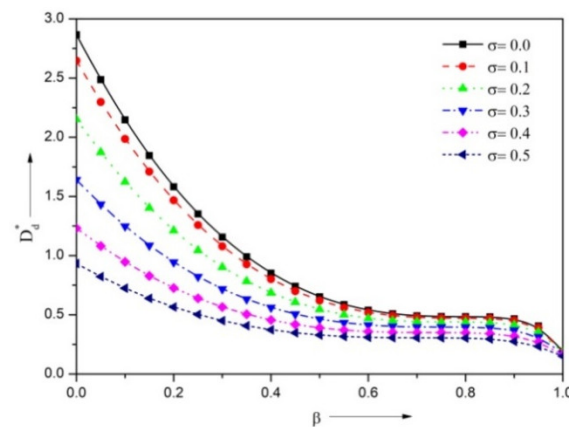


Figure 17. Dynamic damping coefficient D_d^* as a function of β for different σ under $\delta = 1$.

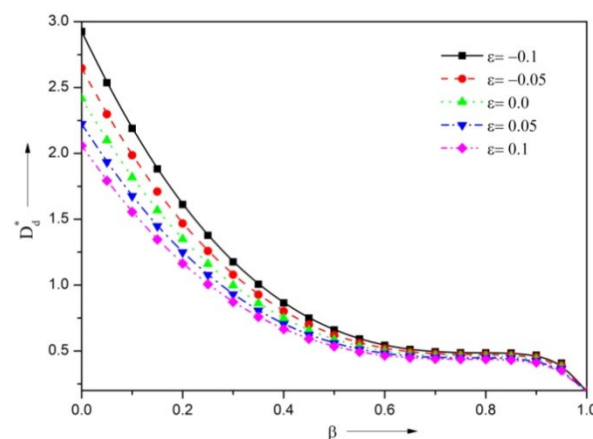


Figure 18. Dynamic damping coefficient D_d^* as a function of β for different ϵ under $\delta = 1$.

The relative percentage increase due to the presence of couple stress fluid and porous medium in static and dynamic performance of a bearing is given by $R = \frac{r_{\text{Newtonian}} - r_{\text{Non-Newtonian}}}{r_{\text{Non-Newtonian}}}$ and $R = \frac{r_{\text{porous}} - r_{\text{non-porous}}}{r_{\text{non-porous}}}$ where r represents VFR, LCC, DSC and DDC.

Table 1 represents the static and dynamic behavior of rough porous RSB and its comparison with the non-porous rough surface. The values for porous rough RSB are tabulated for different values of the shoulder parameter and are compared with the non-porous rough RSB. Notably, when the steady inlet-outlet film thickness difference becomes zero, the step bearing becomes a parallel plate, so the volume flow rate is minimum, the

static LCC and DSC are zero, and the DDC is maximum. As the shoulder parameter increases, the steady flow rate, LCC, and DSC increase until a certain point and then decrease as the value of δ increases.

Table 1. Static and dynamic characteristics of porous rough RSB and its comparison with non-porous rough surface.

$\beta = 0.75$	δ	$l^* = 0.0$				$l^* = 0.4$			
		Non-Porous Rough Surface		Porous Rough Surface		Non-Porous Rough Surface		Porous Rough Surface	
		NSSR	PSSR	NSSR	PSSR	NSSR	PSSR	NSSR	PSSR
Q_s^*	0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
	0.5	0.612961	0.623292	0.621158	0.628646	0.569021	0.610331	0.596764	0.620916
	1	0.626252	0.649364	0.642319	0.661148	0.554155	0.61458	0.588517	0.634438
	1.5	0.609108	0.636499	0.626454	0.650336	0.538447	0.593059	0.5666	0.613143
	2	0.588865	0.615334	0.604549	0.628555	0.528012	0.572678	0.549736	0.589958
	2.5	0.571852	0.595586	0.585248	0.607297	0.521175	0.557179	0.538049	0.57145
	3	0.558584	0.579297	0.569853	0.589395	0.516534	0.545788	0.529902	0.557526
W_s^*	0	0	0	0	0	0	0	0	0
	0.5	0.202712	0.149249	0.18042	0.136804	0.501619	0.260817	0.384162	0.253665
	1	0.226563	0.18081	0.21193	0.171367	0.393575	0.270861	0.351423	0.281003
	1.5	0.195797	0.165237	0.188306	0.15987	0.279421	0.219986	0.26441	0.231931
	2	0.15947	0.139615	0.155687	0.136707	0.203581	0.171807	0.197455	0.180856
	2.5	0.128941	0.11571	0.126945	0.114101	0.153891	0.135169	0.151057	0.141432
	3	0.10513	0.095991	0.104019	0.095064	0.120164	0.108241	0.118714	0.112535
S_d^*	0	0	0	0	0	0	0	0	0
	0.5	0.531458	0.341553	0.420996	0.286967	1.95344	0.682152	1.14572	0.507848
	1	0.44829	0.329102	0.392252	0.295624	0.943611	0.540175	0.752312	0.460935
	1.5	0.298172	0.239054	0.275793	0.223777	0.46872	0.335988	0.419848	0.307854
	2	0.194121	0.164385	0.185019	0.157608	0.261167	0.208737	0.245687	0.198222
	2.5	0.129902	0.113895	0.125912	0.110748	0.159793	0.135509	0.153962	0.131151
	3	0.0901384	0.080881	0.0882435	0.079326	0.104931	0.092197	0.102416	0.090203
D_d^*	0	1.19635	0.807021	0.992749	0.708942	4.84507	1.57596	2.64674	1.24076
	0.5	0.562794	0.421535	0.507631	0.39037	1.33879	0.721327	1.04471	0.632911
	1	0.302087	0.241763	0.28299	0.229856	0.552039	0.36137	0.472845	0.33376
	1.5	0.177866	0.148018	0.169361	0.142651	0.2988	0.202675	0.253901	0.190768
	2	0.114836	0.097212	0.109655	0.094137	0.19713	0.127897	0.159563	0.120346
	2.5	0.0805295	0.068291	0.0765075	0.066079	0.149001	0.089316	0.113463	0.083156
	3	0.0605741	0.050941	0.0570036	0.049083	0.123353	0.067683	0.0884973	0.062033

NSSR: Negatively Skewed Surface Roughness, **PSSR:** Positively Skewed Surface Roughness.

Table 2 provides a numerical example to help with the selection of RSB lubricated with non-Newtonian couple stress fluid. The values for various bearing parameters are obtained. For engineering applications, RSB can be designed using the values obtained for physical quantities.

Table 2. Design example of rough Rayleigh step bearing with couple stress fluid.

Physical Quantity	Symbol	Value of the Physical Quantity
Bearing length	L	1.0×10^{-1} m
Inlet film thickness	h_{in}	2.0×10^{-4} m
Steady outlet film thickness	h_{ms}	1.0×10^{-4} m
Length of the first part of the bearing	βL	0.072 m
Lubricant viscosity	μ	2.45×10^{-6} Pa·s

Table 2. Cont.

Physical Quantity	Symbol	Value of the Physical Quantity
Shoulder parameter	δ	0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0
Riser location parameter	β	0.72
Couple stress parameter	l^*	0, 0.1, 0.2, 0.3, 0.4, 0.5
Couple stress material constant	η	$(0, 2.45, 9.8, 22.05, 39.2, 61.25) \times 10^{-8} \text{ N}\cdot\text{s}$
Roughness parameters	α	−0.05
	σ	0.1
	ε	−0.05

4. Conclusions

We studied the dynamic performance of rough porous RSB on the basis of Darcy's law and Stokes micro-continuum theory. Based on the numerical computation of the results obtained and discussed, the following conclusions are drawn:

- The presence of microstructure additives in the lubricant enhances the LCC, DSC and DDC of the RSB and diminishes the VFR. There is a 7% decrease in VFR, and 14.12%, 16.06%, and 5.52% increase in LCC, DSC and DDC, respectively.
- The porous facing on the Rayleigh step slider bearing structure increases the VFR by 6.2% and decreases the LCC by 10.7%, DSC by 20.2% and DDC by 14.3%.
- The reduction in LCC caused by the porous facing can be compensated for by using lubricants that contain additives of the proper size. With this, bearing performance is enhanced.
- The presence of the negatively skewed surface roughness structure provides a reduction in the VFR, and higher steady LCC, DSC and DDC, whereas the positively skewed roughness enhances the VFR, and diminishes the load, stiffness, and damping coefficient.
- The presence of the surface roughness structure improves the LCC, DSC and DDC in comparison with the smooth surface case.

Compared with the non-porous smooth surface case of wide RSB by Lin et al. [26], a close agreement signifies support for the present study of the porous rough Rayleigh step bearing.

Author Contributions: Conceptualization, Supervision, Validation, Visualization, Writing—review & editing: N.N.; Formal analysis, Methodology, Visualization, Writing—original draft: A.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are thankful to the reviewers for their valuable comments on the earlier manuscripts, which were useful in enhancing the quality of this revised manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

Nomenclature

d	difference between the inlet-outlet film thicknesses, $d = h_{ins} - h_{ms}$
D	bearing width
D_d, D_d^*	DDC, $D_d^* = D_d h_{ms}^3 / \mu L^3 D$
F, F^*	dynamic film force, $F^*(h_m^*, V^*) = F h_{ms}^2 / \mu U L^2 D$
G_1, G_2	integration functions, $G_1 = G_1(h_m^*, V^*) G_2 = G_2(h_m^*, V^*)$
h, h^*	film thickness, $h^*(x^*, t^*) = h(x, t) / h_{ms}$
h_{in}	inlet film thickness, $h_{in}(t) = d + h_m(t)$
h_{ins}	steady inlet film thickness, $h_{ins} = d + h_{ms}$
h_m, h_m^*	outlet film thickness, $h_m^*(t^*) = h_m(t) / h_{ms}$

h_{ms}	steady outlet film thickness
l, l^*	couple stress parameter, $l^* = l/h_{ms}$
L	length of the bearing
p, p^*	dynamic pressure, $p^* = ph_{ms}^2/\mu UL$
Q, Q^*	VFR, $Q^* = Q/Uh_{ms}D$
Q_s, Q_s^*	steady VFR, $Q_s^* = Q_s/Uh_{ms}D$
S_d, S_d^*	DSC, $S_d^* = S_d h_{ms}^3/\mu UL^2D$
t, t^*, t, t^*	time, $t^* = Ut/L$
u, v	velocity components in the x and y directions
U	sliding velocity of the lower part
V^*	non-dimensional squeezing velocity, $V^* = dh_m^*/dt^*$
W_s, W_s^*	steady LCC, $W_s^* = W_s h_{ms}^2/\mu UL^2D$
x, y	Cartesian coordinates
x^*	non-dimensional coordinate, $x^* = x/L$
α	mean defined by $\alpha = E(h_r)$
β	riser location parameter
δ	shoulder parameter, $\delta = d/h_{ms}$
ε	skewness parameter defined by $\varepsilon = E[(h_r - \alpha)^3]$
η	couple stress fluid material constant
μ	lubricant viscosity
σ	standard deviation defined by $\sigma^2 = E[(h_r - \alpha)^2]$

Appendix A

The Associated Functions and Quantities

$$f_{A\beta} = f_A(\beta, h_m^*) \quad (A1)$$

$$f_{C\beta} = f_C(\beta, h_m^*) \quad (A2)$$

$$(f_{B\beta})_s = f_B(\beta, h_{ms}^*) \quad (A3)$$

$$(f_{D\beta})_s = f_D(\beta, h_{ms}^*) \quad (A4)$$

$$F_A(h_m^*) = \int_0^\beta f_A(x^*, h_m^*) dx^* \quad (A5)$$

$$F_B(h_m^*) = \int_0^\beta f_B(x^*, h_m^*) dx^* \quad (A6)$$

$$F_C(h_m^*) = \int_\beta^1 f_C(x^*, h_m^*) dx^* \quad (A7)$$

$$F_D(h_m^*) = \int_\beta^1 f_D(x^*, h_m^*) dx^* \quad (A8)$$

$$\frac{\partial f^*}{\partial h_m^*} = 3h^{*2} + 6h^*\alpha + 3(\alpha^2 + \sigma^2) - 12l^2 \tanh^2\left(\frac{h^*}{2l^*}\right) - \frac{1}{l^*} \tanh^2\left(\frac{h^*}{2l^*}\right) \left(1 - \tanh^2\left(\frac{h^*}{2l^*}\right)\right) (12l^2\alpha - \varepsilon - \alpha - 3\sigma^2\alpha) \quad (A9)$$

$$\frac{\partial f_{B\beta}}{\partial h_m^*} = - \int_0^\beta \frac{1}{f^{*2}} \cdot \frac{\partial f^*}{\partial h_m^*} dx^* \quad (A10)$$

$$\frac{\partial f_{D\beta}}{\partial h_m^*} = - \int_1^\beta \frac{1}{f^2} \cdot \frac{\partial f^*}{\partial h_m^*} dx^* \quad (\text{A11})$$

$$\frac{\partial F_B}{\partial h_m^*} = - \int_0^\beta \int_0^{x^*} \frac{1}{f^2} \cdot \frac{\partial f^*}{\partial h_m^*} dx^* dx^* \quad (\text{A12})$$

$$\frac{\partial F_D}{\partial h_m^*} = - \int_\beta^1 \int_1^{x^*} \frac{1}{f^2} \cdot \frac{\partial f^*}{\partial h_m^*} dx^* dx^* \quad (\text{A13})$$

$$G_{1s} = G_1(h_{m0}^*, 0) = \frac{-6\delta \cdot (f_{D\beta})_s}{(f_{B\beta})_s - (f_{D\beta})_s} \quad (\text{A14})$$

$$\frac{\partial G_1}{\partial h_m^*} = (-6\delta) \cdot \frac{f_{B\beta} \cdot (\partial f_{D\beta} / \partial h_m^*) - f_{D\beta} \cdot (\partial f_{B\beta} / \partial h_m^*)}{(f_{B\beta} - f_{D\beta})^2} \quad (\text{A15})$$

$$\frac{\partial G_1}{\partial V^*} = \frac{12 \cdot (f_{C\beta} - f_{A\beta})}{f_{B\beta} - f_{D\beta}} \quad (\text{A16})$$

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