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Bonferroni Mean Operators of Linguistic Neutrosophic Numbers and Their Multiple Attribute Group Decision-Making Methods

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Abstract: Linguistic neutrosophic numbers (LNN) is presented by Fang and Ye in 2017, which can describe the truth, falsity, and indeterminacy linguistic information independently. In this paper, the LNN and the Bonferroni mean operator are merged together to propose a LNN normalized weighted Bonferroni mean (LNNNWBM) operator and a LNN normalized weighted geometric Bonferroni mean (LNNNWGBM) operator and the properties of these two operators are proved. Further, multi-attribute group decision methods are introduced based on the proposed LNNNWBM and LNNNWGBM operators, and then an example is provided to demonstrate the application and validity of the proposed methods. In addition, in order to consider the effect of the parameters p and q on the decision results, different pairs of parameter values are employed to verify the decision results.

Keywords: linguistic neutrosophic numbers (LNN); LNN normalized weighted Bonferroni mean (LNNNWBM) operator; LNN normalized weighted geometric Bonferroni mean (LNNNWGBM) operator; group decision-making

1. Introduction

In dealing with the complex, unknown, and uncertain decision-making problems, a group of decision-makers are usually employed to analyze a set of alternatives and to get the optimal result in a certain way. Such a decision-making process is called multiple attribute group decision-making (MAGDM) problem. When making decisions, decision-makers tend to use words such as “excellent”, “good”, and “poor” to express their evaluations for objects. Zadeh proposed a linguistic variable set $S = \{S_0, S_1, S_2, S_3, \dots, S_g\}$ (g is an even number) to deal with the approximate reasoning problems [1,2]. The linguistic variable is an effective tool, it improves the reliability and flexibility of classical decision models [3,4]. In recent years, the linguistic variables have been frequently linked to other theories. Liu proposed the intuitionistic linguistic set (ILS) composed of linguistic variables and IFS, where the first component provides its qualitative evaluation value/linguistic value and the second component gives the credibility of its intuitionistic fuzzy value for the given linguistic value [5]. Then, Chen et al. proposed the linguistic intuitionistic fuzzy number (LIFN), which is composed of the intuitionistic fuzzy number (the basic element in IFS) and the linguistic variable [6]. On the other hand, some methods for multiple attribute group decision-making (MAGDM) were proposed based on two-dimension uncertain linguistic variable [7,8]. Some improved linguistic intuitionistic fuzzy aggregation operators and several corresponding applications were given in decision-making [9]. Although the IFS theory considers not only $T(x)$, but also $F(x)$, IFS is still not perfect enough because it ignores the indeterminate and inconsistent information. Thus, the intuitionistic fuzzy number can only

be used for expressing incomplete information, but not for expressing indeterminate and inconsistent information. To make up for the insufficiency of the IFS theory, Smarandache put forward the neutrosophic set (NS) composed of three parts: truth $T(x)$, falsity $F(x)$, and indeterminacy $I(x)$ [10,11]. Wang et al. and Smarandache also proposed the concept of a single-valued neutrosophic set (SVNS) satisfying $T(x), I(x), F(x) \subseteq [0, 1], 0 \leq T(x) + F(x) + I(x) \leq 3$ [10–12]. Ye proposed an extended TOPSIS (technique for order preference by similarity to an ideal Solution) method for MAGDM based on single valued neutrosophic linguistic numbers (SVNLNs), which are basic elements in a single-valued neutrosophic linguistic set (SVNLS) [13]. Liu and Shi presented some neutrosophic uncertain linguistic number Heronian mean operators and their application to MAGDM [14]. Since the Bonferroni mean (BM) is a useful operator in decision-making [15], it was extended to hesitant fuzzy sets, IFSs, and interval-valued IFSs to propose their some Bonferroni mean operators for decision making [16–20]. Then, Fang and Ye proposed the linguistic neutrosophic numbers (LNN) and their basic operational laws [21]. LNN consists of the truth, indeterminacy, and falsity linguistic degrees, which can be expressed as the form $a = \langle l_T, l_I, l_F \rangle$, but the LIFN and SVNLN cannot express such linguistic evaluation value. In [21], Fang and Ye also presented a LNN-weighted arithmetic averaging (LNNWAA) operator and a LNN-weighted geometric averaging (LNNWGA) operator for MAGDM. However, the Bonferroni mean operator is not extended to LNNs so far. Hence, this paper proposes a LNN normalized weighted Bonferroni mean (LNNNWBM) operator, a LNN normalized weighted geometric Bonferroni mean (LNNNWGBM) operator and their MAGDM methods. Compared with the aggregation operators in [14,21], the LNNNWBM and LNNNWGBM operators can calculate the final weights by the relation between attribute values, which can make the information aggregation more objective and reliable.

The rest organizations of this paper are as follows. Section 2 describes some basic concepts of LNN, the basic operational laws of LNNs, and the basic concepts of BM and the normalized weighted BM. Section 3 proposes the LNNNWBM and LNNNWGBM operators and investigates their properties. Section 4 establishes MAGDM methods by using the LNNNWBM operator and LNNNWGBM operator. Section 5 provides an illustrative example with different values of the parameters p and q to demonstrate the application of the proposed methods. Section 6 gives conclusions.

2. Some Concepts of LNNs and BM

2.1. Linguistic Neutrosophic Numbers and Their Operational Laws

Definition 1 [21]. Set $L = \{l_0, l_1, l_2, \dots, l_g\}$ as a language term set, in which g is an even number and $g + 1$ is the particle size of L . If $a = \langle l_T, l_I, l_F \rangle$ is defined for $l_T, l_I, l_F \in L$ and $T, I, F \in [0, g]$, where l_T expresses the truth degree, l_I expresses indeterminacy degree, and l_F expresses falsity degree by linguistic terms, then a is called an LNN.

Definition 2 [21]. Set $a = \langle l_T, l_I, l_F \rangle, a_1 = \langle l_{T_1}, l_{I_1}, l_{F_1} \rangle$, and $a_2 = \langle l_{T_2}, l_{I_2}, l_{F_2} \rangle$ as three LNNs in L , the number $\lambda \geq 0$, they have the follow operational laws:

$$a_1 \oplus a_2 = \langle l_{T_1}, l_{I_1}, l_{F_1} \rangle \oplus \langle l_{T_2}, l_{I_2}, l_{F_2} \rangle = \langle l_{T_1+T_2-\frac{T_1T_2}{g}}, l_{\frac{I_1I_2}{g}}, l_{\frac{F_1F_2}{g}} \rangle; \tag{1}$$

$$a_1 \otimes a_2 = \langle l_{T_1}, l_{I_1}, l_{F_1} \rangle \otimes \langle l_{T_2}, l_{I_2}, l_{F_2} \rangle = \langle l_{\frac{T_1T_2}{g}}, l_{I_1+I_2-\frac{I_1I_2}{g}}, l_{F_1+F_2-\frac{F_1F_2}{g}} \rangle; \tag{2}$$

$$\lambda a = \lambda \langle l_T, l_I, l_F \rangle = \langle l_{g-g(1-\frac{T}{g})^\lambda}, l_{g(\frac{I}{g})^\lambda}, l_{g(\frac{F}{g})^\lambda} \rangle; \tag{3}$$

$$a^\lambda = \langle l_T, l_I, l_F \rangle^\lambda = \langle l_{g(\frac{T}{g})^\lambda}, l_{g-g(1-\frac{I}{g})^\lambda}, l_{g-g(1-\frac{F}{g})^\lambda} \rangle. \tag{4}$$

Definition 3 [21]. Set $a = \langle l_T, l_I, l_F \rangle$ as an LNN in L , then the expectation $E(a)$ and the accuracy $H(a)$ can be defined as follows:

$$E(a) = (2g + T - I - F)/3g \tag{5}$$

$$H(a) = (T - F)/g \tag{6}$$

Definition 4 [21]. Set $a_1 = \langle l_{T_1}, l_{I_1}, l_{F_1} \rangle$ and $a_2 = \langle l_{T_2}, l_{I_2}, l_{F_2} \rangle$ as two LNNs, then:

If $E(a_1) > E(a_2)$, then $a_1 \succ a_2$;

If $E(a_1) = E(a_2)$ then

If $H(a_1) > H(a_2)$, then $a_1 \succ a_2$;

If $H(a_1) = H(a_2)$, then $a_1 \sim a_2$;

If $H(a_1) < H(a_2)$, then $a_1 \prec a_2$.

2.2. Bonferroni Mean Operators

Definition 5 [15]. Let (a_1, a_2, \dots, a_n) be a set of non-negative numbers, the function $BM: R^n \rightarrow R$. If $p, q \geq 0$ and BM satisfies:

$$BM^{p,q}(a_1, a_2, \dots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{j=1 \\ j \neq i}}^n a_i^p a_j^q \right)^{\frac{1}{p+q}} \tag{7}$$

then $BM^{p,q}$ is called a BM operator.

Definition 6 [16]. Let (a_1, a_2, \dots, a_n) be a set of non-negative numbers, the function $NWBM: R^n \rightarrow R$, w_i ($i = 1, 2, \dots, n$) be the relative weight of a_i ($i = 1, 2, \dots, n$), $w_i \in [0, 1]$, and $\sum_{i=1}^n w_i = 1$. If $p, q \geq 0$ and $NWBM$ satisfies:

$$NWBM^{p,q}(a_1, a_2, \dots, a_n) = \left(\sum_{\substack{j=1 \\ j \neq i}}^n \frac{w_i w_j}{1 - w_i} a_i^p a_j^q \right)^{\frac{1}{p+q}} \tag{8}$$

then $NWBM^{p,q}$ is called a normalized weighted BM operator.

Definition 7 [17]. Let (a_1, a_2, \dots, a_n) be a set of non-negative numbers, the function $GBM: R^n \rightarrow R$. If $p, q \geq 0$ and GBM satisfies:

$$GBM^{p,q}(a_1, a_2, \dots, a_n) = \left(\frac{1}{n} \sum_i a_i^p \left(\prod_{j=1, j \neq i}^n a_j^q \right)^{\frac{1}{n-1}} \right)^{\frac{1}{p+q}} \tag{9}$$

then $GBM^{p,q}$ is called a geometric BM operator.

Definition 8 [18–20]. Let (a_1, a_2, \dots, a_n) be a set of non-negative numbers, the function $NWGBM: R^n \rightarrow R$, w_i ($i = 1, 2, \dots, n$) be the relative weight of a_i ($i = 1, 2, \dots, n$), $w_i \in [0, 1]$, and $\sum_{i=1}^n w_i = 1$. If $p, q \geq 0$ and $NWGBM$ satisfies:

$$NWGBM^{p,q}(a_1, a_2, \dots, a_n) = \frac{1}{p+q} \otimes_{i,j=1, j \neq i}^n (pa_i \oplus qa_j)^{\frac{w_i w_j}{1-w_i}} \tag{10}$$

then $NWGBM^{p,q}$ is called a normalized weighted geometric BM ($NWGBM$) operator.

3. Two BM Aggregation Operators of LNNs

3.1. Normalized Weighted BM Operators of LNNs

Definition 9. Set $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ ($i = 1, 2, \dots, n$) as a collection of LNNs in L , then the LNNNWBM operator can be defined as follows:

$$\text{LNNNWBM}^{p,q}(a_1, a_2, \dots, a_n) = \left(\sum_{\substack{i,j=1 \\ j \neq i}}^n \frac{w_i w_j}{1 - w_i} a_i^p a_j^q \right)^{\frac{1}{p+q}} \tag{11}$$

where w_i is the relative weight of a_i , $w_i \in [0, 1]$, and $\sum_{i=1}^n w_i = 1$, w_j is the relative weight of a_j , $w_j \in [0, 1]$, and $\sum_{j=1}^n w_j = 1$.

According to Definitions 2 and 9, we can get the following theorem:

Theorem 1. Set $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ ($i = 1, 2, \dots, n$) as a collection of LNNs in L , then by the Equation (11), the aggregation result obtained is still an LNN, and we can get the following aggregation formula:

$$\begin{aligned} \text{LNNNWBM}^{p,q}(a_1, a_2, \dots, a_n) &= \left(\sum_{\substack{i,j=1 \\ j \neq i}}^n \frac{w_i w_j}{1 - w_i} a_i^p a_j^q \right)^{\frac{1}{p+q}} \\ &= \langle l_{g(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n (1 - (\frac{T_i}{g})^p (\frac{T_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}}}, l_{g - g(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n (1 - (1 - \frac{I_i}{g})^p (1 - \frac{I_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}}}, l_{g - g(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n (1 - (1 - \frac{F_i}{g})^p (1 - \frac{F_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}}} \rangle \end{aligned} \tag{12}$$

where w_i is the relative weight of a_i , $w_i \in [0, 1]$, and $\sum_{i=1}^n w_i = 1$, w_j is the relative weight of a_j , $w_j \in [0, 1]$, and $\sum_{j=1}^n w_j = 1$.

Proof 1.

$$\begin{aligned} (1) \quad a_i^p &= \langle l_{g(\frac{T_i}{g})^p}, l_{g - g(1 - \frac{I_i}{g})^p}, l_{g - g(1 - \frac{F_i}{g})^p} \rangle; \\ (2) \quad a_j^q &= \langle l_{g(\frac{T_j}{g})^q}, l_{g - g(1 - \frac{I_j}{g})^q}, l_{g - g(1 - \frac{F_j}{g})^q} \rangle; \\ (3) \quad a_i^p \otimes a_j^q &= \langle l_{\frac{g(\frac{T_i}{g})^p g(\frac{T_j}{g})^q}{g}}, l_{g - g(1 - \frac{I_i}{g})^p + g - g(1 - \frac{I_j}{g})^q - \frac{(g - g(1 - \frac{I_i}{g})^p)(g - g(1 - \frac{I_j}{g})^q)}{g}}, l_{g - g(1 - \frac{F_i}{g})^p + g - g(1 - \frac{F_j}{g})^q - \frac{(g - g(1 - \frac{F_i}{g})^p)(g - g(1 - \frac{F_j}{g})^q)}{g}} \rangle \\ &= \langle l_{g(\frac{T_i}{g})^p (\frac{T_j}{g})^q}, l_{g - g(1 - \frac{I_i}{g})^p (1 - \frac{I_j}{g})^q}, l_{g - g(1 - \frac{F_i}{g})^p (1 - \frac{F_j}{g})^q} \rangle \\ (4) \quad \frac{w_i w_j}{1 - w_i} a_i^p \otimes a_j^q &= \langle l_{g - g(1 - \frac{g(\frac{T_i}{g})^p (\frac{T_j}{g})^q}{g})^{\frac{w_i w_j}{1 - w_i}}}, l_{g(\frac{g - g(1 - \frac{I_i}{g})^p (1 - \frac{I_j}{g})^q}{g})^{\frac{w_i w_j}{1 - w_i}}}, l_{g(\frac{g - g(1 - \frac{F_i}{g})^p (1 - \frac{F_j}{g})^q}{g})^{\frac{w_i w_j}{1 - w_i}}} \rangle \\ &= \langle l_{g - g(1 - (\frac{T_i}{g})^p (\frac{T_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}}}, l_{g(1 - (1 - \frac{I_i}{g})^p (1 - \frac{I_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}}}, l_{g(1 - (1 - \frac{F_i}{g})^p (1 - \frac{F_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}}} \rangle \\ (5) \quad \bigoplus_{i=1}^n \bigoplus_{\substack{j=1 \\ j \neq i}}^n \frac{w_i w_j}{1 - w_i} a_i^p \otimes a_j^q &= \langle l_{g - g \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n (1 - (\frac{T_i}{g})^p (\frac{T_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}}}, l_{g \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n (1 - (1 - \frac{I_i}{g})^p (1 - \frac{I_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}}}, l_{g \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n (1 - (1 - \frac{F_i}{g})^p (1 - \frac{F_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}}} \rangle \\ (6) \quad (\bigoplus_{i=1}^n \bigoplus_{\substack{j=1 \\ j \neq i}}^n \frac{w_i w_j}{1 - w_i} a_i^p \otimes a_j^q)^{\frac{1}{p+q}} &= \langle l_{g(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n (1 - (\frac{T_i}{g})^p (\frac{T_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}})^{\frac{1}{p+q}}}, l_{g - g(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n (1 - (1 - \frac{I_i}{g})^p (1 - \frac{I_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}})^{\frac{1}{p+q}}}, l_{g - g(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n (1 - (1 - \frac{F_i}{g})^p (1 - \frac{F_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}})^{\frac{1}{p+q}}} \rangle \end{aligned}$$

So $\left(\sum_{\substack{i,j=1 \\ j \neq i}}^n \frac{w_i w_j}{1 - w_i} a_i^p a_j^q \right)^{\frac{1}{p+q}} = \langle l_{g(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n (1 - (\frac{T_i}{g})^p (\frac{T_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}})^{\frac{1}{p+q}}}, l_{g - g(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n (1 - (1 - \frac{I_i}{g})^p (1 - \frac{I_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}})^{\frac{1}{p+q}}}, l_{g - g(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n (1 - (1 - \frac{F_i}{g})^p (1 - \frac{F_j}{g})^q)^{\frac{w_i w_j}{1 - w_i}})^{\frac{1}{p+q}}} \rangle$

The proof of Theorem 1 is completed. \square

Theorem 2. (Idempotency). Set $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ ($i = 1, 2, \dots, n$) as a collection of LNNs in L , if $a_i = a$, then

$$LNNNWBM^{p,q}(a_1, a_2, \dots, a_n) = LNNNWBM^{p,q}(a, a, \dots, a) = a.$$

Proof 2. Since $a_i = a$, i.e., $T_i = T; I_i = I; F_i = F$ for $i = 1, 2, \dots, n$, there are the following result:

$$\begin{aligned} LNNNWBM^{p,q}(a_1, a_2, \dots, a_n) &= LNNNWBM^{p,q}(a, a, \dots, a) = \left(\sum_{\substack{i,j=1 \\ j \neq i}}^n \frac{w_i w_j}{1-w_i} a^p a^q \right)^{\frac{1}{p+q}} \\ &= \left\langle l_{g(1-(1-\frac{T}{g})^p (\frac{T}{g})^q)^{\frac{1}{p+q}}, l_{g-g(1-(1-(1-\frac{I}{g})^p (1-\frac{I}{g})^q))^{\frac{1}{p+q}}, l_{g-g(1-(1-(1-\frac{F}{g})^p (1-\frac{F}{g})^q))^{\frac{1}{p+q}}} \right\rangle \\ &= \left\langle l_{g(1-(1-\frac{T}{g})^{p+q})^{\frac{1}{p+q}}, l_{g-g(1-(1-(1-\frac{I}{g})^{p+q}))^{\frac{1}{p+q}}, l_{g-g(1-(1-(1-\frac{F}{g})^{p+q}))^{\frac{1}{p+q}}} \right\rangle \\ &= \langle l_T, l_I, l_F \rangle = a. \end{aligned}$$

The proof of Theorem 2 is completed. \square

Theorem 3. (Monotonicity). Set $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ and $b_i = \langle l_{T_i'}, l_{I_i'}, l_{F_i'} \rangle$ ($i = 1, 2, \dots, n$) as two collections of LNNs in L , if $T_i \leq T_i', I_i \geq I_i'$, and $F_i \geq F_i'$ then $LNNNWBM^{p,q}(a_1, a_2, \dots, a_n) \leq LNNNWBM^{p,q}(b_1, b_2, \dots, b_n)$.

Proof 3. Since $T_i \leq T_i', I_i \geq I_i'$ and $F_i \geq F_i'$, we can easy obtain:

$$\begin{aligned} 1 - \left(\frac{T_i}{g}\right)^p \left(\frac{T_j}{g}\right)^q &\geq 1 - \left(\frac{T_i'}{g}\right)^p \left(\frac{T_j'}{g}\right)^q \\ 1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n \left(1 - \left(\frac{T_i}{g}\right)^p \left(\frac{T_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}} &\leq 1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n \left(1 - \left(\frac{T_i'}{g}\right)^p \left(\frac{T_j'}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}, \\ g \left(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n \left(1 - \left(\frac{T_i}{g}\right)^p \left(\frac{T_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}} &\leq g \left(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n \left(1 - \left(\frac{T_i'}{g}\right)^p \left(\frac{T_j'}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}. \end{aligned}$$

Similarly

$$\begin{aligned} \left(1 - \frac{I_i}{g}\right)^p \left(1 - \frac{I_j}{g}\right)^q &\leq \left(1 - \frac{I_i'}{g}\right)^p \left(1 - \frac{I_j'}{g}\right)^q, \\ 1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n \left(1 - \left(1 - \frac{I_i}{g}\right)^p \left(1 - \frac{I_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}} &\leq 1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n \left(1 - \left(1 - \frac{I_i'}{g}\right)^p \left(1 - \frac{I_j'}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}, \\ g - g \left(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n \left(1 - \left(1 - \frac{I_i}{g}\right)^p \left(1 - \frac{I_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}} &\geq g - g \left(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n \left(1 - \left(1 - \frac{I_i'}{g}\right)^p \left(1 - \frac{I_j'}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}} \end{aligned}$$

and

$$\begin{aligned} g - g \left(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n \left(1 - \left(1 - \frac{F_i}{g}\right)^p \left(1 - \frac{F_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}} &\geq g - g \left(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n \left(1 - \left(1 - \frac{F_i'}{g}\right)^p \left(1 - \frac{F_j'}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}. \end{aligned}$$

So, $LNNNWBM^{p,q}(a_1, a_2, \dots, a_n) \leq LNNNWBM^{p,q}(b_1, b_2, \dots, b_n)$ is true according to Theorem 3. Therefore, the proof of Theorem 3 is completed. \square

Theorem 4. (Boundedness). Set $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ ($i = 1, 2, \dots, n$) as a collections of LNNs in L , let $a^- = \langle \min(l_{T_i}), \max(l_{I_i}), \max(l_{F_i}) \rangle$ and $a^+ = \langle \max(l_{T_i}), \min(l_{I_i}), \min(l_{F_i}) \rangle$, then:

$$a^- \leq \text{LNNNWBM}^{p,q}(a_1, a_2, \dots, a_n) \leq a^+$$

Proof 4. According Theorem 2, we can obtain:

$$a^- = \text{LNNNWBM}^{p,q}(a^-, a^- \dots a^-) \text{ and } a^+ = \text{LNNNWBM}^{p,q}(a^+, a^+ \dots a^+)$$

According Theorem 3, we can obtain:

$$\text{LNNNWBM}^{p,q}(a^-, a^- \dots a^-) \leq \text{LNNNWBM}^{p,q}(a_1, a_2, \dots, a_n) \leq \text{LNNNWBM}^{p,q}(a^+, a^+ \dots a^+).$$

Then $a^- \leq \text{LNNNWBM}^{p,q}(a_1, a_2, \dots, a_n) \leq a^+$.

The proof of Theorem 4 is completed. \square

3.2. Normalized Weighted Geometric BM Operators of LNNs

Definition 10. Set $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ ($i = 1, 2, \dots, n$) as a collection of LNNs in L , then the LNNNWGBM operator can be defined as follows:

$$\text{LNNNWGBM}^{p,q}(a_1, a_2, \dots, a_n) = \frac{1}{p+q} \otimes_{i=1}^n \otimes_{j=1, j \neq i}^n (pa_i \oplus qa_j)^{\frac{w_i w_j}{1-w_i}} \tag{13}$$

where w_i is the relative weight of a_i , $w_i \in [0, 1]$, and $\sum_{i=1}^n w_i = 1$, w_j is the relative weighted of a_j , $w_j \in [0, 1]$, and $\sum_{j=1}^n w_j = 1$.

According to Definitions 2 and 10, we can get the following theorem:

Theorem 5. Set $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ ($i = 1, 2, \dots, n$) as a collection of LNNs in L , then by the Equation (13) the aggregation result obtained is still an LNN, and we can get the following aggregation formula:

$$\begin{aligned} \text{LNNNWGBM}^{p,q}(a_1, a_2, \dots, a_n) &= \frac{1}{p+q} \oplus_{i=1}^n \oplus_{j=1, j \neq i}^n (pa_i \oplus qa_j)^{\frac{w_i w_j}{1-w_i}} \\ &= \left(l \right. \\ &\quad \left. s^{-s(1-\prod_{i=1}^n \prod_{j=1, j \neq i}^n (1-(\frac{T_i}{s})^p (1-\frac{T_j}{s})^q)^{\frac{w_i w_j}{1-w_i}}}, l \right. \\ &\quad \left. s^{(1-\prod_{i=1}^n \prod_{j=1, j \neq i}^n (1-(\frac{I_i}{s})^p (\frac{I_j}{s})^q)^{\frac{w_i w_j}{1-w_i}}}, l \right. \\ &\quad \left. s^{(1-\prod_{i=1}^n \prod_{j=1, j \neq i}^n (1-(\frac{F_i}{s})^p (\frac{F_j}{s})^q)^{\frac{w_i w_j}{1-w_i}}} \right) \tag{14} \end{aligned}$$

where w_i is the relative weight of a_i , $w_i \in [0, 1]$, and $\sum_{i=1}^n w_i = 1$, w_j is the relative weighted of a_j , $w_j \in [0, 1]$, and $\sum_{j=1}^n w_j = 1$.

The proof of Theorem 5 is similar to that of Theorem 1, so we do not repeat it again.

Theorem 6. (Idempotency). Set $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ ($i = 1, 2, \dots, n$) as a collection of LNNs in L , if $a_i = a$, then

$$\text{LNNNWGBM}^{p,q}(a_1, a_2, \dots, a_n) = \text{LNNNWGBM}^{p,q}(a, a \dots a) = a$$

The proof of Theorem 6 is similar to that of Theorem 2, so we don't repeat it again.

Theorem 7. (Monotonicity). Set $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ and $b_i = \langle l_{T_i'}, l_{I_i'}, l_{F_i'} \rangle$ ($i = 1, 2, \dots, n$) as two collections of LNNs in L , if $T_i \leq T_i', I_i \geq I_i'$ and $F_i \geq F_i'$ then:

$$\text{LNNNWGBM}^{p,q}(a_1, a_2, \dots, a_n) \leq \text{LNNNWGBM}^{p,q}(b_1, b_2, \dots, b_n)$$

The proof of Theorem 7 is similar to that of Theorem 3, so we do not repeat it again.

Theorem 8. (Boundedness). Set $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ ($i = 1, 2, \dots, n$) as a collections of LNNs in L , let $a^- = \langle \min(l_{T_i}), \max(l_{I_i}), \max(l_{F_i}) \rangle$ and $a^+ = \langle \max(l_{T_i}), \min(l_{I_i}), \min(l_{F_i}) \rangle$, then:

$$a^- \leq LNNNWGBM^{p,q}(a_1, a_2, \dots, a_n) \leq a^+$$

The proof of Theorem 8 is similar to that of Theorem 4, so we do not repeat it again.

4. MAGDM Methods Based on the LNNNWBM or LNNNWGBM Operator

In this section, we will use the LNNNWBM or LNNNWGBM operator to deal with the MAGDM problems with LNN information.

In a MAGDM problem, there is a set of several alternatives $A = \{A_1, A_2, \dots, A_m\}$ with a set of some attributes $C = \{C_1, C_2, \dots, C_n\}$. Then, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)^T$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$ are the weights of C_i ($i = 1, 2, \dots, n$). Now, there is a set of t experts $E = \{E_1, E_2, \dots, E_t\}$ to evaluate the MAGDM problem. Assume that $w = (w_1, w_2, \dots, w_t)^T$ with $w_j \geq 0$ and $\sum_{j=1}^t w_j = 1$ is the vector of the weights for E_y ($y = 1, 2, \dots, t$) and $L = \{l_1, l_2, \dots, l_g\}$ is the given linguistic term set. The assessed value of the expert E_y for A_i with attribute C_j is $a_{ij}^{(y)} = \langle l_{T_{ij}}^y, l_{I_{ij}}^y, l_{F_{ij}}^y \rangle \in A$ ($y = 1, 2, \dots, t; i = 1, 2, \dots, m; j = 1, 2, \dots, n$), $l_{T_{ij}}^y, l_{I_{ij}}^y, l_{F_{ij}}^y \in L$. Then, we can get the neutrosophic linguistic decision evaluation matrix R^y , which is shown in Table 1.

Table 1. The neutrosophic linguistic decision matrix R^y of the expert E_y .

	C_1	...	C_n
A_1	$\langle l_{T_{11}}^y, l_{I_{11}}^y, l_{F_{11}}^y \rangle$...	$\langle l_{T_{1n}}^y, l_{I_{1n}}^y, l_{F_{1n}}^y \rangle$
A_2	$\langle l_{T_{21}}^y, l_{I_{21}}^y, l_{F_{21}}^y \rangle$...	$\langle l_{T_{2n}}^y, l_{I_{2n}}^y, l_{F_{2n}}^y \rangle$
...
A_m	$\langle l_{T_{m1}}^y, l_{I_{m1}}^y, l_{F_{m1}}^y \rangle$...	$\langle l_{T_{mn}}^y, l_{I_{mn}}^y, l_{F_{mn}}^y \rangle$

Then, based on the LNNNWBM or LNNNWGBM operator, we propose two decision-making methods, which are described as the following decision steps:

Step 1: According to the weight vector $w = (w_1, w_2, \dots, w_t)^T$ of experts and the LNNNWBM operator, we can obtain the integrated matrix $R = (a_{ij})_{m \times n}$, where the collective LNN a_{ij} can be obtained by the following formula:

$$a_{ij} = LNNNWBM(a_{ij}^1, a_{ij}^2, \dots, a_{ij}^t) = \left(\sum_{\substack{t \\ j \neq i}} \frac{w_t w_j}{1-w_t} a_i^p a_j^q \right)^{\frac{1}{p+q}}$$

$$= \langle l_{s(1-\prod_{i=1}^t \prod_{j=1}^t (1-(\frac{T_i}{g})^p (\frac{T_j}{g})^q)^{\frac{w_i w_j}{1-w_i}})^{\frac{1}{p+q}}, l_{s-g(1-\prod_{i=1}^t \prod_{j=1}^t (1-(1-\frac{I_i}{g})^p (1-\frac{I_j}{g})^q)^{\frac{w_i w_j}{1-w_i}})^{\frac{1}{p+q}}, l_{s-g(1-\prod_{i=1}^t \prod_{j=1}^t (1-(1-\frac{F_i}{g})^p (1-\frac{F_j}{g})^q)^{\frac{w_i w_j}{1-w_i}})^{\frac{1}{p+q}}} \rangle$$
(15)

Step 2: According to the weight vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)^T$ of attributes and the LNNNWBM operator or the LNNNWGBM operator, we can obtain the total collective LNN a_i for A_i ($i = 1, 2, \dots, m$).

$$a_i = LNNNWBM(a_{i1}, a_{i2}, \dots, a_{in}) = \left(\sum_{\substack{n \\ j \neq i}} \frac{w_j \lambda_j}{1-w_j} a_i^p a_j^q \right)^{\frac{1}{p+q}}$$

$$= \langle l_{s(1-\prod_{i=1}^n \prod_{j=1}^n (1-(\frac{T_i}{g})^p (\frac{T_j}{g})^q)^{\frac{w_i w_j}{1-w_i}})^{\frac{1}{p+q}}, l_{s-g(1-\prod_{i=1}^n \prod_{j=1}^n (1-(1-\frac{I_i}{g})^p (1-\frac{I_j}{g})^q)^{\frac{w_i w_j}{1-w_i}})^{\frac{1}{p+q}}, l_{s-g(1-\prod_{i=1}^n \prod_{j=1}^n (1-(1-\frac{F_i}{g})^p (1-\frac{F_j}{g})^q)^{\frac{w_i w_j}{1-w_i}})^{\frac{1}{p+q}}} \rangle$$
(16)

or:

$$\begin{aligned}
 a_i &= \text{LNNNWGBM}(a_{i1}, a_{i2}, \dots, a_{in}) = \frac{1}{p+q} \otimes_{i=1}^n \otimes_{j=1, j \neq i}^n (pa_i \oplus qa_j)^{\frac{w_i w_j}{1-w_i}} \\
 &= \langle l_{g - g(1 - \prod_{i=1}^n \prod_{j \neq i}^n (1 - (1 - \frac{T_i}{g})^p (1 - \frac{T_j}{g})^q)^{\frac{w_i w_j}{1-w_i}}}, l_{g(1 - \prod_{i=1}^n \prod_{j \neq i}^n (1 - (\frac{L_i}{g})^p (\frac{L_j}{g})^q)^{\frac{w_i w_j}{1-w_i}})}, l_{g(1 - \prod_{i=1}^n \prod_{j \neq i}^n (1 - (\frac{F_i}{g})^p (\frac{F_j}{g})^q)^{\frac{w_i w_j}{1-w_i}}} \rangle \quad (17)
 \end{aligned}$$

Step 3: According to the Equation (5) (Equation (6) if necessary), we calculate the expected value $E(a_i)$ and the accuracy $H(a_i)$ of the LNN $A_i (i = 1, 2, \dots, m)$.

Step 4: According to the value $E(a_i)$ ($H(a_i)$ if necessary), then we can rank the alternatives and choose the best one.

5. Illustrative Examples

The decision-making problem used in the literature [21] is considered here. There are four companies as a set of alternatives $A = \{A_1, A_2, A_3, A_4\}$, which are a car company (A_1), a food company (A_2), a computer company (A_3), and an arm company (A_4). An investment company needs to invest the best company, so they invite a set of three experts $E = \{E_1, E_2, E_3\}$ to evaluate these four companies. The evaluation of the alternatives must satisfy a set of three attributes $C = \{C_1, C_2, C_3\}$, which are the risk (C_1), the growth (C_2), and the environmental impact (C_3). The importance of three experts is given as a weight vector $w = (0.37, 0.33, 0.3)^T$ and the importance of three attributes is given as a weight vector $\lambda = (0.35, 0.25, 0.4)^T$. Then, the evaluation criteria are based on the linguistic term set $L = \{l_0 = \text{extremely bad}, l_1 = \text{very bad}, l_2 = \text{bad}, l_3 = \text{slightly bad}, l_4 = \text{medium}, l_5 = \text{slightly good}, l_6 = \text{good}, l_7 = \text{very good}, l_8 = \text{extremely good}\}$. Thus, we can establish the LNN decision matrix $R^i (i = 1, 2, 3)$, which is listed in Tables 2–4.

Table 2. The LNN decision matrix R^1 of the expert E_1 .

	C_1	C_2	C_3
A_1	$\langle l_6^1, l_1^1, l_2^1 \rangle$	$\langle l_7^1, l_2^1, l_1^1 \rangle$	$\langle l_6^1, l_2^1, l_2^1 \rangle$
A_2	$\langle l_7^1, l_1^1, l_1^1 \rangle$	$\langle l_7^1, l_3^1, l_2^1 \rangle$	$\langle l_7^1, l_2^1, l_1^1 \rangle$
A_3	$\langle l_6^1, l_2^1, l_2^1 \rangle$	$\langle l_7^1, l_1^1, l_1^1 \rangle$	$\langle l_6^1, l_2^1, l_2^1 \rangle$
A_4	$\langle l_7^1, l_1^1, l_2^1 \rangle$	$\langle l_7^1, l_2^1, l_3^1 \rangle$	$\langle l_7^1, l_2^1, l_1^1 \rangle$

Table 3. The LNN decision matrix R^2 of the expert E_2 .

	C_1	C_2	C_3
A_1	$\langle l_6^2, l_1^2, l_2^2 \rangle$	$\langle l_6^2, l_1^2, l_1^2 \rangle$	$\langle l_4^2, l_2^2, l_3^2 \rangle$
A_2	$\langle l_7^2, l_2^2, l_3^2 \rangle$	$\langle l_6^2, l_1^2, l_1^2 \rangle$	$\langle l_4^2, l_2^2, l_3^2 \rangle$
A_3	$\langle l_5^2, l_1^2, l_2^2 \rangle$	$\langle l_5^2, l_1^2, l_2^2 \rangle$	$\langle l_5^2, l_4^2, l_2^2 \rangle$
A_4	$\langle l_6^2, l_1^2, l_1^2 \rangle$	$\langle l_5^2, l_1^2, l_1^2 \rangle$	$\langle l_5^2, l_2^2, l_3^2 \rangle$

Table 4. The LNN decision matrix R^3 of the expert E_3 .

	C_1	C_2	C_3
A_1	$\langle l_7^3, l_3^3, l_4^3 \rangle$	$\langle l_7^3, l_3^3, l_3^3 \rangle$	$\langle l_5^3, l_2^3, l_5^3 \rangle$
A_2	$\langle l_6^3, l_3^3, l_4^3 \rangle$	$\langle l_5^3, l_1^3, l_2^3 \rangle$	$\langle l_6^3, l_2^3, l_3^3 \rangle$
A_3	$\langle l_7^3, l_2^3, l_4^3 \rangle$	$\langle l_6^3, l_1^3, l_2^3 \rangle$	$\langle l_7^3, l_2^3, l_4^3 \rangle$
A_4	$\langle l_7^3, l_2^3, l_3^3 \rangle$	$\langle l_5^3, l_2^3, l_1^3 \rangle$	$\langle l_6^3, l_1^3, l_1^3 \rangle$

5.1. The Decision-Making Process Based on the LNNNWBM Operator or LNNNWGBM Operator

Step 1: According to the weight vector $w = (0.37, 0.33, 0.3)^T$ of experts and the LNNNWBM operator (set $p = 1$ and $q = 1$), we can obtain the integrated matrix $R = (a_{ij})_{m \times n}$, which is listed in Table 5.

Table 5. The integrated matrix R.

	C ₁	C ₂	C ₃
A ₁	$\langle l_{6.3176}, l_{1.5682}, l_{2.6129} \rangle$	$\langle l_{6.6819}, l_{1.9641}, l_{1.5682} \rangle$	$\langle l_{5.0059}, l_{2.000}, l_{3.2898} \rangle$
A ₂	$\langle l_{6.7045}, l_{1.9476}, l_{2.6308} \rangle$	$\langle l_{6.0524}, l_{1.6728}, l_{1.6636} \rangle$	$\langle l_{5.7033}, l_{2.000}, l_{2.3074} \rangle$
A ₃	$\langle l_{5.9943}, l_{1.6636}, l_{2.6129} \rangle$	$\langle l_{6.0264}, l_{1.000}, l_{1.6430} \rangle$	$\langle l_{5.9943}, l_{2.6613}, l_{2.6129} \rangle$
A ₄	$\langle l_{6.6819}, l_{1.2955}, l_{1.9641} \rangle$	$\langle l_{5.6926}, l_{1.6636}, l_{1.6728} \rangle$	$\langle l_{6.0264}, l_{1.6824}, l_{1.6170} \rangle$

Step 2: According to the weight vector $\lambda = (0.35, 0.25, 0.4)^T$ of attributes and the LNNNWBM operator (set $p = 1$ and $q = 1$), we can obtain the collective overall LNNs of a_i for $A_i (i = 1, 2, 3, 4)$ as follows:

$$a_1 = \langle l_{5.9328}, l_{1.8388}, l_{2.5784} \rangle, a_2 = \langle l_{6.1489}, l_{1.8908}, l_{2.2399} \rangle,$$

$$a_3 = \langle l_{6.0032}, l_{1.8430}, l_{2.3427} \rangle, \text{ and } a_4 = \langle l_{6.1675}, l_{1.5412}, l_{1.7536} \rangle.$$

Step 3: Calculating the expected values of $E(a_i)$ for $a_i (i = 1, 2, 3, 4)$:

$$E(a_1) = 0.7298, E(a_2) = 0.7508, E(a_3) = 0.7424, \text{ and } E(a_4) = 0.7864.$$

According to the results, we can rank $E(a_4) > E(a_2) > E(a_3) > E(a_1)$, so the company A_4 is the best choice among all the companies.

On the other hand, we also use the LNNNWGBM operator (set $p = 1$ and $q = 1$) to deal with this decision-making problem:

Step 1': Just as step 1;

Step 2': According to the weight vector $\lambda = (0.35, 0.25, 0.4)^T$ of attributes and the LNNNWGBM operator (set $p = 1$ and $q = 1$), we can obtain the collective overall LNNs of a_i for $A_i (i = 1, 2, 3, 4)$ as follows:

$$a_1 = \langle l_{5.9970}, l_{1.8333}, l_{2.5434} \rangle, a_2 = \langle l_{6.1790}, l_{1.8897}, l_{2.2324} \rangle,$$

$$a_3 = \langle l_{6.0032}, l_{1.7928}, l_{2.3332} \rangle, \text{ and } a_4 = \langle l_{6.1824}, l_{1.5362}, l_{1.7500} \rangle.$$

Step 3': Calculating the expected values of $E(a_i)$ for $a_i (i = 1, 2, 3, 4)$:

$$E(a_1) = 0.7342, E(a_2) = 0.7524, E(a_3) = 0.7449, \text{ and } E(a_4) = 0.7873.$$

According to the results, the ranking is $E(a_4) > E(a_2) > E(a_3) > E(a_1)$, so the company A_4 is the best choice among all the companies.

5.2. Analysis the Influence of the Parameters p and q on Decision Results

In order to analyze the effects of different parameters p and q on the decision results, in Steps 1 and 2, we take the different values of p and q , and all the results are shown in the Tables 6 and 7.

Table 6. The ranking based on the LNNNWBM operator with the different values of p and q .

p, q	LNNNWBM Operator	Ranking
$p = 1, q = 0$	$E(a_1) = 0.7528, E(a_2) = 0.7777, E(a_3) = 0.7613, E(a_4) = 0.8060$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 1, q = 0.5$	$E(a_1) = 0.7311, E(a_2) = 0.7534, E(a_3) = 0.7435, E(a_4) = 0.7886$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 1, q = 2$	$E(a_1) = 0.7329, E(a_2) = 0.7545, E(a_3) = 0.7453, E(a_4) = 0.7897$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 0, q = 0$	$E(a_1) = 0.7573, E(a_2) = 0.7766, E(a_3) = 0.7656, E(a_4) = 0.8046$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 0.5, q = 1$	$E(a_1) = 0.7326, E(a_2) = 0.7530, E(a_3) = 0.7449, E(a_4) = 0.7879$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 2, q = 1$	$E(a_1) = 0.7349, E(a_2) = 0.7562, E(a_3) = 0.7463, E(a_4) = 0.7902$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 2, q = 2$	$E(a_1) = 0.7343, E(a_2) = 0.7537, E(a_3) = 0.7458, E(a_4) = 0.7884$	$A_4 \succ A_2 \succ A_3 \succ A_1$

Table 7. The ranking based on the LNNNWGBM operator with the different values of p and q .

p, q	LNNNWGBM Operator	Ranking
$p = 1, q = 0$	$E(a_1) = 0.7397, E(a_2) = 0.7747, E(a_3) = 0.7531, E(a_4) = 0.8035$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 1, q = 0.5$	$E(a_1) = 0.7342, E(a_2) = 0.7545, E(a_3) = 0.7453, E(a_4) = 0.7891$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 1, q = 2$	$E(a_1) = 0.7343, E(a_2) = 0.7548, E(a_3) = 0.7457, E(a_4) = 0.7889$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 0, q = 1$	$E(a_1) = 0.7437, E(a_2) = 0.7730, E(a_3) = 0.7570, E(a_4) = 0.8019$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 0.5, q = 1$	$E(a_1) = 0.7356, E(a_2) = 0.7541, E(a_3) = 0.7467, E(a_4) = 0.7885$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 2, q = 1$	$E(a_1) = 0.7330, E(a_2) = 0.7553, E(a_3) = 0.7445, E(a_4) = 0.7895$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 2, q = 2$	$E(a_1) = 0.7334, E(a_2) = 0.7530, E(a_3) = 0.7441, E(a_4) = 0.7877$	$A_4 \succ A_2 \succ A_3 \succ A_1$

From above two tables, we can see that when the parameters p and q take different values, the sorting results are the same. Therefore, the influence of the two parameters is very little in this decision-making problem.

In the literature [21], the ranking is $A_4 \succ A_2 \succ A_3 \succ A_1$, just according with the ranking result of this paper. Compared with the literature [21], the correlation between attributes is considered by the LNNNWBM operator and the LNNNWGBM operator for MAGDM, which make the information aggregation more objective and reliable. Hence, the proposed MAGDM methods with different p and q values are more flexible than the method in [21]. Compared to the literature [14], on the one hand, the literature [14] cannot express and deal with the decision-making problems with pure linguistic information like LNNs. However, in this paper, the proposed decision-making methods based on the LNNNWBM operator and the LNNNWGBM operator provide a new way for decision-makers under LNN environment.

6. Conclusions

In MADGM, how to tackle the problem of the interdependence between attributes is a challenging issue. Thus, MADGM methods based on the LNNNWGBM and LNNNWGBM operators for LNNs are proposed in this paper. First, a LNN normalized weight Bonferroni mean (LNNNWBM) operator and a LNN normalized weight geometric Bonferroni mean (LNNNWGBM) operator are proposed based on the BM operator, and the related properties of these operators are discussed. Second, based on the LNNNWBM operator and the LNNNWGBM operator, this paper puts forward two methods of MADGM in a LNN setting. Finally, an illustrative example was presented to show that these two methods were used for solving the MADGM problem with LNN information. In addition, the proposed decision-making methods may affect the decision results based on various parameters of p and q in some decision-making problems.

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