SUPPLEMENTARY MATERIALS

Here we develop high-frequency asymptotic expansion of the DSR equation. We show that two-way travel time is governed by a nonlinear eikonal equation while amplitudes obey transport equation similar to that of classic wave-equation ray method. We start from the true-amplitude DSR equation obtained in [11] and follow the derivation presented in [10] with some modifications.

According to [11], true-amplitude DSR equation reads

$$\left(\frac{\partial}{\partial z} - i\Lambda_s - i\Lambda_r - \Gamma_s - \Gamma_r\right) u = 0, \tag{S1}$$

where we used the same notation as in the main text.

As it is shown in [10], principal symbol of $i\Lambda_{s/r}$ from (2) can be recast into

$$i\lambda_{s/r} = \frac{i\omega}{v_{s/r}}\left(1 - \frac{1}{\pi}\int_{-1}^{1} \sqrt{1-\xi^2}\,\frac{v_{s/r}^2 k_{s/r}^2}{\omega^2 - \xi^2 v_{s/r}^2 k_{s/r}^2}\,d\xi\right). \tag{S2}$$

We introduce differential operators $L_{s/r}(\xi)$ and $\left(v_{s/r}\frac{\partial}{\partial x_{s/r}}\right)^2$ with principal symbols being:

$$\begin{aligned} L_{s/r}(\xi) &= \frac{\partial^2}{\partial t^2} - \xi^2\left(v_{s/r}\frac{\partial}{\partial x_{s/r}}\right)^2 \leftrightarrow -\left(\omega^2 - \xi^2 v_{s/r}^2 k_{s/r}^2\right), \\ \left(v_{s/r}\frac{\partial}{\partial x_{s/r}}\right)^2 &= v_{s/r}^2\frac{\partial^2}{\partial x_{s/r}^2} + v_{s/r}\frac{\partial v_{s/r}}{\partial x_{s/r}}\frac{\partial}{\partial x_{s/r}} \leftrightarrow v_{s/r}^2 k_{s/r}^2. \end{aligned} \tag{S3}$$

One may think of symbol $v_{s/r}^2 k_{s/r}^2$ appearing in numerator as of application of operator $\left(v_{s/r}\frac{\partial}{\partial x_{s/r}}\right)^2$. Accordingly, symbol $\omega^2 - \xi^2 v_{s/r}^2 k_{s/r}^2$ in denominator might be interpreted as an inverse operator $L_{s/r}^{-1}(\xi)$. Keeping this in mind and recalling Fourier transform property $\frac{\partial}{\partial t} \leftrightarrow i\omega$ we write down expressions for operators $i\Lambda_{s/r}$ and $\Gamma_{s/r}$ using (2), (S2) and (S3):

$$\begin{aligned} i\Lambda_{s/r} &= \frac{1}{v_{s/r}}\frac{\partial}{\partial t}\left(I - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\xi^2}\,L_{s/r}^{-1}(\xi)\left(v_{s/r}\frac{\partial}{\partial x_{s/r}}\right)^2 d\xi\right), \\ \Gamma_{s/r} &= \frac{1}{2v_{s/r}}\frac{\partial v_{s/r}}{\partial z}\cdot\left(I + L_{s/r}^{-1}(1)\left(v_{s/r}\frac{\partial}{\partial x_{s/r}}\right)^2\right). \end{aligned} \tag{S4}$$

Operator *I* represents identity. Inserting (S4) into (S1) leads to equation

$$\begin{aligned} &\frac{\partial u}{\partial z} - \frac{1}{v_s}\frac{\partial}{\partial t}\left(u - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\xi^2}\,L_s^{-1}(\xi)\left(v_s\frac{\partial}{\partial x_s}\right)^2 u\cdot d\xi\right) \\ &\qquad - \frac{1}{v_r}\frac{\partial}{\partial t}\left(u - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\xi^2}\,L_r^{-1}(\xi)\left(v_r\frac{\partial}{\partial x_r}\right)^2 u\cdot d\xi\right) \\ &- \frac{1}{2v_s}\frac{\partial v_s}{\partial z}\cdot\left(u + L_s^{-1}(1)\left(v_s\frac{\partial}{\partial x_s}\right)^2 u\right) - \frac{1}{2v_r}\frac{\partial v_r}{\partial z}\cdot\left(u + L_r^{-1}(1)\left(v_r\frac{\partial}{\partial x_r}\right)^2 u\right) = 0. \end{aligned} \tag{S5}$$

As in [10] we proceed with a new function

$$q_{s/r}(\xi; x_s, x_r, z, t): \quad L_{s/r}(\xi) q_{s/r} = \left(v_{s/r} \frac{\partial}{\partial x_{s/r}} \right)^2 u \tag{S6}$$

allowing us to recast the DSR equation (S5) into its final form:

$$\begin{aligned} &\frac{\partial u}{\partial z} - \frac{1}{v_s} \frac{\partial}{\partial t} \left(u - \frac{1}{\pi} \int_{-1}^{1} \sqrt{1-\xi^2} q_s(\xi; \dots) \cdot d\xi \right) \\ &\qquad - \frac{1}{v_r} \frac{\partial}{\partial t} \left(u - \frac{1}{\pi} \int_{-1}^{1} \sqrt{1-\xi^2} q_r(\xi; \dots) \cdot d\xi \right) \\ &- \frac{1}{2v_s} \frac{\partial v_s}{\partial z} \cdot \big(u + q_s(1; \dots) \big) - \frac{1}{2v_r} \frac{\partial v_r}{\partial z} \cdot \big(u + q_r(1; \dots) \big) = 0. \end{aligned} \tag{S7}$$

We underline that all functions u, q_s and q_r depend on both x_s and x_r. We also note that ξ is a dummy variable not appearing outside the integrals in the left-hand-side of the equation.

Asymptotic expansions

We postulate asymptotic ray series expansions:

$$\begin{cases} u(x_s, x_r, z, t) = \sum_{k=0}^{\infty} A_k(x_s, x_r, z) F_k\big(t - \tau(x_s, x_r, z)\big), \\ q_{s/r}(\xi; x_s, x_r, z, t) = \sum_{k=0}^{\infty} A_k^{s/r}(\xi; x_s, x_r, z) F_k\big(t - \tau(x_s, x_r, z)\big), \\ \dfrac{dF_k}{dt} = F_{k-1} \quad \forall k \geq 1. \end{cases} \tag{S8}$$

When inserted into the DSR equation (S7), they lead to infinite equation involving terms $\propto F_0'$, F_0, F_1 etc. We shall keep only two first ones. Let us develop each member in the left-hand-side of (S7) separately:

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z} \big(A_0 F_0(t-\tau) + A_1 F_1(t-\tau) + \cdots \big) = -\frac{\partial \tau}{\partial z} A_0 F_0' + \left[\frac{\partial A_0}{\partial z} - \frac{\partial \tau}{\partial z} A_1 \right] F_0 + \cdots, \tag{S9}$$

$$\begin{aligned} &\frac{1}{v_{s/r}} \frac{\partial}{\partial t} \left(u - \frac{1}{\pi} \int_{-1}^{1} \sqrt{1-\xi^2} q_{s/r}(\xi; \dots) \cdot d\xi \right) = \\ &\qquad = \frac{1}{v_{s/r}} \left[A_0 - \frac{1}{\pi} \int_{-1}^{1} \sqrt{1-\xi^2} A_0^{s/r}(\xi; \dots) \cdot d\xi \right] F_0' \\ &\qquad + \frac{1}{v_{s/r}} \left[A_1 - \frac{1}{\pi} \int_{-1}^{1} \sqrt{1-\xi^2} A_1^{s/r}(\xi; \dots) \cdot d\xi \right] F_0 + \cdots, \end{aligned} \tag{S10}$$

$$\frac{1}{2v_{s/r}} \frac{\partial v_{s/r}}{\partial z} \cdot \big(u + q_{s/r}(1; \dots) \big) = \frac{1}{2v_{s/r}} \frac{\partial v_{s/r}}{\partial z} \cdot \big[A_0 + A_0^{s/r}(1; \dots) \big] F_0 + \cdots. \tag{S11}$$

Hence, asymptotic expansion of whole DSR equation reads

$$
\begin{aligned}
(\mathrm{S7}) \Rightarrow -\Bigg[\frac{\partial \tau}{\partial z} A_0 &+ \frac{1}{v_s}\left(A_0 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\xi^2}A_0^s(\xi;\dots)\cdot d\xi\right) \\
&+ \frac{1}{v_r}\left(A_0 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\xi^2}A_0^r(\xi;\dots)\cdot d\xi\right)\Bigg] F_0' + \\
+\Bigg[\frac{\partial A_0}{\partial z} - \frac{\partial \tau}{\partial z}A_1 &- \frac{1}{v_s}\left(A_1 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\xi^2}A_1^s(\xi;\dots)\cdot d\xi\right) \\
&- \frac{1}{v_r}\left(A_1 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\xi^2}A_1^r(\xi;\dots)\cdot d\xi\right) - \frac{1}{2v_s}\frac{\partial v_s}{\partial z}\big(A_0 + A_0^s(1;\dots)\big) \\
&- \frac{1}{2v_r}\frac{\partial v_r}{\partial z}\big(A_0 + A_0^r(1;\dots)\big)\Bigg] F_0 + \dots = 0.
\end{aligned} \tag{S12}
$$

We set the expressions inside square brackets to be zero independently of each other. Zeroing out coefficient before F_0' will lead to the eikonal equation and that before F_0 – to the transport equation.

Auxiliary formulae

Integration with respect to ξ in (S12) requires expressions for $A_0^{s/r}(\xi;\dots)$ and $A_1^{s/r}(\xi;\dots)$ in terms of ξ, A_0 and A_1. One can obtain them by substituting ray series expansions into (S6). Once again, we develop both sides of the equation in turn:

$$
\begin{aligned}
L_{s/r}(\xi)q_{s/r} = &\left[1 - \xi^2 v_{s/r}^2\left(\frac{\partial \tau}{\partial x_{s/r}}\right)^2\right] A_0^{s/r} F_0'' \\
&+ \left[\left(1 - \xi^2 v_{s/r}^2\left(\frac{\partial \tau}{\partial x_{s/r}}\right)^2\right) A_1^{s/r} + \xi^2 B_{s/r}\big(A_0^{s/r}\big)\right] F_0' + \cdots,
\end{aligned} \tag{S13}
$$

$$
\left(v_{s/r}\frac{\partial}{\partial x_{s/r}}\right)^2 u = v_{s/r}^2\left(\frac{\partial \tau}{\partial x_{s/r}}\right)^2 A_0 F_0'' + \left[v_{s/r}^2\left(\frac{\partial \tau}{\partial x_{s/r}}\right)^2 A_1 - B_{s/r}(A_0)\right] F_0' + \cdots, \tag{S14}
$$

where we introduced a new shorthand notation:

$$
B_{s/r}(A) \stackrel{\text{def}}{=} 2v_{s/r}^2 \frac{\partial \tau}{\partial x_{s/r}}\frac{\partial A}{\partial x_{s/r}} + \left(\left(v_{s/r}\frac{\partial}{\partial x_{s/r}}\right)^2 \tau\right) A. \tag{S15}
$$

Equating coefficients before coinciding derivatives of F_0 we obtain

$$
(\mathrm{S6}) \Rightarrow \begin{cases}
\left(1 - \xi^2 v_{s/r}^2\left(\dfrac{\partial \tau}{\partial x_{s/r}}\right)^2\right) A_0^{s/r} = v_{s/r}^2\left(\dfrac{\partial \tau}{\partial x_{s/r}}\right)^2 A_0, \\
\left(1 - \xi^2 v_{s/r}^2\left(\dfrac{\partial \tau}{\partial x_{s/r}}\right)^2\right) A_1^{s/r} + \xi^2 B_{s/r}\big(A_0^{s/r}\big) = \\
\qquad = v_{s/r}^2\left(\dfrac{\partial \tau}{\partial x_{s/r}}\right)^2 A_1 - B_{s/r}(A_0).
\end{cases} \tag{S16}
$$

Hence, the following relations hold true:

$$
\begin{cases}
A_0^{s/r} = \dfrac{v_{s/r}^2\left(\dfrac{\partial\tau}{\partial x_{s/r}}\right)^2}{1-\xi^2 v_{s/r}^2\left(\dfrac{\partial\tau}{\partial x_{s/r}}\right)^2} A_0, \\
A_1^{s/r} = \dfrac{v_{s/r}^2\left(\dfrac{\partial\tau}{\partial x_{s/r}}\right)^2}{1-\xi^2 v_{s/r}^2\left(\dfrac{\partial\tau}{\partial x_{s/r}}\right)^2} A_1 - \dfrac{\xi^2 B_{s/r}\left(A_0^{s/r}\right) + B_{s/r}(A_0)}{1-\xi^2 v_{s/r}^2\left(\dfrac{\partial\tau}{\partial x_{s/r}}\right)^2}.
\end{cases}
\tag{S17}
$$

A few more expressions will be needed in the sequel. One can verify that

$$
J_1^{s/r} = \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-\xi^2}}{1-\xi^2 v_{s/r}^2\left(\dfrac{\partial\tau}{\partial x_{s/r}}\right)^2} \cdot d\xi = \frac{1-\sqrt{1-v_{s/r}^2\left(\dfrac{\partial\tau}{\partial x_{s/r}}\right)^2}}{v_{s/r}^2\left(\dfrac{\partial\tau}{\partial x_{s/r}}\right)^2},
\tag{S18}
$$

$$
J_2^{s/r} = \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-\xi^2}}{\left(1-\xi^2 v_{s/r}^2\left(\dfrac{\partial\tau}{\partial x_{s/r}}\right)^2\right)^2} \cdot d\xi = \frac{1}{2\sqrt{1-v_{s/r}^2\left(\dfrac{\partial\tau}{\partial x_{s/r}}\right)^2}},
\tag{S19}
$$

$$
\begin{aligned}
J_3^{s/r} &= \frac{1}{\pi v_{s/r}}\int_{-1}^{1} \frac{\xi^2 B_{s/r}\left(A_0^{s/r}\right)\sqrt{1-\xi^2}}{1-\xi^2 v_{s/r}^2\left(\dfrac{\partial\tau}{\partial x_{s/r}}\right)^2} \cdot d\xi \\
&= \frac{v_{s/r}\dfrac{\partial\tau}{\partial x_{s/r}}\dfrac{\partial}{\partial x_{s/r}}\left(v_{s/r}^2\left(\dfrac{\partial\tau}{\partial x_{s/r}}\right)^2\right)}{4\left(1-v_{s/r}^2\left(\dfrac{\partial\tau}{\partial x_{s/r}}\right)^2\right)^{\frac{3}{2}}} A_0 + \frac{B_{s/r}(A_0)}{v_{s/r}}\left[J_2^{s/r} - J_1^{s/r}\right].
\end{aligned}
\tag{S20}
$$

Eikonal equation

Now let us consider the coefficient with F_0' in (S12). Substituting expression for $A_0^{s/r}$ from (S17) we obtain:

$$
\begin{aligned}
&\frac{\partial\tau}{\partial z}A_0 + \frac{1}{v_s}\left(A_0 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\xi^2}A_0^s(\xi;\dots)\cdot d\xi\right) \\
&\qquad + \frac{1}{v_r}\left(A_0 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\xi^2}A_0^r(\xi;\dots)\cdot d\xi\right) = \\
&= \frac{\partial\tau}{\partial z}A_0 + \frac{1}{v_s}\left(1 - v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2 J_1^s\right)A_0 + \frac{1}{v_r}\left(1 - v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2 J_1^r\right)A_0 = 0.
\end{aligned}
\tag{S21}
$$

Evaluation of $J_1^{s/r}$ from (S18) yields the eikonal equation:

$$\frac{\partial\tau}{\partial z} = -\frac{1}{v_s}\sqrt{1 - v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2} - \frac{1}{v_r}\sqrt{1 - v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}. \tag{S22}$$

Transport equation

Transport equation involves more elaborate algebra. First, we recall the coefficient before F_0 from (S12) which is set to be zero:

$$\begin{aligned}\frac{\partial A_0}{\partial z} - \frac{\partial\tau}{\partial z}A_1 - \frac{1}{v_s}\left(A_1 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\xi^2}A_1^s(\xi;\dots)\cdot d\xi\right)\\ -\frac{1}{v_r}\left(A_1 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\xi^2}A_1^r(\xi;\dots)\cdot d\xi\right) - \\ -\frac{1}{2v_s}\frac{\partial v_s}{\partial z}\big(A_0 + A_0^s(1;\dots)\big) - \frac{1}{2v_r}\frac{\partial v_r}{\partial z}\big(A_0 + A_0^r(1;\dots)\big) = 0.\end{aligned} \tag{S23}$$

We gather all terms involving A_1 and $A_1^{s/r}$ and denote them by C:

$$\begin{aligned}C \stackrel{\text{def}}{=} -\frac{\partial\tau}{\partial z}A_1 - \frac{1}{v_s}\left(A_1 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\xi^2}A_1^s(\xi;\dots)\cdot d\xi\right)\\ -\frac{1}{v_r}\left(A_1 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\xi^2}A_1^r(\xi;\dots)\cdot d\xi\right).\end{aligned} \tag{S24}$$

Now we substitute $A_1^{s/r}$ from (S17), yielding:

$$\begin{aligned}C = -\frac{\partial\tau}{\partial z}A_1 - \frac{1}{v_s}\left(1 - v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2 J_1^s\right)A_1 - \frac{1}{v_r}\left(1 - v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2 J_1^r\right)A_1\\ -\frac{1}{\pi v_s}\int_{-1}^{1}\frac{\big(\xi^2 B_s(A_0^s) + B_s(A_0)\big)\sqrt{1-\xi^2}}{1-\xi^2 v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}\cdot d\xi\\ -\frac{1}{\pi v_r}\int_{-1}^{1}\frac{\big(\xi^2 B_r(A_0^r) + B_r(A_0)\big)\sqrt{1-\xi^2}}{1-\xi^2 v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}\cdot d\xi.\end{aligned} \tag{S25}$$

Comparing this equation to (S21) it is clear that terms $\propto A_1$ vanish due to the eikonal equation, and, taking into account (S18) and (S20),

$$C = -J_3^s - J_3^r - \frac{B_s(A_0)}{v_s}J_1^s - \frac{B_r(A_0)}{v_r}J_1^r. \tag{S26}$$

Expanding $J_3^{s/r}$ from (S20) we write down final expression for C:

$$C = -\left(\frac{v_s \frac{\partial \tau}{\partial x_s} \frac{\partial}{\partial x_s}\left(v_s^2 \left(\frac{\partial \tau}{\partial x_s} \right)^2 \right)}{4\left(1 - v_s^2 \left(\frac{\partial \tau}{\partial x_s} \right)^2 \right)^{\frac{3}{2}}} A_0 + \frac{B_s(A_0)}{v_s} J_2^s \right) - \left(\frac{v_r \frac{\partial \tau}{\partial x_r} \frac{\partial}{\partial x_r}\left(v_r^2 \left(\frac{\partial \tau}{\partial x_r} \right)^2 \right)}{4\left(1 - v_r^2 \left(\frac{\partial \tau}{\partial x_r} \right)^2 \right)^{\frac{3}{2}}} A_0 + \frac{B_r(A_0)}{v_r} J_2^r \right). \tag{S27}$$

Now we return to the equation (S23). We insert C in its place and expand remaining abbreviations $B_{s/r}(A_0)J_2^{s/r}$ and $A_0^{s/r}(1; \dots)$. The result reads:

$$\frac{\partial A_0}{\partial z} - \frac{v_s \frac{\partial \tau}{\partial x_s} \frac{\partial}{\partial x_s}\left(v_s^2 \left(\frac{\partial \tau}{\partial x_s} \right)^2 \right)}{4\left(1 - v_s^2 \left(\frac{\partial \tau}{\partial x_s} \right)^2 \right)^{\frac{3}{2}}} A_0 - \frac{2 v_s^2 \frac{\partial \tau}{\partial x_s} \frac{\partial A_0}{\partial x_s} + \left(\left(v_s \frac{\partial}{\partial x_s} \right)^2 \tau \right) A_0}{2 v_s \sqrt{1 - v_s^2 \left(\frac{\partial \tau}{\partial x_s} \right)^2}} - \frac{v_r \frac{\partial \tau}{\partial x_r} \frac{\partial}{\partial x_r}\left(v_r^2 \left(\frac{\partial \tau}{\partial x_r} \right)^2 \right)}{4\left(1 - v_r^2 \left(\frac{\partial \tau}{\partial x_r} \right)^2 \right)^{\frac{3}{2}}} A_0 - \frac{2 v_r^2 \frac{\partial \tau}{\partial x_r} \frac{\partial A_0}{\partial x_r} + \left(\left(v_r \frac{\partial}{\partial x_r} \right)^2 \tau \right) A_0}{2 v_r \sqrt{1 - v_r^2 \left(\frac{\partial \tau}{\partial x_r} \right)^2}} \tag{S28}$$

$$- \frac{1}{2 v_s} \frac{\partial v_s}{\partial z} \left(A_0 + \frac{v_s^2 \left(\frac{\partial \tau}{\partial x_s} \right)^2 A_0}{1 - v_s^2 \left(\frac{\partial \tau}{\partial x_s} \right)^2} \right) - \frac{1}{2 v_r} \frac{\partial v_r}{\partial z} \left(A_0 + \frac{v_r^2 \left(\frac{\partial \tau}{\partial x_r} \right)^2 A_0}{1 - v_r^2 \left(\frac{\partial \tau}{\partial x_r} \right)^2} \right) = 0.$$

Developing this equation, we obtain:

$$
\begin{aligned}
&\frac{\partial A_0}{\partial z}-\frac{v_s\frac{\partial\tau}{\partial x_s}\frac{\partial A_0}{\partial x_s}}{\sqrt{1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}}-\frac{v_r\frac{\partial\tau}{\partial x_r}\frac{\partial A_0}{\partial x_r}}{\sqrt{1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}}\\
&\qquad-\frac{A_0}{2}\left[\frac{v_s\frac{\partial^2\tau}{\partial x_s^2}}{\left(1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2\right)^{\frac{3}{2}}}+\frac{\frac{\partial v_s}{\partial x_s}\frac{\partial\tau}{\partial x_s}}{\left(1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2\right)^{\frac{3}{2}}}\right]\\
&\qquad-\frac{A_0}{2}\left[\frac{v_r\frac{\partial^2\tau}{\partial x_r^2}}{\left(1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2\right)^{\frac{3}{2}}}+\frac{\frac{\partial v_r}{\partial x_r}\frac{\partial\tau}{\partial x_r}}{\left(1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2\right)^{\frac{3}{2}}}\right]\\
&\qquad-\frac{A_0}{2}\left[\frac{\frac{1}{v_s}\frac{\partial v_s}{\partial z}}{1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}+\frac{\frac{1}{v_r}\frac{\partial v_r}{\partial z}}{1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}\right]=0.
\end{aligned}
\tag{S29}
$$

This is the transport equation for the DSR equation. In order to make it more similar to its standard ray method counterpart, we multiply this equation by $\frac{\partial\tau}{\partial z}$ while keeping in mind the eikonal equation (S22):

$$\frac{\partial\tau}{\partial z}\frac{\partial A_0}{\partial z}+\frac{\partial\tau}{\partial x_s}\frac{\partial A_0}{\partial x_s}\left(1+\frac{v_s}{v_r}\sqrt{\frac{1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}{1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}}\right)+\frac{\partial\tau}{\partial x_r}\frac{\partial A_0}{\partial x_r}\left(1+\frac{v_r}{v_s}\sqrt{\frac{1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}{1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}}\right)$$
$$+\frac{A_0}{2}\left[\frac{\frac{\partial^2\tau}{\partial x_s^2}+\frac{1}{v_s}\frac{\partial v_s}{\partial x_s}\frac{\partial\tau}{\partial x_s}}{1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}\left(1+\frac{v_s}{v_r}\sqrt{\frac{1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}{1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}}\right)\right.$$
$$\left.+\frac{\frac{\partial^2\tau}{\partial x_r^2}+\frac{1}{v_r}\frac{\partial v_r}{\partial x_r}\frac{\partial\tau}{\partial x_r}}{1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}\left(1+\frac{v_r}{v_s}\sqrt{\frac{1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}{1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}}\right)\right] \tag{S30}$$
$$+\frac{A_0}{2}\left[\frac{\frac{1}{v_s^2}\frac{\partial v_s}{\partial z}}{\sqrt{1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}}\left(1+\frac{v_s}{v_r}\sqrt{\frac{1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}{1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}}\right)\right.$$
$$\left.+\frac{\frac{1}{v_r^2}\frac{\partial v_r}{\partial z}}{\sqrt{1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}}\left(1+\frac{v_r}{v_s}\sqrt{\frac{1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}{1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}}\right)\right]=0.$$

We note that

$$0=\frac{\partial}{\partial z}\left(\frac{\partial\tau}{\partial z}-\frac{\partial\tau}{\partial z}\right)=\frac{\partial}{\partial z}\left(\frac{\partial\tau}{\partial z}+\frac{1}{v_s}\sqrt{1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}+\frac{1}{v_r}\sqrt{1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}\right)=$$
$$\frac{\partial^2\tau}{\partial z^2}-\left[\frac{v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2\frac{\partial^2\tau}{\partial x_s^2}+\frac{1}{v_s}\frac{\partial v_s}{\partial x_s}\frac{\partial\tau}{\partial x_s}}{1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}+\frac{v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2\frac{\partial^2\tau}{\partial x_r^2}+\frac{1}{v_r}\frac{\partial v_r}{\partial x_r}\frac{\partial\tau}{\partial x_r}}{1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}\right]- \tag{S31}$$
$$-\left[\frac{\frac{1}{v_s^2}\frac{\partial v_s}{\partial z}}{\sqrt{1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}}+\frac{\frac{1}{v_r^2}\frac{\partial v_r}{\partial z}}{\sqrt{1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}}\right]-\frac{2v_sv_r\frac{\partial\tau}{\partial x_s}\frac{\partial\tau}{\partial x_r}\frac{\partial^2\tau}{\partial x_s\partial x_r}}{\sqrt{1-v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}\sqrt{1-v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}}=0,$$

where we changed order of differentiation $\frac{\partial}{\partial z}\left(\frac{\partial\tau}{\partial x_{s/r}}\right)=\frac{\partial}{\partial x_{s/r}}\left(\frac{\partial\tau}{\partial z}\right)$ and once again substituted $\frac{\partial\tau}{\partial z}$ from the eikonal equation (S22). This enables us to rewrite the transport equation (S30):

$$
\begin{gathered}
(\mathrm{S30}) + \frac{A_0}{2}\frac{\partial}{\partial z}\left(\frac{\partial\tau}{\partial z} - \frac{\partial\tau}{\partial z}\right) = \frac{\partial\tau}{\partial z}\frac{\partial A_0}{\partial z} + \\
+ \frac{\partial\tau}{\partial x_s}\frac{\partial A_0}{\partial x_s}\left(1 + \frac{v_s}{v_r}\sqrt{\frac{1 - v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}{1 - v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}}\right) + \frac{\partial\tau}{\partial x_r}\frac{\partial A_0}{\partial x_r}\left(1 + \frac{v_r}{v_s}\sqrt{\frac{1 - v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}{1 - v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}}\right) + \\
+ \frac{A_0}{2}\left[\frac{\partial^2\tau}{\partial x_s^2} + \frac{\partial^2\tau}{\partial x_r^2} + \frac{\partial^2\tau}{\partial z^2} - \frac{2 v_s v_r \frac{\partial\tau}{\partial x_s}\frac{\partial\tau}{\partial x_r}\frac{\partial^2\tau}{\partial x_s\partial x_r}}{\sqrt{1 - v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}\sqrt{1 - v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}} + \right. \\
\left. + \frac{\frac{v_s}{v_r}\frac{\partial^2\tau}{\partial x_s^2} + \frac{1}{v_r}\frac{\partial v_s}{\partial x_s}\frac{\partial\tau}{\partial x_s}}{1 - v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}\sqrt{\frac{1 - v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}{1 - v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}} + \frac{\frac{v_r}{v_s}\frac{\partial^2\tau}{\partial x_r^2} + \frac{1}{v_s}\frac{\partial v_r}{\partial x_r}\frac{\partial\tau}{\partial x_r}}{1 - v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}\sqrt{\frac{1 - v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}{1 - v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}}\right] \\
+ \frac{A_0}{2}\left[\frac{\frac{1}{v_s v_r}\frac{\partial v_s}{\partial z}\sqrt{1 - v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}}{1 - v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2} + \frac{\frac{1}{v_s v_r}\frac{\partial v_r}{\partial z}\sqrt{1 - v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}}{1 - v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}\right] = 0.
\end{gathered}
\tag{S32}
$$

Now we introduce a new vector:

$$
\nabla_{DSR}\tau \stackrel{\mathrm{def}}{=} \begin{bmatrix} \frac{\partial\tau}{\partial x_s}\left(1 + \frac{v_s}{v_r}\sqrt{\frac{1 - v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}{1 - v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}}\right) \\ \frac{\partial\tau}{\partial x_r}\left(1 + \frac{v_r}{v_s}\sqrt{\frac{1 - v_s^2\left(\frac{\partial\tau}{\partial x_s}\right)^2}{1 - v_r^2\left(\frac{\partial\tau}{\partial x_r}\right)^2}}\right) \\ \frac{\partial\tau}{\partial z} \end{bmatrix} \tag{S33}
$$

and write down a derivative of its “horizontal” components:

$$\frac{\partial}{\partial x_{s/r}}\left[\frac{\partial \tau}{\partial x_{s/r}}\left(1+\frac{v_{s/r}}{v_{r/s}}\sqrt{\frac{1-v_{r/s}^2\left(\frac{\partial \tau}{\partial x_{r/s}}\right)^2}{1-v_{s/r}^2\left(\frac{\partial \tau}{\partial x_{s/r}}\right)^2}}\right)\right]$$

$$= \frac{\partial^2 \tau}{\partial x_{s/r}^2} + \frac{\frac{v_{s/r}}{v_{r/s}}\frac{\partial^2 \tau}{\partial x_{s/r}^2} + \frac{1}{v_{r/s}}\frac{\partial v_{s/r}}{\partial x_{s/r}}\frac{\partial \tau}{\partial x_{s/r}}}{1-v_{s/r}^2\left(\frac{\partial \tau}{\partial x_{s/r}}\right)^2}\sqrt{\frac{1-v_{r/s}^2\left(\frac{\partial \tau}{\partial x_{r/s}}\right)^2}{1-v_{s/r}^2\left(\frac{\partial \tau}{\partial x_{s/r}}\right)^2}} \tag{S34}$$

$$- \frac{v_s v_r \frac{\partial \tau}{\partial x_s}\frac{\partial \tau}{\partial x_r}\frac{\partial^2 \tau}{\partial x_s x_r}}{\sqrt{1-v_s^2\left(\frac{\partial \tau}{\partial x_s}\right)^2}\sqrt{1-v_r^2\left(\frac{\partial \tau}{\partial x_r}\right)^2}}.$$

Comparing (S34) and (S32) it is clear that transport equation can be written in terms of $\nabla_{DSR}\tau$, namely:

$$\langle \nabla A_0, \nabla_{DSR}\tau\rangle + \frac{1}{2}A_0\langle \nabla, \nabla_{DSR}\tau\rangle +$$

$$\frac{A_0}{2}\left[\frac{\frac{1}{v_s v_r}\frac{\partial v_s}{\partial z}\sqrt{1-v_r^2\left(\frac{\partial \tau}{\partial x_r}\right)^2}}{1-v_s^2\left(\frac{\partial \tau}{\partial x_s}\right)^2} + \frac{\frac{1}{v_s v_r}\frac{\partial v_r}{\partial z}\sqrt{1-v_s^2\left(\frac{\partial \tau}{\partial x_s}\right)^2}}{1-v_r^2\left(\frac{\partial \tau}{\partial x_r}\right)^2}\right] = 0. \tag{S35}$$

This equation accomplishes our derivation. For further development see the main text.