

Article

On Instability Analysis of Linear Feedback Systems

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Abstract: The numerical approximation of the μ -value is key towards the measurement of instability, stability analysis, robustness, and the performance of linear feedback systems in system theory. The MATLAB function `mussv` available in MATLAB Control Toolbox efficiently computes both lower and upper bounds of the μ -value. This article deals with the numerical approximations of the lower bounds of μ -values by means of low-rank ordinary differential equation (ODE)-based techniques. The numerical simulation shows that approximated lower bounds of μ -values are much tighter when compared to those obtained by the MATLAB function `mussv`.

Keywords: doubly stochastic matrices; eigenvalues; singular values; structured singular values; ordinary differential equations

1. Introduction

The μ -value was first introduced by J. C. Doyle in order to deal with robust stability problems underlying structured perturbations [1]. The structured perturbations are used when a system deals with multiple uncertainties. The small gain theorem deals with the robust stability of linear feedback systems and reflects the source of uncertainties originating from the original source of a single reference location within the loop of the system. In this way, all the uncertainties are within a single perturbation to maintain norm boundedness, which is related to a single reference location [2]. An extensive amount of work has been done to study the linear feedback system; for instance, the linear control system on a manifold that is equivalent by means of diffeomorphism to an invariant system has been studied [3].

The size of the uncertainty depends upon the condition number of the nominal matrix when multivariable systems are involved. The condition number corresponding to nominal matrices could be very large at some critical frequencies, which result in the conservation of uncertainties. For such a case, the μ -value is defined in [2], which deals with both robust analysis and synthesis problems.

Consider an $(M - \Delta)$ interaction with a nominal matrix M and an underlying perturbation Δ . The matrix M could be a structured matrix such as an anti-tridiagonal 2-Hankel matrix or anti-tridiagonal Hankel matrix of any order [4]. For more detail, we refer the reader to [5]. The μ -value is the measure of minimum structured perturbations that establish instability in the control system. The μ -value for a given nominal matrix M and admissible perturbation Δ is denoted by $\mu_{\Delta}(M)$. The size of Δ is exactly equal to the quantity $\frac{1}{\mu(M)}$, which means a smaller value of μ shall produce more robustness in the system. The computation of the exact value of μ is very hard—in fact, NP-hard. The NP-hard nature of the exact computation of the μ -value motivates the development of numerical algorithms in order to compute its lower and upper bounds [6]. The well-known numerical algorithms include power algorithms [7] and a balanced/LMI technique to approximate lower and upper boundaries, respectively.

The output feedback stabilization is studied in [8] while considering the linear dynamical controllers for an open loop. The linear dynamical controllers involving point and distributed delays have been studied in great detail. The results presented show that the increase in the dynamic of linear controllers can be used as a stabilization technique for the use of delay in the linear controllers.

The computation of admissible sets of the parametrical multi perturbation for a bounded set is studied in [9]. These sets include the combinations of parametrical multi perturbations across the matrix form from dynamics and in control. Furthermore, various properties such as observability, stabilizability, and detectability are studied as well. The results obtained in the paper are applied to various control systems subject to discrete internal delay and perturbations.

The robustness of a strong delay independent stability analysis for a linear time delay system is studied in [10] by making use of a strong delay independent stability radius. Furthermore, the results presented in this paper show that, in positive time delay systems, real and complex strong stability radii coincide and are computable with the help of simple mathematical formulations.

The linear retarded systems described with the help of generalized linear functional equations are studied in [11]. The results for the computation of both lower and upper bounds of the complex stability radius with respect to multi perturbations are presented. The relationship between the stability radius and the L_σ -gain for linear time invariant systems is studied in [12]. The characterization of L_1 -, L_2 - and L_{inf} -gain for asymptotically stable positive systems are presented in terms of stability radii. Furthermore, it is shown how the structured perturbation corresponding to stable matrices can be treated as a closed-loop system with uncertain structures.

A linear time invariant homogeneous system corresponding to first-order ordinary differential equations (ODEs) is presented in [13]. A family of perturbations consists of coefficient matrices for systems under consideration, and it determines conditions on perturbations such that the internal structure of the system remains unchanged. Sufficient conditions on the robust stability of the system under such perturbations are presented. The robust stability analysis of linear time varying systems with a differential algebraic equation is presented in [14]. The systems under the effect of uncertain dynamical perturbations are considered and studied, and a mathematical formula for structured stability is presented.

The main contribution of this article is to approximate the lower bounds of μ -values by using low-rank ODE-based techniques for a class of matrices presented in [15]. The lower bounds of μ -values obtained with the MATLAB function `mussv` are compared with the results obtained for the approximation of lower bounds with the help of an algorithm found in [16]. We perceive that in most cases the algorithm in [16] approximates tighter lower bounds of μ -values than the `mussv` function does.

This paper is organized as follows: In Section 2, we provide the definition of block diagonal uncertainties and structured singular values (SSVs). Furthermore, in this section we provide the small gain theorem and explain the necessary conditions needed for a linear feedback system to be well-posed and stable. Section 3 discusses a reformulation of the definition of SSVs. Section 4 is about the proposed methodology, which is based on a two-level algorithm. The inner-outer algorithm, like the inner-algorithm, is explained in Section 5. Section 6 explains the outer-algorithm in order to adjust the desired perturbation level. In Section 7, we present numerical experiments to compare the lower bounds of SSVs obtained with the algorithm in [16] to those obtained with the MATLAB function `mussv`. Section 8 summarizes the conclusions.

2. Preliminaries

Definition 1 ([16]). *The set of block diagonal matrices with repeated complex scalar blocks and full complex blocks is defined for all $i = 1 : s$ and $j = 1 : F$ as follows*

$$\mathbb{B} := \{diag(\delta_i I_i; \Delta_j) : \delta_i \in \mathbb{C}(\mathbb{R}), \Delta_j \in \mathbb{C}^{m_j, m_j}(\mathbb{R}^{m_j, m_j})\}.$$

Definition 2 ([16]). For a given square matrix $M \in \mathbb{C}^{n,n}$ and underlying perturbation set \mathbb{B} , the μ -value is defined as

$$\mu_{\mathbb{B}}(M) = \frac{1}{\min\{\|\Delta\|_2 : \Delta \in \mathbb{B}, \det(I - M\Delta) = 0\}}$$

unless no such Δ cause $(I - M\Delta)$ to be singular for which $\mu_{\mathbb{B}}(M) = 0$.

Theorem 1 (Small Gain Theorem [17]). The feedback system is well-posed and stable for an admissible perturbation Δ with the largest singular value bounded above by 1 if and only if

$$\|M\|_{\infty} = \text{Sup}(\|M(j\omega)\|) < 1,$$

and, for any $\omega \in \mathbb{R}^+$, the frequency and Sup is taken over ω .

Theorem 2 ([7]). For two structured uncertainties, $\mathbb{B}_1 \subset \mathbb{B}_2$, where

$$\mu_{\mathbb{B}_1}(\|M(j\omega)\|) < \mu_{\mathbb{B}_2}(\|M(j\omega)\|).$$

Moreover, the feedback system is well-posed and internally stable for $\Delta \in \mathbb{B}$ with $\|\Delta\|_2 \leq 1$ if and only if $\text{Sup}(M(j\omega)) < 1$ for any $\omega \in \mathbb{R}^+$.

3. Reformulation of μ -Values

In this section, we reformulate the μ -values on the basis of structured spectral value sets.

Definition 3. For a given $M \in \mathbb{C}^{n,n}$ and perturbation level $\epsilon > 0$, the structured spectral value set is denoted by $\Lambda_{\epsilon}^{\mathbb{B}}(M)$ and is defined as

$$\Lambda_{\epsilon}^{\mathbb{B}}(M) = \{\lambda \in \Lambda(\epsilon M\Delta), \Delta \in \mathbb{B}, \|\Delta\|_2 \leq 1\},$$

where $\Lambda(\epsilon M\Delta)$ denotes the spectrum of the matrix valued function $(\epsilon M\Delta)$, and is simply a disk centered at origin 0.

Definition 4 ([16]). The structured epsilon spectral value set for a given $M \in \mathbb{C}^{n,n}$ and $\epsilon \geq 0$, is defined as

$$\Sigma_{\epsilon}^{\mathbb{B}}(M) = \{\eta = 1 - \lambda : \lambda \in \Lambda_{\epsilon}^{\mathbb{B}}(M)\}.$$

Definition 2 allows us to express μ -value as given below:

Definition 5 ([16]). For a given $M \in \mathbb{C}^{n,n}$ and an underlying perturbation set \mathbb{B} , the μ -value is defined as

$$\mu_{\mathbb{B}}(M) = \frac{1}{\arg \min_{\epsilon > 0} \{\max|\lambda| = 1, \lambda \in \Lambda_{\epsilon}^{\mathbb{B}}(M)\}}.$$

4. Proposed Methodology

In order to solve the maximization problem discussed in Definition 5, we make use of a numerical method [16] based upon the low-rank ordinary differential equations technique. This numerical method is mainly composed of a two-level algorithm, that is, an inner-algorithm and an outer-algorithm. In the inner-algorithm, the main objective is to first construct and then solve a gradient system of ordinary differential equations. In the outer-algorithm, we vary the perturbation level $\epsilon > 0$ by means of fast Newton iteration. The outer-algorithm computes an exact derivative of an extremizer, say $\Delta(\epsilon)$ for $\Delta \in \mathbb{B}$ and $\epsilon > 0$.

Next, we discuss the computation of an extremizer. For this purpose, we first approximate the derivative of an eigenvalue $\lambda(p)$ of a smooth matrix family, say $A(p)$ for some fixed parameter p .

5. Inner Algorithm

5.1. The Basic Theory

Consider some matrix family $A(p) \in \mathbb{C}^{n,n}$ for a small parameter p . Let $\lambda(p) \in \mathbb{C}$ be an eigenvalue of $A(p)$ and let $x(p) \in \mathbb{C}^{n,1}$ denote the corresponding right eigenvector to $\lambda(p)$. The eigenvalue problem for computing the derivative of $\lambda(p)$ is of the form

$$A(p)x(p) = \lambda(p)x(p), \tag{1}$$

with $\lambda(p)$ and $x(p)$ infinitely differentiable. Let $\lambda(p)$ converge to an eigenvalue $\lambda(0)$ when $p \rightarrow 0$. The eigenvalue $\lambda(0)$ possesses an algebraic multiplicity 1. Furthermore, for eigenvalue problem (1), we have that $x^*(p)x(p) = 1$, where $*$ denotes the complex conjugate transpose of $x(p)$.

The following theorem [18] computes the derivatives of simple eigenvalue $\lambda(0)$ to eigenvalue problem (1).

Theorem 3. *Let $\lambda(0)$ be a simple eigenvalue for a matrix $A(0) \in \mathbb{C}^{n,n}$. Let $x(0)$ be associated with the right eigenvector, so that $A(0)x(0) = \lambda(0)x(0)$ and $\lambda(p)$ and $x(p)$ are defined for all $A(p)$ in the neighborhood $N(A(0)) \in \mathbb{C}^{n,n}$ of $A(0)$ such that*

$$\lambda(A(0)) = \lambda(0), \quad x(A(0)) = x(0)$$

and

$$A(p)x(p) = \lambda(p)x(p), \quad x^*(0)x(0) = 1.$$

Moreover, the function $\lambda(p)$ is differentiable on $N(A(0))$, and the differential at $A(0)$ is

$$\frac{d\lambda(0)}{dt} = \frac{y^*(0)\dot{A}(0)x(0)}{y^*(0)x(0)}, \quad y^*(0)x(0) \neq 0$$

with $y^*(0)$, the left eigenvector of $\lambda(0)$.

Proof. For proof, we refer the reader to [18]. \square

5.2. Approximation of an Extremizer

A matrix valued function $\Delta \in \mathbb{B}$ having the largest singular value bounded above by 1 and the matrix valued function $(I - \epsilon M\Delta)$ having the smallest eigenvalue, which minimizes the modulus of the structured spectral value set $\Sigma_{\epsilon}^{\mathbb{B}}(M)$, is known as an extremizer. The following theorem computes an extremizer for a chosen smallest complex number belonging to the set $\Sigma_{\epsilon}^{\mathbb{B}}(M)$.

Theorem 4 ([16]). *Let $\Delta \in \mathbb{B}$ having the block diagonal structure*

$$\Delta = \{diag(\delta_1 I_1, \dots, \delta_{s'} I_{s'}, \delta_{s'+1} I_{s'+1}, \dots, \delta_s I_s; \Delta_1, \dots, \Delta_F)\},$$

with $\|\Delta\|_2 = 1$, as a local extremizer of a structured spectral value set. For the smallest simple eigenvalue $\lambda = |\lambda|e^{i\theta}$, $\theta \in \mathbb{R}$ of matrix valued function $(I - \epsilon M\Delta)$ having right and left eigenvectors x and y scaled as $S = e^{i\theta}y^*x$, and let $z = M^*y$. The non-degeneracy conditions

$$\begin{aligned} & z_k^* x_k \neq 0, \text{ for any } k = 1 : s' \\ & \text{Re}(z_k^* x_k) \neq 0, \text{ for any } k = s' + 1 : s \\ \text{and } & \|z_{s+h}\| \cdot \|x_{s+h}\| \neq 0, \text{ for any } h = 1 : F \end{aligned}$$

hold. The magnitude of each complex scalar $\delta_i \forall i = 1 : s$ is exactly equal to 1, while each full block possesses a unit 2-norm.

5.3. Gradient System of ODEs

The gradient system of ODEs for an admissible perturbation $\Delta \in \mathbb{B}$ to approximate a local extremizer of smallest eigenvalue $\lambda = |\lambda|e^{i\theta}$ is obtained as

$$\begin{aligned} \dot{\delta}_i &= \nu_i(x_i^* z_i - \text{Re}(x_i^* z_i \bar{\delta}_i) \delta_i); & i = 1 : s' \\ \dot{\delta}_l &= \text{sign}(\text{Re}(z_l^* x_l) \Psi_{(-1,1)}(\delta_l)); & l = s' + 1 : s \\ \dot{\Delta}_j &= \nu_j(z_{s+j} x_{s+j}^* - \text{Re}(\Delta_j; z_{s+j} x_{s+j}^*)); & j = 1 : F, \end{aligned}$$

where $\delta_i \in \mathbb{C}, \forall i = 1 : s', \delta_l \in \mathbb{R}$ for $l = s' + 1 : s$ and $\Psi_{(-1,1)}$, the characteristic function. For more discussion on the construction of a gradient system of ODEs in the above equations, we refer the reader to [16].

6. Outer-Algorithm

In the outer-algorithm, the main aim is to vary $\epsilon > 0$, the perturbation level by means of fast Newton’s iteration. In turn, $\frac{1}{\epsilon}$ will provide us the approximation of the lower bound of the μ -value.

We make use of fast Newton’s iteration in order to solve the problem

$$|\lambda(\epsilon)| = 1. \tag{2}$$

In Equation (2), $\epsilon > 0$. In order to solve Equation (2), we need to compute

$$\frac{d}{d\epsilon} (|\lambda(\epsilon)|).$$

Theorem 5 helps us to compute $\frac{d}{d\epsilon} (|\lambda(\epsilon)|)$, when $\lambda(\epsilon)$ is simple, and $\Delta(0)$ and $\lambda(0)$ are assumed to remain smooth in the neighboring region of perturbation level $\epsilon > 0$

Theorem 5 ([16]). Consider matrix valued function $\Delta \in \mathbb{B}$. Let x and y be functions of perturbation level $\epsilon > 0$ and act as right and left eigenvectors of matrix valued function $(\epsilon M \Delta)$. Consider the scaling of these vectors as given in Theorem 4. Let $z = M^* y$, and assume that the non-degeneracy conclusions given in Theorem 4 hold. Then,

$$\frac{d}{d\epsilon} (|\lambda(\epsilon)|) = \frac{1}{|y(\epsilon)^* x(\epsilon)|} \left(\sum_{i=1}^s |z_i(\epsilon)^* x_i(\epsilon)| + \sum_{j=1}^F \|z_{s+j}(\epsilon)\| \cdot \|y_{s+j}(\epsilon)\| \right) > 0.$$

Choice of a Suitable Initial Value Matrix and Initial Perturbation Level

For a suitable choice of the initial value matrix Δ_0 and an initial perturbation level ϵ_0 , we refer the reader to [16].

7. Numerical Experimentations

In this section, we present numerical results for the lower and upper bounds of μ -values for a class of matrices considered in [15]. The numerical computation of the bounds of SSVs are performed with MATLAB (R2018a, MathWorks, Natick, MA, USA) on system: LAPTOP-FMH097DQ, Intel (R) Core i5-6200U CPU @ 2.30GHz 2.40GHz.

Example 1. Consider a two-dimensional complex valued matrix taken from [15].

$$M = \begin{pmatrix} 4 + i & 0.1 \\ 10 & i \end{pmatrix}.$$

Case 1. We consider the perturbation set $\mathbb{B} = \{diag(\delta_1 I_1, \delta_2 I_1) : \delta_1, \delta_2 \in \mathbb{C}\}$ taken from [15]. By making use of the MATLAB function `mussv`, the perturbation $\hat{\Delta}$ is obtained as

$$\hat{\Delta} = \begin{pmatrix} 0.2236 - 0.0528i & 0 \\ 0 & 0.2236 - 0.0528i \end{pmatrix}.$$

The largest singular value corresponding to $\hat{\Delta}$ is obtained as 0.2298, while the matrix valued function $(I - M\hat{\Delta})$ has eigenvalues 0 and $1 - 0.2361i$. By making use of the algorithm in [16], the perturbation E is obtained as

$$E = \begin{pmatrix} 0.9732 - 0.2298i & 0 \\ 0 & 0.9732 - 0.2298i \end{pmatrix}.$$

The largest singular value corresponding to E is obtained as 1, while the matrix valued function $(I - ME)$ has eigenvalues -3.3525 and $0.0275i$.

Case 2. We consider the perturbation set $\mathbb{B} = \{diag(\delta_1 I_1, \delta_2 I_1) : \delta_1 \in \mathbb{R}, \delta_2 \in \mathbb{C}\}$ taken from [15]. By making use of the MATLAB function `mussv`, the perturbation $\hat{\Delta}$ is obtained as

$$\hat{\Delta} = \begin{pmatrix} 0.2263 & 0 \\ 0 & 0.1002 - 0.5209i \end{pmatrix}.$$

The largest singular value corresponding to $\hat{\Delta}$ is obtained as 0.5305, while the matrix valued function $(I - M\hat{\Delta})$ has eigenvalues 0 and $0.5737 - 0.3265i$. By making use of the algorithm in [16], the perturbation E is obtained as

$$E = \begin{pmatrix} 0.5339 & 0 \\ 0 & 0.0435 - 0.9991i \end{pmatrix}.$$

The largest singular value corresponding to E is obtained as 1, while the matrix valued function $(I - ME)$ has eigenvalues $-1.1457 - 0.0685i$ and $0.0111 - 0.5089i$.

Case 3. We consider the perturbation set $\mathbb{B} = \{diag(\delta_1 I_1, \delta_2 I_1) : \delta_1 \in \mathbb{C}, \delta_2 \in \mathbb{R}\}$ taken from [15]. By making use of the MATLAB function `mussv`, the perturbation $\hat{\Delta}$ is obtained as

$$\hat{\Delta} = \begin{pmatrix} 0.2232 - 0.0556i & 0 \\ 0 & 0.2300 \end{pmatrix}.$$

The largest singular value corresponding to $\hat{\Delta}$ is obtained as 0.2300, while the matrix valued function $(I - M\hat{\Delta})$ has eigenvalues 0 and $1.0516 - 0.2310i$. By making use of the algorithm in [16], the perturbation E is obtained as

$$E = \begin{pmatrix} 0.9704 - 0.2416i & 0 \\ 0 & 1.0000 \end{pmatrix}.$$

The largest singular value corresponding to E is obtained as 1, while the matrix valued function $(I - ME)$ has eigenvalues -3.3472 and $1.2241 - 1.0040i$.

Table 1. Numerical approximation of bounds of structured singular values (SSVs).

\mathbb{B}	<code>mussv</code>	μ_l in [15]	μ_l^{New}
$diag\{(\delta_1 I_1, \delta_2 I_1) : \delta_1, \delta_2 \in \mathbb{C}\}$	[4.3525, 4.3525]	N/A	4.3525
$diag\{(\delta_1 I_1, \delta_2 I_1) : \delta_1 \in \mathbb{R}, \delta_2 \in \mathbb{C}\}$	[1.8851, 2.0840]	N/A	2.0762
$diag\{(\delta_1 I_1, \delta_2 I_1) : \delta_1 \in \mathbb{C}, \delta_2 \in \mathbb{R}\}$	[4.3472, 4.3525]	N/A	4.3472
$diag\{(\delta_1 I_1, \delta_2 I_1) : \delta_1 \in \mathbb{R}, \delta_2 \in \mathbb{R}\}$	[0, 1.8310×10^{-15}]	3.1629	N/A

Table 1 shows the comparison of bounds of SSVs approximated by the MATLAB function `mussv` and the algorithm in [16] for various blocks of \mathbb{B} . Furthermore, *N/A* in each table means that [15] and [16] generate no results for the lower bounds of SSVs.

Example 2. Consider a two-dimensional complex valued matrix taken from [15].

$$M = \begin{pmatrix} 4 + i & 1 \\ -1 & i \end{pmatrix}$$

Case 1. We consider the perturbation set $\mathbb{B} = \{diag(\delta_1 I_1, \delta_2 I_1) : \delta_1, \delta_2 \in \mathbb{C}\}$ taken from [15]. By making use of the MATLAB function `mussv`, the perturbation $\hat{\Delta}$ is obtained as

$$\hat{\Delta} = \begin{pmatrix} 0.2209 - 0.0583i & 0 \\ 0 & -0.1791 - 0.1417i \end{pmatrix}.$$

The largest singular value corresponding to $\hat{\Delta}$ is obtained as 0.2284, while the matrix valued function $(I - M\hat{\Delta})$ has eigenvalues 0 and $0.9165 + 0.1913i$. By making use of the algorithm in [16], the perturbation E is obtained as

$$E = \begin{pmatrix} 0.9690 - 0.2471i & 0 \\ 0 & -0.8556 - 0.5176i \end{pmatrix}.$$

The largest singular value corresponding to E is obtained as 1, while the matrix valued function $(I - ME)$ has eigenvalues -3.3754 and $0.7347 + 0.8749i$.

Case 2. We consider the perturbation set $\mathbb{B} = \{diag(\delta_1 I_1, \delta_2 I_1) : \delta_1 \in \mathbb{R}, \delta_2 \in \mathbb{C}\}$ taken from [15]. By making use of the MATLAB function `mussv`, the perturbation $\hat{\Delta}$ is obtained as

$$\hat{\Delta} = \begin{pmatrix} 0 & 0 \\ 0 & -i \end{pmatrix}.$$

The largest singular value corresponding to $\hat{\Delta}$ is obtained as 1, while the matrix valued function $(I - M\hat{\Delta})$ has eigenvalues 0.9999 and 0. By making use of the algorithm in [16], the perturbation E is obtained as

$$E = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}.$$

The largest singular value corresponding to E is obtained as 1, while the matrix valued function $(I - ME)$ has eigenvalues 1 and 0.

Case 3. We consider the perturbation set $\mathbb{B} = \{diag(\delta_1 I_1, \delta_2 I_1) : \delta_1 \in \mathbb{C}, \delta_2 \in \mathbb{R}\}$ taken from [15]. By making use of the MATLAB function `mussv`, the perturbation $\hat{\Delta}$ is obtained as

$$\hat{\Delta} = \begin{pmatrix} 0.2256 - 0.0507i & 0 \\ 0 & -0.2312 \end{pmatrix}.$$

The largest singular value corresponding to $\hat{\Delta}$ is obtained as 0.2312, while the matrix valued function $(I - M\hat{\Delta})$ has eigenvalues 0 and $1.0469 + 0.2086i$. By making use of the algorithm in [16], the perturbation E is obtained as

$$E = \begin{pmatrix} 0.9756 - 0.2195i & 0 \\ 0 & -1 \end{pmatrix}.$$

The largest singular value corresponding to E is obtained as 1, while the matrix valued function $(I - ME)$ has eigenvalues -3.3249 and $1.2030 + 0.9023i$.

Case 4. We consider the perturbation set $\mathbb{B} = \{diag(\delta_1 I_1, \delta_2 I_1) : \delta_1, \delta_2 \in \mathbb{R}\}$ taken from [15]. By making use of the MATLAB function **mussv**, the perturbation $\hat{\Delta}$ is obtained as

$$\hat{\Delta} = 1.0e + 050 \begin{pmatrix} 1.2127 & 0 \\ 0 & 1.2127 \end{pmatrix}.$$

The largest singular value corresponding to $\hat{\Delta}$ is obtained as 1.2127, while the matrix valued function $(I - M\hat{\Delta})$ has eigenvalues $1.0e + 050(-4.5258 - 1.2127i)$ and $1.0e + 050(-0.3249 - 1.2127i)$. By making use of the algorithm in [16], the perturbation E is obtained

$$E = \begin{pmatrix} 0.5111 & 0 \\ 0 & 1.0000 \end{pmatrix}.$$

The largest singular value corresponding to E is obtained as 1, while the matrix valued function $(I - ME)$ has eigenvalues $-0.7846 - 0.4278i$ and $0.7403 - 1.0833i$.

Table 2 shows the comparison of the bounds of SSVs approximated by the MATLAB function **mussv** and the algorithm in [16] for various blocks of \mathbb{B} .

Table 2. Numerical approximation of bounds of SSVs.

\mathbb{B}	<i>mussv</i>	μ_l in [15]	μ_l^{New}
$diag\{(\delta_1 I_1, \delta_2 I_1) : \delta_1, \delta_2 \in \mathbb{C}\}$	[4.3778, 4.3778]	N/A	4.3754
$diag\{(\delta_1 I_1, \delta_2 I_1) : \delta_1 \in \mathbb{R}, \delta_2 \in \mathbb{C}\}$	[1, 1.4220]	N/A	1
$diag\{(\delta_1 I_1, \delta_2 I_1) : \delta_1 \in \mathbb{C}, \delta_2 \in \mathbb{R}\}$	[4.3249, 4.3375]	N/A	4.3249
$diag\{(\delta_1 I_1, \delta_2 I_1) : \delta_1 \in \mathbb{R}, \delta_2 \in \mathbb{R}\}$	[0, 0.9383]	3.8042	1.9318

Example 3. Consider a three-dimensional complex valued matrix taken from [15].

$$M = \begin{pmatrix} 1 + i & 10 - 2i & -20i \\ 5i & 3 + i & -1 + 3i \\ -2 & i & 4 - i \end{pmatrix}.$$

Case 1. We consider the perturbation set $\mathbb{B} = \{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1) : \delta_1, \delta_2, \delta_3 \in \mathbb{C}\}$ taken from [15]. By making use of the MATLAB function **mussv**, the perturbation $\hat{\Delta}$ is obtained as

$$\hat{\Delta} = \begin{pmatrix} 0.0289 - 0.0784i & 0 & 0 \\ 0 & 0.0817 - 0.0177i & 0 \\ 0 & 0 & 0.0835 - 0.0034i \end{pmatrix}.$$

The largest singular value corresponding to $\hat{\Delta}$ is obtained as 0.0836, while the matrix valued function $(I - M\hat{\Delta})$ has eigenvalues $1.5615 + 0.0786i$ and 0 and $0.7377 + 0.0394i$. By making use of the algorithm in [16], the perturbation E is obtained as

$$E = \begin{pmatrix} 0.4014 & 0 & 0 \\ 0 & 0.9650 - 0.2623i & 0 \\ 0 & 0 & 0.9961 - 0.0877i \end{pmatrix}.$$

The largest singular value corresponding to E is obtained as 1, while the matrix valued function $(I - ME)$ has eigenvalues $3.4800 + 3.3641i$, $-5.8883 - 3.5476i$, and $-2.0471 + 0.9509i$.

Case 2. We consider the perturbation set $\mathbb{B} = \{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1) : \delta_1, \delta_2 \in \mathbb{R}, \delta_3 \in \mathbb{C}\}$ taken from [15].

By making use of the MATLAB function **mussv**, the perturbation $\hat{\Delta}$ is obtained as

$$\widehat{\Delta} = \begin{pmatrix} -0.1054 & 0 & 0 \\ 0 & -0.0846 & 0 \\ 0 & 0 & 0.1087 + 0.0963i \end{pmatrix}.$$

The largest singular value corresponding to $\widehat{\Delta}$ is obtained as 0.1452, while the matrix valued function $(I - M\widehat{\Delta})$ has eigenvalues 0, $1.0866 - 0.4045i$, and $1.7417 + 0.3181i$. By making use of the algorithm in [16], the perturbation E is obtained as

$$E = \begin{pmatrix} -0.9999 & 0 & 0 \\ 0 & -1.0000 & 0 \\ 0 & 0 & 0.7940 + 0.6079i \end{pmatrix}.$$

The largest singular value corresponding to E is obtained as 1, while the matrix valued function $(I - ME)$ has eigenvalues $8.8119 + 4.0861i$, -6.8291 , and $1.2331 - 3.7236i$.

Case 3. We consider the perturbation set $\mathbb{B} = \{diag(\Delta_1, \delta_1 I_1) : \Delta_1 \in \mathbb{C}^{2,2}, \delta_1 \in \mathbb{C}\}$ taken from [15]. By making use of the MATLAB function `mussv`, the perturbation $\widehat{\Delta}$ is obtained as

$$\widehat{\Delta} = \begin{pmatrix} 0.0105 - 0.0288i & -0.0009 - 0.0104i & 0 \\ 0.0595 + 0.0111i & 0.0200 - 0.0052i & 0 \\ 0 & 0 & 0.0716 - 0.0030i \end{pmatrix}.$$

The largest singular value corresponding to $\widehat{\Delta}$ is obtained as 0.07165, while the matrix valued function $(I - M\widehat{\Delta})$ has eigenvalues 0, $1.0000i$, and $0.9433 + 0.1102i$. By making use of the algorithm in [16], the perturbation E is obtained as

$$E = \begin{pmatrix} 0.1465 - 0.4021i & -0.0127 - 0.1453i & 0 \\ 0.8300 + 0.1544i & 0.2787 - 0.0719i & 0 \\ 0 & 0 & 0.9992 - 0.0411i \end{pmatrix}.$$

The largest singular value corresponding to E is obtained as 1, while the matrix valued function $(I - ME)$ has eigenvalues -12.9560 , 1, and $0.2082 + 1.5349i$.

Table 3 shows the comparison of the bounds of SSVs approximated by the MATLAB function `mussv` and the algorithm in [16] for various blocks of \mathbb{B} .

Table 3. Numerical approximation of bounds of SSVs.

\mathbb{B}	<code>mussv</code>	μ_l in [15]	μ_l^{New}
$diag\{(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1) : \delta_1, \delta_2, \delta_3 \in \mathbb{C}\}$	[11.96, 11.96]	11.3156	N/A
$diag\{(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1) : \delta_1, \delta_2, \delta_3 \in \mathbb{R}\}$	[0, 9.9273]	N/A	5.8176
$diag\{(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1) : \delta_1, \delta_2 \in \mathbb{R}, \delta_3 \in \mathbb{C}\}$	[6.88, 10.08]	N/A	7.8291
$diag\{(\Delta_1, \delta_1 I_1) : \Delta_1 \in \mathbb{C}^{2,2}, \delta_1 \in \mathbb{C}\}$	[13.956, 13.956]	N/A	13.956
$diag\{(\Delta_1, \delta_1 I_1) : \Delta_1 \in \mathbb{R}^{2,2}, \delta_1 \in \mathbb{R}\}$	N/A	13.4008	N/A

8. Conclusions

In this article, we have presented the numerical approximation of lower bounds of μ -values. The lower bounds of μ -values show the instability analysis of a linear feedback system in system theory. The numerical experimentation show that the new results for the lower bounds are tighter for some cases than the ones approximated by the MATLAB function `mussv` and Equation (8) in [15]. In general, it is not possible for a new algorithm to provide tight lower bounds of SSVs for all numerical experiments; in some cases, the `mussv` function approximates tight lower bounds.

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