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# A Parametric Family of Triangular Norms and Conorms with an Additive Generator in the Form of an Arctangent of a Linear Fractional Function

**Tatiana Ledeneva**

Department of Computational Mathematics and Applied Information Technologies, Voronezh State University, 394006 Voronezh, Russia; ledeneva-tm@yandex.ru

**Abstract:** At present, fuzzy modeling has established itself as an effective tool for designing and developing systems for various purposes that are used to solve problems of control, diagnostics, forecasting, and decision making. One of the most important problems is the choice and justification of an appropriate functional representation of the main fuzzy operations. It is known that, in the class of rational functions, such operations can be represented by additive generators in the form of a linear fractional function, a logarithm of a linear fractional function, and an arctangent of a linear fractional function. The paper is devoted to the latter case. Restrictions on the parameters, under which the arctangent of a linear fractional function is an increasing or decreasing generator, are defined. For each case, a corresponding fuzzy operation (a triangular norm or a conorm) is constructed. The theoretical significance of the research results lies in the fact that the obtained parametric families enrich the theory of Archimedean triangular norms and conorms and provide additional opportunities for the functional representation of fuzzy operations in the framework of fuzzy modeling. In addition, in fact, we formed a scheme for study functions that can be considered additive generators and constructed the corresponding fuzzy operations.

**Keywords:** linear fractional function; additive generator; triangular norm and conorm



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## 1. Introduction

Fuzzy set theory and fuzzy logic form the basis of the fuzzy modeling methodology, and the following main situations that determine the feasibility of its use in applications can be distinguished: (a) there is an interest in the approximate representation and processing of input on the basis of the concepts of fuzzy and linguistic variables (for example, when choosing the best object, a set of perspective variants is first formed using linguistic assessments, and then, a more thorough quantitative analysis is applied); (b) it is difficult or even impossible to obtain accurate quantitative information, but the involvement of an expert or an expert group solves this problem; (c) the model is constructed according to the "gray-box" principle, when, for a system or process, the dependence of the output variable on the input variables can be described only approximately or at a qualitative level, for example, using if–then rules (this approach allows us to perform a "quick" simulation of complex dynamic systems).

The main directions of the practical use of the fuzzy modeling methodology are the following [1–3]: the development of models and methods of decision making under conditions of uncertainty; technical and medical diagnostics; the improvement of technological processes through the introduction of fuzzy control systems; monitoring and forecasting the state of hazardous facilities; and the management of technical systems and equipment, including transport. The use of fuzzy modeling makes it possible to solve control problems in situations where classical methods are inefficient or even inapplicable because of a lack of sufficiently accurate knowledge about the control object and/or the conditions of its operation. In modern developments, fuzzy models are used together with neural networks, genetic algorithms, and clustering methods. Thus, a mathematical apparatus is formed that is capable of solving complex problems under conditions of uncertainty, which manifests itself both in the behavior of the system or the object being modeled and as a factor in the influence of the external environment.

At present, an extensive arsenal of fuzzy modeling methods is available. One of the most important problems is the problem of choosing the most appropriate way to formalize fundamental logical connectives since any computing system is based on the definitions of basic operations. The results of calculations and their interpretation depend on how these operations are defined, and this fact determines the importance of choosing their functional representation.

Figure 1 shows the well-known problem of parking a mobile object.

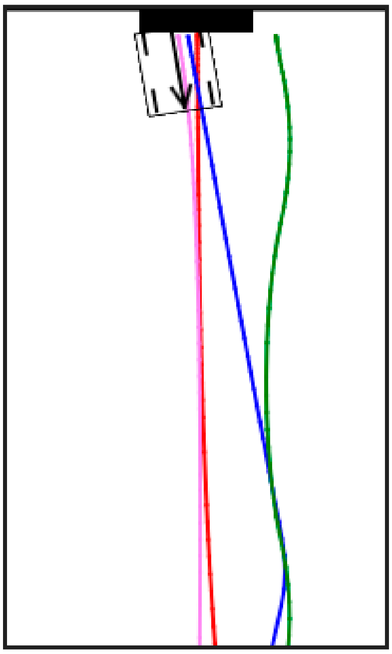

**Figure 1.** The trajectory of movement of a mobile object (Trajectories marked with different colors correspond to inference engine implementations that are based on different representations of fuzzy logic operations).

The object must park up the ramp in a rectangular area (parking zone). The position of the object is defined by the values of the coordinates of the center of its rear part and the angle at which it is located relative to the $y$-axis. Parking on the ramp is performed by moving in reverse at a constant speed. The task of the control module is to select a steering angle for the front wheels that will bring the object to the parking point located at the center of the ramp. The figure shows various object movement trajectories generated using a fuzzy control system. Trajectories in different colors correspond to different implementations of the inference engine. These implementations are based on various representations of fuzzy logical connectives to define the composition operation. The center of gravity method is used as a defuzzification method. It can be seen that an incorrect selection of operation can lead to the control problem not being successfully solved.

Initially, the operation's min and max were used for *and* and *or*, respectively. It should be noted that these operations play a significant role in modern approaches to the construction of fuzzy models in various systems since their physical meaning is obvious, and of the main properties of algebraic operations, only the complementarity property is not satisfied for them. However, a significant drawback of min and max is their "rigidity", which manifests itself in the insensitivity of the result of the operation to small changes in arguments. This problem is partially solved, for example, through algebraic operations: $(xy,\ x + y - xy)$. In the parking problem discussed above, when using the

$(\max - \text{prod})$-composition in the inference engine, the trajectory is smoother and shorter than the inference, which is based on the $(\max - \min)$-composition. This is because the product operation, $xy$, reacts to all input changes in the model, while min only considers the input change with the smallest membership function value.

In our opinion, parametric forms of fuzzy operations—which allow us to consider the characteristic features of a particular application and ensure the flexibility and adaptive properties of fuzzy models by selecting parameters—are of the greatest interest for modeling. This class of fuzzy operations includes triangular norms and conorms, and the theory devoted to them can be considered fully formed. Known families of triangular norms and conorms proposed by various researchers are presented, for example, in [4]. And, of course, there is a question regarding the existence of other families, which are different from those already known. One approach to answering this question is to study the representation of triangular norms and conorms in terms of their additive generators [5], and related problems were discussed in [6]. This representation is connected to the associativity property of binary operations [7]. Knowledge of additive generators is the source of much research on special operations and fuzzy structures [8–11]. Additive generators are used to construct aggregation functions in the class of means and OWA operators [12,13]. In [14], additive generators are used to define special operations in fuzzy numbers and intervals. Based on these, the formalization of the compositional rule of logical inference is conducted when building fuzzy systems [15].

We note that there is no universal method for finding an additive generator or its corresponding triangular norms and conorms. But, in some cases, such procedures are known [16–19]. In particular, the author proposes approaches to finding additive generators for Archimedean triangular norms and conorms represented by a rational function. The choice of the class of rational functions is due to the fact that most of the known families widely cited in theoretical studies and applied in practice belong to this class. In [20], generators in the form of a linear fractional function are considered, and the corresponding families of triangular norms, $T^{LFAG}$, and conorms, $S^{LFAG}$, are constructed. In [21], a family of triangular norms, $T^{\ln LFAG}$, generated by generators in the form of a logarithm of a linear fractional function is proposed. A significant theoretical result of the research is that the resulting families of fuzzy operations include, as special cases, the known families of triangular norms and conorms (Hamacher, Lukasiewicz, Einstein, Weber, Yu), which allows us to speak of systematizing and generalizing the existing results of many years of research in this area.

The purpose of this paper is to study the arctangent of a linear fractional function and to define the corresponding family of Archimedean triangular norms and conorms. The results obtained, in fact, close the issue of the representation of fuzzy operations in the class of rational functions.

Section 2 provides basic theoretical information, including theorems of the representation of triangular norms and conorms using additive generators. Equally important are formulas that establish a relationship between increasing and decreasing generators, which allows us, knowing one of the generators, to define the other. Using generators, we can find negation functions, which is important for defining the de Morgan triple. In Section 3, we shall define restrictions in parameters under which a function in the form of an arctangent of a linear fractional function is an increasing or decreasing generator and find the corresponding fuzzy operations. Section 4 discusses the results obtained. A general scheme of the function study method is provided based on a stage-by-stage verification of the fulfillment of the requirements contained in the corresponding definitions of the generators and fuzzy operations.

## 2. Materials and Methods

Let us introduce the necessary definitions based on [5,22,23].

A *triangular norm* (*t-norm*) is a binary operation, $T : [0, 1] \times [0, 1] \to [0, 1]$, that is non-decreasing with respect to each argument and has the properties of commutativity and associativity and $T(x, 1) = T(1, x) = x$ for each $x \in [0, 1]$.

A *triangular conorm* (*s-conorm*) is a binary operation, $S : [0, 1] \times [0, 1] \to [0, 1]$, that is non-increasing with respect to each argument and also has the properties of commutativity and associativity and $S(x, 0) = S(0, x) = x$ for each $x \in [0, 1]$.

The triangular norm models operations of the multiplication type (intersection of fuzzy sets, conjunction), while the conorm models operations of the addition type (union of fuzzy sets, disjunction). The systems $\langle [0, 1], T, \leq \rangle$ and $\langle [0, 1], S, \leq \rangle$ are Abelian semigroups.

The *negation function* is understood as a continuous, monotonically decreasing operation, $n : [0, 1] \to [0, 1]$, that satisfies the conditions $n(0) = 1$, $n(1) = 0$. If $n(n(x)) = x$, then $n$ is called an *involution*.

Operations $T$ and $S$ are called *dual* with respect to involution, $n$, if the following relations hold: $n(T(x, y)) = S(n(x), n(y)), n(S(x, y)) = T(n(x), n(y))$. This formalizes de Morgan's laws. The tuple, $(T, S, n)$, forms the *de Morgan triple*.

Examples of dual fuzzy operations are the following well-known pairs [4]:

$$T_0(x, y) = \frac{xy}{x + y - xy},\ S_{-1}(x, y) = \frac{x + y - 2xy}{1 - xy};$$

$$T_P(x, y) = xy,\ S_P(x, y) = x + y - xy;$$

$$T_\lambda(x, y) = \max(0, x + y - 1 - \lambda(1 - x)(1 - y)),\ S_\lambda(x, y) = \min(1,\ x + y + \lambda xy)(\lambda > -1).$$

For each pair, $(T, S)$, the standard negation, $n(x) = 1 - x$, "works". However, original negation functions may exist for some pairs. For example, for the pair $(T_\lambda, S_\lambda)$, there is a negation function of the form $n_\lambda(x) = \frac{1-x}{1+\lambda x}$ $(\lambda > -1)$ [5].

A continuous t-norm that satisfies subidempotency ($T(x, x) < x$) is called an Archimedean t-norm. If it also satisfies strict monotonicity, it is called a strict Archimedean t-norm. Similar definitions hold for triangular conorms, but idempotency violation is defined by formula $S(x, x) > x$ (superidempotency).

The notion of an additive generator is fundamental to the representation of associative operations and, in particular, Archimedean t-norms and s-conorms.

The *decreasing generator,* $t :\ [0, 1] \to \mathbb{R}$, is a continuous, strictly decreasing function such that $t(1) = 0$, and its pseudo-inverse, $t^{(-1)} :\ \mathbb{R} \to [0, 1]$, is defined as follows:

$$t^{(-1)}(x) = \begin{cases} 1,\ if\ x < 0, \\ t^{-1}(x),\ if\ x \in [0, t(0)], \\ 0,\ if\ x > t(0). \end{cases}$$

The *Increasing generator,* $s :\ [0, 1] \to \mathbb{R}$, is a continuous, strictly increasing function such that $s(0) = 0$, and its pseudo-inverse, $t^{(-1)} :\ \mathbb{R} \to [0, 1]$, is defined as follows:

$$s^{(-1)}(x) = \begin{cases} 0,\ if\ x < 0, \\ s^{-1}(x),\ if\ x \in [0, s(1)], \\ 1,\ if\ x > s(1). \end{cases}$$

We note that for $x \in [0, 1]$, we have $t^{(-1)}(t(x)) = x$ and $s^{(-1)}(s(x)) = x$.

On the basis of an increasing generator, we can construct a decreasing generator and vice versa [23].

Let $t$ be a decreasing generator; then, the function

$$s(x) = t(0) - t(x), \tag{1}$$

defined in $x \in [0, 1]$, is an increasing generator.

Let $s$ be an increasing generator; then, the function,

$$t(x) = s(1) - s(x), \tag{2}$$

defined in $x \in [0,1]$, is a decreasing generator.

We note that the generators obtained on the basis of the above relations satisfy the condition $s(1) = t(0)$.

It is known [5,23] that each Archimedean t-norm, $T$, can be represented using a decreasing generator, $t$, in the form

$$T(x,y) = t^{(-1)}(t(x) + t(y)), \tag{3}$$

where $t^{(-1)}$ is a pseudo-inverse.

For the Archimedean s-conorm, there is a similar representation,

$$S(x,y) = s^{(-1)}(s(x) + s(y)), \tag{4}$$

where $s$ is an increasing generator, and $s^{(-1)}$ is a pseudo-inverse.

Knowing the additive generator allows us to construct a triangular norm or conorm in accordance with the above formulas. Increasing and decreasing generators are defined up to a positive multiplicative constant [5].

Let $T$ and $S$ be the Archimedean operations generated by the corresponding additive generators, $t$ and $s$, respectively, and $t(1) < \infty$, $s(0) < \infty$; then, $T$ and $S$ induce strong negations, $n_T$ and $n_S$, which are defined by the following formulas [23]:

$$n_S(x) = s^{-1}(s(1) - s(x)), \tag{5}$$

$$n_T(x) = t^{-1}(t(0) - t(x)). \tag{6}$$

If $T$ and $S$ are dual with respect to the negation function, $n$, then $n_T \circ n = n \circ n_S$, where $\circ$ is the superposition operation.

The characterization of the main classes of additive generators is one of the most important problems in the functional representation of fuzzy operations [6]. In [24], we investigated the Archimedean t-norms and s-conorms representable using rational functions, i.e., the relationship between two polynomials or, in a special case, a polynomial. For example, the Hamacher family of t-norms has the following form [4]:

$$T_h^H(x,y) = \begin{cases} \frac{xy}{x+y-xy},\ if\ h=0; \\ xy,\ if\ h=1; \\ \frac{xy}{h+(1-h)(x+y-xy)},\ if\ h \in (0,\infty),\ h \neq 1. \end{cases}$$

For $T_h^H$, the decreasing generator has the form

$$t_h^H(x) = \begin{cases} \frac{1-x}{x},\ if\ h=0; \\ \ln\frac{h+(1-h)x}{x},\ if\ h \in (0,\infty). \end{cases}$$

It is established that, for a rational function of the form,

$$F(x,y) = \frac{a_0 + a_1(x+y) + a_2xy}{b_0 + b_1(x+y) + b_2xy}. \tag{7}$$

There are only three types of functions [24,25],

$$\varphi_1(x) = \frac{ax+b}{cx+d} + C,\ \varphi_2(x) = k \cdot \ln\frac{ax+b}{cx+d} + C,\ \varphi_3(x) = k \cdot \arctan\frac{ax+b}{cx+d} + C,$$

that can be considered additive generators if the conditions of the corresponding definitions are satisfied. This fact can be explained as follows [24]. According to de Finetti, if $F$ is symmetric (possesses the property of commutativity) and has a derivative, then a necessary and sufficient condition for its associativity is the fulfillment of the equality $\frac{F'_x(x,y)}{F'_x(x,y)} = \frac{\varphi'(x)}{\varphi'(y)}$, where $\varphi$ is an additive generator for $F$. For a function, $F$, in form (7) we obtain,

$$\varphi(x) = \pm \int \frac{dx}{(a_1b_0 - a_0b_1) + (a_2b_0 - a_0b_2)x + (a_2b_1 - a_1b_2)x^2} + C.$$

Thus, under the assumption that $F$ has form (7), we find that only $\varphi_1$, $\varphi_2$, $\varphi_3$ can be considered an additive generator, and other variants of generators are impossible. The study of the listed cases will make it possible to describe all variants of t-norms and s-conorms in the class of rational functions.

Via the coefficients of functions based on representations (3) and (4), it is easy to determine the coefficients of function (7) [25]. Since the purpose of the study is the function of $\varphi_3$, we present formulas on the basis of which we can use its coefficients to define the coefficients of the function, $F$ [25]:

$$\begin{aligned} &a_0 = b\left(b^2 + d^2\right),\ a_1 = a\left(b^2 + d^2\right),\ a_2 = 2acd - b\left(c^2 - a^2\right), \\ &b_0 = a\left(d^2 - b^2\right) - 2bcd,\ b_1 = -b\left(a^2 + c^2\right),\ b_2 = -a\left(a^2 + c^2\right). \end{aligned} \tag{8}$$

## 3. Results

### *3.1. Increasing Generator in the Form of an Arctangent of a Linear Fractional Function and the Corresponding Triangular Conorm*

Let us consider a function of the form $\varphi(x) = k \cdot \arctan\frac{ax+b}{cx+d} + C$ and find restrictions on the parameters of $\varphi$, under which it is an increasing generator. The constant $C = -k \cdot \arctan\frac{b}{d}$ is determined from the condition $\varphi(0) = 0$, and then, we obtain

$$\varphi(x) = k \cdot \left(\arctan\frac{ax+b}{cx+d} - \arctan\frac{b}{d}\right) = k \cdot \arctan\frac{x(ad - bc)}{x(ab + cd) + (b^2 + d^2)},$$

where $x \neq -\frac{b^2+d^2}{ab+cd}$, $x \neq -\frac{d}{c}$, $ab + cd \neq 0$, and $d \neq 0$; in this case, according to the formula for the difference in arctangents, the following condition must be satisfied:

$$\frac{ax+b}{cx+d} \cdot \frac{b}{d} > -1. \tag{9}$$

Since the generator is determined up to a positive multiplicative constant, we shall assume that $k > 0$ and further consider the function, $\varphi$, in the following form:

$$\varphi(x) = \arctan\frac{x(ad - bc)}{x(ab + cd) + (b^2 + d^2)}.$$

Dividing the numerator and denominator by $d^2$ under the condition that $d \neq 0$ and introducing the notation $\frac{a}{d} = \alpha$, $\frac{b}{d} = \beta$, $\frac{c}{d} = \gamma$, we obtain the function, $\varphi$, in the form

$$\varphi(x) = \arctan\frac{x(\alpha - \beta\gamma)}{x(\alpha\beta + \gamma) + (1 + \beta^2)}, \tag{10}$$

where $x \neq -\frac{1+\beta^2}{\alpha\beta+\gamma}$ and $\alpha\beta \neq -\gamma$.

Considering the introduced notations, inequality (9) is equivalent to the following inequality:

$$\frac{x(\alpha\beta + \gamma) + \left(1 + \beta^2\right)}{\gamma x + 1} > 0. \tag{11}$$

We note that $x = -\frac{1+\beta^2}{\alpha\beta+\gamma}$ is the vertical asymptote of $\varphi$. Since the additive generator must be determined on $[0,1]$, we require the vertical asymptote to lie outside it. Therefore, one of the inequalities, $-\frac{1+\beta^2}{\alpha\beta+\gamma} < 0$ or $-\frac{1+\beta^2}{\alpha\beta+\gamma} > 1$, must hold. Their analysis allowed us to draw the following conclusion:

(a) If $-\left(1+\beta^2\right) < \alpha\beta+\gamma < 0$, then the asymptote is located to the right of $[0,1]$;
(b) If $\alpha\beta+\gamma > 0$, then the asymptote is located to the left of the given interval (left asymptote).

The function, $\varphi$, is increases if $\varphi'(x) > 0$. We note that the derivative for (10) is defined by the derivative formula for a complex function. The denominator of the derivative for arctan is always positive, so we need to investigate the derivative of the argument. This derivative has the form

$$(\arg(\arctan))'(x) = \frac{(\alpha-\beta\gamma)\left(1+\beta^2\right)}{\left(x(\alpha+\beta\gamma)+\left(1+\beta^2\right)\right)^2}.$$

For $\alpha - \beta\gamma > 0$, we have $\varphi'(x) > 0$, and, therefore, $\varphi$ is an increasing function.

To obtain restrictions on the parameters, under which $\varphi$ is an increasing generator, it is necessary to analyze the system, which includes the inequality $\alpha - \beta\gamma > 0$ and inequality (11), considering the position of the asymptotes. It is important that $x \in [0,1]$, and this fact should be considered in the analysis.

Let the asymptote be located to the right of $[0,1]$. In this case, it is necessary to study the system

$$\begin{cases} \left(x(\alpha\beta+\gamma)+\left(1+\beta^2\right)\right)/(\gamma x+1) > 0, \\ -\left(1+\beta^2\right) < \alpha\beta+\gamma < 0, \\ \alpha-\beta\gamma > 0, \end{cases},$$

which is equivalent to one of the following systems:

$$\text{(i)} \begin{cases} x(\alpha\beta+\gamma)+\left(1+\beta^2\right) > 0, \\ \gamma x+1 > 0, \\ -\left(1+\beta^2\right) < \alpha\beta+\gamma < 0, \\ \alpha-\beta\gamma > 0, \end{cases} \qquad \text{(ii)} \begin{cases} x(\alpha\beta+\gamma)+\left(1+\beta^2\right) < 0, \\ \gamma x+1 < 0, \\ -\left(1+\beta^2\right) < \alpha\beta+\gamma < 0, \\ \alpha-\beta\gamma > 0. \end{cases}$$

Let us consider system (i). Since $\alpha\beta+\gamma < 0$ and $x \in [0,1]$, we have the true inequality $x < -\frac{1+\beta^2}{\alpha\beta+\gamma}$. Let $\gamma > 0$. It follows from the relation $\gamma < -\alpha\beta$ that if $\alpha$ and $\beta$ are positive, then we obtain a contradiction. Therefore, it is advisable to consider the $\alpha$ and $\beta$ of different signs, and then, $\gamma \in (0, -\alpha\beta)$, but we need to consider the inequality $-\left(1+\beta^2\right) - \alpha\beta < \gamma < -\alpha\beta$, which holds for the right asymptote. If $\alpha < 0,\ \beta > 0$, then $\gamma < \frac{\alpha}{\beta} < 0$, which contradicts the assumption. The analysis of the case $\alpha > 0,\ \beta < 0$ made it possible to obtain the following restriction: $\gamma \in \left(\max\left\{0,\ -\alpha\beta - \left(1+\beta^2\right)\right\}, -\alpha\beta\right)$. Let $\gamma < 0$; then, considering that $x \in [0,1]$, we have $\gamma \in (-1, 0)$. In investigations of this case, the following results can be obtained:

(a) $\alpha > 0,\ \beta > 0,\ \alpha\beta < 1,\ \gamma \in \left(\max\left\{-1,\ -\alpha\beta - \left(1+\beta^2\right)\right\}, -\alpha\beta\right)$;
(b) $\alpha > 0,\ \beta < 0,\ \gamma \in \left(\max\left\{\frac{\alpha}{\beta},\ -\alpha\beta - \left(1+\beta^2\right)\right\}, 0\right)$;
(c) $\alpha < 0,\ \beta > 0,\ \alpha > -\beta,\ \gamma \in \left(-1, \frac{\alpha}{\beta}\right)$.

In system (ii), given that $\alpha\beta+\gamma < 0$, the first inequality is rearranged into the form $x > -\frac{1+\beta^2}{\alpha\beta+\gamma} > 1$, but it does not satisfy $x \in [0,1]$.

Now, we consider the case when the asymptote is located to the left of $[0,1]$. The system for determining restrictions on parameters takes the form

$$\begin{cases} \left(x(\alpha\beta+\gamma)+\left(1+\beta^2\right)\right)/(\gamma x+1)>0, \\ \alpha\beta+\gamma>0, \\ \alpha-\beta\gamma>0. \end{cases}$$

Accordingly, it is required to investigate each of the following systems of inequalities:

$$(\mathrm{i}')\begin{cases} x(\alpha\beta+\gamma)+\left(1+\beta^2\right)>0, \\ \gamma x+1>0, \\ \alpha\beta+\gamma>0, \\ \alpha-\beta\gamma>0, \end{cases} \qquad (\mathrm{ii}')\begin{cases} x(\alpha\beta+\gamma)+\left(1+\beta^2\right)<0, \\ \gamma x+1<0, \\ \alpha\beta+\gamma>0, \\ \alpha-\beta\gamma>0, \end{cases}$$

for $\gamma>0$ and $\gamma<0$. We can now investigate system (i$'$) for $\gamma>0$. Provided that $x\in[0,1]$, the first two inequalities are true. The system is rearranged into the form

$$\begin{cases} \gamma>0, \\ \alpha\beta+\gamma>0, \\ \alpha-\beta\gamma>0, \end{cases}$$

from which the following restrictions on the parameters are obtained:

$$\alpha>0,\ \beta>0,\ \gamma\in\left(0,\frac{\alpha}{\beta}\right);\ \alpha<0,\ \beta<0,\ \gamma>\frac{\alpha}{\beta};\ \alpha>0,\ \beta<0,\ \gamma>-\alpha\beta.$$

If $\gamma<0$, then $x<-\frac{1}{\gamma}$, and it is expedient to require that $-\frac{1}{\gamma}>1$, from which we obtain $\gamma\in(-1,0)$. It follows from the second inequality of the system that $\gamma>-\alpha\beta$, and it is reasonable to require that $\alpha$ and $\beta$ be of the same sign and $\alpha\beta<1$. It is established that if $\alpha>0,\ \beta>0$, then $\gamma\in(\max\{-1,-\alpha\beta\},0)$. If $\alpha<0,\ \beta<0$, then this assumption contradicts the last inequality of the system, since, on the one hand, $\gamma\in(-1,0)$; on the other hand, $\gamma>\frac{\alpha}{\beta}>0$.

System (ii$'$) does not make sense for any combination of parameter values since, considering the first and third inequalities, we obtain a contradiction between $x\in[0,1]$ and $x<-\frac{1+\beta^2}{\alpha\beta+\gamma}<0$.

We note that we have a rather complicated system of restrictions.

Thus, the following is proved:

**Assertion 1.** *The function:*

$$s(x)=s_{\alpha,\beta,\gamma}(x)=\arctan\frac{x(\alpha-\beta\gamma)}{x(\alpha\beta+\gamma)+(1+\beta^2)},$$

where $x\neq-\frac{1+\beta^2}{\alpha\beta+\gamma}$ is an increasing generator if the following restrictions on the parameters are satisfied:

(1) $\alpha>0,\ \beta>0, \gamma\in\underbrace{\overbrace{\left(\max\left\{-1,\ -\alpha\beta-\left(1+\beta^2\right)\right\},-\alpha\beta\right)\cup(\max\{-1,-\alpha\beta\},0)}^{\alpha\beta<1}}_{\gamma<0}\cup\underbrace{\left(0,\frac{\alpha}{\beta}\right)}_{\gamma>0};$

(2) $\alpha<0,\ \beta<0,\ \gamma\in\underbrace{\left(\frac{\alpha}{\beta},\infty\right)}_{\gamma>0};$

(3) $\alpha > 0,\ \beta < 0,$
$\gamma \in \underbrace{\left(\max\left\{\frac{\alpha}{\beta},\ -\alpha\beta - \left(1+\beta^2\right)\right\},0\right)}_{\gamma<0} \cup \underbrace{\left(\max\left\{0,\ -\alpha\beta - \left(1+\beta^2\right)\right\},-\alpha\beta\right) \cup (-\alpha\beta,\ \infty)}_{\gamma>0};$

(4) $\alpha < 0,\ \beta > 0,\ \frac{\alpha}{\beta} > -1,\ \gamma \in \left(-1, \frac{\alpha}{\beta}\right) \subset (-1,0).$

**Example 1.** Let $\alpha = 2,\ \beta = -4$ (case 3). Let us construct an interval for $\gamma$. Now, let us calculate $\frac{\alpha}{\beta} = -0.5,\ -\alpha\beta = 8,\ -\alpha\beta - \left(1+\beta^2\right) = 8 - (1+16) = -9$. Let us find the intervals specified for this situation

$$\left(\max\left\{\frac{\alpha}{\beta},\ -\alpha\beta - \left(1+\beta^2\right)\right\},0\right) = (\max\{-0.5,-9\},0) = (-0.5,0);$$

$$\left(\max\left\{0,\ -\alpha\beta - \left(1+\beta^2\right)\right\},-\alpha\beta\right) = (\max\{0,-9\},8) = (0,8);$$

$$(-\alpha\beta,\ \infty) = (8,\ \infty).$$

Combining the obtained intervals, we find that $\gamma \in (-0.5,\ \infty)\backslash\{0,8\}$. Figure 2 shows increasing generators for $\alpha = 2,\ \beta = -4$ and different values of parameter $\gamma$.

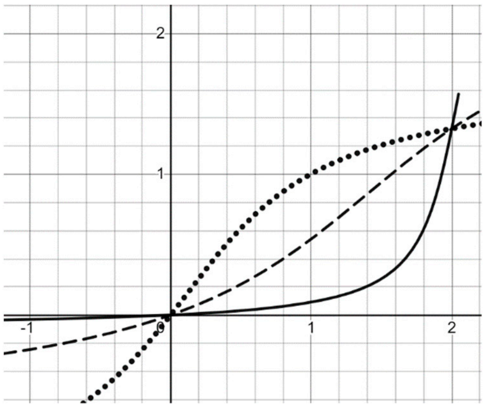


**Figure 2.** Graphs of increasing generators for various combinations of parameter values: $\gamma = -0.3 s(x) = \arctan\frac{0.8x}{-8.3x+17}$ (solid line); $\gamma = 1\ s(x) = \arctan\frac{6x}{-7x+17}$ (wide-spaced dashed line); $\gamma = 5\ s(x) = \arctan\frac{22x}{-3x+17}$ (narrow-spaced dashed line).

Let us find the s-conorm for the increasing generator, $s$. Using Formula (8) for $a = \alpha - \beta\gamma,\ b = 0,\ c = \alpha\beta + \gamma$, and $d = 1 + \beta^2$, we can calculate the coefficients of $F$

$$a_0 = 0,\ a_1 = (\alpha - \beta\gamma)\left(1+\beta^2\right),\ a_2 = 2(\alpha - \beta\gamma)(\alpha\beta + \gamma)\left(1+\beta^2\right),$$
$$b_0 = (\alpha - \beta\gamma)\left(1+\beta^2\right),\ b_1 = 0,\ b_2 = -(\alpha - \beta\gamma)\left((\alpha - \beta\gamma)^2 + (\alpha\beta + \gamma)^2\right).$$

Thus,
$$F(x,y) = \frac{(x+y) + 2(\alpha\beta + \gamma)xy}{1 - (\alpha^2 + \gamma^2)xy}.$$

Let us check whether the definition of the s-conorm holds. Note that $F(0,0) = 0$ and $F(x,0) = F(0,x) = x$. Let us find $f(x) = F(x,x) = \frac{2x + 2(\alpha\beta+\gamma)x^2}{1-(\alpha^2+\gamma^2)x^2}$. It is necessary that the discontinuity points of the second kind, which are determined by the equation $1 - \left(\alpha^2 + \gamma^2\right)x^2 = 0$, do not belong to $[0,1]$. We have $x^2 = \frac{1}{\alpha^2+\gamma^2}$ and, hence, $x = \pm\sqrt{\frac{1}{\alpha^2+\gamma^2}}$. The negative root, $-\sqrt{\frac{1}{\alpha^2+\gamma^2}}$, does not belong to $[0,1]$. For a positive root, the inequality

$\sqrt{\frac{1}{\alpha^2+\gamma^2}} > 1$ must be satisfied, from which we obtain a relation for the parameters of the following form: $\alpha^2 + \gamma^2 < 1$.

Let us find $F(1,1) - 1 = \frac{2+2(\alpha\beta+\gamma)}{1-(\alpha^2+\gamma^2)} - 1 = \frac{(\gamma+1)^2+(\alpha^2+2\alpha\beta)}{1-(\alpha^2+\gamma^2)}$. We note that, considering the previous inequality, the denominator is positive. Then, $F(1,1) > 1$ holds if $(\gamma+1)^2 + (\alpha^2 + 2\alpha\beta) > 0$. If $\alpha$ and $\beta$ have the same sign, then the inequality always holds; otherwise, we require $(\gamma+1)^2 > -(\alpha^2 + 2\alpha\beta)$, where $\alpha^2 + 2\alpha\beta < 0$ and, hence, $\frac{\alpha}{\beta} > -2$. This inequality is valid for restrictions (3) and (4) from Assertion 1.

Thus, the following takes place:

**Assertion 2.** *If there is an increasing generator, $s_{\alpha,\beta,\gamma}$, and the condition $\alpha^2 + \gamma^2 < 1$ is satisfied, then an s-conorm exists:*

$$S_{\alpha,\beta,\gamma}(x,y) = \min\left\{1,\ \frac{(x+y) + 2(\alpha\beta+\gamma)xy}{1-(\alpha^2+\gamma^2)xy}\right\}.$$

Figure 3 shows graphs of functions (a)–(c), obtained on the basis of increasing generators from Example 1. It can be seen that the functions are not s-conorms since the restriction $\alpha^2 + \gamma^2 < 1$ is violated. In (d), we show a graph of the s-conorm for $\alpha = 0.2$, $\beta = 0.4$, $\gamma = -0.05 \in (\max\{-1, -0.08\}, 0)$. In this case, the restriction is met.

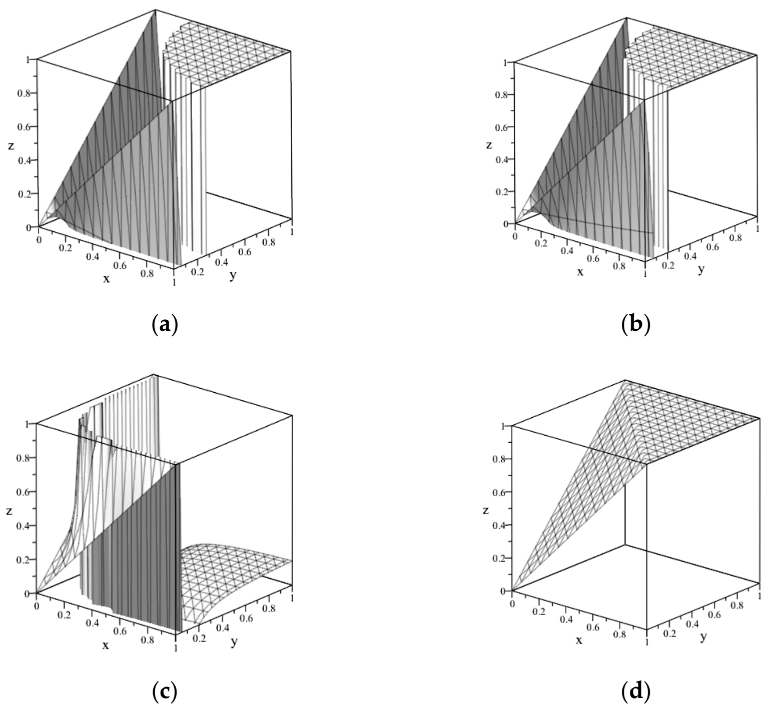

**Figure 3.** Graphs of functions that are obtained on the basis of generators from Example 1: (**a**) $\frac{x+y-16.6xy}{1-4.09xy}$; (**b**) $\frac{x+y-14xy}{1-5xy}$; (**c**) $\frac{x+y-6xy}{1-29xy}$; (**d**) $\min\left\{1, \frac{x+y+0.06xy}{1-0.0225xy}\right\}$-s-conorm.

### *3.2. Decreasing Generator in the Form of an Arctangent of a Linear Fractional Function and the Corresponding Triangular Norm*

Let $s(x) = \arctan\frac{x(\alpha-\beta\gamma)}{x(\alpha\beta+\gamma)+(1+\beta^2)}$ be an increasing generator. According to (2), the function $t(x) = s(1) - s(x)$ is a decreasing generator on $[0,1]$, and $t(0) = s(1)$.

Let us find $s(1) = \arctan\frac{\alpha-\beta\gamma}{(\alpha\beta+\gamma)+(1+\beta^2)}$; then,

$$t(x) = s(1) - s(x) = \arctan\frac{(\alpha-\beta\gamma)(1-x)}{((\alpha\beta+\gamma)+(\alpha^2+\gamma^2))x + ((\alpha\beta+\gamma)+(1+\beta^2))}.$$

We note that $t(0) = \arctan\frac{\alpha-\beta\gamma}{(\alpha\beta+\gamma)+(1+\beta^2)} = s(1)$.

We can investigate the function, $t$, in accordance with the definition of a decreasing generator. To hold the continuity property on $[0,1]$, we require that the vertical asymptote be outside the given interval. The vertical asymptote for $t$ has the form $x = -\frac{(\alpha\beta+\gamma)+(1+\beta^2)}{(\alpha\beta+\gamma)+(\alpha^2+\gamma^2)}$, and it is located to the left of $[0,1]$ if $-\frac{(\alpha\beta+\gamma)+(1+\beta^2)}{(\alpha\beta+\gamma)+(\alpha^2+\gamma^2)} < 0$. We note that if $\alpha\beta+\gamma > 0$, then this inequality is always satisfied. If $-(1+\beta^2) < \alpha\beta+\gamma < 0$ and, therefore, $(\alpha\beta+\gamma)+(1+\beta^2) > 0$, then it is necessary to require that the inequality $(\alpha\beta+\gamma)+(\alpha^2+\gamma^2) > 0$ be satisfied, whence $\alpha\beta+\gamma > -(\alpha^2+\gamma^2)$, and finally we obtain $-\min\{\alpha^2+\gamma^2, 1+\beta^2\} < \alpha\beta+\gamma < 0$.

The vertical asymptote is on the right if $-\frac{(\alpha\beta+\gamma)+(1+\beta^2)}{(\alpha\beta+\gamma)+(\alpha^2+\gamma^2)} > 1$. This inequality is equivalent to one of the following systems:

$$\text{(i)}\begin{cases} \alpha\beta+\gamma < -\frac{1}{2}(\alpha^2+\beta^2+\gamma^2+1), \\ \alpha\beta+\gamma > -(\alpha^2+\gamma^2); \end{cases} \quad \text{or} \quad \text{(ii)}\begin{cases} \alpha\beta+\gamma > -\frac{1}{2}(\alpha^2+\beta^2+\gamma^2+1), \\ \alpha\beta+\gamma < -(\alpha^2+\gamma^2). \end{cases}$$

We note that the first inequality of system (i) can be converted into the inconsistent inequality of the form

$$\alpha\beta+\gamma < -\frac{1}{2}\left(\alpha^2+\beta^2+\gamma^2+1\right) \;\Leftrightarrow\; (\alpha+\beta)^2+(1+\gamma)^2 < 0,$$

and, therefore, this system does not make sense to consider.

In system (ii), the first inequality is converted to the form $(\alpha+\beta)^2+(1+\gamma)^2 > 0$, which is always true for any values of the parameters, so it can be discarded. Thus, the remaining inequality is $\alpha\beta+\gamma < -(\alpha^2+\gamma^2)$. For $\alpha\beta+\gamma > 0$, it is inconsistent, and for $-(1+\beta^2) < \alpha\beta+\gamma < 0$, we obtain the following restriction on the parameters: $-(1+\beta^2) < \alpha\beta+\gamma < -(\alpha^2+\gamma^2)$. Thus, the following restrictions on the parameters are obtained, which guarantee that the vertical asymptote is outside the interval $[0,1]$:

(1) The vertical asymptote is located to the left of $[0,1]$ if $\alpha\beta+\gamma > 0$ or $-\min\{\alpha^2+\gamma^2, 1+\beta^2\} < \alpha\beta+\gamma < 0$;
(2) The vertical asymptote is located to the right of $[0,1]$ if $-(1+\beta^2) < \alpha\beta+\gamma < -(\alpha^2+\gamma^2)$.

When the found restrictions on the parameters are met, the function, $t$, is continuous. Let us move on to the next stage of the study.

According to the formula for the difference in arctangents, the following inequality must hold:

$$\frac{\alpha-\beta\gamma}{(\alpha\beta+\gamma)+(1+\beta^2)} \cdot \frac{(\alpha-\beta\gamma)x}{(\alpha\beta+\gamma)x+(1+\beta^2)} > -1,$$

which is reduced to the form

$$\frac{(1+\beta^2)\left(x\left((\alpha\beta+\gamma)+(\alpha^2+\gamma^2)\right)+\left((\alpha\beta+\gamma)+(1+\beta^2)\right)\right)}{(x(\alpha\beta+\gamma)+(1+\beta^2))\cdot((\alpha\beta+\gamma)+(1+\beta^2))} > 0,$$

from whence we obtain the following systems of inequalities:

$$(1)\begin{cases} x\left((\alpha^2+\gamma^2)+(\alpha\beta+\gamma)\right)+(1+\beta^2)+(\alpha\beta+\gamma) > 0, \\ \left((\alpha\beta+\gamma)x+(1+\beta^2)\right)\cdot\left((\alpha\beta+\gamma)+(1+\beta^2)\right) > 0; \end{cases} \quad (2)\begin{cases} x\left((\alpha^2+\gamma^2)+(\alpha\beta+\gamma)\right)+(1+\beta^2)+(\alpha\beta+\gamma) < 0, \\ \left((\alpha\beta+\gamma)x+(1+\beta^2)\right)\cdot\left((\alpha\beta+\gamma)+(1+\beta^2)\right) < 0. \end{cases}$$

System (1) is equivalent to one of the following inequalities:

$$(1\text{i})\begin{cases} x\left((\alpha^2+\gamma^2)+(\alpha\beta+\gamma)\right) > -\left((1+\beta^2)+(\alpha\beta+\gamma)\right), \\ (\alpha\beta+\gamma)x > -(1+\beta^2), \\ (\alpha\beta+\gamma) > -(1+\beta^2); \end{cases} \quad (1\text{ii})\begin{cases} x\left((\alpha^2+\gamma^2)+(\alpha\beta+\gamma)\right) > -\left((1+\beta^2)+(\alpha\beta+\gamma)\right), \\ (\alpha\beta+\gamma)x < -(1+\beta^2), \\ (\alpha\beta+\gamma) < -(1+\beta^2). \end{cases}$$

Similarly, for system (2), the following cases take place:

$$(2\mathrm{i})\begin{cases} x\big(\big(\alpha^2+\gamma^2\big)+(\alpha\beta+\gamma)\big) < -\big(\big(1+\beta^2\big)+(\alpha\beta+\gamma)\big), \\ (\alpha\beta+\gamma)x < -\big(1+\beta^2\big), \\ \alpha\beta+\gamma > -\big(1+\beta^2\big); \end{cases} \quad (2\mathrm{ii})\begin{cases} x\big(\big(\alpha^2+\gamma^2\big)+(\alpha\beta+\gamma)\big) < -\big(\big(1+\beta^2\big)+(\alpha\beta+\gamma)\big), \\ (\alpha\beta+\gamma)x > -\big(1+\beta^2\big), \\ \alpha\beta+\gamma < -\big(1+\beta^2\big). \end{cases}$$

We shall study these systems considering the already found restrictions on the parameters. If $\alpha\beta+\gamma>0$, then system (1ii) is inconsistent, and in system (1i), all inequalities are true for $x\in[0,1]$

$$\begin{cases} (\alpha\beta+\gamma) > -\big(1+\beta^2\big), \\ x > -\frac{(\alpha\beta+\gamma)+\left(1+\beta^2\right)}{(\alpha\beta+\gamma)+(\alpha^2+\gamma^2)}, \\ x > -\frac{1+\beta^2}{\alpha\beta+\gamma}. \end{cases} \tag{12}$$

If $\alpha\beta+\gamma>0$, then system (2i) is inconsistent because of the second inequality, and system (2ii) is inconsistent because of the third inequality.

Now let $-\min\{\alpha^2+\gamma^2, 1+\beta^2\} < \alpha\beta+\gamma < 0$ and, therefore, simultaneously, $(\alpha\beta+\gamma)+\big(\alpha^2+\gamma^2\big)>0$ and $(\alpha\beta+\gamma)+\big(1+\beta^2\big)>0$. It follows from the last inequality that $\alpha\beta+\gamma > -\big(1+\beta^2\big)$, which contradicts system (1ii). System (1i) is converted into form (12) and is consistent since $-\frac{1+\beta^2}{\alpha\beta+\gamma}>1$, which, as in the first inequality, there is an expression for the vertical asymptote on the right side, which satisfies the requirement $x\in[0,1]$.

Let us consider (2), subject to the same restrictions. It can be seen that system (2ii) does not satisfy the restriction on $\alpha\beta+\gamma$ and is, therefore, inconsistent. For (2i), we obtain

$$(2\mathrm{i})\begin{cases} \alpha\beta+\gamma > -\big(1+\beta^2\big), \\ x < -\frac{\left(1+\beta^2\right)+(\alpha\beta+\gamma)}{(\alpha^2+\gamma^2)+(\alpha\beta+\gamma)}, \\ x > -\frac{1+\beta^2}{\alpha\beta+\gamma}. \end{cases}$$

Since $x\in[0,1]$, the first inequality is not true, so this system is inconsistent.

Now, we consider the case $-\big(1+\beta^2\big) < \alpha\beta+\gamma < -\big(\alpha^2+\gamma^2\big)$. Here, the following inequalities hold simultaneously: $(\alpha\beta+\gamma)+\big(1+\beta^2\big)>0$ and $(\alpha\beta+\gamma)+\big(\alpha^2+\gamma^2\big)<0$.

We can investigate system (1). System (1ii) contradicts the inequality $-\big(1+\beta^2\big)<\alpha\beta+\gamma$. System (1i) is equivalent to the following system:

$$\begin{cases} (\alpha\beta+\gamma) > -\big(1+\beta^2\big), \\ x < -\frac{\left(1+\beta^2\right)+(\alpha\beta+\gamma)}{(\alpha^2+\gamma^2)+(\alpha\beta+\gamma)}, \\ x < -\frac{1+\beta^2}{\alpha\beta+\gamma}. \end{cases}$$

The first inequality is true for $x\in[0,1]$ because its right side contains the expression for the right asymptote. Since $(\alpha\beta+\gamma)+\big(1+\beta^2\big)>0$, $-\frac{1+\beta^2}{\alpha\beta+\gamma}>1$, which is also true for $x\in[0,1]$. Thus, this system is consistent.

Now we consider (2) under the same restrictions. Since $-\big(1+\beta^2\big)<\alpha\beta+\gamma$, it is expedient to consider only system (2i). Here $(\alpha\beta+\gamma)+\big(\alpha^2+\gamma^2\big)<0$, so it takes the form of (12). Since, in this case, $-\frac{1+\beta^2}{\alpha\beta+\gamma}>1$, the second inequality contradicts the assumption $x\in[0,1]$, so this system is inconsistent. Thus, the found restrictions on the parameters that ensure the continuity of the function, $t$, do not contradict its definition in the form of the difference in arctangents.

The function, $t$, is a decreasing function if $t'(x) < 0$. The denominator of the derivative for $t$ is always positive, so we need to investigate the derivative of the argument. This derivative has the form

$$(\arg(\arctan))'(x) = -\frac{(\alpha - \beta\gamma)\left((\alpha+\beta)^2 + (1+\gamma)^2\right)}{(((\alpha\beta+\gamma)+(\alpha^2+\gamma^2))x + ((\alpha\beta+\gamma)+(1+\beta^2)))^2}.$$

Then, $t'(x) < 0$ if $\alpha - \beta\gamma > 0$. Thus, the following is proved:

**Assertion 3.** *The function:*

$$t(x) = t_{\alpha,\beta,\gamma}(x) = \arctan\frac{(\alpha-\beta\gamma)(1-x)}{((\alpha\beta+\gamma)+(\alpha^2+\gamma^2))x + ((\alpha\beta+\gamma)+(1+\beta^2))},$$

where $x \neq -\frac{(\alpha\beta+\gamma)+\left(1+\beta^2\right)}{(\alpha\beta+\gamma)+\left(\alpha^2+\gamma^2\right)}$ is a decreasing generator if $\alpha - \beta\gamma > 0$ and one of the following conditions is met:

(1) $\alpha\beta + \gamma > 0$;
(2) $-\min\left\{\alpha^2+\gamma^2, 1+\beta^2\right\} < \alpha\beta+\gamma < 0$;
(3) $-\left(1+\beta^2\right) < \alpha\beta+\gamma < -\left(\alpha^2+\gamma^2\right)$.

**Example 2.** Let $\alpha - \beta\gamma > 0$ and $\alpha\beta + \gamma > 0$ (case 1). We have the system $\begin{cases} \gamma > -\alpha\beta, \\ \alpha > \beta\gamma. \end{cases}$. Next, we list all possible cases for the signs of parameters $\alpha$ and $\beta$ and the obtained restrictions for $\gamma$: if $\alpha > 0,\ \beta > 0$, then $-\alpha\beta < \gamma < \frac{\alpha}{\beta}$; if $\alpha > 0,\ \beta < 0$, then $\gamma > -\alpha\beta$; if $\alpha < 0,\ \beta > 0$, the values of $\gamma$ do not exist; if $\alpha < 0,\ \beta < 0$, then $\gamma > \frac{\alpha}{\beta}$. Figure 4 shows decreasing generators for different values of the parameters.

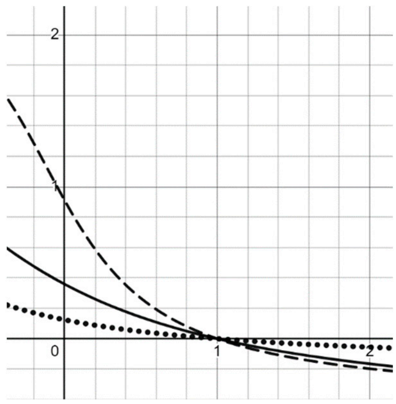


**Figure 4.** Graphs of decreasing generators for various combinations of parameter values: $\alpha = 1,\ \beta = 2,\ \gamma = -1,\ t_1(x) = \arctan\frac{3(1-x)}{5x+8}$ (solid line); $\alpha = 1,\ \beta = -2,\ \gamma = 4,\ t_2(x) = \arctan\frac{9(1-x)}{19x+7}$ (wide-spaced dashed line); $\alpha = -1,\ \beta = -2,\ \gamma = 1,\ t_3(x) = \arctan\frac{1-x}{5x+8}$ (narrow-spaced dashed line).

Let us find the corresponding t-norm for (13). To simplify the transformations, we introduce the following notation:

$$u = \frac{\alpha-\beta\gamma}{\alpha\beta+\gamma},\ v = 1 + \frac{\alpha^2+\gamma^2}{\alpha\beta+\gamma},\ w = 1 + \frac{1+\beta^2}{\alpha\beta+\gamma};$$

then, the generator, $t$, is converted into the form

$$t(x) = \arctan\frac{u(1-x)}{vx+w},$$

where $x \neq -\frac{w}{v}$. Let us find the coefficients of the function, $F$, using Formula (8), and as a result, we obtain the function.

$$F(x,y) = \frac{xy - \frac{u^2+w^2}{(v+w)^2}(1-x)(1-y)}{1 - \frac{u^2+v^2}{(v+w)^2}(1-x)(1-y)}.$$

We note that $F(x,1) = F(1,x) = x$. To find out if the function, $F$, is continuous, we move on to the function

$$f(x) = F(x,x) = \frac{x^2 - \frac{u^2+w^2}{(v+w)^2}(1-x)^2}{1 + \frac{u^2+v^2}{(v+w)^2}(1-x)^2},$$

which is rational, so the points of discontinuity correspond to the roots of the equation $1 - \frac{v^2+u^2}{(v+w)^2}(1-x)^2 = 0$; hence, $1 - x = \pm\sqrt{\frac{(v+w)^2}{v^2+u^2}}$, and then, $x_1 = 1 + \sqrt{\frac{(v+w)^2}{v^2+u^2}}$ and $x_2 = 1 - \sqrt{\frac{(v+w)^2}{v^2+u^2}}$. The root, $x_1$, is located to the right of 1. The expression $\frac{(v+w)^2}{v^2+u^2}$ in terms of $\alpha,\ \beta,\ \gamma$ is reduced to the form $\left(1 + \frac{2(\alpha\beta+\gamma)+(1+\beta^2)}{\alpha^2+\gamma^2}\right)$, and hence, in accordance with Assertion 3, it exceeds 1. Thus, the root $x_2$ is located to the left of 0.

Let us find $F(0,0) = \frac{-(u^2+w^2)}{(v+w)^2+u^2+v^2}$. Note that $F(0,0) < 0$, so a design operation is required. Thus, returning to the parameters $\alpha, \beta, \gamma$, we can formulate the following:

**Assertion 4.** *If a decreasing generator, $t_{\alpha,\beta,\gamma}$, exists and, consequently, the restrictions from Assertion 3 are satisfied, then a t-norm of the form exists:*

$$T_{\alpha,\beta,\gamma}(x,y) = \max\left\{0, \frac{xy - \frac{1+\beta^2}{(\alpha+\beta)^2+(1+\gamma)^2}(1-x)(1-y)}{1 + \frac{\alpha^2+\gamma^2}{(\alpha+\beta)^2+(1+\gamma)^2}(1-x)(1-y)}\right\}.$$

Figure 5 shows graphs of the t-norm, $T_{\alpha,\beta,\gamma}$, for various parameter values.

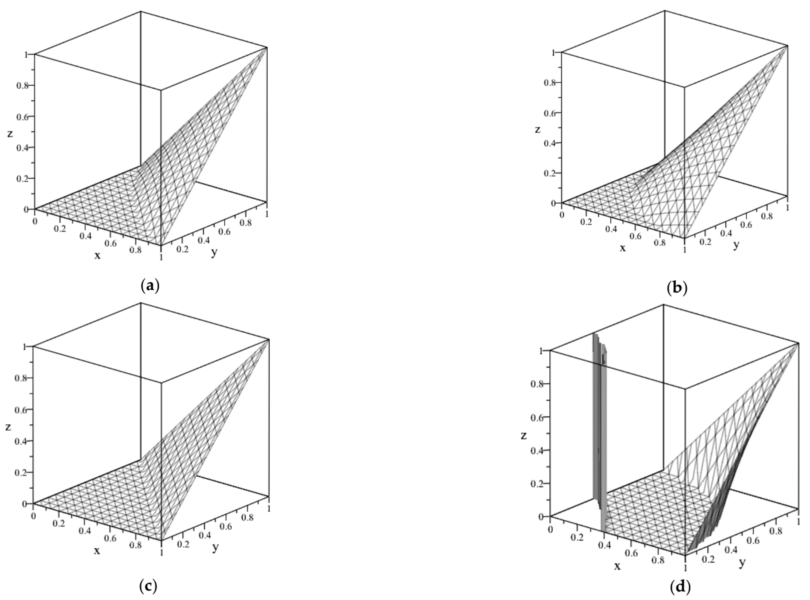

**Figure 5.** The t-norm, $T_{\alpha,\beta,\gamma}$, for various combinations of parameter values (Example 2). (**a**) $T_{1,2,-1}(x,y)$;

(**b**) $T_{1,-2,4}(x,y)$; (**c**) $T_{0.2,0.4,-0.05}(x,y)$; (**d**) For $\alpha = -3,\ \beta = 5,\ \gamma = 0.3$, the function, $T_{-3,\,5,\,0.3}$, is not a t-norm ($\alpha - \beta\gamma < 0$).

## 4. Discussion

Within the framework of the study, we can speak about a sequence of actions, the result of which is the found additive generator and the corresponding t-norm or s-conorm. Let the parametric function, $\varphi$, be given in one of the forms $\varphi_1,\ \varphi_2,\ \varphi_3$, possibly with a multiplicative factor, as well as with an additive constant that can play the role of an additive generator. To investigate this function and construct the corresponding t-norm or s-conorm, the following procedure is proposed:

1. For the function, $\varphi$, ensure the continuity, strict decrease (or increase), and fulfillment of condition $\varphi(1) = 0$ (or condition $\varphi(0) = 0$) by adjusting the parameters. If at least one of the requirements is not met, then the given function cannot be considered a generator.
2. The construction of a t-norm or s-conorm from the class of rational functions is conducted on the basis of the commutative and associative function of form (7). Based on the constructed generator, $\varphi$, one can find the coefficients of the function, $F$.
3. Define restrictions on the parameters of the function, $F$, that ensure the fulfillment of the boundary conditions from the definition of a triangular norm or conorm.
4. Investigate the continuity of $F$ on the basis of the function $f(x) = F(x,x)$. At this step, those parameter values for which the function, $f$, has points of discontinuity on $[0,1]$ are excluded.
5. Determine whether the design operation is needed: if $f(1) = 1$, then $S = F$. Otherwise, $(f(1) > 1)$, and the design operation $S(x,y) = \min\{1, F(x,y)\}$ must be used. Similar reasoning takes place for the t-norm: if $f(0) = 0$, then $T = F$; if $f(0) < 0$, then $T(x,y) = \max\{0, F(x,y)\}$.

In the process of using the proposed procedure, it may turn out that the generator exists but the corresponding t-norm or s-conorm does not. Moreover, if $T$ and $S$ exist, they are not necessarily dual in the sense of de Morgan's laws. For each of the operations, $T$ and $S$, we can construct a dual norm or a conorm using the standard negation. If both generators and the corresponding t-norm and s-conorm are constructed on the basis of the selected function, $\varphi$, then it is possible to define the negation function, as well as the conditions that ensure the fulfillment of de Morgan's laws.

In fact, this scheme was used in [20,21], although it was not explicitly described.

To define the restrictions on parameters, we can sequentially consider the properties of a function that can potentially act as a generator, each time "cutting off" inappropriate parameter values. On the other hand, it is possible to form systems of restrictions, considering all the requirements of the corresponding definitions, and analyze them. It is impossible to predict in advance which path will be shorter.

On the basis of the generators, Formulas (5) and (6) can be used to define the negation functions corresponding to fuzzy operations.

Let us find the negation function for the s-conorm, $S_{\alpha,\beta,\gamma}$. The inverse function for the generator, $s = s_{\alpha,\beta,\gamma}$, has the form $s^{-1}(x) = \frac{(1+\beta^2)\tan x}{(\alpha-\beta\gamma)-(\alpha\beta+\gamma)\tan x}$; then, in accordance with Formula (5), we obtain

$$n_S(x) = \left(1 + \frac{\left((\alpha+\beta)^2 + (1+\gamma)^2\right)x}{(1+\beta^2)(1-x)}\right)^{-1}.$$

Figure 6 shows graphs of negation functions generated by increasing generators from Example 1 for $\alpha = 2,\ \beta = -4$ and various values of parameter $\gamma$.

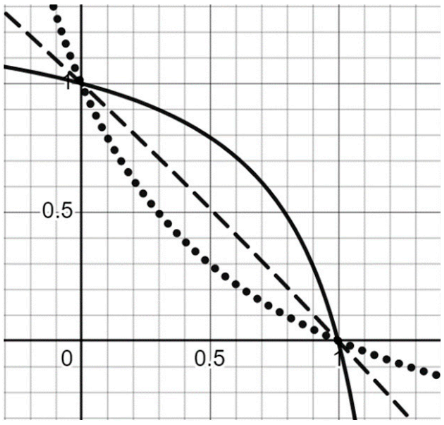


**Figure 6.** The functions, $n_S$, for generators from Example 1: $\gamma = -0.3$, $n_1(x) = \left(1 + \frac{4.49x}{17(1-x)}\right)^{-1}$ (solid line); $\gamma = 1$, $n_2(x) = \left(1 + \frac{16x}{17(1-x)}\right)^{-1}$ (wide-spaced dashed line); $\gamma = 5$, $n_3(x) = \left(1 + \frac{40x}{17(1-x)}\right)^{-1}$ (narrow-spaced dashed line).

Let us find the negation function for the t-norm $T_{\alpha,\beta,\gamma}$. For $t(x) = \arctan\frac{u(1-x)}{vx+w}$, the inverse function has the form $t^{-1}(x) = \frac{u-w\tan x}{u+v\tan x}$. According to (6), we obtain $n_T(x) = t^{-1}(t(0) - t(x)) = \frac{1-x}{1+\left(\frac{(v+w)^2}{w^2+u^2}-1\right)x}$ or, returning to variables $\alpha$, $\beta$, $\gamma$,

$$n_T(x) = \frac{1-x}{1+\left(\frac{(\alpha+\beta)^2+(1+\gamma)^2}{1+\beta^2}-1\right)x}.$$

Figure 7 shows graphs of negation functions generated by decreasing generators from Example 2 for various parameter values.

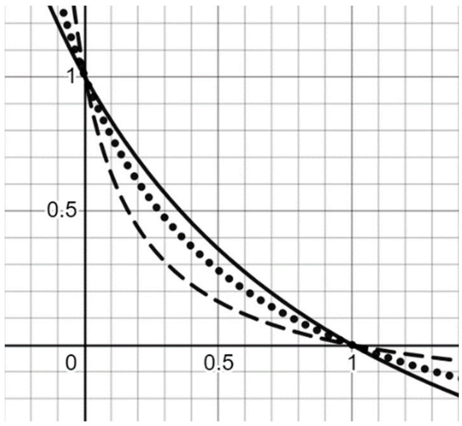


**Figure 7.** The functions, $n_T$, for generators from Example 2: $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $n_1(x) = \frac{1-x}{1+0.8x}$ (solid line); $\alpha = 1$, $\beta = -2$, $\gamma = 4$, $n_2(x) = \frac{1-x}{1+4.2x}$ (wide-spaced dashed line); $\alpha = -1$, $\beta = -2$, $\gamma = 1$, $n_3(x) = \frac{1-x}{1+1.6x}$ (narrow-spaced dashed line).

It is shown in [24] that t-norms and s-conorms that are representable using rational functions can be obtained using only three types of additive generators. Considering the results of this paper, it can be argued that this class of Archimedean triangular norms is most basically completely described. It is important that the resulting formulas for fuzzy operations have a certain structure. Thus, in [20], a s-conorm of the form $S_\rho(x,y) = \min\left\{1, \frac{(x+y)+2\rho xy}{1-\rho^2 xy}\right\}$, with the increasing generator $s_\rho(x) = \frac{x}{\rho x+1}$ ($\rho \in (-1,1]$), is presented. On the other hand, in this study, we obtained the s-conorm $S_{\alpha,\beta,\gamma}(x,y) = \min\left\{1, \frac{(x+y)+2(\alpha\beta+\gamma)xy}{1-(\alpha^2+\gamma^2)xy}\right\}$, with the increasing generator $s_{\alpha,\beta,\gamma}(x) = \arctan\frac{x(\alpha-\beta\gamma)}{x(\alpha\beta+\gamma)+(1+\beta^2)}$. We note that $S_\rho$ and $S_{\alpha,\beta}$ have a similar structure. It follows from this that, with a suitable choice of parameters, the same fuzzy operation can be obtained using different generators.

## 5. Conclusions

This paper considers a particular case of representing an additive generator in the form of an arctangent of a linear fractional function. The study of this case is aimed at solving the problem of characterizing additive generators that generate continuous Archimedean fuzzy operations in the class of rational functions. Other types of additive generators (in the form of a linear fractional function and the logarithm of a linear fractional function) were studied in [20,21]. Knowledge of additive generators makes it possible to solve a number of related problems in the theory of Archimedean norms and conorms: (a) if $T$ is a continuous Archimedean t-norm with additive generator $t$, then the function, $e^{-t(x)}$, is a multiplicative generator of $T$ [5]; (b) let $t(x)$ be an additive generator of Archimedean t-norm $T$, and then, for each $\lambda \in (0,\infty)$, the function $t^\lambda(x) = (t(x))^\lambda$ is an additive generator [4]; (c) on the basis of an increasing generator, we can define a decreasing generator and vice versa. In addition, we can find the corresponding negation functions (these properties are demonstrated in this paper). Using the additive generator, we can construct other logical operations, such as the implication [23], which is important for inference systems, or the indistinguishability operator [26], which is used, for example, in classification/clustering problems. In [27], continuous Archimedean triangular norms and pseudo-inverses of their additive generators are used to construct fuzzy metrics. There are known approaches to determining operations for fuzzy numbers in terms of additive generators, which, according to the authors, simplifies calculations and provides additional opportunities for analyzing the results. Also, additive generators are used to construct aggregation functions and operations (for example, an associative weighted mean), and the generator can be considered a transformation function of the initial aggregated variables, which converts them into a dimensionless scale in accordance with a certain principle.

Against the background of the variety of functional representations of fuzzy operations, the problem of choosing the most appropriate representation arises. In our opinion, there are no universal criteria for choosing the type of fuzzy operation. Such criteria can be defined within a particular application. For example, a fuzzy system can be considered a universal approximator; then, when using some parametric t-norm to formalize the inference mechanism, it is necessary to select such a parameter value in the training set so that the approximation accuracy is a maximum. In the problem of choosing transitively nearest subsets (the fuzzy clustering problem), the t-norm, $T$, is used to formalize the $(\max - T)$-transitivity property. When choosing a suitable t-norm or adjusting parameters, we should consider not only the quality criteria of clustering but also the quality of the decomposition tree, which contains all possible set partitions of a given set of subsets. Different t-norms or different parameter values of the chosen parametric t-norm allow us to find in this problem a different number of clusters for the same value of the decomposition parameter, which can be a significant argument when choosing a suitable functional representation for a fuzzy operation. Triangular norms and conorms are used in multi-objective (multi-attribute) decision-making models, which are based on the aggregation of partial estimates in accordance with some aggregation strategy. The choice of a suitable representation for the aggregation function can be determined with the initial hypotheses of the study. For example, if, in order to reduce the set of alternatives, it is required to obtain the most different generalized estimates of alternatives with minimum partial estimates using indicators or criteria, then, for example, the t-norm presented in Figure 3d) can be used.

A promising area of research is to determine the partition of the parameter space into regions, each of which is "responsible" for a certain type of generator and the corresponding fuzzy operation. The formulas that define t-norms for three types of generators, as noted in this paper, have the same structure, but the coefficients of fuzzy operations are expressed in different ways through the generator coefficients. In addition, it is important to study the algebraic properties of fuzzy operations, which will allow us, from a mathematical point of view, to justify the choice of an operation for a particular application.

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