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Improved Craig–Bampton Method Implemented into Durability Analysis of Flexible Multibody Systems

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Abstract: The Craig–Bampton method is frequently applied in most commercial multibody dynamic software. Nevertheless, the Craig–Bampton modes only represent the free-free modes in flexible multibody systems. However, the free-free modes are incapable of all engineering applications. Hence, a rational set of reference conditions must be correctly chosen to define a unique displacement field. Firstly, a simple 2D beam with two revolute joints is taken as an example to prove that the free-free modes are not suitable for all engineering applications, and the results are validated by ANSYS and the analytical solution. Secondly, the Craig–Bampton method is improved by two different methods: (i) the reference conditions are added to the original Craig–Bampton matrix and (ii) the reference conditions are applied to the shape functions to redefine the mass and stiffness matrices before constructing the original Craig–Bampton matrix. This implementation illustrates that the improved Craig–Bampton matrix can not only generate the free-free modes but is also suitable for the non-free-free modes. Finally, two discrepant reference conditions are imposed to obtain the dynamic response of the flexible connecting rod based on the improved Craig–Bampton method, which is validated using the normal mode approach. Simulations show that the improved Craig–Bampton method can be used as a general-purpose method in durability analysis.

Keywords: Craig–Bampton method; flexible multibody system; reference conditions; component mode synthesis

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1. Introduction

With recent developments in science and technology, the requirement for mechanical manufacturing efficiency and machining precision is continuously increasing in the industrial environment [1]. This requirement will force designers in different research fields to cooperate so as to integrate the different mechanical modules into complicated machines [2,3]. Therefore, the structure and dimensions of mechanical systems have become larger and more complicated [4]. In addition, the connection between mechanical modules with different functions has become extremely sophisticated as well [5]. Further, due to the developing trend indicating that mechanical systems will become more lightweight and high-speed in the future, the components in the mechanical system will be prone to deformation [6]. These aforementioned reasons are the main motivations to make the dynamics models of complicated systems highly nonlinear and to increase the complexity of the numerical simulation of the durability analysis [7–9].

To study the dynamic performance of flexible multibody systems, especially when small deformation occurs, the floating frame of reference (FFR) formulation [10] is frequently applied to the durability analysis by many industry sectors [11]. To obtain the exact displacement of the node on the flexible body, the flexible body is meshed using many finite elements. This leads to a significant increase in both the number of degrees of freedom of the system [12] and the difficulty of the system durability analysis [13]. This is mainly because when the number of nodes on the flexible body is increased, the elements in the mass matrix become smaller, and the elements in the stiffness matrix become larger [14]. This will lead to very high-frequency modes, which significantly restrict the selection of the time step during numerical simulation [15–17], and leads to a loss of computational efficiency and an increased computational cost [18].

The sub-structuring technique component mode synthesis (CMS) presented by Hurty [19] and Gradwell [20] improves the computational efficiency of a complex structure by reducing the number of elastic coordinates of the flexible components. This is due to the CMS transferring the physical coordinates to the modal coordinates. In the model space, the linear superposition of the eigenvectors corresponding to the low-frequency modes [21] can represent the elastic coordinates of the nodes on the flexible components. During this process, the low-frequency modes play an important role in the deformation of the node on the flexible body; in contrast, the high-frequency modes are negligible in the displacement of the flexible system [22]. Hence, only low-frequency modes are included to ensure the convergence of the displacement, while the high-frequency modes are removed from the modal space. Therefore, the computational efficiency of the flexible multibody system is conspicuously enhanced by the CMS [23].

A complex multibody system can be divided into several substructures, and the dynamics model of each substructure can be independently formulated and solved in parallel [24]. The eigenvectors of each substructure can be obtained by the finite element method, experimental data, and analytical solutions, and the computational efficiency after using a sub-structuring technique is higher than that of the original system [25]. The equations of motion of the system as a whole can be formulated by using the interface conditions of each substructure based on their low-frequency modes from each substructure [26] and the physical elastic coordinate of the node can be obtained by the inverse modal transformation when the modal dynamics model is solved. In general, since the equations of the motion of the system are formulated according to the constrained definition of the interface condition [27], the substructure CMS can be classified into four different types: free interface methods, fixed interface methods, the mixed interface method comprising free and fixed interface conditions [28], and the loaded interface method [1], in which the Craig–Bampton (CB) matrix [29] is employed into the CMS based on the fixed interface conditions and the original CB method [30] is improved to avoid the cumbersome process of identifying the degrees of freedom of the substructure interface. This method is suitable for programming when solving the dynamics model of complex multibody systems. Therefore, the CB matrix [31] is commonly applied in popular commercial software, such as Adams, RecurDyn, and Nastran.

In order to improve the simulation accuracy and computational efficiency of the CB method, and especially expand its application area, many scholars are focusing on this topic. Boo et al. [32] noted that the original CB method is incapable of simulating complicated structural systems with many degrees of freedom; thereby, they used an algebraic sub-structuring method and interface boundary reduction to improve computational accuracy and efficiency. The effect of this method was proven by several large structural finite element models. Gruber et al. [33] considered that the boundary degrees of freedom of the substructures would still remain when many substructures are assembled, and the number of the interface degrees of freedom is large; however, the structure of the system is still complicated, especially larger structures which include many substructures [34]. Hence, they provided a method to reduce the interface degrees of freedom and used numerical analysis to validate its efficacy. Junge et al. [30] studied the low-frequency vibroacoustic behavior of a fluid-structure coupled system using the CB method and the Rubin method with large coupling interfaces. Rixen [35] proposed a novel CMS method to study the dynamic analysis of a structure based on free interface vibration and residual flexibility components. The computational efficiency of this new method was validated by numerical examples in a three-dimensional frame. Kim and Lee [36] provided an accurate error estimator for the CB method; this error estimator could accurately predict the relative eigenvalue errors in the finite element models reduced by the CB method. Kuether and Allen [37] provided a new method for geometrically nonlinear finite element models based on the CB method; this reduced model can be expressed as the cubic and quadratic polynomials of the modal coordinates. Kim et al. [38] introduced a new CMS that is the enhanced CB method. This new method considered the residual sub-structural modes neglected in the original CB method and can be used to obtain a much more precise construction. Kim and Lee [34] proposed a new CMS method to enhance the simulation accuracy of the CB method, including the effect of residual sub-structural modes, which is proven by several numerical examples. Carassale and Maurici [39] indicated two drawbacks of the CB method when the number of the interface degree of freedom is large. First, the reduction step of the modes may become extremely time-consuming. Additionally, after reducing high-frequency modes, the model may still be complicated because the number of original substructures of the system is large. Therefore, a simple interface-reduction technique based on a preferential choice of interface modes was provided by a set of orthogonal basis functions [40]. In order to accurately predict the maximum dynamic response of the satellites in the case of a coupled dynamic load, Lim et al. [26] increased the number of residual modes to reduce the predicted error from the dynamic response, increasing the accuracy of the simulation compared to the original CB method. Kim et al. [6] improved the precision of the original CB method by considering the higher-order effect of the residual substructure modes through residual flexibility. The accuracy of this new method was proven by numerical examples. Fang et al. [41] proposed an adaptive numerical scheme based on the CB method to predict the dynamic response of tall buildings. Because the distribution of nonlinear components is initially unknown due to the randomness of earthquake inputs, the degrees of freedom of the linear substructures in the dynamics model were reduced using the CB method. This numerical method was validated by the numerical solution of a 20-story building.

However, the aforementioned studies all ignored an important concept, which is the reference conditions. In the literature [13], Shabana and Wang emphasized the importance of the reference conditions and demonstrated the difference between boundary conditions and reference conditions. The reference conditions are applied to remove the rigid body modes in the shape functions of the flexible component. The different basis deformation vectors can be obtained when the different reference conditions are added to the shape functions. However, the reference conditions do not impose any constraint on the multi-body system structure and do not change the system topology. Conversely, the boundary conditions are independent concepts regarding the reference conditions; they not only eliminate degrees of freedom of the system to define the topology but also change the frequency information of the flexible components. In general, the free-free modes are obtained under the reference conditions (mean axis conditions). One of the most important methods of the calculated free-free modes is the CB method [19,31]. However, the free-free modes are incapable of all engineering applications [13]. The objective of this paper, therefore, is to expand the application area of the CB method and to make this method usable for any engineering application by adding the reference conditions to the original CB matrix [42].

The contribution and novelty of this paper can be summarized as follows: (i) the original Craig–Bampton method may result in the wrong solution, which is validated by a simple example; (ii) the Craig–Bampton method is improved by imposing the reference conditions on the original Craig–Bampton modal transformation matrix or the shape functions; (iii) the correctness of the improved Craig–Bampton method is proved by statics and dynamics models; and (iv) the improved CB method is shown to serve as a general-purpose method suitable for all applications rather than restricted only to free-free modes.

This paper consists of the following sections: Section 1 describes the developed history and research background of the CMS; the small deformation theory of flexible bodies is simply introduced based on FFR formulation in Section 2; the CB modal transformation matrix is derived in Section 3; the limitation of the free-free modes is proven by a simple two dimension beam in Section 4; the CB method is improved by two different methods in Section 5; and Section 6 presents the conclusions of this investigation.

2. Small Deformation Theory of Floating Frame of Reference Formulation

The FFR formulation [10] based on the finite element method used to study the small deformation of flexible multibody systems has a long history [43] and results in a highly nonlinear inertia matrix that exhibits a strong inertial coupling between elastic deformation and reference motion. The elastic deformation of the multibody system can be described as a unique set of inertia-shape integrals that depend on the assumed displacement field [44]. However, the early FFR formulation was only suited for describing the dynamics of beams and plates with simple shapes, not for the flexible body with discontinuous structures. The main reason is that an infinitesimal rotation serves as one of the nodal coordinates of the FFR elements, which restricts the dynamics model of systems with discontinuous structures. In 1997, Shabana [44] introduced the intermediate coordinate system into the FFR to effectively describe the deformation of the flexible multibody system with discontinuous and complicated structures. Moreover, Shabana [13] first proposed the reference conditions, which are distinct from the boundary conditions. The reference condition is necessary in order to eliminate the rigid body modes in the shape functions, in order to eliminate the redundant nodal coordinates in the element and define a unique displacement field. Considering the example of a simple straight two-dimensional beam without discontinuous structure, the transformation matrix between the intermediate coordinate system and body coordinate system is the identity matrix. Thereby, the intermediate coordinate system is not introduced into this system model. For more information, the interested reader can refer to the literature [45].

2.1. Position Equation of an Arbitrary Point on the Flexible Body

In Figure 1, the position vector $\mathbf{r}_p^i$ of the arbitrary point p of the element i in the global coordinate system π_g can be expressed as

$$\mathbf{r}_p^i = \mathbf{R}_0 + \mathbf{A}\overline{\mathbf{u}}_p^i \ (i = 1, 2, \ldots n) \tag{1}$$

where $\mathbf{R}_0$ is the position vector of the origin of the body coordinate system π_l in the global coordinate system π_g, $\mathbf{A}$ is the transformation matrix between the body coordinate system and the global coordinate system, in which the body coordinate system is the reference frame of all elements and the deformation of the flexible body is defined in this coordinate system. $\overline{\mathbf{u}}_p^i$ is the position vector of the arbitrary point p on the element in the body coordinate system π_l and n is the total number of the elements in the flexible body.

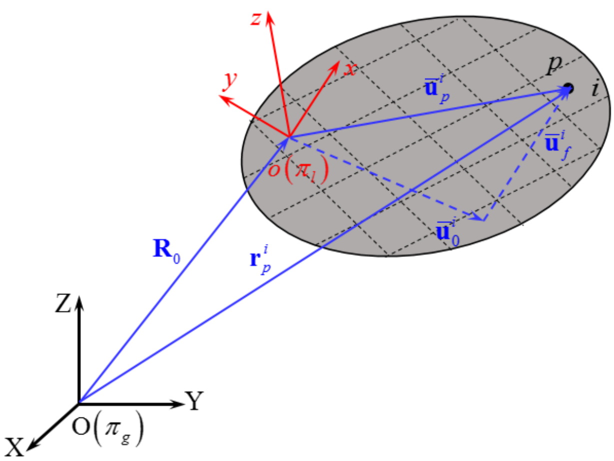

Figure 1. Vector description of floating frame of reference.

In the deformed case, the coordinate vector of the node on the element of coordinate i can be written as

$$\mathbf{e}^i = \mathbf{e}_0^i + \mathbf{e}_f^i \tag{2}$$

in which $\mathbf{e}_0^i$ is the coordinate vector of the node of element i in the un-deformed case and $\mathbf{e}_f^i$ is the deformation coordinate vector of the node of element i in the deformed case.

The displacement vector of an arbitrary point on the element i can be calculated based on the nodal coordinate vector of the flexible body

$$\overline{\mathbf{u}}^i = \mathbf{S}^i \mathbf{e}^i \tag{3}$$

where $\mathbf{S}^i$ is the shape function matrix of element i.

Hence, the nodal coordinate vector of each element can be described using a Boolean matrix and can be written as

$$\mathbf{e}^i = \mathbf{B}_b^i \mathbf{e}_b \tag{4}$$

in which $\mathbf{B}_b^i$ is the Boolean matrix and $\mathbf{e}_b$ is the nodal coordinate vector of the flexible body.

Likewise, the nodal coordinate vector of the flexible body can be rewritten as

$$\mathbf{e}_b = \mathbf{e}_{b0} + \mathbf{e}_{bf} \tag{5}$$

where $\mathbf{e}_{b0}$ is the nodal coordinate vector in the un-deformed state of the flexible body and $\mathbf{e}_{bf}$ is the nodal coordinate vector in the deformed state.

In order to eliminate the rigid body modes in the shape function matrix and eliminate the redundant coordinates of the nodal deformation coordinate vector, the reference conditions are imposed on the displacement field of the flexible body to define a unique displacement field. Thus, a new nodal coordinate vector $\mathbf{e}_f$ of the flexible body can be defined as

$$\mathbf{e}_{bf} = \mathbf{B}_r \mathbf{e}_f \tag{6}$$

where $\mathbf{B}_r$ is a linear transformation that arises from imposing the reference conditions [45].

Thereby, the position vector of the arbitrary point p on the element i of the flexible body in the body coordinate system π_l can be expressed as

$$\overline{\mathbf{u}}_p^i = \mathbf{S}^i \mathbf{B}_b^i \left(\mathbf{e}_{b0} + \mathbf{B}_r \mathbf{e}_f \right) = \overline{\mathbf{u}}_0^i + \overline{\mathbf{u}}_f^i \tag{7}$$

Substituting Equation (7) into Equation (1), the position vector of an arbitrary point p on element i of the flexible body in the global coordinate system π_g can be expressed as

$$\mathbf{r}_p^i = \mathbf{R}^i + \mathbf{A} \left(\overline{\mathbf{u}}_0^i + \overline{\mathbf{u}}_f^i \right) \tag{8}$$

2.2. Equation of Motion of the Flexible Multibody System

According to the position equation of an arbitrary point on the flexible body in the global coordinate system, the equation of motion of the flexible multibody system can be obtained by the virtual work theory in order to clearly describe the coupling relationship between rigid body coordinates and flexible body coordinates. The generalized coordinates of the system are partitioned as reference coordinates and elastic coordinates. Hence, the equation of motion of the flexible multibody system can be written as [45]

$$\begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{rf} \\ \mathbf{M}_{fr} & \mathbf{M}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\mathbf{q}}_f \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{ff} \end{bmatrix} \begin{bmatrix} \mathbf{q}_r \\ \mathbf{q}_f \end{bmatrix} = \begin{bmatrix} (\mathbf{Q}_e)_r + (\mathbf{Q}_v)_r \\ (\mathbf{Q}_e)_f + (\mathbf{Q}_v)_f \end{bmatrix} \tag{9}$$

where $\mathbf{M}_{rr}$ is the mass matrix corresponding to the reference motion, $\mathbf{M}_{rf} = \mathbf{M}_{fr}^{\mathbf{T}}$ is the mass matrix representing the coupling between the reference motion and elastic deformation, $\mathbf{M}_{ff}$ is the mass matrix of a flexible body, $\mathbf{q}_r$ is the reference coordinates, $\mathbf{q}_f$ is the elastic coordinates, $\mathbf{K}_{ff}$ is the stiffness matrix of the flexible body, $\mathbf{Q}_e$ is the generalized external forces, and $\mathbf{Q}_v$ is the quadratic velocity including the centrifugal force and Coriolis force.

3. Orthogonal Craig–Bampton Modal Transformation Matrix

In general, since the displacement of a considerable number of finite elements in the system needs to be calculated during durability analysis in engineering, there is a significant computational cost [46]. Hence, the CMS is used to reduce the elastic coordinates in the equation of motion in which the reduced model is constructed by synthesizing the dominating substructural modes, which constitute a very small portion of the total substructural modes and constraint modes [47]. The reduced model significantly reduces the computational cost of the flexible multibody system. In 1968, Craig and Bampton proposed the well-known Craig–Bampton modal transformation matrix [31] which is applied to durability analysis in practical engineering.

3.1. Component Mode Synthesis (Normal Mode Approach)

Based on Equation (9), the free vibration equation of the flexible body in the body coordinate system can be written as

$$\mathbf{M}_{ff}\ddot{\mathbf{q}}_f + \mathbf{K}_{ff}\mathbf{q}_f = \mathbf{0} \tag{10}$$

where $\mathbf{M}_{ff}$ is the mass matrix corresponding to the flexible body.

The eigenvalue problem of this equation can be defined as

$$\left[\mathbf{K}_{ff} - \omega_j^2\mathbf{M}_{ff}\right]\mathbf{A}_j = \mathbf{0}\ (j\ =\ 1,2,\ldots m) \tag{11}$$

The natural frequencies ω_j and mode shapes $\mathbf{A}_j$ can be obtained by solving Equation (11), where m is the number of modes.

The high-frequency modes, which have a negligible effect on the nodal displacement, are truncated, and therefore only a few low-frequency modes are preserved so that the degrees of freedom of the system are reduced. The remaining low-frequency modes comprise the modal transformation matrix and can be expressed as

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \ldots & \mathbf{A}_j \end{bmatrix} \tag{12}$$

By this transformation matrix $\mathbf{\Phi}$, the elastic coordinates $\mathbf{q}_f$ can be transformed into the modal coordinates $\mathbf{p}$, which can be written as

$$\mathbf{q}_f = \mathbf{\Phi}\mathbf{p} \tag{13}$$

Based on Equation (13), Equation (9) can be rewritten as

$$\begin{bmatrix} \mathbf{M}_{rr} & \bar{\mathbf{M}}_{rf} \\ \bar{\mathbf{M}}_{fr} & \bar{\mathbf{M}}_{ff} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\mathbf{p}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{K}}_{ff} \end{bmatrix}\begin{bmatrix} \mathbf{q}_r \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} (\mathbf{Q}_e)_r + (\mathbf{Q}_v)_r \\ \left(\bar{\mathbf{Q}}_e\right)_f + \left(\bar{\mathbf{Q}}_v\right)_f \end{bmatrix} \tag{14}$$

in which

$$\begin{cases} \bar{\mathbf{M}}_{rf} = \bar{\mathbf{M}}_{fr}^{\mathrm{T}} = \mathbf{M}_{rf}\mathbf{\Phi}, \bar{\mathbf{M}}_{ff} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{M}_{ff}\mathbf{\Phi},\ \overline{\mathbf{K}}_{ff} = \mathbf{\Phi}^{\mathrm{T}}\overline{\mathbf{K}}_{ff}\mathbf{\Phi} \\ \left(\bar{\mathbf{Q}}_e\right)_f = \mathbf{\Phi}^{\mathrm{T}}(\mathbf{Q}_e)_f, \left(\bar{\mathbf{Q}}_v\right)_f = \mathbf{\Phi}^{\mathrm{T}}(\mathbf{Q}_v)_f \end{cases} \tag{15}$$

The process from Equation (9) to Equation (15) describes how to transform the elastic coordinates in the equation of motion to modal coordinates using the modal transformation matrix. The dynamic response of the flexible body is approximated by a few low-frequency modes, which significantly enhances computational efficiency and saves costs. This process is called the normal mode approach [13].

When a sufficient number of low-frequency modes remain, the normal mode approach can be used for all applications, but the premise is that a set of appropriate reference conditions are applied to the shape function matrix to define a unique displacement field and eliminate the rigid body modes. The reference conditions must be used in the durability analysis of the system. The reference conditions matrix $\mathbf{B}_r$ in Equation (6) can be used to obtain the different types of modes that are suited for all applications.

3.2. Craig–Bampton Modal Transformation Matrix

The CB matrix with orthogonal modes can be obtained by solving the eigenvalue problem twice [36]. The first time, the eigenvalue is obtained for the normal modes in the original CB matrix, and the second time, the eigenvalue analysis is solved for the orthogonal modes in the CB matrix because the static correction modes in the first eigenvalue problem solution affect the orthogonal characteristic of the original CB matrix. Therefore, the CB matrix with orthogonal modes can be used to decouple the equation of the motion of the system, which is the generalized form used in most commercial software [48].

In order to obtain the static correction modes and fixed interface modes in the original CB matrix, the elastic coordinates $\mathbf{q}_f$ in Equation (10) are rewritten as boundary coordinates $\left(\mathbf{q}_f\right)_b$ and internal coordinates $\left(\mathbf{q}_f\right)_i$ using the matrix partition technique as

$$\mathbf{q}_f = \left[\left(\mathbf{q}_f\right)_b^{\mathrm{T}} \quad \left(\mathbf{q}_f\right)_i^{\mathrm{T}} \right]^{\mathrm{T}} \tag{16}$$

Therefore, Equation (10) can be rewritten based on Equation (16) as

$$\begin{bmatrix} \left(\mathbf{M}_{ff}\right)_{bb} & \left(\mathbf{M}_{ff}\right)_{bi} \\ \left(\mathbf{M}_{ff}\right)_{ib} & \left(\mathbf{M}_{ff}\right)_{ii} \end{bmatrix} \begin{bmatrix} \left(\ddot{\mathbf{q}}_f\right)_b \\ \left(\ddot{\mathbf{q}}_f\right)_i \end{bmatrix} + \begin{bmatrix} \left(\mathbf{K}_{ff}\right)_{bb} & \left(\mathbf{K}_{ff}\right)_{bi} \\ \left(\mathbf{K}_{ff}\right)_{ib} & \left(\mathbf{K}_{ff}\right)_{ii} \end{bmatrix} \begin{bmatrix} \left(\mathbf{q}_f\right)_b \\ \left(\mathbf{q}_f\right)_i \end{bmatrix} = \mathbf{0} \tag{17}$$

3.2.1. Static Correction Modes

The static correction modes can be obtained when the inertial force in the second equation of Equation (17) is neglected

$$\left(\mathbf{K}_{ff}\right)_{ib}\left(\mathbf{q}_f\right)_b + \left(\mathbf{K}_{ff}\right)_{ii}\left(\mathbf{q}_f\right)_i = \mathbf{0} \tag{18}$$

The internal coordinates $\left(\mathbf{q}_f\right)_i$ can be expressed using boundary coordinates $\left(\mathbf{q}_f\right)_b$ when the matrix $\left(\mathbf{K}_{ff}\right)_{ii}$ is a non-singularity matrix.

$$\left(\mathbf{q}_f\right)_i = -\left(\mathbf{K}_{ff}\right)_{ii}^{-1}\left(\mathbf{K}_{ff}\right)_{ib}\left(\mathbf{q}_f\right)_b \tag{19}$$

in which the static correction modes can be obtained as

$$\mathbf{\Phi}_c = -\left(\mathbf{K}_{ff}\right)_{ii}^{-1}\left(\mathbf{K}_{ff}\right)_{ib} \tag{20}$$

3.2.2. Fixed Interface Modes

To solve the fixed interface modes, the second equation in Equation (17) can be expressed as the following by considering the free vibration of the internal nodes with respect to the boundary nodes,

$$\left(\mathbf{M}_{ff}\right)_{ii}\left(\ddot{\mathbf{q}}_f\right)_i + \left(\mathbf{K}_{ff}\right)_{ii}\left(\mathbf{q}_f\right)_i = \mathbf{0} \tag{21}$$

The eigenvalue is solved first to estimate the fixed interface modes

$$\left[\left(\mathbf{K}_{ff}\right)_{ii} - \omega_k^2\left(\mathbf{M}_{ff}\right)_{ii}\right]\mathbf{A}_k^{\mathrm{I}} = \mathbf{0} \tag{22}$$

where ω_k is the natural frequency, $\mathbf{A}_k^{\mathrm{I}}$ is the mode shape (k = 1,2, … l), l is the total number of modes, and I depicts the first eigenvalue.

After removing the high-frequency modes, the fixed interface modes are expressed as

$$\mathbf{\Phi}_f = \begin{bmatrix} \mathbf{A}_1^{\mathrm{I}} & \mathbf{A}_2^{\mathrm{I}} & \ldots & \mathbf{A}_k^{\mathrm{I}} \end{bmatrix} \tag{23}$$

3.2.3. Original CB Modal Transformation Matrix

Once the static correction modes and fixed interface modes are obtained, the original non-orthogonal CB matrix can be assembled as

$$\mathbf{\Phi}_{CB} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{\Phi}_c & \mathbf{\Phi}_f \end{bmatrix} \tag{24}$$

The physical elastic coordinates of the flexible body can be written in terms of modal coordinates using Equation (24) as

$$\mathbf{q}_f = \mathbf{\Phi}_{CB}\mathbf{p}_{\mathrm{I}} \tag{25}$$

in which $\mathbf{p}_{\mathrm{I}}$ are the modal coordinates corresponding to the original CB matrix.

3.2.4. Orthogonal CB Modal Transformation Matrix

The static correction modes are not really normal modes due to solving the eigenvalue of the flexible system, which results in the modal mass and stiffness matrices no longer being diagonal matrices, and the decoupled nature of the original CB matrix is absent. Hence, the second eigenvalue must be found to form the orthogonal CB matrix. Substituting Equation (25) into Equation (10), one can write

$$\left(\mathbf{\Phi}_{CB}^{\mathrm{T}}\mathbf{M}_{ff}\mathbf{\Phi}_{CB}\right)\ddot{\mathbf{p}}_{\mathrm{I}} + \left(\mathbf{\Phi}_{CB}^{\mathrm{T}}\mathbf{K}_{ff}\mathbf{\Phi}_{CB}\right)\mathbf{p}_{\mathrm{I}} = \mathbf{0} \tag{26}$$

in which the modal mass matrix is $\hat{\mathbf{M}}_{ff} = \mathbf{\Phi}_{CB}^{\mathrm{T}}\mathbf{M}_{ff}\mathbf{\Phi}_{CB}$, and the modal stiffness matrix is $\hat{\mathbf{K}}_{ff} = \mathbf{\Phi}_{CB}^{\mathrm{T}}\mathbf{K}_{ff}\mathbf{\Phi}_{CB}$.

By the modal mass and stiffness matrices, the second eigenvalue solution can be written as

$$\left[\hat{\mathbf{K}}_{ff} - \omega_h^2\hat{\mathbf{M}}_{ff}\right]\mathbf{A}_h^{\mathrm{II}} = \mathbf{0} \tag{27}$$

in which, ω_h is the natural frequency, $\mathbf{A}_h^{\mathrm{II}}$ is the mode shape (k = 1,2, … h), h is the total number of modes, and II is the second time eigenvalue problem.

Likewise, the modal transformation matrix consists of the low-frequency modes after truncating the high-frequency modes and can be expressed as

$$\mathbf{N} = \begin{bmatrix} \mathbf{A}_1^{\mathrm{II}} & \mathbf{A}_2^{\mathrm{II}} & \ldots & \mathbf{A}_h^{\mathrm{II}} \end{bmatrix} \tag{28}$$

The relationship between the first and second eigenvalue problems is expressed as

$$\mathbf{p}_{\mathrm{I}} = \mathbf{N}\mathbf{p}_{\mathrm{II}} \tag{29}$$

where $\mathbf{p}_{\mathrm{II}}$ can be obtained by solving the second eigenvalue problem.

Substituting Equation (29) into Equation (25), one can write

$$\mathbf{q}_f = \mathbf{\Phi}_{CB}\mathbf{N}\mathbf{p}_{\text{II}} \tag{30}$$

Consequently, the orthogonalized CB matrix is given by

$$\overline{\mathbf{\Phi}}_{CB} = \mathbf{\Phi}_{CB}\mathbf{N} \tag{31}$$

This modal transformation matrix is the first time improved CB method.

4. Applied Limitation of the Free-Free Modes

Based on the literature [13], there is no set of reference conditions that are suited for all applications. This section shows the limitation of the free-free modes again by considering a simple planar beam and implements a comparison between the CB method, the normal mode approach, and ANSYS. The normal mode approach is a general method, mainly because it can be incorporated with any reference conditions to model all applications. However, the CB method without any reference conditions is only used to derive the free-free modes, which will result in the wrong solution for the special case in Figure 2b.

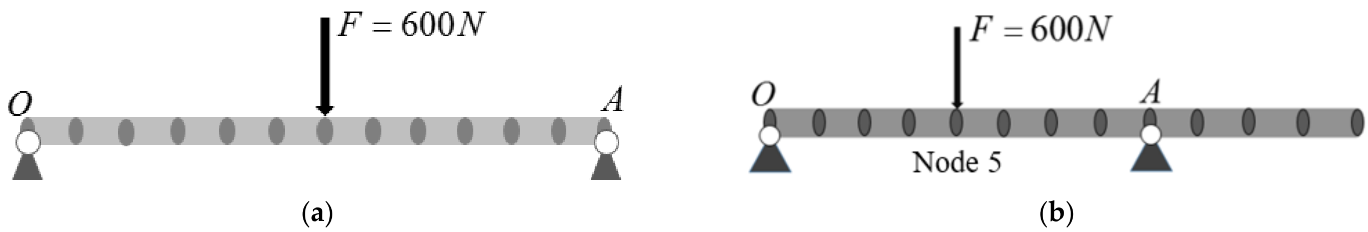

Figure 2. Planar beam. (**a**) simply-support beam; (**b**) extended beam.

As shown in Figure 2, beams with two different structures are considered; Figure 2a shows a planar beam with two revolute joints at the end nodes and Figure 2b shows an extended beam with two revolute joints, in which one revolute joint is at the left end of the beam (point O), and the other revolute joint is at the ninth node (point A). The reference conditions $u_0 = v_0 = v_A = 0$ (where u is the axial deformation and v is the transverse deformation) should be imposed on the constraint joints of the beam. Thus, the simply-supported modes are selected by imposing the corresponding reference conditions. The parameters of the planar beam in Figure 2 are listed in Table 1. However, in the CB method, there are not any reference conditions or boundary conditions to impose, which leads to free-free modes that include the rigid body modes, as shown in Tables 2 and 3.

Table 1. Structural parameters.

Parameters	Value
Length	4.572×10^{-1} m
Mass	1.135×10^{-1} kg
Density	7.840×10^{3} kg/m^3
Young's modulus	2.068×10^{11} Pa
Cross-section	3.167×10^{-5} m^2
Second moment of the area	7.981×10^{-11} m^4
Radius of cross-section	3.175×10^{-3} m
Modal damping coefficient	3.000×10^{-2}
Number of elements	12

Table 2. Comparative analysis of the frequencies from different methods (Hz).

CB Method	Normal Mode Approach		ANSYS	
Free-Free	Free-Free	Simply-Supported	Free-Free	Simply-Supported
0	0	61.28	0	61.27
0	0	245.11	0	245.05
0	0	551.62	0	551.32
138.91	138.91	981.19	138.86	980.25
382.94	382.94	1534.85	382.69	1532.60
750.97	750.97	2214.60	750.12	2209.90

Table 3. Comparative analysis of the frequencies from different methods (Hz).

CB Method	Normal Mode Approach		ANSYS	
Free-Free	Free-Free	Simply-Supported	Free-Free	Simply-Supported
0	0	88.64	0	88.62
0	0	209.40	0	209.36
0	0	606.26	0	605.88
138.91	138.91	1014.15	138.86	1012.70
382.94	382.94	1403.73	382.69	1401.80
750.97	750.97	2318.76	750.12	2313.40

The frequencies of the first six modes of the beam in Figure 2a are listed in Table 2, and a comparative analysis of the CB method, the normal mode approach, and ANSYS is implemented. There are two different cases, including free-free modes and simply-supported modes, in which the frequencies based on the normal mode approach can be verified by ANSYS. The frequencies of the free-free modes from the CB method are validated by the normal mode approach and ANSYS. This conclusion can then be used to validate the normal mode approach, which is a general-purpose method when imposing the appropriate reference conditions based on the engineering structure. However, the CB method can only be used to obtain the free-free modes in Table 2.

Similarly, in Table 3, the frequencies of the first six modes of the beam in Figure 2b, corresponding to the free-free modes and simply-supported modes based on the normal mode approach, are validated by ANSYS. The free-free modes from the CB method are validated by ANSYS and the normal mode approach as well in the following content. Moreover, the frequencies of the free-free modes of the beam in Figure 2a are the same as the frequencies of the beam in Figure 2b, because the free-free modes are not affected by constraints from joints. Hence, only if the physical parameters of the system are the same, the frequencies of the free-free modes are the same, despite the fact that the constraints are different, which is the physical meaning of the free-free modes. This conclusion has proven again that the normal mode approach is the general-purpose method, and the CB method is only suited for free-free modes in Table 3.

Since the frequencies of the free-free modes based on the normal mode approach are the same compared to the frequencies obtained using the CB method, the frequencies of the simply-supported modes based on the normal mode approach are identical to the frequencies obtained from ANSYS. For brevity, the static analysis of the beam in Figure 2 is implemented using the CB method and ANSYS. The displacement of the middle node of the beam in Figure 2a using the CB method is the same as that obtained using ANSYS, which can be seen in Figure 3. The convergence value of both cases is -7.24×10^{-2} m. The final deformed shape of the beam in Figure 2a under the case of constant load can be seen in Figure 4. The deformed shape resulting from both methods is the same, which accounts for the fact that the free-free modes derived from the CB method can be used to obtain the exact displacement for the example of the beam in Figure 2a, and this result is verified by

ANSYS, demonstrating that the free-free modes can be used in this simple beam example. However, when considering the extended beam shown in Figure 2b, the displacement of node 5 calculated using the CB method is very different from the ANSYS result, as shown in Figure 5. The convergence value from the CB method corresponding to the free-free modes is -7.12×10^{-3} m, but the convergence value from ANSYS corresponding to simply-supported modes is -2.14×10^{-2} m. The displacement of the free end can be seen in Figure 6 where the convergence value from the CB method is -7.17×10^{-3} m, and the convergence value from ANSYS is 3.22×10^{-2} m. This conclusion proves that the free-free modes from the CB method are not suited for the extended beam. Figure 7 shows that the final deformed shape from the CB method does not match the result from ANSYS. The maximum relative error between the CB method and ANSYS is 122.3%, which is not acceptable. This is due to the fact that the free-free modes based on the CB method do not impose the reference conditions to eliminate the rigid body modes from the shape function in order to define the unique displacement field. This conclusion has also been shown analytically in the literature [13].

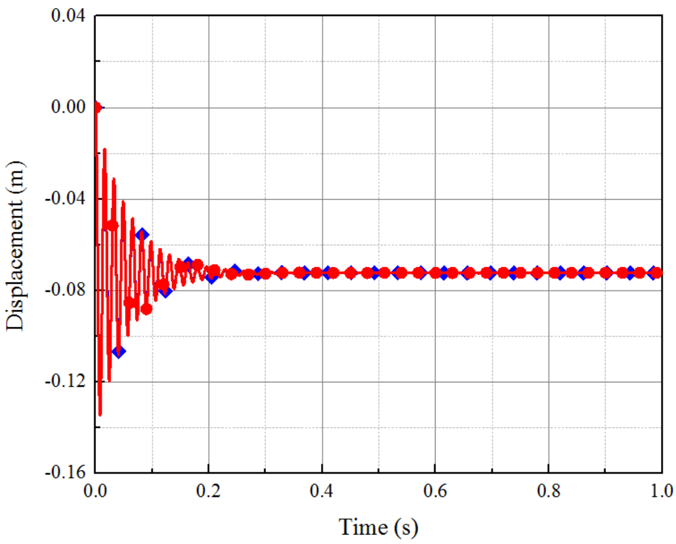

Figure 3. Transverse displacement of the middle node. (CB method (free-free modes) and ANSYS (simply-supported modes)).

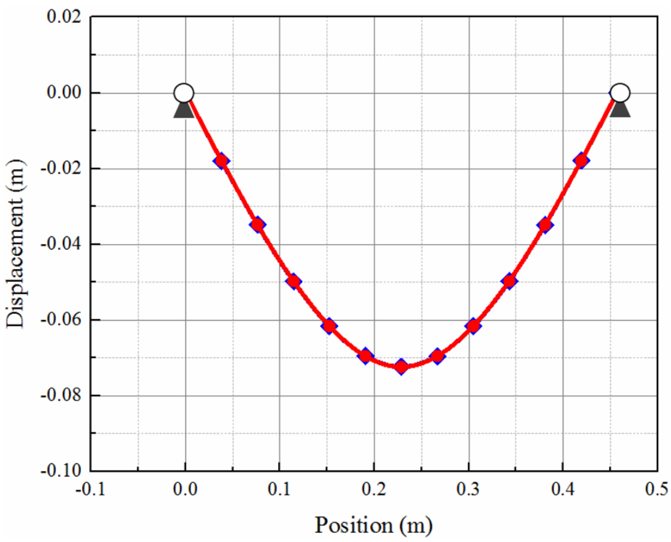

Figure 4. Deformed shape of the beam. (CB method (free-free modes) and ANSYS (simply-supported modes)).

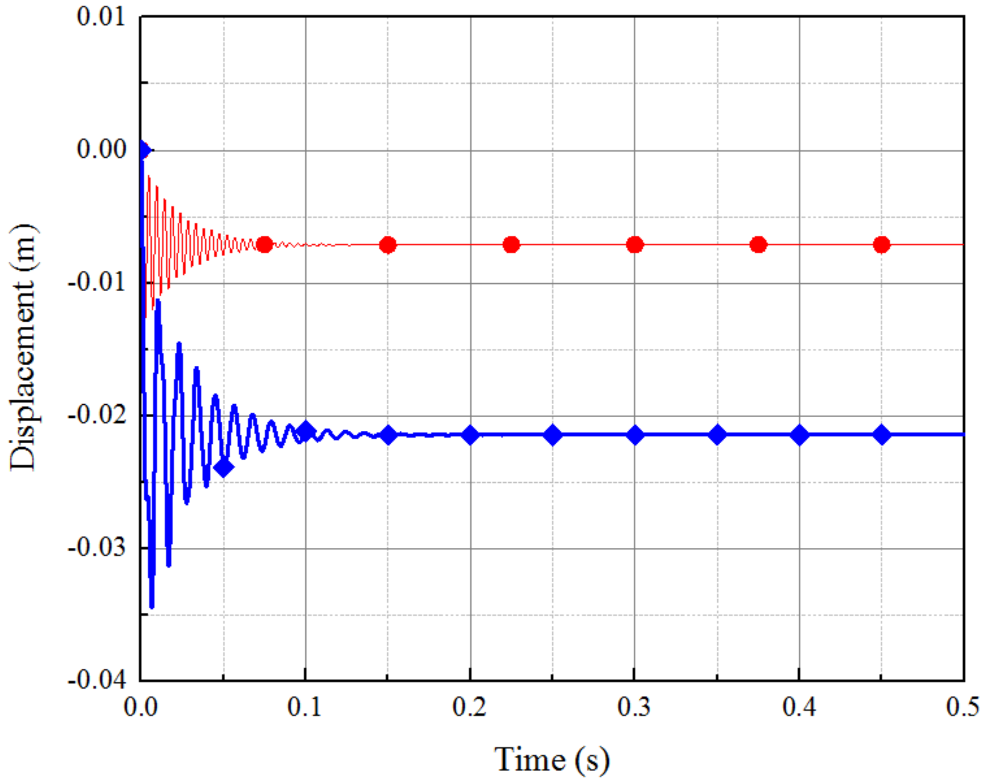

Figure 5. Transverse displacement of node 5. (CB method (free-free modes) and ANSYS (simply-supported modes)).

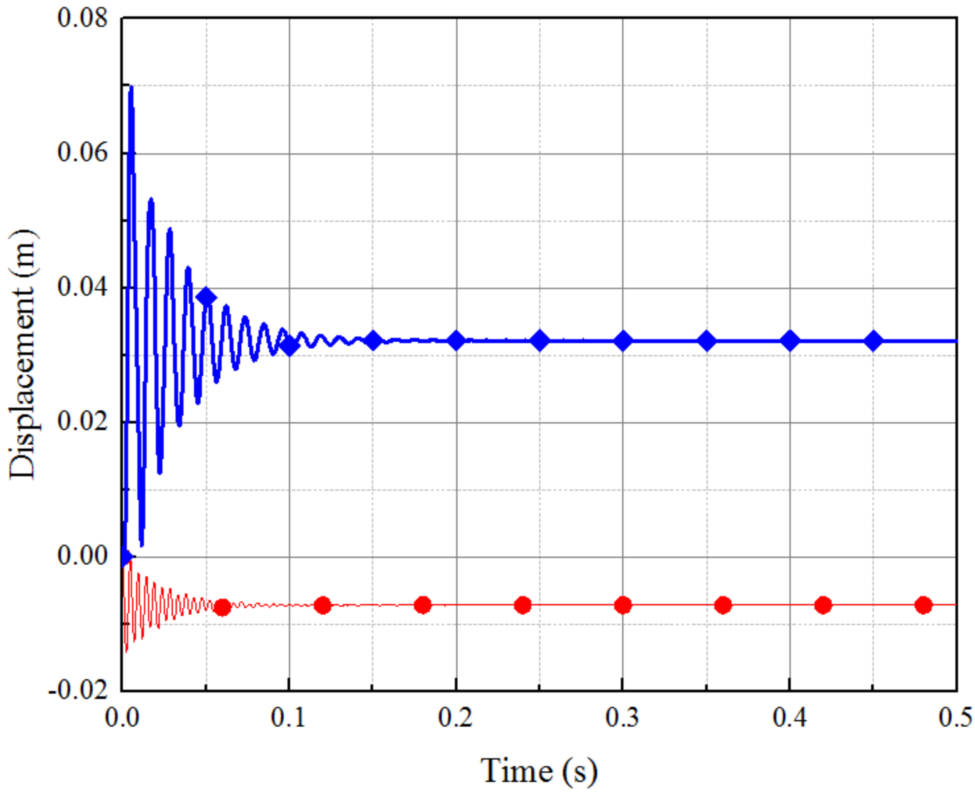

Figure 6. Transverse displacement of the free end node. (CB method (free-free modes) and ANSYS (simply-supported modes)).

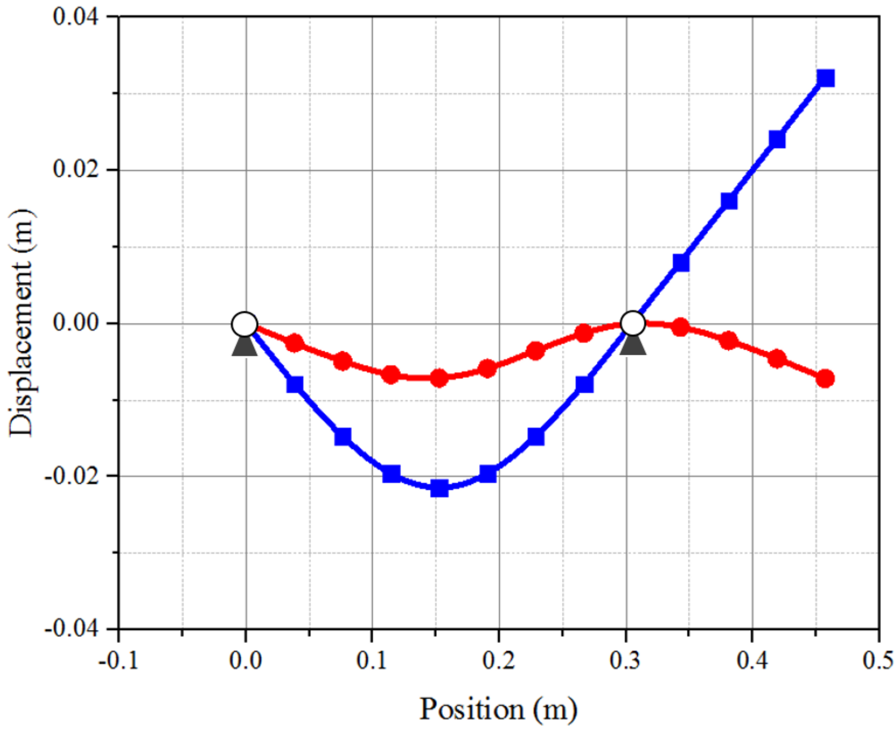

Figure 7. Deformed shape of the beam. (CB method (free-free modes) and ANSYS (simply-supported modes)).

5. Improved Craig–Bampton Modal Transformation Matrix

As shown by the comparative analysis in Section 4, the free-free modes derived from the CB method are not suitable for all engineering applications, even simple static problems [13]. However, this conclusion does not imply that using the CB method only derives free-free modes. Actually, the CB method has a flexible change form [49]. In this section, the CB method will be improved by two different methods: (i) by imposing the reference conditions into the original CB matrix and (ii) by imposing the reference conditions on the shape function to satisfy the requirement of a unique displacement field, which can be used to form mass and stiffness matrices. The difference between the two methods lies in the fact that the improved CB matrix from the first method cannot be orthogonalized; however, the improved CB matrix from the second method can be orthogonalized. Regardless, the simulation results are the same when the improved CB matrix from the above two methods is used for the durability analysis.

When adding the reference conditions, the CB method can be used for all applications rather than only being suited for free-free modes. Similar to the normal mode approach, the improved CB method can be treated as a general-purpose CMS. In addition, the CB method has a clear physical meaning compared to the normal mode approach. Namely, the information on the topology structure and constraints can be directly obtained from the CB modal transformation matrix.

5.1. Imposing the Reference Conditions on the Original CB Matrix

Based on the constraint conditions of the extended beam in Figure 2b, the identity matrix $\mathbf{I}$ in the original CB modal transformation matrix in Equation (24) corresponds to the unit displacement of the boundary nodes, including the first node (point O) and the ninth node (point A). The static correction modes $\mathbf{\Phi}_c$ are obtained by assuming the unit displacement on the boundary nodes corresponding to the constraint joints. The fixed interface modes $\mathbf{\Phi}_f$ are obtained corresponding to the internal nodes of the extended beam. The zero matrix $\mathbf{0}$ accounts for the fact that the internal nodes are free without any constraints, which leads to the normal modes of the internal nodes that correspond to the fixed interface modes. By the aforementioned description, the topology structure and constraint information of the extended beam can be clearly obtained from the CB modal transformation matrix. The free-free modes will be obtained when using the original CB method because there are no reference conditions in this matrix. In order to obtain the exact nodal deformation of the flexible body, the reference conditions must be imposed on the original CB modal transformation matrix.

Based on Figure 2b, the original CB modal transformation matrix can be rewritten as

$$\left\{\begin{aligned} \mathbf{\Phi}_{CB} &= \begin{bmatrix} \begin{bmatrix} u_1 & & & & & \\ & v_1 & & & & \\ & & \theta_1 & & & \\ & & & u_9 & & \\ & & & & v_9 & \\ & & & & & \theta_9 \end{bmatrix} & \mathbf{0} \\ \mathbf{\Phi}_c & \mathbf{\Phi}_f \end{bmatrix} \\ \mathbf{I} &= \begin{bmatrix} u_1 & & & & & \\ & v_1 & & & & \\ & & \theta_1 & & & \\ & & & u_9 & & \\ & & & & v_9 & \\ & & & & & \theta_9 \end{bmatrix} \end{aligned}\right. \tag{32}$$

in which the unit displacement of the first node is $u_1 = v_1 = \theta_1 = 1$, and the unit displacement of the ninth node is $u_9 = v_9 = \theta_9 = 1$.

However, to solve the simply-supported modes, the reference conditions ($u_1 = v_1 = v_9 = 0$) should be imposed on the boundary nodes in the structure, which can be expressed as

$$\mathbf{I}_{CB}^{r} = \mathbf{B}_{CB}^{r}\mathbf{I} \tag{33}$$

where the matrix $\mathbf{B}_{CB}^{r}$ is the reference condition matrix, which can be written as

$$\mathbf{B}_{CB}^{r} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{34}$$

After that, the original CB matrix is given by

$$\mathbf{\Phi}_{CB}^{rc} = \begin{bmatrix} \mathbf{I}_{CB}^{r} & \mathbf{0} \\ \mathbf{\Phi}_c & \mathbf{\Phi}_f \end{bmatrix} \tag{35}$$

Substituting Equation (35) into Equation (26), the new modal mass and modal stiffness matrices can be written as,

$$\left.\begin{aligned} \tilde{\mathbf{M}}_{ff} &= {\mathbf{\Phi}_{CB}^{rc}}^{\mathrm{T}}\mathbf{M}_{ff}\mathbf{\Phi}_{CB}^{rc} \\ \tilde{\mathbf{K}}_{ff} &= {\mathbf{\Phi}_{CB}^{rc}}^{\mathrm{T}}\mathbf{K}_{ff}\mathbf{\Phi}_{CB}^{rc} \end{aligned}\right\} \tag{36}$$

Therefore, the simply-supported modes can be obtained by,

$$\left[\tilde{\mathbf{K}}_{ff} - \omega_s^2\tilde{\mathbf{M}}_{ff}\right]\mathbf{A}_s = \mathbf{0} \tag{37}$$

where ω_s is the natural frequency, $\mathbf{A}_s$ is the modal shape, and s is the total number of modes. The process starting at Equation (32) to reach Equation (37) is referred to as the first improved CB method. Similarly, any other modes can be obtained using the improved CB method only if the reference conditions are exactly selected and imposed.

5.2. Imposing Reference Conditions Prior to Forming the Craig–Bampton Matrix

It must be stated that the mass and stiffness matrices corresponding to the CB method, which can be written as $\mathbf{M}_{ff}^{CB}$ and $\mathbf{K}_{ff}^{CB}$ respectively, do not include any reference conditions. Because the reference conditions must be added to the shape functions to remove the rigid body modes, the mass and stiffness matrices are related to the shape functions. Hence, the new mass and stiffness matrices corresponding to the elastic coordinates can be expressed as

$$\begin{cases} \mathbf{M}^N = \mathbf{B}_f^{r\,\mathrm{T}} \mathbf{M}_{ff}^{CB} \mathbf{B}_f^r \\ \mathbf{K}^N = \mathbf{B}_f^{r\,\mathrm{T}} \mathbf{K}_{ff}^{CB} \mathbf{B}_f^r \end{cases} \tag{38}$$

in which $\mathbf{B}_f^r$ is the reference conditions matrix. For example, in Figure 2b, in order to add the simply-supported reference conditions, the identity matrix $\mathbf{I}_{39\times 39}$ must be changed, such that all elements corresponding to the first, second, and twenty-sixth columns in the reference conditions matrix $\mathbf{B}_f^r$ are equal to zero.

According to the new mass and stiffness matrices a, Equation (17) is rewritten as

$$\begin{bmatrix} \left(\mathbf{M}^N\right)_{bb} & \left(\mathbf{M}^N\right)_{bi} \\ \left(\mathbf{M}^N\right)_{ib} & \left(\mathbf{M}^N\right)_{ii} \end{bmatrix} \begin{bmatrix} \left(\ddot{\mathbf{q}}_f\right)_b \\ \left(\ddot{\mathbf{q}}_f\right)_i \end{bmatrix} + \begin{bmatrix} \left(\mathbf{K}^N\right)_{bb} & \left(\mathbf{K}^N\right)_{bi} \\ \left(\mathbf{K}^N\right)_{ib} & \left(\mathbf{K}^N\right)_{ii} \end{bmatrix} \begin{bmatrix} \left(\mathbf{q}_f\right)_b \\ \left(\mathbf{q}_f\right)_i \end{bmatrix} = \mathbf{0} \tag{39}$$

Based on Equation (39), the static correction modes can be expressed as

$$\mathbf{\Phi}_c^N = -\left(\mathbf{K}^N\right)_{ii}^{-1} \left(\mathbf{K}^N\right)_{ib} \tag{40}$$

Similarly, the fixed interface modes can be obtained from the following equation

$$\left(\mathbf{M}^N\right)_{ii} \left(\ddot{\mathbf{q}}_f\right)_i + \left(\mathbf{K}^N\right)_{ii} \left(\mathbf{q}_f\right)_i = \mathbf{0} \tag{41}$$

Hence, the fixed interface modes are rewritten as $\mathbf{\Phi}_f^N$.

After the static correction modes and fixed interface modes are obtained, the original CB matrix after imposing the reference conditions can be expressed as,

$$\mathbf{\Phi}_{CB}^N = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{\Phi}_c^N & \mathbf{\Phi}_f^N \end{bmatrix} \tag{42}$$

The matrix in Equation (42) can be orthogonalized to generate the diagonal modal mass and stiffness matrices; the orthogonalization process is the same as before described between Equations (26)–(31). Thus, the improved CB matrix after orthogonalizing is given by

$$\overline{\mathbf{\Phi}}_{CB}^N = \mathbf{\Phi}_{CB}^N \mathbf{N}^N \tag{43}$$

where $\mathbf{N}^N$ is the new modal transformation matrix from solving the second eigenvalue. The process from Equations (38)–(43) is also referred to as the second improved CB method. Similarly, any other modes can be obtained using this improved CB method only if the reference conditions are exactly selected and imposed.

In Table 4, the comparison between the normal mode approach, CB matrix, improved CB matrix, and ANSYS is provided. The results in Table 4 illuminate the fact that the simply-supported modes can be obtained by the CB method after imposing the reference conditions, and the results are validated by ANSYS. Moreover, the nodal displacement of node 5 and the free end of the extended beam in Figure 2b are recalculated based on the improved CB method. As shown in Figures 8 and 9, while the nodal displacements of node 5 and the free end of the extended beam from the original CB method are identical to the results from the normal mode approach based on free-free modes, the nodal displacements of node 5 and the free end of the extended beam using the improved CB methods are the same as ANSYS. Figure 10 shows the final deformed shape of the extended beam based on the four different cases. The deformed shape from the original CB method is the same compared to the normal mode approach based on free-free modes, and the deformed shape from the improved CB method is identical to ANSYS. Hence, the improved CB method is not only suited for free-free modes but also can be used to represent the simply-supported modes if the appropriate reference conditions are imposed on the original CB modal transformation matrix. Similarly, other types of modes, such as pinned-pinned modes and fixed-free modes, can be obtained using the same procedure based on the CB method. This investigation expands the scope of application of the CB method as a general-purpose method in the structural engineering community.

Table 4. Comparative analysis of the frequencies from different methods (Hz).

CB Method	Improved CB		Normal Mode Approach		ANSYS
Free-Free	**Simply-Supported (First Method)**	**Simply-Supported (Second Method)**	**Free-Free**	**Simply-Supported**	**Simply-Supported**
0	88.67	88.67	0	88.64	88.62
0	209.47	209.47	0	209.40	209.36
0	606.47	606.47	0	606.26	605.88
138.91	1014.51	1014.51	138.91	1014.15	1012.70
382.94	1404.22	1404.22	382.94	1403.73	1401.80
750.97	2319.56	2319.56	750.97	2318.76	2313.40

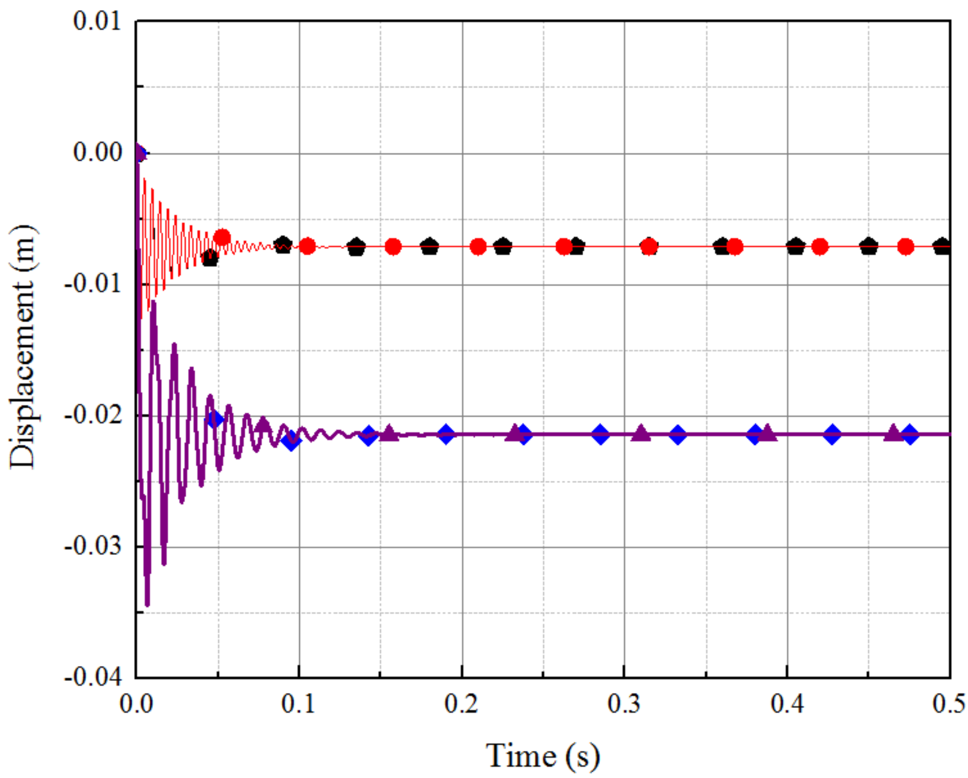

Figure 8. Transverse displacement of node 5. (Normal mode approach (free-free modes), CB matrix (free-free modes), ANSYS (simply-supported modes), and Improved CB matrix (simply-supported modes)).

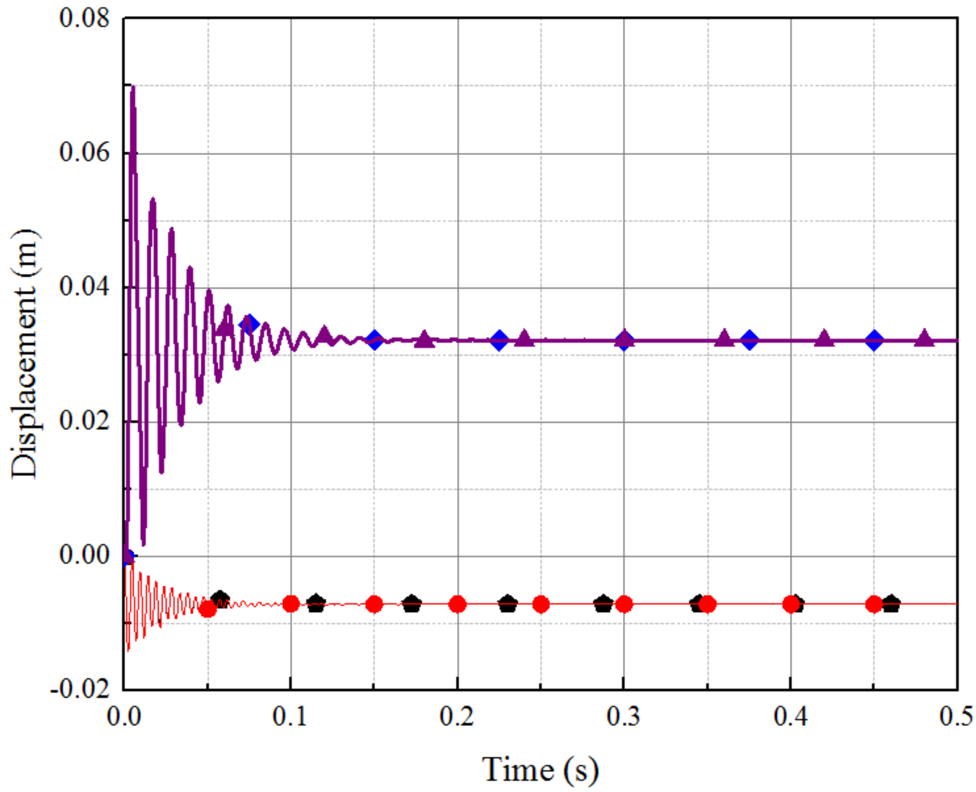

Figure 9. Transverse displacement of the free end node. (Normal mode approach (free-free modes), CB matrix (free-free modes), ANSYS (simply-supported modes), and Improved CB matrix (simply-supported modes)).

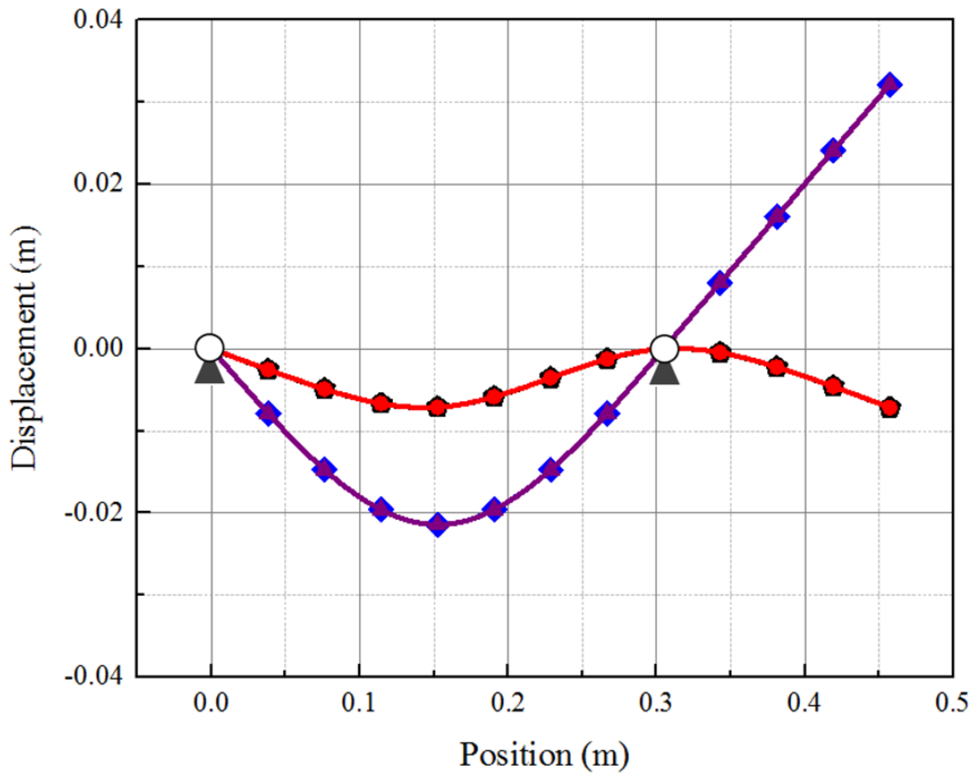

Figure 10. Deformed shape of the beam. (Normal mode approach (free-free modes), CB matrix (free-free modes), ANSYS (simply-supported modes), and Improved CB matrix (simply-supported modes)).

6. Conclusions

In this paper, the floating frame of reference formulation applied to the flexible multibody system is discussed, and the necessity of imposed reference conditions during durability analysis is described. The reference conditions must be used to eliminate the rigid body modes from the shape functions and define a unique displacement field. The normal mode approach is a general-purpose CMS in the durability analysis of the system. In general, after imposing the appropriate reference conditions, this method is straightforward for determining the deformation basis vectors which have been validated by ANSYS in the static analysis. Hence, the normal mode approach is taken as a standard method to test the correctness of the improved CB method from both improved methods in the dynamic analysis. Moreover, the derived process of the CB matrix is specifically introduced, and its limitation of the free-free modes is proven by the simple planar beam example so as to explain the motivation for its modification. In addition to free-free modes, other modes can be obtained using the improved CB matrix after imposing the appropriate reference conditions. The main conclusions of this paper can be summarized as follows:

(1) The CB method only generates the free-free modes; however, the free-free modes can lead to the wrong solution in some scenarios, as was demonstrated in this paper by a simple planar beam example. If the CB method is to be used for many applications, reference conditions must be imposed to improve the CB method. In this paper, two different methods are proposed to improve the CB method. The first method is to directly impose the reference condition on the unit matrix of the original CB matrix. The second method is to impose the reference conditions on the shape functions to calculate the new mass and stiffness matrices; subsequently, the improved CB matrix can be derived from the new mass and stiffness matrices. Although these two different methods are adopted to improve the CB matrix to make it suitable for all applications, the simulation results from both methods in the durability analysis are the same.

(2) The CB method is not only used to derive the free-free modes but can be suited for deriving the simply-supported modes (or other modes, such as pinned-pinned, fixed-fixed, fixed-free, etc.) only if the appropriate reference conditions are imposed on the original CB matrix or on the shape functions prior to forming the improved CB matrix. Otherwise, the wrong solution will be obtained in some special cases, and it may be difficult to determine the reasons that lead to the wrong solution. Hence, the reference condition concept should be paid more attention to prior to implementing the static or dynamic analysis of the flexible multibody systems. The application area of the CB method is thus expanded from only free-free modes to any other mode.

(3) Although the normal mode approach is a more direct method to obtain the normal modes compared to the improved CB method, the improved CB method has a much clearer physical meaning compared to the normal mode approach. The topology structure and constraint information of the flexible multibody system can be directly obtained from the CB modal transformation matrix, which is better than the generalized normal mode approach. In addition, it is very convenient to impose the reference conditions on the original CB modal transformation matrix or shape functions; hence, the structure of the improved CB method is beneficial for programming code during the durability analysis of the flexible multibody system so as to improve computational efficiency.

In conclusion, the improved CB method should be treated as a general-purpose method suitable for all applications rather than restricted only to free-free modes.

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