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# Entanglement Control of Two-Level Atoms in Dissipative Cavities

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**Abstract:** An open quantum bipartite system consisting of two independent two-level atoms interacting nonlinearly with a two-mode electromagnetic cavity field is investigated by proposing a suitable non-Hermitian generalization of the Hamiltonian. The mathematical procedure of obtaining the corresponding wave function of the system is clearly given. Pancharatnam phase is studied to give a precise information about the required initial system state, which is related to artificial phase jumps, to control the degree of entanglement (DEM) and get the highest concurrence. We discuss the effect of time-variation coupling, and dissipation of both atoms and cavity. The effect of the time-variation function appears as frequency modulation (FM) effect in the radio waves. Concurrence rapidly reaches the disentangled state (death of entanglement) by increasing the effect of field decay. On the contrary, the atomic decay has no effect.

**Keywords:** concurrence; control; entanglement; Pancharatnam phase; two-two level atoms

## 1. Inroduction

Quantum systems promise enhanced capabilities in sensing, communications, and computing beyond what can be achieved with classical-based conventional technologies rather than quantum physics. Mathematical models are essential for analyzing these systems and building suitable quantum models from empirical data is an important research topic. In Dirac theory of radiation [1], he considered atoms and the radiation field with which they interact as a single system whose energy is represented by the frequency/energy of each atom solely, the frequency/energy of every mode of the applied laser field alone, and a small term is to the coupling energy between atoms and field modes. The interaction term is necessary if atoms and field modes are to affect each other. A simple model is that we consider a pendulum of resonant frequency $\omega_0$, which corresponds to an atom, and a vibrating string of resonant frequency $\omega_1$ which corresponds to the radiation field. Jaynes–Cummings model (JCM) [2] is the first solvable analytical model to represent the atom–field interaction with experimental verification [3]. JCM has been subjected to intensive research in the last decades with many interesting phenomena explored [4–7]. The matter–field coupling term may be constant [8–10] or time-dependent [11,12], and that depends on the considered physical situation. Parametric coupling

for superconducting qubits [13,14] is studied [15]. In our case, the atoms are moving while interacting with the laser field, this topic has been investigated for different quantum systems [16–20].

Using a non-Hermitian generalization of the Hamiltonian (NHH) is now considered as a model to describe an open quantum system [21–23], we may obtain complex-energy eigenvalues. These NHHs are justified as an approximate and phenomenological description of an open quantum system such as radioactive decay processes [24]. Driving a quantum system with the output field from another driven quantum system, and a quantum trajectory theory for cascaded open systems were studied by proposing NHH in [25,26], respectively. Investigating the dynamics of three-level systems has allowed the discussion of teleportation and non-classical properties [27,28], concurrence, and Shannon information entropy [29,30]. Entanglement is robust under decoherence, independent of the initial state of the qubits, is relatively easy to generate, and has a system-independent value, such systems could be used as "entanglement batteries". This entanglement can be harvested and further transmitted in a quantum circuit [31]. The entanglement is maximized for intermediate values of the cavity damping rates and the intensity of the white noise field, vanishing both for small and for large values of these parameters and thus exhibiting a stochastic-resonance-like behaviour [32].

In this work, we propose a new technique to control the entanglement. Stimulated Raman adiabatic passage (STIRAP) is a process that allows transfer of a population between two states via at least two coherent field pulses by inversely engineering the Hamiltonian parameters via Lewis-Riesenfeld phases [33]. To investigate the essential requirements, such as, efficiency, selectivity robustness and flexibility, one may use what is called STIRAP [34]. The goal was reached after implementing changes which were motivated by the following consideration, based again on a rate-equation picture. The laser beam S, generated by optical pumping , is necessarily confined to the axis of the cavity, and thus, the driving laser-field P must also propagate along that axis. New molecules traveling towards the cavity to further feed the laser action find themselves first in the wings of these two lasers, P and S. The P laser is already strong enough to excite molecules in level 1 to the upper laser level (level 2), but the S laser is not yet strong enough for the rate of stimulated emission to the target level to compete successfully with the rate of spontaneous emission. Thus, before reaching the center of the cavity, many molecules are lost by spontaneous emission to levels other than the target level, thereby spoiling efficiency and selectivity. After model calculations had confirmed the validity of that view, a remedy was quickly found: remove the cavity and bring in a second external laser S, with its center shifted upstream of the axis of laser P. The molecules would then be interacting with a strong S field as they reached the wings of the P laser. Therefore, the rate of stimulated emission, induced by the S field, would be sufficiently large to compete with spontaneous emission as soon as molecules have traveled far enough to be excited by the P laser. Modification of the molecular-beam-laser modeling code (using a density-matrix approach) [35] to include the spatial shift of the two laser fields (a timing offset between the two fields affecting the molecules) confirmed that fully transfer efficiency should be achievable. This prediction was in fact experimentally demonstrated soon afterwards [36], where it was noted that the proposed and demonstrated technique does not require laser intensity modulation, laser frequency chirping, or level shifting. STIRAP has been explained chemically and physically [37], and its protocols have been applied to various models to speed up adiabatic passage techniques and achieve fast and robust population control in two-level atomic systems [38,39], in a three-level system STIRAP [40–42] can be used to transfer population from the ground state $|0\rangle$ to the second excited state $|2\rangle$ without having excitations in state $|1\rangle$ during the process. In this situation, it is possible to realize $0-2$ population transfer even in systems where that would otherwise be a forbidden transition, as is the case of the transmon [43], and four-level atom [44]. The system in [45] consisted of a fixed-frequency transmon qubit embedded in a three-dimensional superconducting aluminium cavity and if the coupling strength is much smaller than the qubit–cavity detuning, the undriven qubit–cavity system is well described by the dispersive Jaynes–Cummings Hamiltonian. The authors of [46] measured the modulation response of several QC lasers, packaged and processed as just described, by driven them with the output of synthesized signal generator and light output was detected with a GaAs/AlGaAs quantum-well infrared photodetector

(QWIP), also packaged for high-speed operation and the resulting photocurrent was amplified and fed to a microwave spectrum analyzer, where the modulation amplitude was measured. In [47], the authors review the physical phenomena that arise when quantum mechanical energy levels are modulated in time. The dynamics resulting from changes in the transition frequency is a problem studied since the early days of quantum mechanics. It has been of constant interest both experimentally and theoretically, with the simple two-state model providing an inexhaustible source of novel concept. The choice of initial system parameters as we propose is related to the artificial phase jumps of Pancharatnam phase. Phase jumps are promising points such that they generate better entanglement degrees, and its successive repetition inside any system dynamics reflects a good sign of system capability to transfer information, as the geometric phases can be altered by changing the relative delay of the laser pulses [48].

Our work here is oriented around the interaction of an open quantum system of two independent two-level atoms with a quantization (non-classical) of electromagnetic field in a dissipative cavity in the multi-photon process. In Section 2, the considered physical scenario is introduced, the corresponding Hamiltonian is investigated, and the mathematical procedure for obtaining the solution of the wave equation is clearly given. In Section 3, we discuss the proposed technique to control the entanglement by properly choosing the initial values of the atomic state. Concurrence is also discussed to determine the effect of other parameters in the system. In Section 4, a brief conclusion and results are given.

## 2. Physical Scenario

The theoretical model as illustrated in Figure 1 can be written as a non-Hermitian Hamiltonian

$$\begin{aligned}\hat{H} = \sum_{j=1}^{2}\sum_{\ell=1}^{2}\frac{(-1)^{j+1}}{2}(\Omega_j^{\ell} - i\Gamma_j^{\ell})\hat{\sigma}_{jj}^{\ell} + \sum_{j=1}^{2}(\omega_j - i\gamma_j)\hat{a}_j^{\dagger}\hat{a}_j \\ + \sum_{\ell=1}^{2}\bar{\lambda}_{\ell}(t)\left(\hat{a}_{\ell}^{\kappa}\hat{\sigma}_{12}^{\ell} + \hat{a}_{\ell}^{\dagger\kappa}\hat{\sigma}_{21}^{\ell}\right).\end{aligned} \tag{1}$$

where $\Omega_j^{\ell}$ is the associated frequency/energy of level $j$ of the corresponding atom $\ell$, with $\Gamma_j^{\ell}$ is the atomic corresponding decay rate, and $\hat{\sigma}_{ij}^{\ell} = |i\rangle\langle j|$, $(i, j = 1, 2)$ are the atomic-flip operators for $|j\rangle \to |i\rangle$, they satisfy the commutation relation $[\hat{\sigma}_{ij}, \hat{\sigma}_{\alpha\beta}] = \hat{\sigma}_{i\beta}\delta_{\alpha j} - \hat{\sigma}_{\alpha j}\delta_{i\beta}$. $\omega_j$ is the energy/frequency of the quantized electromagnetic cavity field mode $j$ with a corresponding decay rate $\gamma_j$ and $\hat{a}_j$ ($\hat{a}_j^{\dagger}$) is the annihilation (creation) operator for the field mode $j$, and they obey the commutation relation $[\hat{a}_i, \hat{a}_j^{\dagger}] = \delta_{ij}$. Here, we consider that $\Omega_j^{\ell} >> \Gamma_j^{\ell}$, and $\omega_j^{\ell} >> \gamma_j^{\ell}$ [49]. $\bar{\lambda}_{\ell}(t)$ is the time-dependent coupling torque of the matter–field interaction. It is more realistic to consider that the interaction intensity is not uniform, and in the following calculations we consider that $\bar{\lambda}_{\ell}(t) = \lambda_{\ell}\cos(\varpi_{\ell}t)$. $\kappa$ refers to photon number process. To study the dynamics and properties of this model, we need to get the corresponding wave function $|\psi(t)\rangle$, which can be formulated in the following form,

$$\begin{aligned}|\psi(\phi, t)\rangle = \sum_{n_1,n_2=0}^{\infty}\Big(&A_1^{(n_1,n_2)}(\phi, t)e^{-i\alpha_1 t}|n_1, n_2, 1, 1\rangle \\ &+ A_2^{(n_1,n_2)}(\phi, t)e^{-i\alpha_1 t}|n_1, n_2 + \kappa, 1, 2\rangle \\ &+ A_3^{(n_1,n_2)}(\phi, t)e^{-i\alpha_1 t}|n_1 + \kappa, n_2, 2, 1\rangle \\ &+ A_4^{(n_1,n_2)}(\phi, t)e^{-i\alpha_1 t}|n_1 + \kappa, n_2 + \kappa, 2, 2\rangle\Big)\end{aligned} \tag{2}$$

where $A_m^{(n_1,n_2)}(t)$ $(m = 1, 2, 3, 4)$ are functions of time and field modes, called the probability amplitudes. $\alpha_m$ are field-dependent functions, and can be defined as follows,

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}(\Omega_1^1 + \Omega_1^2) + n_1\omega_1 + n_2\omega_2,\\
\alpha_2 &= \frac{1}{2}(\Omega_1^1 - \Omega_2^2) + n_1\omega_1 + (n_2+\kappa)\omega_2,\\
\alpha_3 &= \frac{1}{2}(\Omega_1^2 - \Omega_2^1) + (n_1+\kappa)\omega_1 + n_2\omega_2,\\
\alpha_4 &= \frac{-1}{2}(\Omega_2^1 + \Omega_2^2) + (n_1+\kappa)\omega_1 + (n_2+\kappa)\omega_2.
\end{aligned} \tag{3}$$

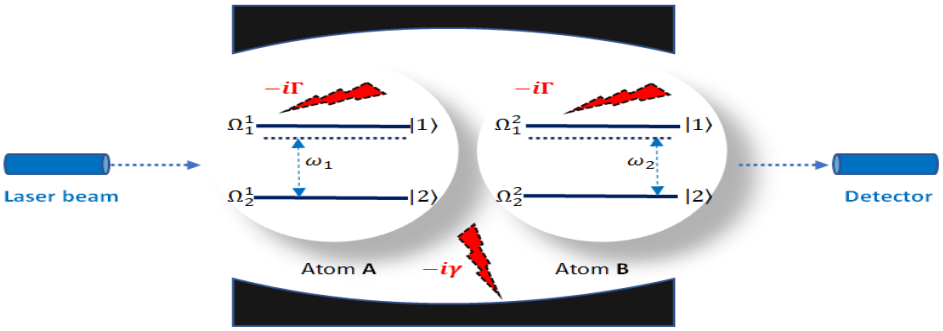



**Figure 1.** Energy level diagram of two two-level atoms coupled to two-mode field in a dissipative cavity.

By applying the time-dependent Schrödinger equation to the system, we get the following coupled differential equations. The trigonometric function in $\bar{\lambda}_\ell(t)$ can be reformulated in an exponential form. There exist exponential terms with two different powers in the differential equations, $e^{\pm i(\Delta+\omega)t}$ and $e^{\pm i(\Delta-\omega)t}$. Approximately, we can ignore the counter oscillating terms $e^{\pm i(\Delta+\omega)t}$. This approximation is similar to the RWA and has been accepted physically for numerous models [29,50]. Therefore, the differential equations can be recomposed to be

$$\begin{aligned}
i\begin{bmatrix} \dot{A}_1^{(n_1,n_2)}(t)\\ \dot{A}_2^{(n_1,n_2)}(t)\\ \dot{A}_3^{(n_1,n_2)}(t)\\ \dot{A}_4^{(n_1,n_2)}(t)\end{bmatrix} &= \begin{bmatrix} k_1 & g_2e^{i(\Delta-\omega)t} & g_1e^{i(\Delta-\omega)t} & 0\\ g_2e^{-i(\Delta-\omega)t} & k_2 & 0 & g_1e^{i(\Delta-\omega)t}\\ g_1e^{-i(\Delta-\omega)t} & 0 & k_3 & g_2e^{i(\Delta-\omega)t}\\ 0 & g_2e^{-i(\Delta-\omega)t} & g_1e^{-i(\Delta-\omega)t} & k_4\end{bmatrix}\times\\
&\times\begin{bmatrix} A_1^{(n_1,n_2)}(t) & A_2^{(n_1,n_2)}(t) & A_3^{(n_1,n_2)}(t) & A_4^{(n_1,n_2)}(t)\end{bmatrix}^T
\end{aligned} \tag{4}$$

After using the method in [29], we get

$$i\begin{bmatrix} \dot{B}_1^{(n_1,n_2)}(t)\\ \dot{B}_2^{(n_1,n_2)}(t)\\ \dot{B}_3^{(n_1,n_2)}(t)\\ \dot{B}_4^{(n_1,n_2)}(t)\end{bmatrix} = \begin{bmatrix} k_1 & g_2 & g_1 & 0\\ g_2 & k_2 & 0 & g_1\\ g_1 & 0 & k_3 & g_2\\ 0 & g_2 & g_1 & k_4\end{bmatrix}\begin{bmatrix} B_1^{(n_1,n_2)}(t)\\ B_2^{(n_1,n_2)}(t)\\ B_3^{(n_1,n_2)}(t)\\ B_4^{(n_1,n_2)}(t)\end{bmatrix} \tag{5}$$

where

$$\begin{aligned}
k_1 &= \frac{1}{2i}\left(\Gamma_1^1 + \Gamma_1^2 + n_1\gamma_1 + n_2\gamma_2\right),\\
k_2 &= \frac{1}{2i}\left(\Gamma_1^1 - \Gamma_2^2 + n_1\gamma_1 + (n_2+\kappa)\gamma_2\right),\\
k_3 &= \frac{1}{2i}\left(\Gamma_1^2 - \Gamma_2^1 + (n_1+\kappa)\gamma_1 + n_2\gamma_2\right),\\
k_4 &= \frac{1}{2i}\left(-\Gamma_2^1 - \Gamma_2^2 + (n_1+\kappa)\gamma_1 + (n_2+\kappa)\gamma_2\right),\\
\Delta &= \Omega_j^1 - \kappa\omega_1 = \Omega_j^2 - \kappa\omega_2,\\
g_1 &= \frac{\bar{\lambda}_1}{2}\sqrt{\frac{(n_1+\kappa)!}{n_1!}},\\
g_2 &= \frac{\bar{\lambda}_2}{2}\sqrt{\frac{(n_2+\kappa)!}{n_2!}},
\end{aligned} \tag{6}$$

$$\begin{aligned}
B_1^{(n_1,n_2)}(t) &= A_1^{(n_1,n_2)}(t)e^{i(\Delta-\omega)t},\\
B_2^{(n_1,n_2)}(t) &= A_2^{(n_1,n_2)}(t),\\
B_3^{(n_1,n_2)}(t) &= A_3^{(n_1,n_2)}(t),\\
B_4^{(n_1,n_2)}(t) &= A_4^{(n_1,n_2)}(t)e^{-i(\Delta-\omega)t}.
\end{aligned}$$

$$i\dot{\mathcal{F}}(n_1,n_2,t) = \mathcal{M}(n_1,n_2)\mathcal{F}(n_1,n_2,t), \tag{7}$$

The solution of the coupled system in Equation (7) depends on the initial states of the system. We pay an attention for determining the formulas of probability amplitudes by preparing the atoms initially to be made in superposition of the states

$$|\psi(0)\rangle_{atom} = \cos(\theta)\,|e_1,e_2\rangle + e^{-i\phi}\sin(\theta)\,|g_1,g_2\rangle\,,$$

where $\phi$ is the corresponding phase of the two states. If we take $\theta = 0$, both atoms are in excited states. Whereas if we take $\theta = \frac{1}{2}\pi$, the ground states are considered. The solution of this system is given by

$$\mathcal{F}(n_1,n_2,t) = \exp\big[-i\mathcal{M}(n_1,n_2)t\big]\mathcal{F}(n_1,n_2,0). \tag{8}$$

where $\mathcal{M}(n_1,n_2)$ and $\mathcal{F}(n_1,n_2,t)$ take the following forms,

$$\mathcal{M}(n_1,n_2) = \begin{bmatrix} k_1 & g_2 & g_1 & 0\\ g_2 & k_2 & 0 & g_1\\ g_1 & 0 & k_3 & g_2\\ 0 & g_2 & g_1 & k_4 \end{bmatrix}, \tag{9}$$

$$\mathcal{F}(n_1,n_2,t) = \begin{bmatrix} B1^{(n_1,n_2)}(t) & B_2^{(n_1,n_2)}(t) & B_3^{(n_1,n_2)}(t) & B_4^{(n_1,n_2)}(t) \end{bmatrix}^T \tag{10}$$

This coupled system of differential equations can be solved numerically or analytically be using a method to get the exponential of the $4\times 4$ matrix in Equation (5) as illustrated in [51] and applied in [52–54]. By applying Newton interpolation method [51] for getting the matrix exponential, which states that for a matrix $A$ with eigenvalues $\lambda_j$, $(j = 1,2,..,n)$, $n$ is the dimension of the matrix,

$$e^{tA} = e^{\lambda_1 t}\mathcal{I} + \sum_{j=2}^{n} \left[\lambda_1, ..., \lambda_j\right] \Pi_{\kappa=1}^{j-1} \left(A - \lambda_\kappa \mathcal{I}\right), \tag{11}$$

where $\mathcal{I}$ is the unitary matrix and the divided differences $\left[\lambda_1, ..., \lambda_j\right]$ depend on $t$ and defined recursively by

$$[\lambda_1, \lambda_2] = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \tag{12}$$

$$[\lambda_1, ..., \lambda_{\kappa+1}] = \frac{[\lambda_1, ..., \lambda_\kappa] - [\lambda_2, ..., \lambda_\kappa]}{\lambda_1 - \lambda_{\kappa+1}}, \qquad \kappa \geq 2. \tag{13}$$

Thus, by using the above method to $e^{-i\mathcal{M}t}$, where $A = -i\mathcal{M}$ and the eigenvalues $\mathcal{E}_m(t)$ of $A$ are defined in Equation (5), then

$$\mathcal{E}_m(t) = -\frac{b}{4} \pm s \pm \sqrt{\frac{q}{s} - 2p - 4s^2}, \tag{14}$$

with

$$\begin{aligned}
&s = \frac{1}{2}\sqrt{\frac{1}{3}\left(\mathcal{Q} + \frac{\Delta_0}{\mathcal{Q}}\right)}, \qquad \mathcal{Q} = \left(\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}\right)^{1/3}, \\
&p = \frac{8c - 3b^2}{8}, \qquad q = \frac{b^3 + 8d - 4bc}{8}, \\
&\Delta_0 = c^2 - 3bd + 12e, \\
&\Delta_1 = 2c^3 - 9bcd + 27(b^2 e + d^2) - 72ce,
\end{aligned} \tag{15}$$

$$\begin{aligned}
b &= -i(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4), \\
c &= \Delta^2 + 2g_1^2 + 2g_2^2 + \Delta\kappa_1 - \kappa_1\kappa_2 - \kappa_1\kappa_3 - \kappa_2\kappa_3 - \Delta\kappa_4 \\
&\quad - \kappa_1\kappa_4 - \kappa_2\kappa_4 - \kappa_3\kappa_4 - 2\Delta\varpi - \varpi\kappa_1 + \varpi\kappa_4 + \varpi^2, \\
d &= -i\Big(\Delta\kappa_2 + \Delta\kappa_1\kappa_2 + \Delta^2\kappa_3 + \Delta\kappa_1\kappa_3 - \kappa_1\kappa_2\kappa_3 - \Delta\kappa_2\kappa_4 - \kappa_1\kappa_2\kappa_4 \\
&\quad - \Delta\kappa_3\kappa_4 - \kappa_1\kappa_3\kappa_4 - \kappa_2\kappa_3\kappa_4 + g_1^2(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4) \\
&\quad + g_2^2(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4) - (\kappa_2 + \kappa_3)(2\Delta + \kappa_1 - \kappa_4)\varpi \\
&\quad + \varpi^2(\kappa_2 + \kappa_3)\Big), \\
e &= g_1^4 + \Big(g_2^2 - \kappa_2(\Delta + \kappa_1 - \varpi)\Big)\Big(g_2^2 + \kappa_3(\Delta - \kappa_4 - \varpi)\Big) \\
&\quad - g_1^2\Big(2g_2^2 + \Delta(\kappa_3 - \kappa_2) + \kappa_3(\kappa_1 - \varpi) + \kappa_2(\kappa_4 + \varpi)\Big).
\end{aligned} \tag{16}$$

$$\begin{aligned}
e^{-it\mathcal{M}} = e^{\mathcal{E}_1 t}\mathcal{I} &+ [\mathcal{E}_1, \mathcal{E}_2]\left(-i\mathcal{M} - \mathcal{E}_1\mathcal{I}\right) + [\mathcal{E}_1, \mathcal{E}_3]\left(-i\mathcal{M} - \mathcal{E}_1\mathcal{I}\right)\left(-i\mathcal{M} - \mathcal{E}_2\mathcal{I}\right) \\
&+ [\mathcal{E}_1, \mathcal{E}_4]\left(-i\mathcal{M} - \mathcal{E}_1\mathcal{I}\right)\left(-i\mathcal{M} - \mathcal{E}_2\mathcal{I}\right)\left(-i\mathcal{M} - \mathcal{E}_3\mathcal{I}\right) \\
&+ [\mathcal{E}_1, \mathcal{E}_5]\left(-i\mathcal{M} - \mathcal{E}_1\mathcal{I}\right)\left(-i\mathcal{M} - \mathcal{E}_2\mathcal{I}\right)\left(-i\mathcal{M} - \mathcal{E}_3\mathcal{I}\right)\left(-i\mathcal{M} - \mathcal{E}_4\mathcal{I}\right),
\end{aligned} \tag{17}$$

where the divided differences are formulated as follows,

$$[\mathcal{E}_1, \mathcal{E}_2] = \frac{e^{\mathcal{E}_1 t} - e^{\mathcal{E}_2 t}}{\mathcal{E}_1 - \mathcal{E}_2}, \tag{18}$$

$$[\mathcal{E}_1, \mathcal{E}_3] = \frac{[\mathcal{E}_1, \mathcal{E}_2] - [\mathcal{E}_2, \mathcal{E}_3]}{\mathcal{E}_1 - \mathcal{E}_3} = \frac{e^{\mathcal{E}_1 t} - e^{\mathcal{E}_2 t}}{(\mathcal{E}_1 - \mathcal{E}_2)(\mathcal{E}_1 - \mathcal{E}_3)} - \frac{e^{\mathcal{E}_2 t} - e^{\mathcal{E}_3 t}}{(\mathcal{E}_1 - \mathcal{E}_2)(\mathcal{E}_2 - \mathcal{E}_3)}, \tag{19}$$

$$
\begin{aligned}
[\mathcal{E}_1, \mathcal{E}_4] &= \frac{[\mathcal{E}_1, \mathcal{E}_3] - [\mathcal{E}_3, \mathcal{E}_4]}{\mathcal{E}_1 - \mathcal{E}_4} \\
&= \frac{e^{\mathcal{E}_1 t} - e^{\mathcal{E}_2 t}}{(\mathcal{E}_1 - \mathcal{E}_2)(\mathcal{E}_1 - \mathcal{E}_3)(\mathcal{E}_1 - \mathcal{E}_4)} - \frac{e^{\mathcal{E}_2 t} - e^{\mathcal{E}_3 t}}{(\mathcal{E}_1 - \mathcal{E}_2)(\mathcal{E}_2 - \mathcal{E}_3)(\mathcal{E}_1 - \mathcal{E}_4)} \\
&\quad - \frac{e^{\mathcal{E}_3 t} - e^{\mathcal{E}_4 t}}{(\mathcal{E}_1 - \mathcal{E}_4)(\mathcal{E}_3 - \mathcal{E}_4)}.
\end{aligned} \tag{20}
$$

After the derivation of the matrix exponential, we can calculate the formulas of the probability amplitudes of the wave function of the sytem. The atoms are initially in superposition of states, and the initial field is oriented in the coherent states. Then, the final form of the probability amplitudes be formulated to be

$$
\begin{bmatrix} A_1^{(n_1,n_2)}(t)e^{i(\Delta-\omega)t} \\ A_2^{(n_1,n_2)}(t) \\ A_3^{(n_1,n_2)}(t) \\ A_4^{(n_1,n_2)}(t)e^{-i(\Delta-\omega)t} \end{bmatrix} = e^{-it\mathcal{M}} \begin{bmatrix} A_1^{(n_1,n_2)}(0) \\ A_2^{(n_1,n_2)}(0) \\ A_3^{(n_1,n_2)}(0) \\ A_4^{(n_1,n_2)}(0) \end{bmatrix} \tag{21}
$$

For simplicity, in the next calculations we consider $\lambda_1 = \lambda_2 = \lambda$, $\omega_1 = \omega_2 = \omega$, $\Gamma_1 = \Gamma_2 = \Gamma$, and $\gamma_1 = \gamma_2 = \gamma$.

## 3. Pancharatnam Phase and Concurrence

We need to estimate a certain parameter for controlling the dynamics and entanglement of the system. A special attention is paid for the value of the initial latter phase parameter $\phi$. To reach that goal, we investigate the evolution of Pancharatnam phase $\Phi(\phi, t) = \arg(\langle\psi(\phi, 0)|\psi(\phi, t)\rangle)$. To control the phase $\phi$, we plot $\Phi(\phi, t)$ vs. $\phi$ for three different values of the scaled time $\lambda t$, as in Figure 2. The red, black, and blue curves are plotted for $\lambda t = \pi/3$, $\pi/4$, and $\pi/2$, respectively. In the red curve, we note that there is a smooth evolution of the phase, whereas for the black and blue curves, they exhibit two artificial phase jumps for two different values of $\phi$. The phase jump for the blue curve ($\lambda t = \pi/2$) is repeated every period of $\pi$ and in-between the jumps the evolution is semi-parabolic shaped and reflects a slow variation of the system. The two phase jumps of the black curve are repeated every $\approx \frac{13}{20}\pi$ and in-between the two jumps the variation is very slow, smooth and is separated by $\approx \frac{7}{20}\pi$.

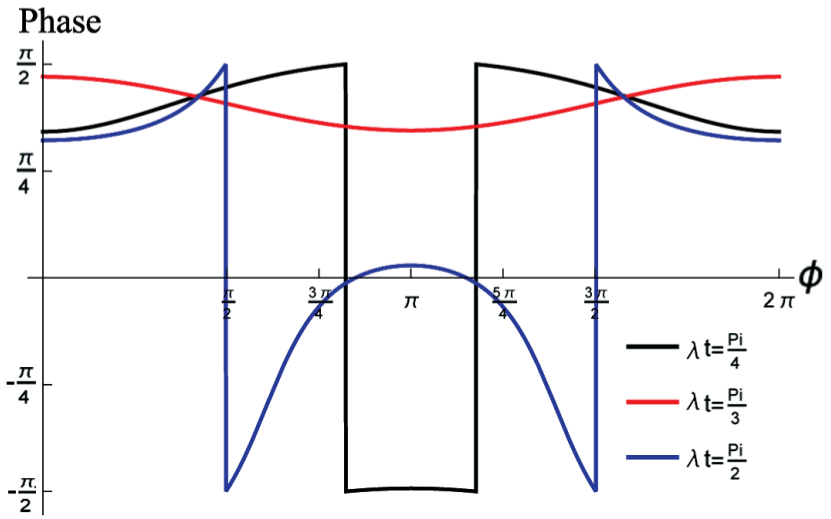


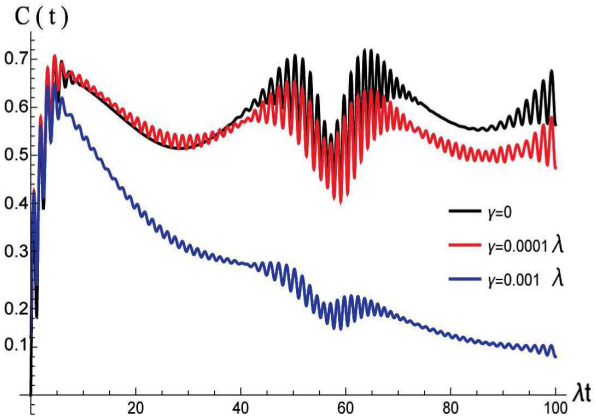

**Figure 2.** The evolution of Pancharatnam phase $\Phi(\phi)$.

Now, we can detect the dynamical behavior of the considered mutipartite system, by investigating the degree of entanglement (DEM) by using the *concurrence* measure, which was formulated as a

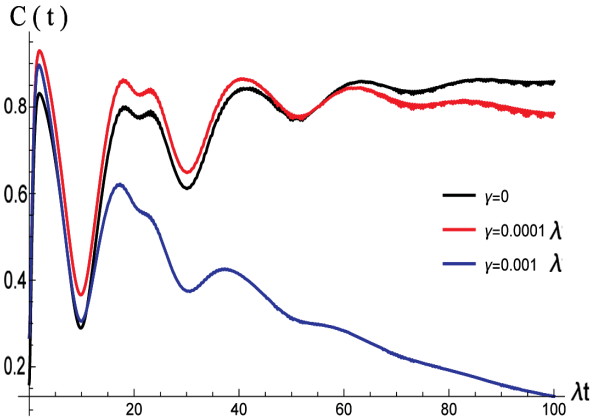
convex measure to amount the DEM for two qubits in pure states by Wootters and Hills [55]. For two qubits in pure states, concurrence is $\mathcal{C}(t) = \sqrt{2\left(1 - Tr\hat{\varrho}^2\right)}$, where $\hat{\varrho} = |\psi(t)\rangle\langle\psi(t)|$ is the reduced density operator. The definition of concurrence has been extended to include multiple qubits [56], and can be calculated generally by

$$\mathcal{C}(t) = \sqrt{2\sum_{i,j=1,i\neq j}^{4}\left(\varrho_{ii}\rho_{jj} - \varrho_{ij}\varrho_{ji}\right)},$$

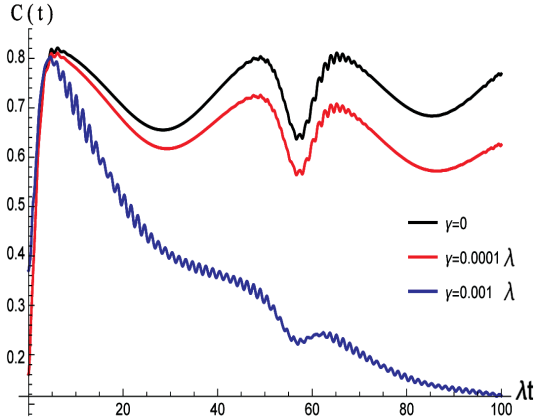
where $\varrho_{ij}$ are the elements of reduced density $\varrho$ in matrix form. Figure 3 sketches the evolution of *concurrence* $\mathcal{C}(t)$ against the scaled time $\lambda t$.




(**a**) $\varpi = \pi\lambda$, $\theta = 0$.




(**b**) $\varpi = 0$, $\theta = \frac{\pi}{4}$.

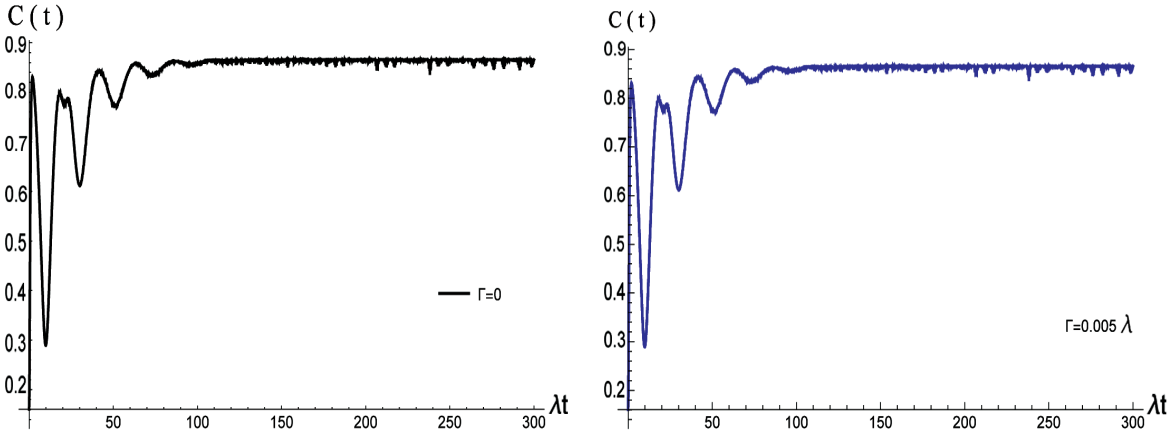



(**c**) $\varpi = \pi\lambda$, $\theta = \frac{\pi}{4}$.

**Figure 3.** The evolution of *concurrence* $\mathcal{C}(t)$ vs. the scaled time, for the one-photon process ($\kappa = 1$), $\omega_j = \Omega_j = 0.1\lambda$, $\Gamma_j = 0$, and $\bar{n}_j = 10$.

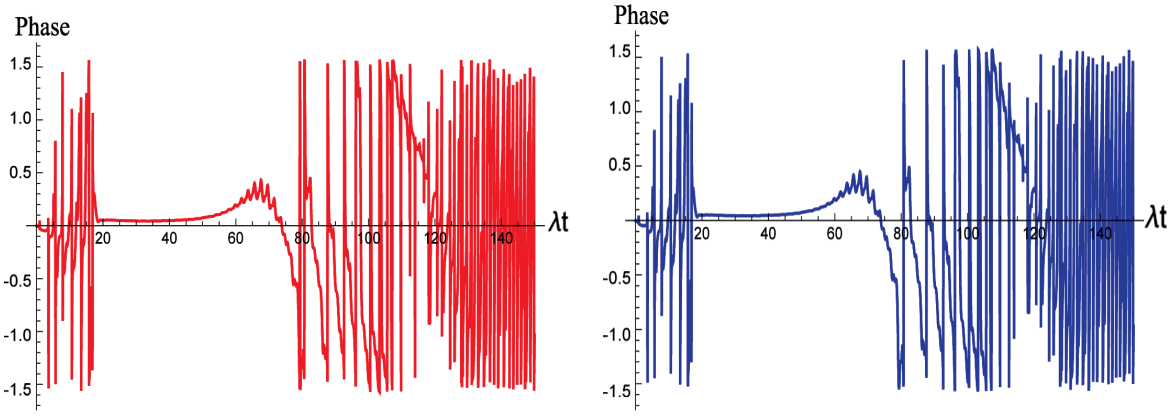
In Figure 3, we plot $\mathcal{C}(t)$ versus the scaled time by using the estimated initial value for the latter phase $\phi = \frac{\pi}{2}$, which is chosen due to the existence of the artificial phase jump at this value in the geometric phase in Figure 2. In Figure 3a, we set the atoms initially to be in excited (upper-most) states $\theta = 0$, and $\varpi = \pi\lambda$, we note that DEM $\leq \ln 2$, which is less than the standard result in models initially prepared in superposition of states. In the next figures, we examine the results of considering superposition of atomic states. In Figure 3b, we set the coupling variation parameter $\varpi = 0$, and take three various values for the decay parameter $\gamma$ of the field. We observe that, in the beginning of the interaction between the two atoms and the coherent field $\lambda t \leq 3\pi$, the effect of the decay parameter is not noticed and the concurrence curves are very similar, but at a drastic point of change,

it differs dramatically, as we see that the black curve $\gamma = 0$, and then it fluctuates till reaching a stable case of concurrence to be $\geq 0.8$; the red curve $\gamma = (10^{-4}\lambda)$ has a chaotic behavior, as in the beginning. It evolves to give a higher rate of concurrence compared with the absence of decay case (black curve), and after a sufficient time it decreases. The blue curve $\gamma = (10^{-3}\lambda)$ represents the system when concurrence rapidly reaches the disentangled state (death of entanglement). In Figure 3c, we set $\varpi = \pi\lambda$, and we note the effect of the oscillation in the matter–field coupling as proposed in the considered model. The effect of that function is clearly noted in the higher case of the decay rate (blue curve) as the interaction has become very weak and fluctuations affect the system evolution. The effect of the time variation function appears as the frequency modulation (FM) effect in the radio waves. FM is a method to encode information in a laser field by varying the instantaneous frequency of the coupling between matter and laser. Also, we note that the presence of $\varpi$ or its absence, the system has reached a disentangled state in the same period of scaled time, but the evolution itself changes by the presence of $\varpi$. In Figure 4, by taking into consideration the effect of the decay in the atomic energy levels $(\Gamma_j)$, the concurrence has not been affected.




(**a**) $\varpi = 0$ (**b**) $\varpi = \pi\lambda$

**Figure 4.** The evolution of *concurrence* $\mathcal{C}(t)$ vs. the scaled time, for the one-photon process ($\kappa = 1$), $\theta = \pi/4$, $\omega_j = \Omega_j = 0.1\lambda$, $\gamma_j = 0$, and $\bar{n}_j = 10$.




(**a**) $\varpi = \pi\lambda$, and $\gamma = 10^{-3}\lambda$ (**b**) $\varpi = \pi\lambda$, and $\Gamma = 10^{-4}\lambda$

**Figure 5.** The evolution of Pancharatnam $\Phi(t)$ vs. the scaled time, for the one-photon process ($\kappa = 1$), $\theta = \frac{\pi}{4}$, $\omega_j = \Omega_j = 0.1\lambda$, and $\bar{n}_j = 10$.

In Figure 5, we display the evolution of Pancharatnam phase $\Phi(t)$ vs. the scaled time, for various values of the system decay parameters $\gamma$ and $\Gamma$. Both curves approximately exhibit the same behavior and for $\lambda t \leq 6\pi$ the phases exhibits a quick subsequent artificial phase jumps, then take a dominate

saturation period till $\lambda t \leq 16\pi$, which is followed by a slow fluctuation that evolves to start another subsequent artificial phase jumps but less quick than the previous evolution.

## 4. Conclusions

The interaction between atoms with field of the system has been investigated by taking into consideration that cavity leaks energies of both atoms and field while the laser field couples the atom as a cosine wave function of time with a parameter $\varpi$ in the multi-photon process. The RWA has been applied twice to approximate the interaction part of the system. By solving the coupled differential equations resulting by applying the time-dependent Schrödinger equation, we get the wave vector and the corresponding eigenenergies. To control the Degree of the Entanglement (DEM) of the system, we determine the initial latter phase by plotting the Pancharatnam phase for three different time points, and investigate the concurrence between the two atoms according to the best value of the latter phase. By increasing the effect of field decay parameter $\gamma$, the concurrence rapidly reaches to the disentangled state (death of entanglement). On the contrary, the atomic decay parameter $\Gamma$ has no effect on the concurrence. The effect of the time-variation function appears as FM effect in the radio waves. FM is used to encode information between atom and field. The system reaches disentangled state in the same period of scaled time, but the evolution itself changes by the presence of $\varpi$. We note that for various values of the system decay parameters $\gamma$ and $\Gamma$, the evolution of Pancharatnam phase in both curves approximately exhibits the same behavior.

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