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The Fading Memory Formalism with Mittag-Leffler-Type Kernels as A Generator of Non-Local Operators

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Abstract: Transient heat conduction problems are systematically applied to the fading memory formalism with different Mittag-Leffler-type memory kernels. With such an approach, using various memories naturally results in definitions of various fractional operators. Six examples are given and interpreted from a common perspective, covering the most well-liked versions of the Mittag-Leffler function. The fading memory approach was used as a template and demonstrated that, if the constitutive equations are correctly built, it is also possible to directly determine where the hereditary terms are located in the models.

Keywords: fading memory formalism;heat conduction; Mittag-Leffler functions; memory kernels; fractional derivatives

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1. Introduction

1.1. Motivation of This Study

In recent years, there are "hot discussions" on definitions and applicability of fractional operators with different kernels [1–7]. In general, these discussions are from a mathematical point of view [4,8–11], because as a rule, the new operators are formulated constitutively, declaring their structures, with various kernels: correctly defined but depending only on the author's imagination, without any physical backgrounds. The "mischance" in fractional modeling begins when operators with kernels not related to certain physical problems are implemented by replacements in already established integer-order models that make the results questionable.

Moreover, there are quests to see in the new operators what we already know without understanding that each operator has its physical basis and new properties. Moreover, there are attempts to "frame" all operators in already existing theories of fractional operator generalizations, not related to any real-world problems; This is an approach that, to some extent, is self-motivated and avoids the fact that fractional calculus is a branch of the applied mathematics; any steps to make it unified and generalized shrink the areas of applications and suppress its multifaceted nature, still developing in applied mathematical modeling.

Motivated by the idea to show some steps toward fractional modeling by applying constitutive equations with memories, obeying the causality principle, and having thermodynamic consistency, the following study was developed, using transient heat conduction with memory as an example.

1.2. Heat Conduction as a Transport Process with Memory

Transport phenomena with hereditaries that can be described by fractional operators (derivatives) are hot topics in modeling, and the approach to constructing physically adequate models is of primary importance. Models with memories can describe behaviors of elastic and viscoelastic solids, Non-Newtonian fluids, electromagnetic materials [12], as well as transport processes as flux models of diffusion and heat conduction [12–15].

It is well known that the earlier attempt to model transport processes with hereditaries comes from the works of Boltzmann [16] and Volterra [17]. We will skip the analysis of the

Volterra constitutive equation but refer to [15] for a comprehensive analysis showing the differences between the Volterra approach and the fractional calculus modeling, which is the main task of this study. Precisely we address heat conduction as a physical problem where the constitutive equation of the flux can be modeled by fractional derivatives. Moreover, the main goal is to demonstrate that from the constitutive flux equation, following the fading memory principle (formalism) [18–21] and assuming different memory kernels, we may derive several operators widely used in fractional modeling. Hence, we will try to show that fractional operators appear logically from the correctly constructed constitutive equations rather than constituted as is generally done in mathematical papers- see the sequel for more details and reference quotations.

The first step in building the study's exposition logically is to demonstrate the background in constitutive modeling of heat flux with a finite speed. Because of that, we will briefly present two examples: (i) How to create a constitutive equation for heat flux using the Caputo derivative [15] (Section 1.2.1) and (ii) constitutive equation formulation for heat conductors with fading memories [22] (Section 1.2.2).

1.2.1. Constitutive Fractional Heat Flux with Caputo Derivative [15]

Let us start with the energy balance (the First Law of Thermodynamics) in a rigid heat conductor

$$\frac{\partial e(x,t)}{\partial t} = -\frac{\partial q(x,t)}{\partial x} + r \tag{1}$$

where $e = \rho C_p T$ is the internal energy, $q(x,t)$is the heat flux, and r is the heat generation (per unit volume). With the Fourier constitutive equation

$$q(x,t) = -k(T(x,t))\frac{\partial T(x,t)}{\partial x}, \quad k(T(x,t)) \tag{2}$$

where $k(T(x,t))$is the thermal conductivity. This construction of the heat flux *has no memory*, propagates with *infinite speed*, and with the energy balance equation results in *the parabolic Fourier model*.

Attempts to model heat flux propagation with a finite speed were done by Cattaneo [23,24] and Vernotte [25] with slight modifications of the heat flux (via a Taylor series expansion of the heat flux, in time, and as a first-order approximation), namely

$$\frac{\partial q(x,t)}{\partial t} = q(x,t) + k(T(x,t))\frac{1}{\tau}\frac{\partial q(x,t)}{\partial x}, \quad \tau > 0 \tag{3}$$

with exponential memory (Jeffrey's memory kernel) $R(t) = \exp(-t/\tau\,)$, and a finite relaxation time τ, named the Cattaneo equation (see more details in Section 4.2).

As Fabrizio [15] demonstrated, replacing the time derivative in (2) by a fractional (Caputo) derivative, we get (in the original notations)

$${}_{-\infty}^{C}D_t^{\alpha} q(x,t) = q(x,t) + k(T(x,t))\frac{\partial T(x,t)}{\partial x}, \quad \alpha > 0, \quad t > 0 \tag{4}$$

The thermodynamic restriction from the Second Law of Thermodynamics for a rigid heat conductor needs the following inequality to be obeyed [14]

$$\frac{\partial \eta(x,t)}{\partial t} \geq \frac{\frac{\partial e(x,t)}{\partial t}}{T(x,t)} + \frac{1}{(T(x,t))^2}\frac{\partial T(x,t)}{\partial x} \tag{5}$$

where the internal energy $e(x,t)$ and the entropy $\eta(x,t)$ are state functions.

In this framework, the following extension of (2) was proposed [15]

$$\begin{aligned} q(x,t) = {}_{-\infty}^{C}D_t^{\alpha}\left[\frac{\partial q(x,t)}{\partial x}\right] &= -k(x)\frac{1}{\Gamma(1-\alpha)}\int\limits_{-\infty}^{t}\frac{1}{(t-\tau)^{\alpha}}\frac{\partial}{\partial t}\left[\frac{\partial T(x,\tau)}{\partial x}\right]d\tau = \\ &= -k(x)\frac{\alpha}{\Gamma(1-\alpha)}\int\limits_{-\infty}^{t}\frac{1}{(t-\tau)^{\alpha+1}}\left[\frac{\partial T(x,\tau)}{\partial x} - \frac{\partial T(x,\tau)}{\partial x}\right]d\tau \end{aligned} \tag{6}$$

The second version (in the second row) in (6) is an equivalent presentation of the Caputo derivative.

Finally, with the assumption that the internal energy $e(x,t)$ depends only on $T(x,t)$ (i.e., the density ρ and the heat capacity C_p are temperature independent) and applying the energy balance equation the following model was obtained

$$\rho C_p \frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x} k(x) \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{1}{(t-\tau)^{\alpha+1}} \left[\frac{\partial T(x,\tau)}{\partial x} - \frac{\partial T(x,\tau)}{\partial x} \right] d\tau + r(x,t) \tag{7}$$

or in a more convenient form, for the presentation in this article, as

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x} a(x) \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{1}{(t-\tau)^{\alpha}} \frac{d}{d\tau} \left[\frac{\partial T(x,\tau)}{\partial x} \right] d\tau + r(x,t), \quad a(x) = \frac{k(x)}{\rho C_p}, \quad t > 0 \tag{8}$$

Remark 1. *It is worth noting and important for the entire discussion developed in this article that the time derivative in the energy balance equation (sometimes termed a continuity equation) is not replaced by a fractional derivative. The flux constitutive model is satisfying the causality principle [26–28], that between the cause (temperature gradient) and the result (output of the system), there is a time shift. This time shift in (7) is assured by the fractional Caputo derivative. This comment is of great importance for the model's development in this article, where similar constructions of heat flux constitutive equations are developed with contributions of different fractional derivatives.*

Remark 2. *The construction (6), in the light of the fractional diffusion model (we have a fractional diffusion model of thermal energy), does not consider a stationary state, that is, when the model reduces to the Fourier constitutive Equation (2) which is valid for long times where the relaxation of the heat flux vanishes. In addition, the analysis in [29] provides sufficient information on fractional modeling of transient heat conduction.*

1.2.2. Constitutive Heat Flux with Linear Memory [22]

Following Giorgi and Gentili [22], the internal energy $e(x,t)$ and heat flux $q(x,t)$, and following the fading memory principle [18–21] are described by the following linearized constitutive equations (with notations used in this work)

$$e(x,t) = e_0 + C_p \theta(x,t) + \int_0^{\infty} R_e(s)\theta(t-s)ds \tag{9}$$

$$q(x,t) = -k_0 \frac{\partial T(x,t)}{\partial x} - \int_0^{\infty} R_q(s)\theta(t-s)ds \tag{10}$$

$\theta(x,t) = T(x,t) - T_0(x,t), \quad T_0(x,t) = T(x,0)$ is temperature variation field. Further, $R_e(t)$ and $R_q(t)$ are relaxation functions (fading memories), and following the arguments of Day [21], the fading memory principle is equivalent to $R_e(t), R_q(t) \in L^1(\mathbb{R}^+)$ and $R_q(t)$.

In this context, the heat capacity $C_p(t)$, and thermal conductivity $k(t)$ are defined as [22]

$$C_p(t) = C_{p0} + \int_0^{t} R_e(s)ds, \quad k(t) = k_0 + \int_0^{t} R_q(s)ds, \quad C_{p0} > 0 \tag{11}$$

and $C_0 = \lim_{t\to\infty} C_p(t)$, and $k_0 = \lim_{t\to\infty} k(t)$ are *equilibrium heat capacity* , and the *equilibrium thermal conductivity*, that is when the memory integrals in (11) are vanishing.

At this point, we will discontinue the discussion of the thermodynamic principles of fading memory (it will be revisited in Section 3 devoted to model development with fractional operators), but it is critical to emphasize two principles in the constitutive model formulation:

1. The memory integral is an additive part (in the cases discussed here, a linear contribution) to the constitutive heat flux model. This gives us a great opportunity to apply fractional modeling through the use of adequately formulated relaxation functions (memory kernels), as we will do in the sequel to this article.

2. The energy balance equation remains unchanged since the First Law of Thermodynamics holds. This directly means that the time derivative of internal energy cannot be mechanistically replaced by fractional operators.

Remark 3. *We can see that we have two versions of constitutive heat flux formulations, but the energy balance equation remains unmodified since this is the First Law of Thermodynamics. Any changes in the balance equation, which is a very common step in fractional calculus publications, caused by replacing the time derivatives with fractional counterparts are violations of the First Law of Thermodynamics. The main task of mathematical modeling, as well as fractional modeling, is to solve correctly constructed models with physically interpretable results, which are often found in formalistic fractional models obtained by replacement.*

Remark 4. *Before starting with the development of fractional operators with various Mittag-Leffler kernels, we have to say that there is no mathematics only for mathematics in this study. Fractional calculus is applied mathematics invoked by reality since the non-localities are natural phenomena in the transport processes. Because of that, the following exposition stresses the attention to the physical background of the modeled problems and their adequate interpretations.*

1.3. The Main Task of This Article

The main task of this article is threefold:

1. To start the introduction of correctly formulated models of heat conductors with memory (actually, these are diffusion models, not diffusion models commonly modeled in heterogeneous media by application of fractional derivatives), applying the fading memory formalism.

2. To introduce different fractional operators in a logical way by using different relaxation functions (completely monotone Mittag-Leffler functions) and how these operators are related. From our point of view, this approach is of high importance for fractional modeling, where there are too many operators constituted without any physical background and then incorrectly applied in the modeling of real-world problems.

3. To analyze the developed constructions of diffusion (heat conduction) models and perform a comparative analysis regarding existing equations from the literature involving fractional operators based on various Mittag-Leffler-type kernels.

1.4. Paper Organization in the Sequel

The following part of this article is organized as follows: Section 2.1 provides the basic definitions and properties of the classical (one-parameter) Mittag-Leffler function (MLF) Section 2.1.1, two-parameter MLF Section 2.1.2, three-parameter MLF (Prabhakar function) Section 2.1.3, Prabhakar kernel Section 2.1.4. The main fractional operators based on Mittag-Leffler type kernels and their interrelations are briefly outlined in Section 2.2. The fading memory formalism with the concept for model build-up with different kernels are explained in Sections 3 and 3.2. Constitutive heat flux models with different memories are developed in Section 4: Exponential memory Section 4.2, with Mittag-Leffler (one-parameter) memory kernel Section 4.3, with Prabhakar memory kernel Section 4.4. Additional experiments with some special well-kernels are developed in the Section 5: Rzanitsyn kernel Section 5.1, Miller–Ross kernel Section 5.2 and Rabotnov kernel Section 5.3, thus, demonstrating how new operators can be generated from a common viewpoint. An analysis of the developed models in parallel to some available equations from the literature is developed in Section 6.

2. Preliminaries on Mittag-Leffler Functions, Their Properties and Available Fractional Operators

To support the further development of the problems envisaged in this article, we need some preliminary information on the Mittag-Leffler functions and related operators to be briefly outlined.

2.1. Mittag-Leffler Functions and Related Kernels

2.1.1. One-Parameters Mittag-Leffler Function

The Mittag-Leffler function is defined as a power-series convergent in the whole complex plane [30,31]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0 \tag{12}$$

and is an entire function of order $1/\alpha$

It is a completely monotone (CM) function [30,32] that implies $(-1)^m \frac{d^m}{dz^m} E_\alpha(z) \geq 0$, i.e., a negative function of class C^∞ for all $m \in \mathbb{N}$, that it is a Bernstein function since its first derivative is CM. Our interest is oriented toward the function

$$E_\alpha(-t^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{\alpha k}}{\Gamma(\alpha k + 1)}, \quad 0 < \alpha < 1, \quad t > 0 \tag{13}$$

with a Laplace transform

$$\mathcal{L}[E_\alpha(-t^\alpha)] = \frac{s^{\alpha-1}}{s^\alpha + 1}, \quad \mathcal{L}\left[\frac{d}{dt} E_\alpha(-t^\alpha)\right] = \frac{1}{s^\alpha + 1}, \quad \alpha > 0 \tag{14}$$

For $t \to 0^+$ the asymptotic expansion matches the stretched exponential [30]

$$E_\alpha(-t^\alpha) = 1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \cdots \sim \exp\left[-\frac{t^\alpha}{\Gamma(\alpha+1)}\right], \quad t > 0 \tag{15}$$

while for $t \to \infty$ the asymptote is a negative power-law $\equiv t^{-\alpha}$

$$E_\alpha(-t^\alpha) \approx \sum_{k=0}^{\infty} (-1)^{k-1} \frac{t^{-\alpha k}}{\Gamma(1 - \alpha k)}, \quad t \to \infty \tag{16}$$

2.1.2. Two-Parameters Mittag-Leffler Function

The two-parameter of the Mittag-Leffler type is defined as a series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \quad \alpha > 0, \quad \beta > 0, \quad \beta \in \mathbb{C} \tag{17}$$

For $\beta = 1$ we get the classical one-parameter Mittag-Leffler function $E_{\alpha,1}(z)$ defined by (12).

From the definitions (12) and (17) it follows that [30,31,33]

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \tag{18}$$

$$E_{1,2} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)} = \frac{e^z - 1}{z} \tag{19}$$

The corresponding Laplace transforms are [30,31,33]

$$\mathcal{L}\left[t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\right] = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda} = \frac{s^{-\beta}}{1 + \lambda s^{-\alpha}} \tag{20}$$

$$\mathcal{L}\left[t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(-\lambda t^{\alpha})\right]=\frac{k!s^{\alpha-\beta}}{(s^{\alpha}-\lambda)^{k+1}},\quad k=0,1,2,\ldots \tag{21}$$

$E_{\alpha,\beta}(-z)$ is completely monotonic [34,35] for any $\alpha\in(0,1]$, precisely if and only if $\alpha\in(0,1]$ and $\beta\geq\alpha$ [36].

The integral of $E_{\alpha,\beta}(t)$ is

$$\int_0^t E_{\alpha,\beta}(\lambda t^{\alpha})t^{\beta-1}dt=t^{\beta}E_{\alpha,\beta+1}(\lambda t^{\alpha}),\quad \beta>0 \tag{22}$$

Applying the Riemann–Liouville fractional derivative to (17) we get [33]

$${}_0D_t^{\mu}\left[t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\lambda t^{\alpha})\right]=t^{\alpha k+\beta-\mu-1}E_{\alpha,\beta-\mu}^{(k)}(\lambda t^{\alpha}) \tag{23}$$

When, $k=0$, $\lambda=1$ and $\mu=m$ is an integer we get [33]

$$\frac{d^m}{dt^m}\left[t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})\right]=t^{\beta-m-1}E_{\alpha,\beta-m}(t^{\alpha}),\quad m=1,2,3,\ldots.. \tag{24}$$

For $m=1$ we have

$$\frac{d}{dt}\left[t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})\right]=t^{\beta-2}E_{\alpha,\beta-1}(t^{\alpha}) \tag{25}$$

When $\beta=1$ (one parameter Mittag-Leffler function), then (25) yields

$$\frac{d}{dt}[E_{\alpha,1}(t^{\alpha})]=\frac{E_{\alpha,0}(t^{\alpha})}{t} \tag{26}$$

In this context, the first derivative of $E_{\alpha}(-t^{\alpha})$, defined by (13), and the first derivative of $E_{\alpha,\beta}(-t^{\alpha})$ are related as follows [30]

$$\frac{d}{dt}[E_{\alpha,\alpha}(-t^{\alpha})]=t^{-(1-\alpha)}E_{\alpha,\alpha}(-t^{\alpha})=-\frac{d}{dt}E_{\alpha}(-t^{\alpha}) \tag{27}$$

For $\lambda\neq1$ we have [33]

$$\frac{d}{dt}[E_{\alpha,1}(\lambda t^{\alpha})]=\frac{E_{\alpha,0}(\lambda t^{\alpha})}{t} \tag{28}$$

That is

$$\int_0^t\frac{d}{dt}[E_{\alpha,1}(\lambda t^{\alpha})]=\frac{E_{\alpha,0}(\lambda t^{\alpha})}{t} \tag{29}$$

2.1.3. Three-Parameter Mittag-Leffler Function

Prabhakar [37] introduced the function (see also [38])

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{k=0}^{\infty}\frac{\gamma_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!},\quad \alpha,\beta,\gamma\in\mathbb{C},\quad \mathrm{Re}(\alpha)>0 \tag{30}$$

where γ_k is the Pochhammer symbol

$$\gamma_0=1,\quad \gamma_k=\gamma(\gamma+1)\cdots(\gamma+k-1)=\frac{\Gamma(\gamma+k)}{\Gamma(\gamma)},\quad k=1,2,\ldots. \tag{31}$$

For $\gamma=1$ (30) reduces to (17). For $\gamma=\beta=1$ we get the one-parameter Mittag-Leffler function (12).

The three-parameter Mittag-Leffler function $E_{\alpha,\beta}^{\gamma}(-z)$ is completely monotone (CM) for $0<\alpha\leq1$ and $\beta\geq\alpha\gamma$ [39,40], precisely when $0<\alpha\gamma\leq\beta\leq1$ [40].

The differentiation of the generalized Mittag-Leffler function (30) [38] through the product $z^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda z^{\alpha})$ leads to

$$\frac{d^m}{dz^m}\left[z^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda z^{\alpha})\right] = z^{\beta-m-1}E^{\gamma}_{\alpha,\beta-m}(\lambda z^{\alpha}) \tag{32}$$

For $\gamma = 1$ we have

$$\frac{d^m}{dz^m}\left[z^{\beta-1}E_{\alpha,\beta}(\lambda z^{\alpha})\right] = z^{\beta-m-1}E_{\alpha,\beta-m}(\lambda z^{\alpha}) \tag{33}$$

and for $m = 1$ the differentiation of $E^{\gamma}_{\alpha,\beta-m}(z)$ is [41]

$$\frac{d}{dz}\left[z^{\beta-1}E_{\alpha,\beta}(\lambda z^{\alpha})\right] = z^{\beta-2}E_{\alpha,\beta-1}(\lambda z^{\alpha}) \tag{34}$$

For $\beta = 1$ we get

$$\frac{d}{dt}\left[E^{\gamma}_{\alpha,1}(\lambda t^{\alpha})\right] = \frac{E^{\gamma}_{\alpha,0}(\lambda t^{\alpha})}{t} \tag{35}$$

and for $\gamma = 1$ we get (27).

2.1.4. Prabhakar Kernel

Prabhakar [37] studied a generalized Mittag-Leffler function (hereafter termed as Prabhakar kernel) (see also [40])

$$e^{\gamma}_{\alpha,\beta}(t) = t^{\beta-1}E^{\gamma}_{\alpha,\beta}(-t^{\alpha}), \quad t \geq 0 \tag{36}$$

The Laplace transform of $e^{\gamma}_{\alpha,\beta}(t)$, for some particular values of the parameters is [40,42,43]

$$\mathcal{L}\left[e^{\gamma}_{\alpha,\beta}(t)\right] = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}+1)^{\gamma}}, \quad \Re(s) > 0, \quad |s^{\alpha}| > 1 \tag{37}$$

For $\lambda \neq 1$ the Laplace transform and its inverse are [44]

$$\mathcal{L}\left[e^{\gamma}_{\alpha,\beta}(\pm\lambda t^{\alpha})\right] = \frac{s^{\alpha\gamma-\beta}}{(s\mp\lambda)^{\gamma}}, \quad \mathcal{L}^{-1}\left[\frac{s^{\alpha\gamma-\beta}}{(s\mp\lambda)^{\gamma}}\right] = t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\pm\lambda t^{\alpha}) \tag{38}$$

The function $e^{\gamma}_{\alpha,\beta}(t)$ is locally integrable and completely monotone if the conditions $0 < \alpha \leq 1$ and $0 < \alpha\gamma \leq \beta \leq 1$ are satisfied [45].

Asymptotically, for $t \to \infty$ we have [40]

$$e^{\gamma}_{\alpha,\beta}(t) \sim \begin{cases} \dfrac{t^{\beta-\alpha\gamma-1}}{\Gamma(\beta-\alpha\gamma)}, & 0 < \alpha\gamma < \beta \leq 1 \\ -\gamma\dfrac{t^{\beta-\alpha\gamma-1}}{\Gamma(\beta-\alpha\gamma)}, & 0 < \alpha\gamma = \beta \leq 1 \end{cases} \tag{39}$$

The integral of $e^{\gamma}_{\alpha,\beta}(\lambda t)$ for any $t \in \mathbb{R}_+$ and $\lambda \in \mathbb{C}$ is [43]

$$\int_0^t u^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda u^{\alpha})du = t^{\beta}E^{\gamma}_{\alpha,\beta-1}(\lambda t^{\alpha}) \tag{40}$$

The $m-th$ repeated integration of $e^{\gamma}_{\alpha,\beta}(\lambda t) = t^{\beta-1}E^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha})$ [41]

$$I^m_0\left[E^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha})\right] = t^{\beta+m-1}E^{\gamma}_{\alpha,\beta+m}(\lambda t^{\alpha}) \tag{41}$$

In addition, for any $m \in \mathbb{N}$ the differentiation is is defined by (32) and in terms of time t is [43]

$$\frac{d^m}{dt^m}\left[t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha})\right] = t^{\beta-m-1}E^{\gamma}_{\alpha,\beta-m}(\lambda t^{\alpha}) \tag{42}$$

For $m = 1$ the differentiation of $t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha})$ is (33) [41]

$$\frac{d}{dt}\left[t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha})\right] = t^{\beta-2}E^{\gamma}_{\alpha,\beta-1}(\lambda t^{\alpha}) \tag{43}$$

2.2. Fractional Operators with Mittag-Leffler-Type Kernels: Definitions and Interrelationships

Here, features of the main "artists" in this study employing different members of the Mittag-Leffler function family will be briefly encompassed. It is worth mentioning for a better understanding of the approach taken in the current work that all these fractional operators have been introduced constitutively, without proof of where they come from [38,40–43,45–49]. To be correct, the procedure is straightforward: we select a kernel, then insert it into constructions such as the Riemann–Liouville integral or the Caputo derivative [31,50] to obtain new operators. The sources contain no words that explain how these memories (kernels) are linked to certain physical models.

The three-parameter Mittag-Leffler function (30), as we mentioned, is CM's entire function, and we focus the attention on the so-called Prabhakar kernel (36), which in particular reduces to [51]

$$e^{1}_{1,1}(\lambda t) = \exp(\lambda t), \quad e^{0}_{1,1}(\lambda t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad e^{\gamma}_{\alpha,\beta}(\lambda = 0, t) = \frac{t^{\beta-1}}{\Gamma(\beta)} \tag{44}$$

2.2.1. Prabhakar Integral

The Prabhakar integral with base point 0 is defined as [41,51] (in a Riemann–Liouville sense)

$$^{P}I^{\gamma}_{\alpha,\beta,\lambda}f(t) = \int_0^t e^{\gamma}_{\alpha,\beta}(\lambda, t-\tau)f(\tau)d\tau, \quad f(t) \in L^1(0,1) \tag{45}$$

Moreover, $^{P}I^{\gamma}_{\alpha,\beta,\lambda}f(t)$, for $f(t) \in L^1(0,1)$ can be presented as [51]

$$^{P}I^{\gamma}_{\alpha,\beta,\lambda}f(t) = \sum_{k=0}^{\infty}\frac{\gamma_k}{k!}\lambda^k I^{\alpha k+\beta}f(t) \tag{46}$$

That is $^{P}I^{\gamma}_{\alpha,\beta,\lambda}f(t)$ is linear and bounded from $L^p(0,1)$ into $L^p(0,1)$ for any $1 \leq p \leq \infty$ [51]. Recalling that for $\lambda = -1$ we get $e(-1,t) = t^{\beta-1}E^{\gamma}_{\alpha,\beta}(-t^{\alpha})$ as CM function, if $0 < \alpha\gamma \leq \beta \leq 1$ as well as [41]

$$^{P}I^{\gamma}_{\alpha,\beta,\lambda}\left[e^{\omega}_{\alpha,\mu}(\lambda t)\right] = e^{\omega+\gamma}_{\alpha,\mu+\beta}(\lambda t) \tag{47}$$

2.2.2. Prabhakar Derivatives

The Prabhakar derivative in the Riemann–Liouville sense is defined as [41,49,51]

$$^{RLP}D^{\gamma}_{\alpha,\beta,\lambda}f(t) = \frac{d}{dt}\left[^{P}I^{-\gamma}_{\alpha,1-\beta,\lambda}f(t)\right] \tag{48}$$

The Prabhakar derivative in the Caputo sense is defined as [41,49,51]

$$^{CP}D^{\gamma}_{\alpha,\beta,\lambda}f(t) = {}^{P}I^{-\gamma}_{\alpha,1-\beta,\lambda}\frac{df(t)}{dt} \tag{49}$$

In this context, for $\gamma = 0$ and $\lambda = 0$ the Prabhakar integral reduces to the Riemann–Liouville integral, [51].

$$ {}^{P}I^{0}_{\alpha,\beta,0}f(t) = \int_0^t e^{\gamma}_{\alpha,\beta}(0,t-\tau)f(\tau)d\tau = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}f(\tau)d\tau = I^{\beta}f(t) \tag{50} $$

The Laplace transform of ${}^{CP}D^{\gamma}_{\alpha,\beta,\lambda}f(t)$ is [51]

$$ \mathcal{L}\left[{}^{CP}D^{\gamma}_{\alpha,\beta,\lambda}f(t)\right] = s^{\beta-\alpha\gamma}(s^{\alpha}-\lambda)^{\gamma}\left\{\mathcal{L}\left[f(t) - m = \sum_{k=0}^{1} s^{-k-1}f^{(k)}(0+)\right]\right\} \tag{51} $$

With m defined as the integer part of β. For $m = 0$ (also for $m = 1$) we have [51]

$$ \mathcal{L}\left[{}^{CP}D^{\gamma}_{\alpha,\beta,\lambda}t^{n}\right] = s^{\beta-\alpha\gamma}(s^{\alpha}-\lambda)^{-\gamma}\frac{\Gamma(n+1)}{s^{n+1}} \tag{52} $$

2.2.3. Caputo Derivative

Considering the Caputo derivative [31,50]

$$ {}^{C}D^{\alpha}_{t} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{df(\tau)}{d\tau}d\tau \tag{53} $$

and its Laplace transform

$$ \mathcal{L}\left[{}^{C}D^{\alpha}_{t}t^{n}\right] = \frac{\Gamma(n+1)}{s^{n-\alpha+1}}, \quad \alpha > 0 \tag{54} $$

we can see that for $f(t) = t^{n}$, we get [51]

$$ \mathcal{L}\left[{}^{CP}D^{\gamma}_{\alpha,\beta,0}t^{n}\right] = \mathcal{L}\left[{}^{CP}D^{0}_{\alpha,\beta,\lambda}f(t)\right] = \frac{\Gamma(n+1)}{s^{n-\beta+1}} = \mathcal{L}\left[{}^{C}D^{\beta}t^{n}\right] \tag{55} $$

That is, comparing the Laplace transforms (52) and (54), we see that the Caputo derivative (53) is a particular case of Prabhakar fractional derivative (49).

2.2.4. Atangana–Baleanu Derivative

Further, the Atangana–Baleanu derivative in the Caputo sense [52]

$$ {}^{ABC}D^{\alpha}_{t}f(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right)\frac{df(\tau)}{d\tau}d\tau \tag{56} $$

With a normalization function $M(\alpha)$ such that $M(0) = M(1) = 1$, and we may see that $\lambda = \alpha/(1-\alpha)$, has a Laplace transform, when $f(t) = t^{n}$, can be presented as [51]

$$ \mathcal{L}\left[{}^{ABC}D^{\alpha}_{t}t^{n}\right] = \frac{M(\alpha)}{1-\alpha}\frac{1}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\frac{\Gamma(n+1)}{s^{n-\alpha+1}} \tag{57} $$

Thus, we can see that ${}^{ABC}D^{\alpha}_{t}f(t)$ is a particular case of ${}^{CP}D^{\gamma}_{\alpha,\beta,\lambda}f(t)$. Recall that $E^{1}_{\alpha,1}(z) = e^{1}_{\alpha,1}(z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+1)}$ (12).

2.2.5. Caputo–Fabrizio Derivative

Finally, let us consider the simplest case with the Caputo–Fabrizio derivative [53] (with $M(\alpha) = 1$ [54]

$$ {}^{CF}D^{\alpha}_{t} = \frac{1}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)\frac{df(\tau)}{d\tau}d\tau \tag{58} $$

has a Laplace transform in case with $f(t) = t^n$ [51]

$$\mathcal{L}\left[{}^{CF}D_t^{\alpha} t^n\right] = \frac{\Gamma(n+1)}{s^n} \frac{1}{\alpha + (1-\alpha)s} \tag{59}$$

Then, considering $\mathcal{L}\left[{}^{CP}D^{\gamma}_{\alpha,\beta,\lambda} t^n\right]$ for $\alpha = 1$, $\beta = 0$, $\gamma = -1$, and $\lambda = \alpha/(1-\alpha)$ we get [51].

$$\mathcal{L}\left[{}^{CP}D^{1}_{1,0,\alpha/(1-\alpha)}\, t^n\right] = \frac{\Gamma(n+1)}{s^n} \frac{1}{s + \frac{\alpha}{1-\alpha}} = (1-\alpha)\mathcal{L}\left[{}^{CF}D_t^{\alpha} t^n\right] \tag{60}$$

That is, the Caputo–Fabrizio derivative is a particular case of the Prabhakar derivative of the Caputo-sense. Recall that $\exp(z) = E^1_{1,1}(z) = e^1_{1,1}(z)$.

All these interrelationships between the derivatives of the Caputo sense naturally come from the interrelationships between the Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}$ and the Prabhakar kernel $e^{\gamma}_{\alpha,\beta}$ when the parameters α ,β and γ take different values as was discussed in Section 2.1.

In the sequel, we can see how these fractional derivatives logically appear through constitutive equations based on the fading memory formalism when the primary relaxation function (memory) is defined as the first derivative of a certain kernel based on the Mittag-Leffler type function.

After this brief exposition of interrelationships between fractional operators based on the Mittag-Leffler function, we highly recommend the analyses in [55–59] where more deep relations are developed. At this end, the naturally arising question is: Despite the mathematical correctness of the generalized operators, their applicability to real-world physical problems is still questionable, albeit some successful steps in this direction have been done [40,41,48,49,60–62].

2.3. The Fractional Order in Caputo–Fabrizio and Atangana–Baleanu Derivatives

Caputo–Fabrizio Fractional Operator: The Fractional Parameter

In the definition of the Caputo–Fabrizio operator, the stretched time $(t-s)$ is multiplied by a dimensional factor $\lambda = \alpha/(1-\alpha)$ which should have a dimension s^{-1} while actually, it should be dimensionless because physically α is a dimensionless parameter. By a nondimesionalization of the exponential function with the help of characteristic time of the relaxation process t_0, namely

$$\exp\frac{(t-s)}{\tau} = \exp\frac{(t/t_0 \; - s/t_0\;)}{\tau/t_0} = \exp\frac{(\bar{t}-\bar{s})}{\bar{\tau}} \tag{61}$$

where τ is the relaxation time and in the context used earlier $\tau = 1/\lambda$ in the definition (58) (see also the explanations about the construction of (59) and has a dimension $1/time$.

This nondimesionalization does not change the meaning of the exponential relaxation function but avoids any doubts about the definition of the fractional order α as [63].

$$\frac{1-\alpha}{\alpha} = \frac{\tau}{t_0} \Rightarrow \alpha = \frac{1}{1+\tau/t_0} \tag{62}$$

The Atangana–Baleanu derivative of Caputo sense (ABC) with $B(\alpha) = 1$ can be rescaled as [64]

$$\begin{gathered} {}^{ABC}D^{\alpha}_{a+} f(t) = \frac{1}{1-\alpha}\int_0^z \frac{df(\bar{s})}{d\bar{s}} E_{\alpha}\left[-\left(\frac{\bar{t}-\bar{s}}{\bar{\tau}}\right)^{\alpha}\right] d\bar{s}, \\ \frac{1-\alpha}{\alpha} = \left(\frac{\tau}{t_0}\right)^{\alpha} = (\bar{\tau})^{\alpha}, \quad \bar{t} = \frac{t}{t_0}, \quad z = s/t_0 \end{gathered} \tag{63}$$

In detail, we have

$$
{}^{ABC}D^{\alpha}_{a+}f(t) = \frac{1}{1-\alpha}\int_0^z \frac{df(\bar{s})}{d\bar{s}}\left\{\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+1)}\left[-\left(\frac{\bar{t}-\bar{s}}{\bar{\tau}}\right)^{\alpha}\right]^k\right\}d\bar{s} \tag{64}
$$

Therefore, the argument of the Mittag-Leffler kernel $E_\alpha(-Z)$ is $Z = [\alpha/(1-\alpha)](t-s)^\alpha$ and consequently we get [64]

$$
\left(\frac{1-\alpha}{\alpha}\right)^k = \left(\frac{\tau}{t_0}\right)^k \Rightarrow \alpha = \frac{1}{1+(\tau/t_0\ )} \tag{65}
$$

The results are the same as established for the Caputo–Fabrizio operator [64–66].

Hence, the fractional order α introduced through $\lambda = \alpha/(1-\alpha)$ in the generalized formulation of the derivatives has a physical meaning and is strongly related to the characteristic time scales of the relaxation process modeled. In this context, the case with the Caputo–Fabrizio operator is more physically clear due to the large background of exponentially decaying processes.

3. Fading Memory Approach (Formalism) and Concept of Model Development

3.1. Fading Memory Concept

The Boltzmann linear superposition functional [16] for hereditary viscoelasticity with a time-dependent memory kernel (correlation function) $R(t,\tau)$ is

$$
\varphi(x,t) = A_1[v_x(x,t)] + A_2\int_0^t R(x,t-\tau)\,v_x(\tau)d\tau \tag{66}
$$

The fading memory concept relating the flux to its gradient, for simple materials [67–69] is modeled by the following integrodifferential equation

$$
j(x,t) = -A_1\frac{\partial C(x,t)}{\partial x} - A_2\int_{-\infty}^t R(t-\tau)\frac{\partial C(x,\tau)}{\partial x}d\tau \tag{67}
$$

as a manifestation of the Boltzmann linear superposition functional 66). In (67) the transport coefficients A_1 and A_2 are diffusivities. The history value problem for (67) addresses the following integral [69]

$$
d(t) = \int_{-\infty}^0 R(t-\tau)\frac{\partial C(x,\tau)}{\partial x}d\tau \tag{68}
$$

allowing to give a function $C(x,t)$ on $(-\infty < t \le 0)$ From (67) and (68) it follows that

$$
\frac{\partial}{\partial x}j(x,t) = -A_1\frac{\partial^2 C(x,t)}{\partial x^2} - A_2\int_0^t R(t-\tau)\frac{\partial^2 C(x,\tau)}{\partial x^2}d\tau + \frac{\partial}{\partial x}d(t) \tag{69}
$$

Since $C(x,t)$ is a causal function (vanishing for $t < 0$) and considered only for $0 < t < \infty$ we accept $d(t) = 0 \Rightarrow \frac{\partial}{\partial x}d(t) = 0$, and therefore (69) can be rewritten as

$$
\frac{\partial}{\partial x}j(x,t) = -A_1\frac{\partial C(x,t)}{\partial x} - A_2\int_0^t R(t-\tau)\frac{\partial C(x,\tau)}{\partial x}d\tau \tag{70}
$$

Reducing (69) and setting the lower terminal of the memory integral to zero have deep physical meaning when applying hereditary integrals. The essence is well expressed by Hilfer in [70] that in fractional operators, the time *is not the chronological time* (*instant time*) but *the intrinsic time* of the process (the *time of duration*), *starting at the point accepted as* $t = 0$. Hence, in the context of the reduction of (69) to (70), *there is no process before* $t = 0$.

The deep thermodynamic sense of the fading memory formulation is that the non-locality represented by the convolution term works for short times, while for long times, we get the first term in (70), i.e., the instant reaction of the system. Moreover, models constructed with the fading memory principle obey: the *causality principle* (through the convolution term) [26–28], thermodynamic consistency [71,72], and *model observability* (objectivity) [73–75].

Further, if $A_1 = 0$, that is, the process modeled has no long-time asymptotically stable state, the result is a model close to the Continuum Time Random Walk (CTRW) without a stationary state [76,77], and applying the continuity equation to such flux, we get

$$\frac{\partial C(x,t)}{\partial t} = \frac{\partial}{\partial x} j(x,t) \Rightarrow \frac{\partial C(x,t)}{\partial t} = A_2 \int_0^t R(t-\tau) \frac{\partial C(x,\tau)}{\partial x} d\tau \tag{71}$$

To a greater extent, the model (71) mimics the master equation [77,78]; this is only a remark, and we will not elaborate on it further in this study but refer readers to [79–81] where this issue was developed in detail regarding Prabhakar-type memory.

Remark 5. *Fading memory concept in the case of heat conduction is well applicable to the so-called* simple materials *[21,82] where* the flux is proportional to the temperature gradient *(the term was coined after the work of Storm [82]), as in all examples studied here.*

3.2. Model Build-Up Concept

Let us take a look at the flux definition of (67) where the relaxation (memory) function can be presented as $M(t) = \delta(t) + R(t)$ such that (67) expressed as a convolution product

$$j(x,t) = \left[-\frac{\partial C(x,t)}{\partial x}\right] * M(t) = \left[-\frac{\partial C(x,t)}{\partial x}\right] * [A_1\delta(t) + A_2 R(t)] \tag{72}$$

The two main requirements for $R(t)$ are: it *should be completely monotone*, regardless of the fact that it is singular or non-singular, and that *its first derivative* $dR(t)/dt = R_t(t)$, should also be completely monotone.

The two main requirements for $R(t)$ are: it *should be completely monotone*, regardless of whether it is singular or non-singular at $t0+$, and that *its first derivative*$dR(t)/dt = R_t(t)$ should also be a completely monotone function.

Then, the main step in the construction of the flux constitutive model is the use $R_t(t)$ instead $R(t)$, that is

$$j(x,t) = \left[-\frac{\partial C(x,t)}{\partial x}\right] * M(t) = \left[-\frac{\partial C(x,t)}{\partial x}\right] * [A_1\delta(t) + A_2 R_t(t)] \tag{73}$$

Let us consider the convolution term alone and use integration by parts to clarify things.

$$\begin{aligned} &\int_{-\infty}^{t} R_t(t-\tau)\frac{\partial C(x,\tau)}{\partial x} d\tau = \int_{-\infty}^{t} \frac{\partial C(x,t)}{\partial x} dR(t-\tau) = \\ &= \frac{\partial C(x,\tau)}{\partial x} dR(t-\tau)\Big|_{-\infty}^{t} - \int_{-\infty}^{t} R(t-\tau)\frac{d}{d\tau}\left[\frac{\partial C(x,\tau)}{\partial x}\right] \end{aligned} \tag{74}$$

The term $= \frac{\partial C(x,t)}{\partial x} dR(t-\tau)\Big|_{-\infty}^{t}$ is zero when we have: $R(t)$ is a *causal function* and the *initial conditions corresponding to a virgin material* (medium) are $C(x,0) = C(0,0) = C(\infty,t) = C(-\infty,t)$, $C_x(x,0) = C_{xx}(x,0) = 0$ (signaling problem). As a result, we get

$$\int_{-\infty}^{t} R_t(t-\tau)\frac{\partial C(x,\tau)}{\partial x}d\tau = -\int_{-\infty}^{t} R(t-\tau)\frac{d}{d\tau}\left[\frac{\partial C(x,\tau)}{\partial x}\right]d\tau \tag{75}$$

Denoting $\frac{\partial}{\partial x}C(x,t) = f(x,t)$ and setting the lower terminal at zero, we get a constriction of Caputo-type fractional derivative that is

$${}^{C}D_t^{\alpha} f(x,t) = \frac{M(\alpha)}{N(\alpha)}\int_{0}^{t} R(t-\tau)\frac{df(x,\tau)}{d\tau}d\tau \tag{76}$$

where the pre-factor $M(\alpha)/N(\alpha)$ that only takes into account fractional order, but it is determined by the type of memory kernel (see Section 2.2 and the developments in the sequel).

The flux can therefore be expressed as

$$j(x,t) = -A_1\frac{\partial C(x,t)}{\partial x} - A_2{}^{C}D_t^{\alpha}\left[\frac{\partial C(x,t)}{\partial x}\right] \tag{77}$$

In this case, the hereditary term is generically expressed as a fractional derivative of the Caputo type without any kernel-specific information. Then, applying the continuity equation $\frac{\partial C}{\partial t} = -\frac{\partial q}{\partial x}$ we get

$$\frac{\partial C(x,t)}{\partial t} = A_1\frac{\partial C(x,t)}{\partial x} + A_2{}^{C}D_t^{\alpha}\left[\frac{\partial C(x,t)}{\partial x}\right] \tag{78}$$

For $\alpha = 1$, this model reduces to the local diffusion equation.

Remark 6. *Specific comment is required regarding the following exposition before moving on to the modeling method. Assuming that the kernel of the most straightforward scenario is built first, the model build-ups are demonstrated inductively. The model build-ups are illustrated deductively, which suggests that the kernel of the most straightforward scenario is constructed first. When the same approach is used to handle ever-more complex kernels, the resulting fractional operator structures gradually become more complex. The approach taken here is more instructive than the one frequently used in the literature, which starts with the most complex kernel and gradually creates simpler kernels by lowering the complexity parameters.*

4. Heat Conduction Models with Mittag-Leffler-Type Memories: Examples

4.1. Heat Conduction with Infinite Flux Speed (Local in Time)

Here we start with simple heat conduction, local in time, which will allow us to see how the approach developed here upgrades it towards non-local versions expressed in terms of fractional operators. The energy conservation (continuity) equation is

$$\frac{\partial\left(\rho C_p T\right)}{\partial t} = -\frac{\partial q}{\partial x} \tag{79}$$

If the constitutive equation is (Fourier law)

$$q(x,t) = -k\frac{\partial T(x,t)}{\partial x} \tag{80}$$

we get the local in-time heat conduction model (Fourier model)

$$\frac{\partial\left(\rho C_p T\right)}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \tag{81}$$

or with constant density $\rho = const.$ and heat capacity $C_p = const.$ as

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}, \quad a = \frac{k}{\rho C_p} \tag{82}$$

where $a = \frac{k}{\rho C_p}$ is the thermal diffusivity.

4.2. Heat Flux with Exponential Memory Kernel

Consider a virgin semi-infinite material (medium) with initial and boundary conditions (signaling problem)

$$T(x,0) = T(\infty,t) = T(-\infty,t) = 0, \quad T_x(x,0) = T_{xx}(x,0) = 0, \quad T(0,t) = T_s \tag{83}$$

These conditions will be valid for all examples developed in the sequel.

We start with the simplest case when the kernel is

$$e^1_{1,1}(-\lambda t) = \exp(-\lambda t), \quad \frac{d}{dt}\left[e^1_{1,1}(-\lambda t)\right] = -\lambda \exp(-\lambda t) \tag{84}$$

which will allow an easier understanding of the technology of the solution. This development follows the results of [66].

Hence, we consider an exponential memory (Jeffrey's kernel) $R(t)$ [83] with a finite relaxation time τ (and $\lambda = 1/\tau$)

$$R(t-s) = \exp\left(-\frac{t-s}{\tau}\right), \quad R_t(t) = -\frac{1}{\tau}\exp\left(-\frac{(t-s)}{\tau}\right) \tag{85}$$

If we now apply the energy balance (79) (with $\rho C_p = const.$), the result is the Cattaneo equations [23] modeling only the relaxation part of the heat conduction (the RHS of (86))

$$\frac{\partial T(x,t)}{\partial t} = -\frac{a_2}{\tau}\int_0^t \exp\left[-\left(\frac{t-s}{\tau}\right)\right]\frac{\partial T(x,s)}{\partial x}ds, \quad a_2 = \frac{k_2}{\rho C_p} \tag{86}$$

The Fourier law is the Cattaneo equation's limit $\tau \to 0$ resulting in (82)

Now, let us accept the construction of the relaxation kernel (73), namely

$$M(t) = k_1\delta(t) + k_2 R_t(t) \Rightarrow M(t-s) = k_1\delta(s) + k_2\frac{1}{\tau}\exp\left(-\frac{(t-s)}{\tau}\right) \tag{87}$$

In (87) $\delta(s)$ is the Dirac delta function, while a_1 and a_2 are the *effective thermal conductivity* and the *elastic conductivity*. Then, the heat flux can be expressed as

$$q(x,t) = -k_1\frac{\partial T(x,t)}{\partial x} - \frac{k_2}{\tau}\int_{-\infty}^t e^{-\left(\frac{t-s}{\tau}\right)}\frac{\partial T(x,s)}{\partial x}ds \tag{88}$$

The energy balance (79) yields

$$\frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial^2 T(x,t)}{\partial x^2} + a_2\lambda\int_{-\infty}^t e^{-\lambda(t-s)}\frac{\partial^2 T(x,s)}{\partial x^2}ds, \quad \lambda = \frac{1}{\tau} \tag{89}$$

Here $a_1 = k_1/\rho C_p$, $a_2 = k_2/\rho C_p$ are thermal diffusivities: they have equal dimensions $\left[m^2/s\right]$ because the exponential second term in (87) is dimensionless.

Denoting $F(x,t) = \partial^2 T(x,t)/\partial x^2$, for the sake of the simplicity of the expressions, and after integration by parts in the second term in (89) we get [66].

$$\lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} F(x,s)ds = e^{-\lambda(t-s)}[F(x,s) - F(x,t)]\Big|_{-\infty}^{t} + \\ + \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)}[F(x,t) - F(x,s)]ds \tag{90}$$

The first term in the RHS of (90), is zero, but the second one matches the definition of the Caputo–Fabrizio fractional derivative [53] (see also the general construction (75) of a Caputo-type derivative). Precisely, we have the formulation [53]

$$^{CF}D_t^{\alpha} = \frac{\alpha}{(1-\alpha)^2} \int_{-\infty}^{t} [f(t) - f_a(s)] \exp\left[-\frac{\alpha}{1-\alpha}(t-s)\right] ds, \quad t > 0 \tag{91}$$

This can be considered as a *pro-Caputo* (non-normalized) derivative denoted as $_{PC}D_t^{\lambda}$. It can be expressed in two equivalent forms (following the notations used in [53] namely [66]

$$_{PC}D_t^{\lambda} F(x,t) = \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)}[F(x,t) - F(x,s)]ds = \\ = \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} \frac{dF(x,s)}{dt} ds \tag{92}$$

where the rate constant $\lambda \in (0,\infty)$ controls the kernel. If we like to satisfy the conditions: $\alpha \in [0.1] \Rightarrow 1/\lambda \in [0,\infty]$ we get $\lambda(\alpha) = \alpha/(1-\alpha)$: the desired properties are obtained, namely

$$\frac{1}{\lambda} = \frac{1-\alpha}{\alpha} \in [0,\infty], \quad \alpha = \frac{1}{1+1/\lambda} = \frac{\lambda}{1+\lambda} \in [0,1], \quad \frac{\alpha}{(1-\alpha)^2} = \frac{\lambda}{(1-\alpha)} \tag{93}$$

Further, following the definition of the Caputo–Fabrizio derivative [53] and considering the lower limit of integral at zero, we have [66]

$$^{CF}D_t^{\alpha} T(x,t) = \frac{N(\sigma)}{\sigma}\, {}_{PC}D_t^{\beta} T(x,t) = \frac{N(\sigma)}{\sigma} \int_{0}^{t} e^{\frac{-\alpha}{1-\alpha}(t-s)} \frac{dF(x,s)}{dt} ds = \\ = \frac{M(\alpha)}{1-\alpha} \int_{0}^{t} e^{\frac{-\alpha}{1-\alpha}(t-s)} \frac{dF(x,s)}{dt} \tag{94}$$

In the terms used here $\sigma = 1/\lambda = (1-\alpha)/\alpha \in [0,\infty]$, while $N(\alpha)$ and $M(\alpha)$ are normalization functions [53] (see (58) and (76)).

Turning on (89) in terms of $T(x,t)$ and taking into account the last expression of (94), we get [66]

$$\frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,t)}{\partial x^2} + a_2(1-\alpha)^{CF}D_t^{\alpha} \frac{\partial^2 T(x,t)}{\partial x^2}, \quad t > 0 \tag{95}$$

Equation (95) models transient heat conduction with a damping term expressed through the Caputo–Fabrizio fractional derivative. For $\alpha = 1$ we get the Fourier Equation (82).

If the Jeffrey kernel is only considered (in the present context (84)), for instance, then the equivalent form of (95) is (86) accounting only for the elastic part of heat diffusion and the result in terms of $_{PC}D_t^{\alpha}$ is [66].

$$\frac{\partial T(x,t)}{\partial t} = -a_2\beta \int_0^t \exp[-\beta(t-s)]\frac{\partial T(x,s)}{\partial x}ds \Rightarrow$$
$$\Rightarrow \frac{\partial T(x,t)}{\partial t} = -a_2\frac{\alpha}{1-\alpha}{}_{PC}D_t^{\alpha}\left[\frac{\partial T(x,s)}{\partial x}\right] \tag{96}$$

As a result, we could see that the Caputo–Fabrizio derivative of a damping term was produced directly by the fading memory approach.

Remark 7. *It is worth noting and drawing attention to an incorrect interpretation of the Caputo–Fabrizio derivative based on its exponential kernel. As stated by Tarasov and Tarasova [84]: " Note that the memoryless property of the exponential distribution allows us to state that the differential operator (eq.6 in [84]) (i.e., the Caputo–Fabrizio derivative) cannot be used to describe processes with memory " (sic!).*

As to the memoryless properties of the exponential function, this is a well-known fact from the students' textbooks. However, the memory kernel should be a complete monotone, causal function, *and thermodynamically consistent* **noting more**. *The exponential function obeys all of these requirements [14].* The memory properties of the non-local operator do not depend on whether the function used as a kernel has a memory or not, outside the context of the construction of the convolution integral.

The opinion about the Caputo–Fabrizio derivative in [84] could be interpreted either as a deep misunderstanding of the meaning of non-local operators or as a manipulative statement in the context of the confrontation [1–7] between some trends in fractional calculus (see the very beginning of the Introduction) rather than from a serious scientific perspective.

In addition, without continuing the discussion on the use of exponential memories, we refer to the book of Uchaikin [85] (in section 2.1.3 Memory) where the same standpoint based on memoryless behavior of the exponential function is expressed.

4.3. Heat Flux with Mittag-Leffler (One-Parameter) Memory Kernel

Now, using the relation (26) we constitute a new relaxation (memory) function for the heat flux [86]

$$R_{ML} = k_1\delta(t) + k_2\lambda\frac{E_{\alpha,0}(-\lambda t^{\alpha})}{t} = k_1\delta(t) + k_2\frac{d}{dt}[E_{\alpha,1}(-\lambda t^{\alpha})] \tag{97}$$

The second term in (97) is a monotonically decaying function, singular at $t \to 0$, and thus, obeys all necessary properties to be used as a memory kernel. Then, the heat flux with memory can be expressed as [86]

$$q(x,t) = -k_1\frac{\partial T(x,t)}{\partial x} - k_2\lambda\int_{-\infty}^{t}\frac{\partial T(x,s)}{\partial x}\frac{d}{ds}\left[E_{\alpha,1}\left(\lambda(t-s)^{\alpha}\right)\right]ds \tag{98}$$

Now, the integration by parts of the integral (denoting $F(x,t) = \partial T(x,t)/\partial x$) in (98) yields

$$\int_{-\infty}^{t} E_{\alpha,1}\left(\lambda(t-s)^{\alpha}\right)F(x,s)d\lambda s = E_{\alpha,1}\left((t-s)^{\alpha}\right)[F(x,s)-F(x,t)]\big|_{-\infty}^{t} +$$
$$+\int_{-\infty}^{t} E_{\alpha,1}\left(\lambda(t-s)^{\alpha}\right)\frac{\partial F(x,s)}{\partial s}d\lambda s \tag{99}$$

The first term in the RHS of (99) is zero, while the second one matches the definition of the Atangana–Baleanu derivative fractional derivative (56).

The coefficient $\lambda \in (\infty, 0)$ can be mapped by (see the same relations in [66] (see (93) used earlier)

$$\lambda = \frac{\alpha}{1-\alpha} \in [0,\infty], \quad \alpha = \frac{1}{1+1/\lambda} = \frac{\lambda}{1+\lambda} \in [0,1] \tag{100}$$

Then, assuming the lower limit in the Stieltjes integral at zero (the causality principle), we have a new expression for the second term of the heat flux with a memory [86]

$$q_e = -a_2 \int_0^t E_{\alpha,1}\left[-\frac{\alpha}{1-\alpha}(t-s)^\alpha\right]\frac{d}{ds}\left[\frac{\partial T(x,s)}{\partial x}\right]ds \tag{101}$$

We can recast the integral in (101) as the elastic component of the flux q_e because it resembles how the ABC derivative is constructed. [86].

$$\begin{aligned} q_e &= -a_2(1-\alpha)\left\{\frac{1}{1-\alpha}\int_0^t E_{\alpha,1}\left[-\frac{\alpha}{1-\alpha}(t-s)^\alpha\right]\frac{d}{ds}\left[\frac{\partial T(x,s)}{\partial x}\right]ds\right\} = \\ &= -a_2(1-\alpha)^{ABC}D_t^\alpha\left[\frac{\partial T(x,t)}{\partial x}\right] \end{aligned} \tag{102}$$

Consequently

$$\begin{aligned} -\frac{\partial q_e}{\partial x} &= a_2(1-\alpha)\int_0^t E_{\alpha,1}\left[-\frac{\alpha}{1-\alpha}(t-s)^\alpha\right]\frac{d}{ds}\left[\frac{\partial^2 T(x,s)}{\partial x^2}\right]ds = \\ &= -a_2(1-\alpha)^{ABC}D_t^\alpha\left[\frac{\partial^2 T(x,t)}{\partial x^2}\right] \end{aligned} \tag{103}$$

Finally, the energy conservation equation yields [86]

$$\frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial^2 T(x,t)}{\partial x^2} + a_2(1-\alpha)^{ABC}D_t^\alpha\left[\frac{\partial^2 T(x,t)}{\partial x^2}\right] \tag{104}$$

For $\alpha = 1$, it reduces to the Fourier equation.

Based on the fading memory formalisms for the heat flux, we found that the ABC derivative spontaneously manifests itself when a suitable memory kernel (97) is incorporated into the constitutive equation.

4.4. Heat Flux with Prabhakar Memory Kernel

After the two prior examples, which used relatively simple kernels, we now attempt to create a fractional derivative using the Prabhakar kernel and the fading memory formalism. Therefore, applying (36) and its first derivative (43) we get for the heat flux with relaxation function defined as

$$R_{PB} = k_1\delta(t) + k_2\frac{d}{dt}\left[e^{\gamma}_{\alpha,\beta}(-\lambda t^\alpha)\right] = k_1\delta(t) + k_2\left[t^{\beta-2}E^{\gamma}_{\alpha,\beta-1}(-\lambda t^\alpha)\right] \tag{105}$$

Then, the heat flux can be formulated as

$$q(x,t) = -k_1\frac{\partial T(x,t)}{\partial x} - k_2\int_{-\infty}^t \frac{\partial T(x,u)}{\partial x}\frac{d}{du}\left[e^{\gamma}_{\alpha,\beta}(t,-\lambda u^\alpha)\right]du \tag{106}$$

Assuming for the sake of simplicity $\lambda = 1$, and applying integration by parts in the second term of (106), and what *mutatis mutandis*, we get

$$q(x,t) = -k_1\frac{\partial T(x,t)}{\partial x} - k_2\int_0^t e^{\gamma}_{\alpha,\beta}(t,-u^\alpha)\frac{d}{du}\left[\frac{\partial T(x,u)}{\partial x}\right]du \tag{107}$$

In general, the second term in (107) mimics the fractional derivative of the Caputo type.

Alternately, by using the Laplace transform on (106) and accounting for the convolution product in the second term, we obtain (see (38))

$$\mathcal{L}[q(x,t)] = \mathcal{L}\left[-k_1\frac{\partial T(x,t)}{\partial x}\right] - k_2\mathcal{L}\left[\frac{d}{dt}e^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha})\right] \times \mathcal{L}\left[\frac{\partial T(x,t)}{\partial x}\right] \tag{108}$$

that is

$$\begin{aligned} q(x,s) &= -k_1\frac{\partial T(x,s)}{\partial x} - k_2\left[\frac{s^{\alpha\gamma-\beta+1}}{(s+\lambda)^{\gamma}}\right] \times \frac{\partial T(x,s)}{\partial x} = \\ &= -k_1\frac{\partial T(x,s)}{\partial x} - k_2\left[\frac{s^{\alpha\gamma-\beta}}{(s+\lambda)^{\gamma}}\right] \times s\frac{\partial T(x,s)}{\partial x} \end{aligned} \tag{109}$$

The inverse transformation $\mathcal{L}^{-1}[q(x,s)]$ yields

$$q(x,t) = -k_1\frac{\partial T(x,t)}{\partial x} - k_2\int_0^t e^{\gamma}_{\alpha,\beta}(t,-\lambda u^{\alpha})\frac{d}{du}\left[\frac{\partial T(x,u)}{\partial x}\right]du \tag{110}$$

The second term in (110) has the construction of Caputo-type with the Prabhakar memory kernel (see (49)).

Assuming a normalization function $N(\alpha) = (1-\alpha)$ and setting$\lambda = \alpha/(1-\alpha)$ (110) can be reformulated as follows without loss of generality

$$\begin{aligned} &q(x,t) = -k_1\frac{\partial T(x,t)}{\partial x} - k_2(1-\alpha)\left\{\frac{1}{1-\alpha}\int_0^t e^{\gamma}_{\alpha,\beta}(t,-\lambda u^{\alpha})\frac{d}{du}\left[\frac{\partial T(x,u)}{\partial x}\right]du\right\} \\ &\lambda = \frac{\alpha}{1-\alpha}, \quad t>0 \end{aligned} \tag{111}$$

Now, applying the energy balance equation, we have

$$\begin{aligned} &\frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial T(x,t)}{\partial x} + a_2(1-\alpha)\left\{\frac{1}{1-\alpha}\int_0^t e^{\gamma}_{\alpha,\beta}(t,-\lambda u^{\alpha})\frac{d}{du}\left[\frac{\partial T(x,u)}{\partial x}\right]du\right\}, \\ &\lambda = \frac{\alpha}{1-\alpha}, \quad t>0 \end{aligned} \tag{112}$$

For $\alpha = 1$, the non-locality in (112) is absent, and it reduces to the Fourier model.

The construction

$${}^{CP}_{H}\mathfrak{D}^{\gamma}_{\alpha,\beta}f(t) = \frac{1}{1-\alpha}\int_0^t e^{\gamma}_{\alpha,\beta}(t,-\lambda u^{\alpha})\frac{d}{du}f(u)du, \quad \lambda = \frac{\alpha}{1-\alpha}, \quad 0<\alpha<1, \quad t>0 \tag{113}$$

is a fractional operator (derivative) that uses a base point zero but does not exactly match the definition (49). The differences result from the fact that (49) is formulated constitutively, mathematically correct (but with no physics behind it), and similar to the traditional Caputo derivative. They also result from the definition of the Prabhakar fractional integral (45) (in a Riemann–Liouville sense) [38,41,43,45,49].

Hence, we may present (112) as

$$\frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial T(x,t)}{\partial x} + a_2(1-\alpha)\left\{{}^{CP}_{H}\mathfrak{D}^{\gamma}_{\alpha,\beta}\left[\frac{\partial T(x,t)}{\partial x}\right]\right\} \tag{114}$$

It reduces to the Fourier model for $\alpha = 1$.

The Laplace transform of ${}^{CP}_{H}\mathfrak{D}^{\gamma}_{\alpha,\beta}f(t)$, with the assumption $\sum\limits_{k=0}^{1} s^{-k-1}f^{(k)}(0^{+}) = 0$, for the sake of simplicity, yields

$$\mathcal{L}\left[{}^{CP}_{H}\mathfrak{D}^{\gamma}_{\alpha,\beta}f(t)\right] \doteq \mathcal{L}\left[e^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha})\right] \times \mathcal{L}[f(t)] \doteq \frac{s^{\beta-\alpha\gamma}}{(s+\lambda)^{\gamma}} \times f(s) \tag{115}$$

Comparing to (38) (see also (52)), we see that the Laplace transform is the same, which is a nice result.

Furthermore, as it was shown in Section 2.2, it is easy to see, and straightforwardly developed, that for some particular cases, the kernel of ${}^{CP}_{H}\mathfrak{D}^{\gamma}_{\alpha,\beta}f(t)$ reduces to memory functions of well-known fractional operators. For instance, we obtain the Atangana–Baleanu derivative for $\gamma = \beta = 1$.

Additionally, it is evident that for $\gamma = 1$ we have $e^{1}_{\alpha,\beta,\lambda}(-\lambda t^{\alpha})$, a kernel based on the two-parameter Mittag-Leffler function

$$e^{1}_{\alpha,\beta,\lambda} = t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha}) \tag{116}$$

This enables the definition of a novel derivative (which has not yet been studied), namely

$${}^{CP}_{H}\mathfrak{D}^{1}_{\alpha,\beta}f(t) = \frac{1}{1-\alpha}\int\limits_{0}^{t} t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha},\tau)\frac{df(\tau)}{d\tau}d\tau \tag{117}$$

If $\lambda = \alpha/(1-\alpha)$, it can be aligned to the ABC definition and reduced to it for $\beta = 1$.

5. Experiments with Other Known Kernels

In this section, we clearly show how well-known memory kernels related to the Mittag-Leffler functions can be incorporated into constructions of fractional operators of Caputo-type.

5.1. Rzanitsyn Kernel

This is a composite function defined as [87–89] (singular at t_{0+})

$$M_R(\alpha,t;\lambda) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\exp(-\lambda t), \quad 0<\alpha<1 \tag{118}$$

which is a product of a particular case, the Sonine kernel $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ (known as *the kernel of fractional integration* or *Gel'fand-Shilov distribution*) [90,91] and the exponential decaying function.

It can be presented as a product of two particular versions of the Prabhakar kernel (see (44))

$$M_R(\alpha,t;\lambda) = e^{\gamma}_{\alpha,\beta}(\lambda=0,t) \times e^{1}_{1,1}(-\lambda t) \tag{119}$$

or in terms of the two-parameter Mittag-Leffler function as

$$M_R = \frac{t^{\alpha-1}}{\Gamma(\alpha)}E^{1}_{1,1}(-\lambda t) \tag{120}$$

The Laplace transform of M_R is (see Equation (25) for the special case $\alpha = \beta = 1$)

$$\mathcal{L}[M_R(\alpha,\lambda t)] = \frac{1}{\Gamma(\alpha)}\frac{1}{s+\lambda} \tag{121}$$

The derivative $\frac{d}{dt}M_R(\alpha,t;\lambda)$ is (recall the definition (25) with $\alpha = \beta = 1$) is

$$\frac{d}{dt}[M_R(\alpha,t;\lambda)] = \frac{t^{-1}}{\Gamma(\alpha)} E_{1,0}(\alpha,t;\lambda) \tag{122}$$

which is singular at t_{0+} with a Laplace transform (see (121))

$$\mathcal{L}\left[\frac{d}{dt} M_R(\alpha,\lambda,t)\right] = \frac{1}{\Gamma(\alpha)} \frac{s}{s+\lambda} \tag{123}$$

Assuming $\lambda = \frac{\alpha}{1-\alpha}$, we get

$$M_R(\alpha,t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} E^1_{1,1}\left(-\frac{\alpha}{1-\alpha}t\right), \quad 0 < \alpha < 1 \tag{124}$$

$$\frac{d}{dt}[M_R(\alpha,t)] = \frac{t^{-1}}{\Gamma(\alpha)} E_{1,0}\left(\alpha,t;\frac{-\alpha}{1-\alpha}\right), \quad 0 < \alpha < 1 \tag{125}$$

Now, we may write the heat flux constitutive equation as

$$q_R(x,t) = -k_0 \frac{\partial T(x,t)}{\partial x} - k_2 \int_0^t \frac{(t-\tau)^{-1}}{\Gamma(\alpha)} E_{1,0}\left(\alpha,t;\frac{-\alpha}{1-\alpha}\right) \frac{\partial T(x,t)}{\partial x} d\tau \tag{126}$$

Then, what is *mutatis mutandis*, we get

$$q_R(x,t) = -k_0 \frac{\partial T(x,t)}{\partial x} - k_2 \int_0^t M_R\left(\alpha,t;\frac{-\alpha}{1-\alpha}\right) \frac{d}{d\tau}\left[\frac{\partial T(x,\tau)}{\partial x}\right] d\tau \tag{127}$$

and a heat diffusion equation

$$\frac{\partial T(x,t)}{\partial t} = a_0 \frac{\partial T(x,t)}{\partial x} + a_2(1-\alpha) \int_0^t M_R\left(\alpha,t;\frac{-\alpha}{1-\alpha}\right) \frac{d}{d\tau}\left[\frac{\partial T(x,\tau)}{\partial x}\right] d\tau \tag{128}$$

With the Rzhanitsyn kernel, we, therefore, defined a fractional derivative of the Caputo type while assuming a normalization function $N(\alpha) = 1-\alpha$, namely

$$^{RC}D^{\alpha}_t f(t) = \frac{1}{1-\alpha} \int_0^t M_R\left(\alpha,t;\frac{-\alpha}{1-\alpha}\right) \frac{d}{d\tau}[f(\tau)] d\tau \tag{129}$$

5.2. Miller–Ross Kernel

The Miller–Ross function (MR) function is defined as

$$M_{MR}(\alpha,\lambda t) = t^{\alpha} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{\Gamma(\alpha+k+1)} = \sum_{k=0}^{\infty} \frac{\lambda^k t^{k+\alpha}}{\Gamma(\alpha+k+1)} t^{\alpha} \tag{130}$$

It is defined as the $D^{-\alpha}_t[e(\lambda t)] = I^{\alpha}_t[e(\lambda t)]$ as the $\alpha - th$ integral of the exponential function$e(\lambda t)$

It can be represented as the two-parameter Mittag-Leffler function by [31]

$$M_{MR}(\alpha,\lambda t) = t^{\alpha} E_{1,\alpha+1}(\lambda t), \quad 0 < \alpha < 1 \tag{131}$$

The Laplace transform of the MR function is

$$\mathcal{L}[M_{MR}(\alpha,\lambda t^{\alpha})] = \frac{1}{s^{\alpha-1}}\left(1-\frac{\lambda}{s}\right)^{-1}, \quad \left|\frac{\lambda}{s}\right| < 1 \tag{132}$$

The time derivative of the MR function is

$$\frac{d}{dt}[M_{MR}(\alpha,\lambda t)] = \frac{d}{dt}\sum_{k=0}^{\infty}\frac{\lambda^k t^{k+\alpha}}{\Gamma(\alpha+k+1)} = t^{\alpha-1}\sum_{k=0}^{\infty}\frac{(k+\alpha)\lambda^k t^{k+\alpha}}{\Gamma(\alpha+k+1)} \tag{133}$$

and through the Laplace transform , with the rule $\mathcal{L}\left[\frac{d}{dt}f(t)\right] = s\mathcal{L}[f(t)] - f(0) = sf(s) - f(0)$

$$\mathcal{L}\left[\frac{d}{dt}M_{MR}(\alpha,-\lambda t^\alpha)\right] = s\left[\frac{1}{s^{\alpha-1}}\left(1+\frac{\lambda}{s}\right)^{-1}\right] = s^{2-\alpha}\left(1-\frac{\lambda}{s}\right)^{-1}, \quad \left|\frac{\lambda}{s}\right| < 1 \tag{134}$$

because $M_{MR}(\alpha,-\lambda t^\alpha)(t=0) = 0$.

Then, if we define a relaxation function $\frac{d}{dt}[M_{MR}(\alpha,-\lambda t)] = \frac{d}{dt}[t^\alpha E_{1,\alpha+1}(-\lambda t)] = R_{MR}(\alpha,-\lambda t)$ such that $R_{MR}(\alpha,-\lambda t)$ and $M_{MR}(\alpha,-\lambda t)$ are monotonically decaying functions, we may write the heat flux constitutive equation as

$$q_{MR}(x,t) = -k_0\frac{\partial T(x,t)}{\partial x} - k_2\int_0^t (t-\tau)^\alpha E_{1,\alpha+1}\left(-\lambda(t-\tau)^\alpha\right)\frac{\partial T(x,\tau)}{\partial x}d\tau \tag{135}$$

Then, following the integration by parts in the second term of (135), and applying all steps as in the preceding examples, as well as faster by the Laplace transform of $q_{MR}(x,t)$

$$q_{MR}(x,s) = -k_0\frac{\partial T(x,s)}{\partial x} - k_2\int_0^t s\left[\frac{1}{s^{\alpha-1}}\left(1+\frac{\lambda}{s}\right)^{-1}\right]\frac{\partial T(x,s)}{\partial x}ds, \quad 0<\alpha<1 \tag{136}$$

Then, a rearrangement of the second term in (136) yields

$$\int_0^t s\left[\frac{1}{s^{\alpha-1}}\left(1-\frac{\lambda}{s}\right)^{-1}\right]\frac{\partial T(x,s)}{\partial x}ds = \int_0^t \left[\frac{1}{s^{\alpha-1}}\left(1+\frac{\lambda}{s}\right)^{-1}\right]\left[s\frac{\partial T(x,s)}{\partial x}\right] \tag{137}$$

remembering that $\frac{\partial T(x,0)}{\partial x} = 0$, as the initial condition defined at the beginning. This allows expressing the hereditary term with the MR function as a memory kernel, namely

$$q_{MR}(x,t) = -k_0\frac{\partial T(x,t)}{\partial x} - k_2\int_0^t M_{MR}\left[-\lambda(t-\tau)^\alpha\right]\frac{\partial T(x,t)}{\partial x}d\tau, \quad 0<\alpha<1, \quad t>0 \tag{138}$$

Now, without loss of generality, we may assume $-\lambda = -\frac{\alpha}{1-\alpha}$, as well as defining the normalization function $N(\alpha) = (1-\alpha)$. Thus, we may formulate a Miller–Ross fractional operator of Caputo-type, namely

$$^{MRC}D_t^\alpha f(t) = \frac{1}{1-\alpha}\int_0^t M_{MR}\left[-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right]\frac{d}{d\tau}[f(\tau)]d\tau. \tag{139}$$

Applying the energy conservation equation with the flux expressed as (138) we get

$$\frac{\partial T(x,t)}{\partial t} = a_0\frac{\partial^2 T(x,t)}{\partial x^2} + (1-\alpha)a_2\,{}^{MRC}D_t^\alpha\left[\frac{\partial T(x,t)}{\partial x}\right] \tag{140}$$

For $\alpha = 1$, Equation (140) reduces to the local Fourier equation.

To further explain the recently developed result, it is important to draw your attention to the fact that a fractional integral with MR function is defined in [92] as (in a Riemann–Liouville style) (see also [93,94])

$$ {}^{MR}I_{0+}^{\alpha,\lambda}f(t) = \int_0^t MR\left[-\lambda(t-\tau)^{\alpha}\right]f(\tau)d\tau \tag{141} $$

and based on it, a fractional Liouville–Sonine derivative was proposed as[92]

$$ {}_{LS}^{MR}D_t^{\alpha}f(t) = \frac{d}{dt}\left[{}^{MR}I_{0+}^{\alpha,\lambda}f(t)\right] = \frac{d}{dt}\int_0^t MR\left[-\lambda(t-\tau)^{\alpha}\right]f(\tau)d\tau \tag{142} $$

5.3. Rabotnov Kernel

The Rabotnov function is defined originally as [89,95]

$$ M_{Rab}(\alpha,\lambda t) = \frac{t^{\alpha-1}}{\left(\frac{1}{\lambda}\right)^{\alpha}}\sum_{k=0}^{\infty}\frac{(-1)^k(\lambda t)^{\alpha k}}{\Gamma[\alpha(k+1)]}, \quad \lambda = \frac{1}{\tau_k}, -1 < \alpha < 0 \tag{143} $$

and can be presented alternatively as [31]

$$ M_{Rab}(\alpha,\lambda) = t^{\alpha}\sum_{k=0}^{\infty}\frac{\lambda^k t^{k(\alpha+1)}}{\Gamma[(k+1)(\alpha+1)]}], \quad -1 < \alpha < 0 \tag{144} $$

or in terms of two-parameter Mittag-Leffler functions as [31]

$$ M_{Rab}(\alpha,\lambda) = t^{\alpha}E_{\alpha+1,\alpha+1}\left(\lambda t^{\alpha+1}\right), \quad M_{Rab}(\alpha,\lambda) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) \tag{145} $$

For $\alpha = 1$ the function $M_{Rab}(1,\lambda)$ reduces to the exponential function. However, when $\alpha \to 0$ it goes to $\frac{1}{2}\delta(t)$ [89].

Its time derivative, applying (25) to the second version of (145), can be expressed as

$$ \frac{d}{dt}[M_{Rab}(\alpha,\lambda)] = t^{\alpha-2}E_{\alpha,\alpha}(\lambda t^{\alpha}) \tag{146} $$

The original Rabotnov function (143) is a growing in time function, conceived to model creep (deformation, extension) of viscoelastic and elastoplastic materials [89], and in such a case, the kernel in a hereditary integral should be

$$ M_{Rab}(-\alpha,\lambda) = (t-\tau)^{-\alpha}\sum_{k=0}^{\infty}\frac{(-\lambda)^k(t-\tau)^{k(1-\alpha)}}{\Gamma[(k+1)(1-\alpha)]} \tag{147} $$

A decaying (monotonic) version was used in a new fractional operator known as the Yang–Abdel-Aty–Cattani (YAC) derivative [92,96,97] (see also [98–100])

$$ M_{YAC}(-\lambda,t^{\alpha}) = \sum_{k=0}^{\infty}\frac{(-\lambda)^k t^{(k+1)(\alpha+1)-1}}{\Gamma[(k+1)(\alpha+1)]} \tag{148} $$

Its Laplace transform is [98,99]

$$ \mathcal{L}[M_{Rab}(\lambda t^{\alpha})] = \frac{1}{s^{\alpha+1}}\frac{1}{1-\frac{\lambda}{s^{\alpha+1}}}, \quad \left|\frac{\lambda}{s^{\alpha+1}}\right| < 1 \tag{149} $$

The time derivative of $M_{Rab}(\lambda t^{\alpha})$ through the Laplace transform is

$$ \mathcal{L}\left[\frac{d}{dt}M_{Rab}(\lambda t^{\alpha})\right] = s\left(\frac{1}{s^{\alpha+1}}\frac{1}{1-\frac{\lambda}{s^{\alpha+1}}}\right) = \frac{1}{s^{\alpha}}\frac{1}{1-\frac{\lambda}{s^{\alpha+1}}}, \quad \left|\frac{\lambda}{s^{\alpha+1}}\right| < 1 \tag{150} $$

Now, constructing the heat flux constitutive equation, we get

$$q_{YAC}(x,t) = -k_1 \frac{\partial T(x,t)}{\partial x} - k_2 \int_0^t \left[\frac{d}{dt} M_{Rab}(\lambda t^\alpha) \right] \frac{\partial T(x,\tau)}{\partial x} d\tau \tag{151}$$

Then, applying the Laplace transform to both sides of (151) and taking into account (149), we have

$$q_{YAC}(x,s) = -k_1 \frac{\partial T(x,s)}{\partial x} - k_2 \int_0^t \left[\left(\frac{1}{s^{\alpha+1}} \frac{1}{1 - \frac{\lambda}{s^{\alpha+1}}} \right) \right] \left[s \frac{\partial T(x,s)}{\partial x} \right] ds \tag{152}$$

The inverse Laplace transform yields

$$q_{YAC}(x,t) = -k_1 \frac{\partial T(x,t)}{\partial x} - k_2 \int_0^t M_{Rab}(-\lambda(t-\tau)^\alpha) \frac{d}{dt} \left[\frac{\partial T(x,\tau)}{\partial x} \right] ds \tag{153}$$

Consequently, the application of the energy balance equation results in

$$\frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,t)}{\partial x^2} + a_2 \int_0^t M_{Rab} \left(-\frac{\alpha}{1-\alpha} (t-\tau)^\alpha \right) \frac{d}{dt} \left[\frac{\partial^2 T(x,\tau)}{\partial x^2} \right] d\tau \tag{154}$$

We may suggest again that $\lambda = -\frac{\alpha}{1-\alpha}$ and normalization function $N(\alpha) = 1 - \alpha$. The result is a heat conduction equation

$$\frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,t)}{\partial x^2} + a_2 (1-\alpha) {}_C^{Rab} D_t^\alpha \left[\frac{\partial^2 T(x,t)}{\partial x^2} \right] \tag{155}$$

with damping terms expressed through fractional of Caputo-type with Rabotnov kernel, defined as

$${}_C^{Rab} D_t^\alpha [f(t)] = \frac{1}{1-\alpha} \int_0^t M_{Rab} \left(-\frac{\alpha}{1-\alpha} (t-\tau)^\alpha \right) \frac{d}{dt} f(\tau) d\tau \tag{156}$$

This definition differs from the YAC derivative, where the approach uses a fractional integral of a Riemann–Liouville type [92,96,97]

$${}^{Rab} I_t^\alpha f(t) = \int_0^t M_{Rab}(-\lambda(t-\tau)^\alpha) f(\tau) d\tau, \quad f(t) \in L(0,\infty) \tag{157}$$

and a left-sided fractional derivative in the classical (termed Liouville–Caputo derivative) style is *declared, without any derivation from a physical model*, namely

$${}^{Rab} D_t^\alpha f(t) = \int_{-\infty}^t M_{Rab}(-\lambda(t-\tau)^\alpha) f^{(1)}(\tau) d\tau \tag{158}$$

and without any specification of the parameter λ.

Remark 8. *This section of the study can be summarized by pointing out that the new derivatives naturally appear when the constitutive equation for the heat flux is constructed using a proper Mittag-Leffler memory kernel and the fading memory formalism. The integer-order time derivative of the energy balance equation is unaffected by the new constitutive equation, and the fractional derivatives appear as damping terms.*

6. Heat Conduction Model Analysis

Now, after the development of six models with various time-fractional derivatives as damping terms, we have to discuss:

What are the differences between the models using some of the fractional operators discussed in this article and those published in the literature?

6.1. A Trivial Example That Is Correct but Might Be Misleading

At the beginning of this discussion, we like to stress the attention on a trivial for the fractional calculus example. Let us consider again Equation (6), assuming that $k(x) = k = const.$(time and space independent thermal conductivity, for the sake of simplicity) precisely the first version of its RHS, namely

$$q(x,t) = -k^C D_t^\alpha \left[\frac{\partial T(x,t)}{\partial x}\right] \tag{159}$$

and represent it through its corresponding fractional integral of order $1-\alpha$ as

$$q(x,t) = -kI_t^{1-\alpha} \left[\frac{\partial T(x,t)}{\partial x}\right] \tag{160}$$

This is a completely casual relationship, in a convolution form, due to the time shift and history (memory) of the gradient related to the time evolution of the flux. In addition, this formulation does not consider a long time term (that is, thermal conductivity $k_1 = 0$, while $k_2 \neq 0$). Moreover, taking into account that ${}^C D_t^{-\gamma} f = {}^C I^\gamma f$ and with $\gamma = 1-\alpha$, we get an alternative form in terms of the left Caputo derivative, namely

$$q(x,t) = -{}^C D_t^{\alpha-1} \left\{ k\left[\frac{\partial T(x,t)}{\partial x}\right]\right\} \tag{161}$$

Taking into account that ${}^C D_t^{-\gamma} f = {}^C I^\gamma f$ and with $\gamma = 1-\alpha$ we got (161) Further, since ${}^C D_t^\gamma$ is an operator left inverse of the fractional integral, we have ${}^C D_t^\gamma I^\gamma = {}^C D_t^{\gamma C} D_t^{-\gamma} f = D_t^0 f = f$.

Thus, the application of the continuity equation yields

$$\frac{\partial T(x,t)}{\partial t} = kI_t^{1-\alpha}\left[\frac{\partial^2 T(x,\tau)}{\partial x^2}\right](x,t) = k\left\{{}^C D_t^{\alpha-1}\left[\frac{\partial^2 (x,t)}{\partial x^2}\right]\right\} \tag{162}$$

Applying to both sides of (162) the Caputo operator of order $\alpha - 1$ we get

$$\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = a\frac{\partial^2 T(x,t)}{\partial x^2}, \quad a = \frac{k}{\rho C_p} \tag{163}$$

This is the well-known time-fractional diffusion equation. It was correctly derived from a causal relationship (160). It is correct from both the mathematical and thermodynamic points of view.

However, when mathematical modeling skills are lacking, this simple equation may be misleading because it can be formalistically written based on the normal diffusion equation (when $\alpha = 1$) by a simple fractional replacement of the integer-order time derivative. This is the well-known "replacement fractionalization," which is generally an incorrect operation. To be precise, such a replacement means that the continuity equation is written as

$$\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = -k\frac{\partial q(x,t)}{\partial x} \Rightarrow T(x,t) = -kI^\alpha\left[\frac{\partial q(x,t)}{\partial x}\right] \tag{164}$$

which contradicts the First Law of Thermodynamics.

In the case presented here, the manipulations were correct mathematically, but the replacement is incorrect, and the starting point should be (160) as an element of the fading memory constitutive equation.

In a general sense, when a *long time behavior is not considered* (when the memory kernel goes to zero) as a model of the process (see the comments on CTRW models where stationary states are missing in the concept), we have to use the general constitutive equation

$$\frac{\partial u(x,t)}{\partial t} = -k\frac{\partial j(x,t)}{\partial x} = \frac{\partial}{\partial x}\left\{-k\int_0^t R(\alpha,t-\tau)\frac{d}{d\tau}\left[\frac{\partial u(x,\tau)}{\partial x}\right]d\tau\right\} \tag{165}$$

The more general flux formulation with $k_1 \neq 0$, *when the long time behaviour should be taken into account*, is

$$j = -k_1\frac{\partial u(x,t)}{\partial x} - k_2\int_0^t R(\alpha,t-\tau)\frac{d}{d\tau}\left[\frac{\partial u(x,\tau)}{\partial x}\right]d\tau \tag{166}$$

Then, after the application of the continuity equation, we have

$$\frac{\partial u(x,t)}{\partial t} = k_1\frac{\partial^2 u(x,t)}{\partial x^2} + k_2\int_0^t R(\alpha,t-\tau)\frac{d}{d\tau}\left[\frac{\partial^2 u(x,\tau)}{\partial x^2}\right]d\tau \tag{167}$$

In (165)–(167) $u(x,t)$ could be concentration, temperature, or another variable (such as stress or deformation), and the non-locality is modeled by the *memory kernel* (*relaxation function*, also termed a *correlation function*), and *it is strongly dependent on the physics of the modeled process*. It would be a nice situation if the resulting fractional operator had semigroup properties, but this is not possible in some cases. The most important issue is *the correct correspondence of the memory kernel to the physical relaxation process* [64,65].

In the following section, we will examine some existing diffusion models from the literature to determine whether or not they confirm the results of the fading memory formalism.

6.2. Heat Conduction and Diffusion Models with Mittag-Leffler Kernels: An Analysis

Garra–Garrappa's Non-Linear Heat Conduction Model

In the general context of this study, we address a model of a non-linear heat equation formulated as (section 4 in [43])

$$D^{\gamma}_{\alpha,\beta,\lambda}T(x,t) = K(T)\frac{\partial^2 T(x,t)}{\partial x^2} - \beta T, \quad K(T) = k_0 T^{\xi}, \quad \xi > 0, \quad t > 0, \quad x \in \mathbb{R} \tag{168}$$

The derivative $D^{\gamma}_{\alpha,\beta,\lambda}$ was applied in two versions: a non-regularized (Riemann–Liouville type) derivative (169), and a regularized (Caputo-type) derivative (170), defined as (in the original notations established in [101]) (see also the definitions in Section 2.2.2),

$$({}_0D^{\alpha}_t + \lambda)^{\gamma} f(t) \equiv \frac{d^m}{dt^m}\int_0^t e^{\gamma}_{\alpha,\alpha\gamma}(t-u;-\lambda)f(u)du \tag{169}$$

$${}^{C}({}_0D^{\alpha}_t + \lambda)^{\gamma} f(t) \equiv \int_0^t e^{\gamma}_{\alpha,\alpha\gamma}(t-u;-\lambda)f^{(m)}(u)du \tag{170}$$

with a Prabhakar integral (see also the definition (45))

$$({}_0I^{\alpha}_t + \lambda)^{\gamma} f(t) \equiv e^{\gamma}_{\alpha,\alpha\gamma}(t;\lambda) * f(t) = \int_0^t e^{\gamma}_{\alpha,\alpha\gamma}(t-u;-\lambda)f(u)du \tag{171}$$

In general, this type of non-linearity in the diffusion term, when the equation is local in time, makes the heat conduction model a degenerate parabolic equation. It is well-known, since the first solutions in the 50s of the last century, that such degenerate parabolic problems have solutions moving with finite speeds, in contrast to the case with $\xi = 0$ where the flux speed is infinite (see [102] and the reference therein).

Now, for the sake of the coherence of the exposition, we return to the common symbol used here (let the definitions (169)–(171) remain, demonstrating that different versions of the symbols used are also possible; in general, there are various notations of these operators, so the reader should carefully understand the meaning in each particular case).

If we like to construct (derive) such a heat conduction model, we may define the flux-gradient relationship as

$$\begin{aligned} q(x,t) &= -\int_0^t e^{\gamma}_{\alpha,\beta}(\lambda, t-\tau)\left[k_0 T^{\xi}\frac{\partial T(x,t)}{\partial x}\right] = \\ &= -\frac{k_0}{1+\xi}\int_0^t e^{\gamma}_{\alpha,\beta}(\lambda, t-\tau)\left[\frac{\partial T^{\xi+1}(x,t)}{\partial x}\right] \end{aligned} \tag{172}$$

If the energy balance equation is (1) is applied (neglecting the sink term for the sake of simplicity), then the heat conduction model would be

$$\frac{\partial T(x,t)}{\partial t} = \frac{k_0}{1+\xi}\int_0^t e^{\gamma}_{\alpha,\beta}(\lambda, t-\tau)\left[\frac{\partial^2 T^{\xi+1}(x,\tau)}{\partial x^2}\right] \tag{173}$$

or in a more compact form as

$$\frac{\partial T(x,t)}{\partial t} = \frac{k_0}{1+\xi}\, {}^{P}I^{\gamma}_{\alpha,\beta}\left[\frac{\partial^2 T^{\xi+1}(x,t)}{\partial x^2}\right] \tag{174}$$

In the local case when memory is $\delta(t)$ it will be

$$\frac{\partial T(x,t)}{\partial t} = \frac{k_0}{1+\xi}\frac{\partial^2 T^{\xi+1}(x,t)}{\partial x^2} \tag{175}$$

and the solutions are mainly approximate (see [102]) with convex temperature profiles (with almost infinite gradient close to the solution front for high values of ξ), in contrast to the case for $\xi = 0$ with concave temperature distributions.

Now, the principle question is: how to transform (173) to (168)? Taking into account that the regularized Caputo-type derivative ${}^{C}D^{\gamma}_{\alpha,\beta,\lambda}$ acts as left-inverse of the Prabhakar fractional integral [41,49] , as well as that ${}^{C}D^{\gamma}_{\alpha,\beta,\lambda}f(t) = {}^{RLC}D^{\gamma}_{\alpha,\beta,\lambda}f(t)$ at zero initial conditions (see eq. 5.11 in [41]), we may apply ${}^{C}D^{\gamma}_{\alpha,\beta,\lambda}$ to both sides of (173) or (174). The result of this operation is

$${}^{C}D^{\gamma}_{\alpha,\beta,\lambda}\left[\frac{\partial T(x,t)}{\partial t}\right] = \frac{k_0}{1+\xi}\frac{\partial^2 T^{\xi+1}(x,t)}{\partial x^2} \tag{176}$$

We can see that the left side of (176) is not the same as in (168). That is, the operation that was successful with the Caputo derivative in Section 6.1 does not work here. The origin is that (173) is a natural consequence of the constructive flux equation, while (168) is postulated and could be explained if the continuity equation is defined as

$${}^{C}D^{\gamma}_{\alpha,\beta,\lambda}[T(x,t)] = -\frac{\partial q(x,t)}{\partial x} \tag{177}$$

that is an imitation of the First Law of Thermodynamics, where the causality principle is violated. Despite this, Equation (168) was solved elegantly even though a physical analysis on its basis is impossible.

Remark 9. *We can see that when the models are developed by the "copy-paste" technology, known as "replacement" in fractional modeling, the outcomes are non-physical. Especially for the Prabhakar derivative, it is important to stress the fact that all efforts were oriented towards " framing " in Kochubei's " general fractional calculus ", [49], thus, attempting to make it " unipolar fractional calculus". High merits, with good results, but physically inapplicable because Kochubei's calculus is almost " sterile ", and not related to solutions of physical problems: Bright mathematics where the results are not provoked by nature and the consequence in fractional modeling is evident: nice mathematics, as it was done in [43], but with inexplicable results. We may consider all these studies, and others of the same style (products of replacement fractionalizations), as mathematical experiments, with elegant mathematics and still expecting relevant physical interpretations.*

Remark 10. *Despite the strong verdict in the previous remarks, there are some good cases when the results are adequate and provide new information, such as modeling relaxations in anomalous dielectrics [60,101], diffusion with non-static stochastic resetting [103], generalized Langevin equation [104], anomalous diffusion [105], even though the models already exist, just the memory kernels are changed.*

After this analysis, we will skip similar models with Miller–Ross [94] and Rabotnov derivatives [96,97,100], where the models are obtained by replacements.

7. Fading Memory Approach or Volterra Equations?

The results developed the fading memory formalism and the construction of the Stieltjes integral

$$\int_0^t R(t-\tau)\frac{df(\tau)}{d\tau}d\tau \tag{178}$$

lead to the formulation of normalized fractional operators of Caputo type, following the definition of Stieltjes integral that $f(\tau)$ is a function of bounded variation.

However, if the constitutive equations are of the Volterra type, such as (10), the hereditary term is

$$\int_0^t R(t-\tau)f(\tau)d\tau \tag{179}$$

The construction (179) allows formulations of both fractional integrals and fractional derivatives of a Riemann–Liouville type (as in the studies devoted and analyzes here on the Prabhakar calculus)

$$I_t^\alpha f(\tau) \equiv \int_0^t R(t-\tau)f(\tau)d\tau \tag{180}$$

$$D_t^\alpha f(\tau) \equiv \frac{d^m}{dt^m}\int_0^t R(t-\tau)f(\tau)d\tau, \quad m = 1, 2, 3 \ldots\ldots \tag{181}$$

The kernel $R(t)$ in both formulations depends on the physics of the modeled relaxation process.

Taking into account that the "driving force," i.e., the reason the heat flux to "flow" is the temperature gradient $\partial T/\partial x$ (for simple materials [18–21,67,69,71]), then the construction (178) of the hereditary term is physically motivated. Hence, the physics of the process motivates the model, not the formal and voluntary use of fractional operators.

The application of Volterra equations in the process of generation of fractional operators with different kernels is beyond the scope of this study, but we refer to [89], where there are many examples of this approach applied in the early studies on fractional modeling in viscoelasticity.

8. Final Comments and Outcomes

Now, after this long study with many analyses and remarks, the main questions are

- What the main achievements are?
- How the systematic approach applying the fading memory formalism allows constructions of physically adequate fractional models?

Since developing this study, many explanations and specific remarks were made; the answers to these questions can be briefly outlined as

1. One of the principal achievements of this study is that it shows how different fractional operators appear in physical models, avoiding unmotivated replacement fractionalizations. The emphasis was on different versions of the Mittag-Leffler functions as memory kernels.

2. We demonstrated only six examples, but there is no limit to developing new ones if the constitutive equations are properly formulated.

3. We emphasize the importance of correct formulations of constitutive equations because they express physical laws relating to causes and effects in mathematical forms.

4. Therefore, the first step is to correctly formulate a constitutive equation with a hereditary term involving a memory kernel connected to the underlying physics of the modeled process. Then, the balance equation yields the final model (to which various initial and boundary conditions can be applied). However, the final models are not direct expressions of physical laws where hereditary terms can be inserted by replacements.

5. Some comments on existing heat conduction models confirm the standpoint of point 4. There are many articles with questionable results (see, for instance, the comments of Hilfer in [10] on physics of fractional models) where the final model is fractionalized by replacement, and the results that cannot be interpreted because the basic physics of the modeled process is violated.

We believe that this study will provide necessary and enough instructive information on how fractional modeling should be done. There is no need for high mathematics to formulate the constitutive equation correctly with a certain memory term and to get a correct model. High mathematics applied to incorrect models does not provide useful information and could be related to mathematical experiments, not to mathematical modeling of real-world phenomena.

The example of heat conduction used here is merely one to illustrate the general idea behind the physical basis of the formulation of fractional operators. This technique can be applied to a wide variety of other physical processes that exhibit memories and have properly formulated constitutive equations.

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