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Discrete Competitive Lotka–Volterra Model with Controllable Phase Volume

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Abstract: The simulation of population dynamics and social processes is of great interest in nonlinear systems. Recently, many scholars have paid attention to the possible applications of population dynamics models, such as the competitive Lotka–Volterra equation, in economic, demographic and social sciences. It was found that these models can describe some complex behavioral phenomena such as marital behavior, the stable marriage problem and other demographic processes, possessing chaotic dynamics under certain conditions. However, the introduction of external factors directly into the continuous system can influence its dynamic properties and requires a reformulation of the whole model. Nowadays most of the simulations are performed on digital computers. Thus, it is possible to use special numerical techniques and discrete effects to introduce additional features to the digital models of continuous systems. In this paper we propose a discrete model with controllable phase-space volume based on the competitive Lotka–Volterra equations. This model is obtained through the application of semi-implicit numerical methods with controllable symmetry to the continuous competitive Lotka–Volterra model. The proposed model provides almost linear control of the phase-space volume and, consequently, the quantitative characteristics of simulated behavior, by shifting the symmetry of the underlying finite-difference scheme. We explicitly show the possibility of introducing almost arbitrary law to control the phase-space volume and entropy of the system. The proposed approach is verified through bifurcation, time domain and phase-space volume analysis. Several possible applications of the developed model to the social and demographic problems' simulation are discussed. The developed discrete model can be broadly used in modern behavioral, demographic and social studies.

Keywords: Lotka–Volterra; competitive model; adaptive discrete maps; discrete nonlinear systems; reproductive behavior simulation; stable marital problem

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1. Introduction

Population dynamics, being the branch of science that studies the size and age composition of populations as dynamic systems [1], has been and continues to be a dominant branch of mathematical biology. Several models have been proposed and formulated to study population dynamics. In this paper, we focus on the family of dynamic systems developed by Alfred J. Lotka and Vito Volterra, usually referred to as the Lotka–Volterra models (LVM).

Originally proposed by A. Lotka in 1910 when developing the theory of autocatalytic chemical reactions, it was extended in 1920 to organic systems [2,3]. The same set of equations was independently developed by V. Volterra in 1926 [4].

Many scholars have numerically studied the LV equations and highlighted their various applications. A. Korobenkov and G.C. Wake studied the global properties of the three-dimensional predator–prey model and, through the correlation between the results they obtained with the interaction of the indigenous species of New Zealand and introduced ones, we gain valuable insight into the mechanism of natural selection [5].

The most common application of the LV model in economics [6] is the description of concurrence in fund markets [7,8], technological competition [9,10], marketing [11–13], trade relationships [14], etc.

A macroeconomic framework to evaluate the social and economic consequences generated by a shift of investment to low-carbon options based on the Lotka–Volterra model was studied by Bernardo and D'Alessandro [15].

In urbanistics, the dynamic model of urbanistic habitat growth was studied by D.-J.F. Kamann and P. Nijkamp [16]. The role covered by dynamic models as support for the decision-making process in the evaluation of policies and actions for increasing the resilience of cities and territories was considered by Assumma et al. [17].

The most interesting applications of the Lotka–Volterra model can be found in sociology. For instance, the simulation of political behavior described by some researchers [18,19], ecologically responsible behavior [20,21], racial groups' interactions [18,22], cross-country convergence [23,24] etc. Moreover, Lotka–Volterra models were successfully applied to the modeling of class and military conflicts [25,26].

In the current demographic situation, the applications of Lotka–Volterra models to population reproduction processes are of great interest. One of the most interesting studies is the work of Yoshikazu Hata and Yoshiteru Ishida [27], who applied the Lotka–Volterra model to the stable marriage problem (SMP). SMP is a matching problem that tends to find a stable matching between N women and men [27]. The authors studied an approach for SMP by mapping SMP as a dynamic system. However, using a predator–prey model for the SMP is quite a simplified approach. This model focuses on the interactions between pairs and is capable of choosing a match (a set of pairs) among all possible pairs. Yoshikazu and Yoshiteru simulated the situation where blocking pairs "feed" upon the pairs being blocked, and therefore pairs sharing the same person must be excluded [27].

In our study, we propose a broader concept of reproductive problems' simulation. Thus, considering marital behavior, one should simulate not only the match between people in pairs (men and women) and "open/closed" relationships. Marital and reproductive behavior nowadays is determined by a broad set of factors [28]. Therefore, while simulating reproductive behavior, it is necessary to consider these factors.

For example, there is a fairly large number of options for sexual orientation in the modern world. Considering the Lotka–Volterra model, homosexual and heterosexual orientation and bisexuality should be considered as targeting ("preying on") a certain type of sexual partner. In this case, men can be partners for both women and men and vice versa.

The second essential factor in the modeling of modern processes of reproduction is childhood settings. Today there is a great variety in strategies of reproduction: single parenthood, large families, child-free, foster parenting, surrogacy, etc. Thus, even if a presumed partner has a "suitable" sexual orientation, the reproductive strategy can significantly influence the future of the partnership.

Taking into consideration other factors (e.g., nonbinary gender identities, cultural or racial differences, differences in lifestyle and value systems, marriage types) one can see that we need a more complex model to simulate and predict the creation and stability of relationships in pairs or marriage groups. Thus, more complicated models than the predator–prey are to be introduced. In our study, we will use the competitive Lotka–Volterra (CLV) model as a basic system. However, the CLV model is not a ready-to-go solution to simulate complex marital behavior.

The abovementioned basic factors (gender, sexual orientation, reproductive plans, cultural and racial characteristics, etc.) are characteristics of the elements of the system. However, when developing complex dynamic models of mating behavior, it is necessary to take into account external factors that both directly or indirectly affect the reproductive behavior of people. These include certain objectively and independently existing environmental conditions. For example, these can be political and cultural discourse, socio-economic conditions, the information field, the level of technological progress, etc. These factors are external to the dynamic system and cannot be directly introduced as system parameters or state variables without affecting the basic principles of its dynamics. Recently, some authors proposed the concept of adaptive symmetry as a technique to control the dynamics of discrete nonlinear maps in an almost linear way [29,30]. Based on the idea of semi-implicit [31] and semi-explicit integration [32], this approach allows one to introduce external control law into the discrete model dynamics. In our paper, we propose the discrete controllable symmetric competitive Lotka–Volterra model (CS CLVM), which is able to reproduce an arbitrary set of external factors in marital behavior simulations.

The rest of the paper is organized as follows. In Section 2 the general form of the competitive Lotka–Volterra model is presented and a semi-implicit integration technique is given to construct the proposed discrete competitive model with controllable phase volume. In Section 3, some numerical simulation results are given, showing the capability to change the quantitative properties of the system without breaking the established regime of oscillations. Finally, in Section 4 some discussion and conclusions are given.

2. Materials and Methods

2.1. Competitive Lotka–Volterra Model

The general form of the competitive Lotka–Volterra model (CLVM) for i species is as follows [33]:

$$\frac{dx_i}{dt} = r_i x_i \left(1 - \frac{\sum\limits_{j=1}^{N} \alpha_{ij} x_j}{K_j} \right) \tag{1}$$

Here r_i represents the growth rate of species i and a_{ij} the competitive effect species j has on i as they compete for resources. The competitive Lotka–Volterra system is based on the logistic population model, and K_i represents the carrying capacity of species i.

The competitive model uses to be applied in social and economic science but it is not common as a predator–prey model. It is also known to possess chaotic behavior under some conditions [34]. In this paper, we consider the competitive model as it given by Equation (1).

One can see that the original model takes into account only effects that species have on each other, the quantity of resources and growth rates. To introduce the external factor, let us apply the semi-implicit numerical integration technique to the competitive Lotka–Volterra model and obtain a discrete model with controllable symmetry.

Nowadays, types of marriages are not limited by stable (monogamous) and unstable (open) relationships as the original Yoshikazu and Yoshiteru model suggests [27]. The modern sociocultural environment in developed societies, in contrast to, for example, the more traditional communities, allows for a huge number of variants (sequential or parallel) of mating and reproductive behavior. The simulation of the contemporary marriage market must take into account the far greater number of possible alliance options. For example, one has to consider not only the (relatively) monogamous and polygamous heterosexual relationships but also homosexual and bisexual partnerships, as well as the voluntary refusal of relationships altogether (the relationship with the "zero" partner) and radical forms of gender movements. However, even under these conditions, one cannot simultaneously be in two ideologically opposing unions. If one of the partners allows polygamy, the relationship which he is in, in fact, ceases to be monogamous, even if his/her partner remains monogamous. This allows us to consider the marriage market in the framework of the competitive Lotka–Volterra model, where the prevalence of one species (namely, the type of the partnership) destroys the other. The resource, in this case, is the population size, since the number of possible pairs is limited by the number and resource of individuals in the population. Even with the formation of polygamous unions, the individual's ability to form an infinite number of pairs is physically limited (time, resources, health, etc.).

Thus, we come to an expanded model of the marriage market based on the competitive Lotka–Volterra model, reclaiming its parameters as follows:

r_i—the growth rate of the number of partnerships
x_n—partnerships of various types
n—the number of types of partnerships.

The proposed model is an updated model of the marriage market with advanced modeling capabilities for such a complex behavioral system as human marital and reproductive relationships.

As we mentioned above, the sociocultural discourse should be considered as an external factor of this system. One of the possible scenarios to introduce this factor can be the recently discovered adaptive symmetry phenomenon [29].

2.2. Finite-Difference Models with Controllable Symmetry

2.2.1. Semi-Implicit Integration as a Tool to Obtain Adaptive Discrete Systems

Semi-implicit integration initially appeared in numerical solutions of Hamiltonian systems, and proved to be an efficient technique to simulate chaotic systems [35]. Later it was proven to be a unique tool to construct nonlinear discrete maps with adaptive symmetry [29,30]. We will use this technique to synthesize the adaptive discrete competitive Lotka–Volterra model.

For a second-order initial value problem in scalar form,

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, t) \\ \dot{x}_2 &= f(x_1, x_2, t) \end{aligned} \tag{2}$$

semi-implicit numerical integration method *D1* is as follows [36]:

$$\begin{aligned} x_1^{n+1} &= x_1^n + hf_1(x_1^{n+1}, x_2^n, t) \\ x_2^{n+1} &= x_2^n + hf_2(x_1^{n+1}, x_2^{n+1}, t) \end{aligned} \tag{3}$$

The order of operations of (3) can be reversed to obtain an *adjoint* method *D2* [37]

$$\begin{aligned} x_2^{n+1} &= x_2^n + hf_1(x_1^n, x_2^n, t) \\ x_1^{n+1} &= x_1^n + hf_2(x_1^n, x_2^{n+1}, t) \end{aligned} \tag{4}$$

Using a pair of adjoint methods (3) and (4) of order 1 with increment functions Φ_{D1} and Φ_{D2} [31], one can obtain the symmetric *composition CD method* of order 2

$$\Phi_{CD} = \Phi_{\mathrm{D1}} \circ \Phi_{\mathrm{D2}} \tag{5}$$

In the case where the diagonally implicit variables cannot be found analytically, one can use a simple iterations method for solving these equations numerically. It is known that the semi-explicit method converges to the semi-implicit after very few simple iterations [32]. We will call the model based on this type of numerical integration, an SED (semi-explicit diagonal) model, and the model based on the original semi-implicit integration - a CD (compositional diagonally implicit) model.

2.2.2. Discrete CLVM Model with Controllable Symmetry

Let us apply method given by Equation (5) to the system (1) to obtain a semi-implicit finite-difference model of the competitive Lotka–Volterra system. We introduce symmetry parameter *S*, setting the local stepsize *h* = *H* * *S* where *H* is a global stepsize. Thus, the first adjoint of the model is:

$$\begin{aligned} X[1]^{n+1} &= \frac{-(K_1+hr_1(a_{12}X[2]^n+\ldots+a_{1N}X[N]^n))+\sqrt{(K_1+hr_1(a_{12}X[2]^n+\ldots+a_{1N}X[N]^n))^2+4hr_1X[1]^n}}{-2hr_1} \\ X[2]^{n+1} &= \frac{-(K_2+hr_2(a_{21}X[1]^{n+1}+a_{23}X[3]^n+\ldots+a_{2N}X[N]^n))+\sqrt{(K_2+hr_2(a_{21}X[1]^n+a_{23}X[3]^n+\ldots+a_{2N}X[N]^n))^2+4hr_2X[2]^n}}{-2hr_2} \\ X[N]^{n+1} &= \frac{-(K_N+hr_N(a_{N1}X[1]^{n+1}+\ldots+a_{N(N-1)}X[N-1]^{n+1}))+\sqrt{(K_N+hr_N(a_{N1}X[1]^{n+1}+\ldots+a_{N(N-1)}X[N-1]^{n+1}))^2+4hr_NX[N]^n}}{-2hr_N} \end{aligned} \quad (6)$$

where r_i is the growth rate of number of partnerships, x_n—number of partnerships of various types, *n*—the number of existing types of partnerships. As was previously mentioned, the direct calculation in Equation (6) can be replaced by the simple iterations algorithm [32], which results in the so-called semi-explicit (SED) method.

To keep the sum of local stepsizes equal 1, we recalculated the stepsize for the second part of the model as *h* = *H* * *(1 − S)*, and the order of operations was reversed in accordance with Equation (4):

$$\begin{aligned} X[N]^{n+1} &= X[N]^n + h\left(\frac{r_N(X[N]^n(1-(X[N]^n+a_{N1}X[1]^n+\ldots+a_{N(N-1)}X[N-1]^n}{K_N}\right) \\ X[2]^{n+1} &= X[2]^n + h\left(\frac{r_2(X[2]^n(1-(X[2]^n+a_{21}X[1]^n+\ldots+a_{2N}X[N]^{n+1}}{K_2}\right) \\ X[1]^{n+1} &= X[1]^n + h\left(\frac{r_1(X[1]^n(1-(X[1]^n+a_{12}X[2]^{n+1}+\ldots+a_{1N}X[N]^{n+1}}{K_1}\right) \end{aligned} \quad (7)$$

The constructed model is an updated model of the marriage market with advanced modeling capabilities for such a complex behavioral system as human marital and reproductive relationships.

At the same time, in our opinion, the sociocultural discourse that generates, allows or does not allow a variety of different types of partnerships in society should be considered an external factor in the management of this system. The obtained finite-difference model is an algebraic composition of Equations (6) and (7) and possesses controllable symmetry *S*. We will call the competitive Lotka–Volterra model with controllable symmetry, based on CD integration, the CD CSCLV model, and the model based on SED integration the SED CSCLV model. Both models are time-symmetric and possess the possibility of controlling their properties by an external value as the symmetry coefficient function. We will explicitly show this feature in the experimental section.

3. Results

3.1. Phase Space Analysis

In this section, we investigate the CD CSCLV model of order 4, which can be written as following a finite-difference scheme based on Equations (6) and (7):

$$
\begin{aligned}
x[1]^{n+1} &= -\frac{1+h\left(K_1+r_1\left(-K_1+r_1a_{12}x[2]^n+r_1a_{13}x[3]^n+r_1a_{14}x[4]^n\right)\right)}{2hr_1}+ \\
&\quad+\frac{\sqrt{\left(1+h\left(K_1+r_1\left(-K_1+r_1a_{12}x[2]^n+r_1a_{13}x[3]^n+r_1a_{14}x[4]^n\right)\right)\right)^2+4hr_1x[1]^n}}{2hr_1} \\
x[2]^{n+1} &= -\frac{1+h\left(K_2+r_2\left(-K_1+r_2a_{21}x[1]^{n+1}+r_2a_{23}x[3]^n+r_2a_{24}x[4]^n\right)\right)}{2hr_2}+ \\
&\quad+\frac{\sqrt{\left(1+h\left(K_2+r_2\left(-K_1+r_2a_{21}x[1]^{n+1}+r_2a_{23}x[3]^n+r_2a_{24}x[4]^n\right)\right)\right)^2-4hr_2x[2]^n}}{2hr_2} \\
x[3]^{n+1} &= -\frac{1+h\left(K_3+r_3\left(-K_3+r_3a_{31}x[1]^{n+1}+r_3a_{32}x[2]^{n+1}+r_3a_{34}x[4]^n\right)\right)}{2hr_3}+ \\
&\quad+\frac{\sqrt{\left(1+h\left(K_3+r_3\left(-K_3+r_3a_{31}x[1]^{n+1}+r_3a_{32}x[2]^{n+1}+r_3a_{34}x[4]^n\right)\right)\right)^2-4hr_3x[3]^n}}{2hr_3} \\
x[4]^{n+1} &= -\frac{1+h\left(K_4+r_4\left(-K_4+r_3a_{41}x[1]^{n+1}+r_3a_{42}x[2]^{n+1}+r_3a_{43}x[3]^{n+1}\right)\right)}{2hr_4}+ \\
&\quad+\frac{\sqrt{\left(1+h\left(K_4+r_4\left(-K_4+r_3a_{41}x[1]^{n+1}+r_3a_{42}x[2]^{n+1}+r_3a_{43}x[3]^{n+1}\right)\right)\right)^2-4hr_4x[4]^n}}{2hr_4} \\
x[4]^{n+1} &= x[4]^n+h\left(\frac{r_4x[4]^n(1-(x[4]^n+a_{41}x[1]^n+a_{42}x[2]^n+a_{43}[3]^n))}{K_4}\right) \\
x[3]^{n+1} &= x[3]^n+h\left(\frac{r_3x[3]^n(1-(x[3]^n+a_{31}x[1]^n+a_{32}x[2]^n+a_{34}[4]^{n+1}))}{K_3}\right) \\
x[2]^{n+1} &= x[2]^n+h\left(\frac{r_2x[2]^n(1-(x[2]^n+a_{21}x[1]^n+a_{23}x[3]^{n+1}+a_{24}[4]^{n+1}))}{K_2}\right) \\
x[1]^{n+1} &= x[1]^n+h\left(\frac{r_1x[1]^n(1-(x[1]^n+a_{12}x[2]^{n+1}+a_{13}x[3]^{n+1}+a_{14}[4]^{n+1}))}{K_1}\right)
\end{aligned}
\tag{8}
$$

where the parameters are as follows:

$$
r_i = \begin{bmatrix} 1 \\ 0.72 \\ 1.53 \\ 1.27 \end{bmatrix},\ a_{ij} = \begin{bmatrix} 1 & 1.09 & 1.52 & 0 \\ 0 & 1 & 0.44 & 1.36 \\ 2.33 & 0 & 1 & 0.47 \\ 1.21 & 0.51 & 0.35 & 1 \end{bmatrix}
$$

Carrying capacity K_i was set to 1 in all of the experiments.

The CLVM system possesses chaotic behavior [33] for a chosen set of parameters. To illustrate the phenomena of a discrete chaotic system with controllable symmetry, we plotted the chaotic attractors of the CSCLV model in Figure 1. One can see that the phase space of the system under investigation shrinks and expands following the changes of symmetry value. We can use this property to simulate various natural phenomena such as seasonal changes in population dynamics or global warming not affecting the chaotic regime of the system but changing the populations of species. Speaking of social sciences, such factors can be the influence of the mass media, and economic and general sociocultural discourse that generates, allows or does not allow a variety of different types of partnerships.

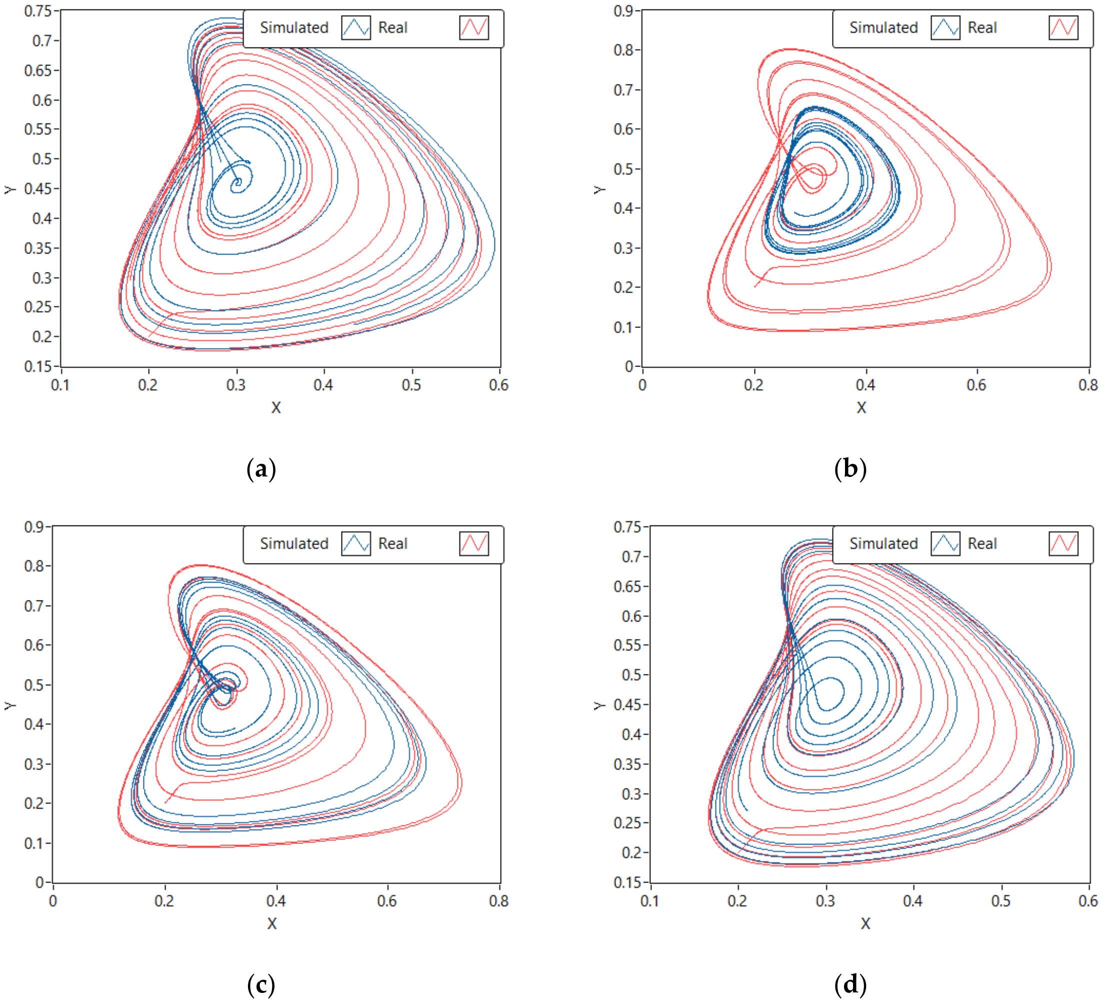

Figure 1. (**a**) Phase space of proposed competitive Lotka–Volterra model with controllable symmetry (CSCLV) model with S = 2; (**b**) Phase space of proposed CSCLV model with S = 1; (**c**) Phase space of proposed CSCLV model with S = 1.5; (**d**) Phase space of proposed CSCLV model with S = 1.1. The red line shows the attractor of the competitive Lotka–Volterra system with no phase volume control ("real volume").

To satisfy the abovementioned requirements, one needs not only the possibility to change the symmetry of the discrete system arbitrarily but to control the phase properties by a known law. To further illustrate the possibility to control the phase properties of CD CSCLV models (8), we will calculate the phase volume of the system [38] as follows:

$$V = \prod_{i=1}^{N} \left(x_i^{\max} - x_i^{\min} \right)$$

where N is the order of the system. The changes in phase space volume, shown in Figures 2 and 3, reflect both the entropy and energy dynamics of the system under investigation on a long-term simulation. The phase volume is calculated on a sliding window of 1000 s.

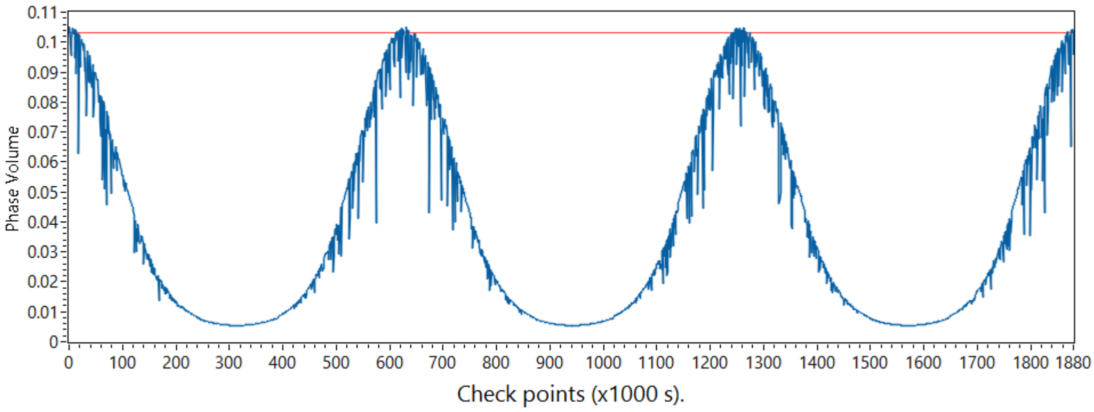

Figure 2. Phase volume changes of CD-based CSCLV model while symmetry is varied as $S = cos\ (wt)$. The red line shows the phase space volume of CSCVLM with fixed symmetry $S = 0.5$.

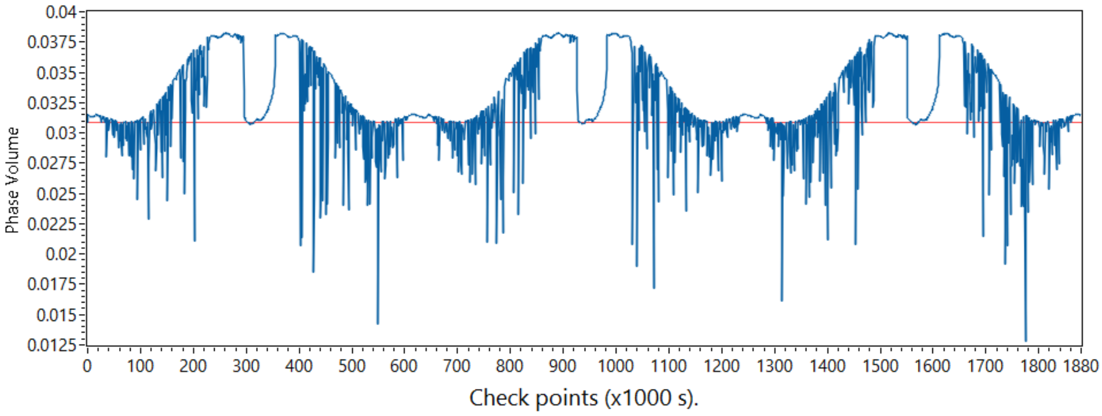

Figure 3. Phase-volume diagram of SED CSCLVM with the cosine function applied as symmetry law. The red line shows initial phase volume for a system with fixed symmetry $S = 0.5$.

One can see that both models possess the possibility of applying control laws to their dynamics. The SED CSCLV model provides less stable behavior, while the CD CSCLV model is better controllable and suggests richer dynamics. In real applications, the choice between models will highly depend on the simulated system's features, but we recommend considering the CD CSCLV model as a default choice. Let us perform the bifurcation analysis to analyze how changes in symmetry affect the chaotic behavior of the CSCLV system.

3.2. Bifurcation Analysis

Another claimed feature of the presented CSCLV model is the preservation of chaotic behavior while changing the symmetry of the system. To prove this fact, we performed the bifurcation analysis changing both nonlinearity parameters and symmetry of the system Equation (8).

Figures 4–6 represent the obtained set of bifurcation diagrams for the discrete Lotka–Volterra competitive model with controllable symmetry. One can see from Figure 4 that changing the bifurcation parameter *r2* influences the behavior of the CSCLV model, providing a cascade of bifurcations. Despite the fact the phase volume is obviously changing with different values of the nonlinearity parameter, this does not allow one to separately control the phase and chaotic properties of the system.

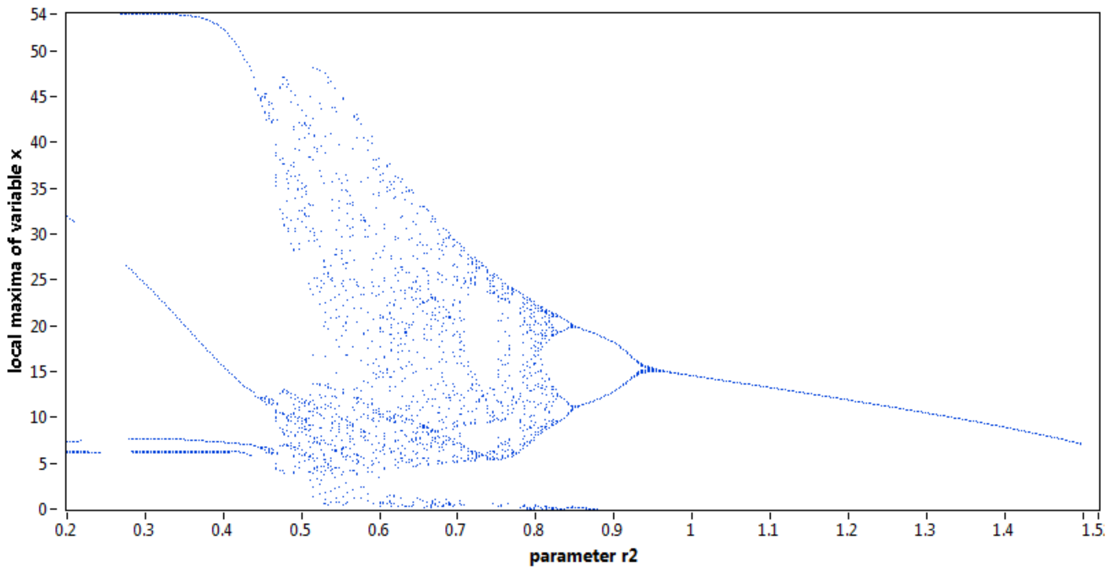

Figure 4. Bifurcation diagram for CSCLV model by parameter r_2. S = 0.5.

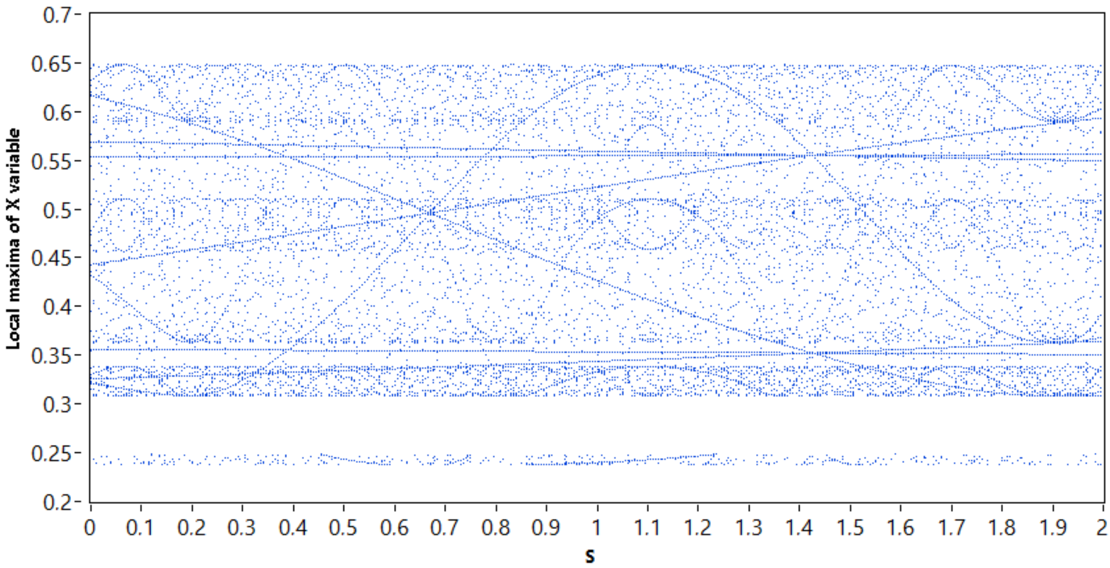

Figure 5. Bifurcation diagram for SED-based CSCLVM. The system is chaotic in the whole range of the symmetry parameter *S*.

However, the situation is different when we plot the diagram changing the symmetry coefficient (Figures 5 and 6). Here, the phase space can expand and shrink according to the given law, but the system preserves the chaotic regime of oscillations and general dynamics of the species' interaction.

The oscillating lines in the diagram (Figure 5) show the phase shift caused by symmetry change. Nevertheless, in the whole range of [0, 2], no non-chaotic "windows" or cascades of bifurcations can be observed. The system preserves chaotic behavior. One should note that here we study the chaotic mode of the system (8) just as the most representative example. It is known that numerical errors and digital noise can influence chaos, thus, chaotic modes of oscillations are most vulnerable to the side effects appearing in control and simulation techniques. In our simulations we followed the idea that phase-volume control should not affect the nature of oscillations mode, changing it only quantitatively as an external factor.

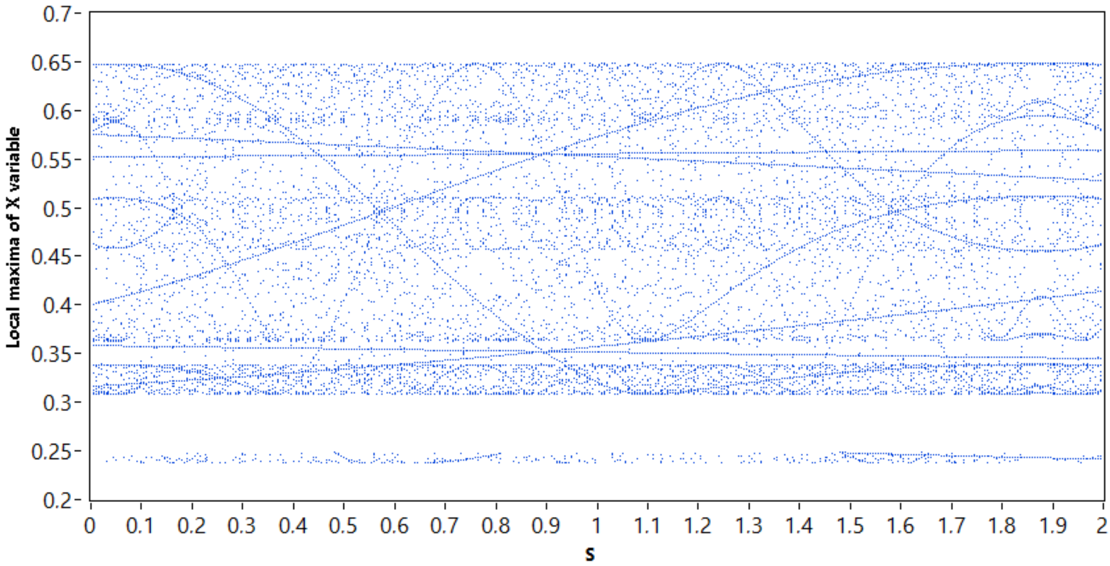

Figure 6. Bifurcation diagram for CD-based CSCLVM. The system is chaotic for all values of the symmetry parameter *S*.

Thus, the proposed model can change its phase-space properties without canceling the chaotic modes, which is extremely important in the simulation of various external factors. One does not need a wide variation of the symmetry parameter due to the fact that phase-space volume does not change instantly and requires relatively long transient time. Thus, we recommend changing the symmetry coefficient smoothly. Moreover, the speed of the transient process and quantitative parameters of the simulation can be adjusted by the applied control law as well.

4. Discussion

The original predator–prey Lotka–Volterra model provides limited applicability to the social processes' simulation [39]. The competitive Lotka–Volterra model seems more promising for this role, but also possesses several major drawbacks, including the inability to take into account multiple external factors of reproductive behavior [40]. To overcome this issue, we introduced the modified discrete competitive Lotka–Volterra model with controllable symmetry. We explicitly show through the phase-space and bifurcation analysis that the proposed technique allows controlling the phase-space volume of the discrete competitive Lotka–Volterra model without affecting the nonlinear nature of the system's behavior. The proposed model can be of interest in many practical cases when simulating the modern situation of partnership establishment. One of the main external factors here to simulate is the informational background, which influences the relationships and values systems. This defines the spectrum of possible reproductive behavior and its social consequences.

The intensiveness of interaction between system counterparts can increase when economic conditions get better and stabilize (people spend more time on recreation and socialize more). The second important factor is the informational, law and cultural field, which forms more or less tolerant relations to the various partnerships. Some factors, however, can be negative, such as an epidemic, natural disasters, war conflicts and political instability. The presented model does not allow applying multiple factors at once. In our further studies, we plan to overcome this limitation by creating composition models with several independent compartments, each possessing its own symmetry value, correlated with selected external factors.

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