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# Hausdorff Distance and Similarity Measures for Single-Valued Neutrosophic Sets with Application in Multi-Criteria Decision Making

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**Abstract:** Hausdorff distance is one of the important distance measures to study the degree of dissimilarity between two sets that had been used in various fields under fuzzy environments. Among those, the framework of single-valued neutrosophic sets (SVNSs) is the one that has more potential to explain uncertain, inconsistent and indeterminate information in a comprehensive way. And so, Hausdorff distance for SVNSs is important. Thus, we propose two novel schemes to calculate the Hausdorff distance and its corresponding similarity measures (SMs) for SVNSs. In doing so, we firstly develop the two forms of Hausdorff distance between SVNSs based on the definition of Hausdorff metric between two sets. We then use these new distance measures to construct several SMs for SVNSs. Some mathematical theorems regarding the proposed Hausdorff distances for SVNSs are also proven to strengthen its theoretical properties. In order to show the exact calculation behavior and distance measurement mechanism of our proposed methods in accordance with the decorum of Hausdorff metric, we utilize an intuitive numerical example that demonstrate the novelty and practicality of our proposed measures. Furthermore, we develop a multi-criteria decision making (MCDM) method under single-valued neutrosophic environment using the proposed SMs based on our defined Hausdorff distance measures, called as a single-valued neutrosophic MCDM (SVN-MCDM) method. In this connection, we employ our proposed SMs to compute the degree of similarity of each option with the ideal choice to identify the best alternative as well as to perform an overall ranking of the alternatives under study. We then apply our proposed SVN-MCDM scheme to solve two real world problems of MCDM under single-valued neutrosophic environment to show its effectiveness and application.

**Keywords:** fuzzy sets; neutrosophic sets (NSs); single-valued NSs; hausdorff metric; distance; similarity measure; single-valued neutrosophic decision matrix; multi-criteria decision making

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## 1. Introduction

The idea of fuzzy sets (FSs) was first proposed by Zadeh [1] in 1965 that had been applied in many fields to handle uncertainty arising due to vagueness and partial belongingness of an element in a set [2–4]. A fuzzy set (FS) is different from probability for representing uncertainty, where the membership degree of an element in the FS is defined as a value ranging between 0 and 1, and with the non-membership degree as a subtracting value from 1. Since then, various generalizations had been extended by researchers to handle imprecise, uncertain and incomplete information in a more comprehensive way. Among those extensions, Atanassov's intuitionistic fuzzy sets (IFSs) [5] had been found to be useful in coping with vagueness and uncertainty. In IFSs, both the membership degree and non-membership degree are defined between 0 and 1 that their sum should be belonging to the unit interval [0,1]. Furthermore, the degree of hesitancy can be obtained by subtracting the sum of membership and non-membership values from 1. These FSs and IFSs have been applied in various areas in the literature [6–9].

IFSs have been shown to be a powerful tool to handle the vagueness, but it is out of its scope to handle indeterminate information as well as inconsistent information, that commonly exists in belief system and real world problems. Realizing this, Samarandache [10] proposed the idea of neutrosophic sets (NSs) to present the indeterminate, incomplete and inconsistent information in a comprehensive way. NSs are a formal framework that generalizes classical sets, FSs and IFSs [11]. A neutrosophic set (NS) is characterized by truth membership, falsity membership, and indeterminacy membership, in which they are independent. In addition, these membership functions of truth, falsity and indeterminacy can be belonging to a subset of a real standard or non-standard unit interval. That is, the sum of these memberships of truth, falsity and indeterminacy need not to be contained in the unit interval [0,1]. Thus, the generalization idea of NS is in respect with the philosophical point of view, and so it is difficult to implement it in real problems of engineering and scientific applications. Therefore, Wang et al. [12] introduced the idea of single-valued NSs (SVNSs), a subclass of NSs, where each value in truth membership, falsity membership, and indeterminacy membership lies in the unit interval [0,1], and the sum of these three memberships should be between 0 and 1. According to its theoretical elegance and practical competency, it can be used to exemplify the information which is imprecise, incomplete and inconsistent, in a smarter way as well as to address real, scientific and engineering problems. Afterward, SVNSs had attracted many researchers to develop and devise various methods to handle the real world problems related to diverse fields with their unique requirements and field specifications [13–15].

The study of distance measures (DMs) and similarity measures (SMs) has been an important research topic for demonstrating the distinction and likeliness between various objects. Many researchers had developed DMs and SMs for fuzzy sets and its various extensions [16,17]. Grzegorzewski first [18] proposed distances of IFSs and interval-valued FSs based on Hausdroff metric [19], and then Yang and Hussian [20] gave distance and similarity measures of hesitant FSs based on Hausdroff metric. However, there is no any distance and similarity measure for SVNSs based on Hausdroff metric. We mention that, although Xu et al. [21] had proposed the so-called Hausdroff distance for single-valued neutrosophic numbers (SVNNs), it was used to determine the distance between two numbers (points), not between two sets. On the other hand, it did not exactly follow the calculation behavior and definition of Hausdroff metric between two sets. In this sense, there is no any Hausdroff distance for SVNSs in the literature. This motivates us for designing the Hausdroff distance for SVNSs and its corresponding SMs for SVNSs. Therefore, we propose the novel methods to calculate Hausdroff distance between two SVNSs with the true spirit of Hausdroff metric between two sets. We also develop some SMs based on our proposed Hausdorff distance to compute the degree of similarity between two SVNSs. In addition, we also utilize our proposed SMs to construct a multi-criteria decision making (MCDM) method to address the problems of MCDM under single-valued neutrosophic environment, called single-valued neutrosophic MCDM (SVN-MCDM) method. The major contributions of this study are as follows:

1. Proposing two novel forms of Hausdroff distance for SVNSs.
2. Developing some SMs for SVNSs using our newly defined Hausdorff distances.
3. Construction of an MCDM method under single-valued neutrosophic environment.
4. Identification/selection of best alternative among available options as well as the overall ranking of alternatives under study by using our proposed SVN-MCDM.
5. Application of our proposed SVN-MCDM scheme to solve two real problems of MCDM under SVN environment.

The rest of this paper is organized as follows. In Section 2, we first give the literature review. In Section 3, we recall some of the basic concepts and properties of NSs and SVNSs. We also give the definition of Hausdorff metric. We then give the existing so-called Hausdorff distance for SVNSs. Section 4 is devoted to present our proposed Hausdorff distance and its corresponding similarity measures between SVNSs based on Hausdorff metric. We further prove some properties and theorems related to the proposed Hausdorff distance for SVNSs to strengthen its theoretical properties. Furthermore, a numerical analysis of our proposed methods has been presented to demonstrate the practicality of our methods by using an intuitive numerical example. In Section 5, we demonstrate the feasibility and application of our proposed methods in multi-criteria group decision making with real problems. Section 6 is dedicated to elaborate the managerial insight and benefits of our research study. We finally give our conclusions in Section 7.

## 2. Literature Review

Multi-criteria decision making (MCDM) under neutrosophic environment and its various special cases are one of research domains, which had attracted many researchers to contribute their valuable ideas and best practices to solve real world problems and to extend the frontiers of knowledge and skills. Some of the precious works regarding theory, practices and applications of MCDM under various frameworks of neutrosophic domains can be furnished in the followings. In Bhaumik et al. [22], authors proposed a multi-objective scheme under single-valued neutrosophic environment (SVNE) with a linguistic design to solve the problem of MCDM regarding tourism management, in which the authors had implemented their proposed method to identify and to recommend the best hotel for accommodation among the alternatives under consideration. Another MCDM method had been considered to address a real life problem in the domain of transportation under neutrosophic environment in [23]. In De et al. [24], a robust intelligent decision making system had been developed under doubt fuzzy environment to solve an economic production quantity problem. Two methods had been formulated by [25] to solve a multi-objective green 4D fixed charge transportation problem. A MCDM method for two-layer supply chain had been proposed in Roy et al. [26] under doubt fuzzy framework. In Ye [27], author had introduced a MCDM method using weighted correlation coefficient under SVNE that was implemented to handle an MCDM issue and to select the best choice. In Das and Roy [28], authors developed a multi-objective decision making scheme to handle a transportation facility location problem under neutrosophic environment.

A MCDM under the single-valued neutrosophic environment (SVNE) had been presented using TOPSIS method by Biswas et al. [29], and a divergence measure with its based TOPSIS for MCDM under SVNE was proposed by Garg [30]. In Zhang and Wu [31], they developed a novel MCDM under SVNE with incomplete weight information that had been applied to evaluate the characteristics of some initially nominated global suppliers on the basis of desirable capabilities. Jana and Pal [32] coined a MCDM method using Dombi power aggregation operators under SVNE that had been applied in a MCDM problem to select the best road construction company among the available options. Xu et al. [33] proposed a MCDM method using TODIM approach and implemented it for selecting the best emerging technology enterprise (ETE) among the top five ETEs. In [34], Borah and Dutta constructed a MCDM scheme by using their proposed vector similarity measures (SMs) in which they applied them to solve few of McDM problems as well as for the selection of appropriate face mask among the alternatives available in the market under study to contribute in preventing the spread of COVID-19. A correlation coefficient (CC) with its based TOPSIS method was introduced by Zeng et al. [35] to address some MCDM problems that had also been employed to identify and select the best IT softer development enterprise. In [36], Ye developed a MCDM method using the proposed wieghted SMs for SVNSs with unknown criteria weights, and then utilized the MCDM method to identify the best investment choice among top five available options. We had mentioned that, Xu et al. [21] proposed the so-called Hausdroff distance for SVNNs, but it was used to determine the distance between two numbers (points), not between two sets, and so it did not exactly follow the original definition of Hausdroff metric between two sets. That is, no one had considered Hausdorff distance to construct the MCDM for SVNSs under SVNE. Therefore, we formulate a MCDM under SVNE using our proposed Hausdorff distance for SVNSs in this research study. We summarize a tabular representation of the research gap along with the connection between MCDM and Hausdourff distance as well as existing methods along with the contribution of the authors in the domain of MCDM under SVNE in Table 1.

**Table 1.** Different measures and contributions regarding MCDM and Hausdorff distance.

| | Authors | MCDM | Hausdorff Distance | Multiple Comparisons (for Nearest Point) | Measures | Application |
|---|---|---|---|---|---|---|
| 1 | Ye [27] | Yes | No | No | CC | MCDM |
| 2 | Ye [36] | Yes | No | No | SM | MCDM |
| 3 | Zang and Wu [31] | Yes | No | No | DM | TOPSIS |
| 4 | Biswas et al. [29] | Yes | No | No | SM | TOPSIS |
| 5 | Xu et al. [33] | Yes | No | No | DM | TODIM |
| 6 | Garg [30] | Yes | No | No | Div-M | TOPSIS |
| 7 | Zeng et al. [35] | Yes | No | No | CC | TOPSIS |
| 8 | Xu et al. [21] | Yes | So-called | No | SM | TOPSIS |
| 9 | Borah and Dutta [34] | Yes | No | No | Vector SMs | MCDM |
| 10 | Jana and Pal [32] | Yes | No | No | DPAO | MCDM |
| 11 | Our propose method | Yes | Yes | Yes | Hausdorff distance based SM | The proposed SVN-MCDM |

CC: correlation coefficient; SM: similarity measure; DM: distance measure; Div-M: divergence measure; DPAO: Dombi power averaging operator.

## 3. Preliminaries

In this section, we give a brief review of neutrosophic sets (NSs) and single-valued neutrosophic sets (SVNSs). We then give the existing so-called Hausdorff distance and similarity measures for NSs and SVNSs which are related to our research study.

**Definition 1 [11].** *Let $X$ be a space of objects with a basic element denoted by $x \in X$. A NS $A$ in $X$ is described by a truth-membership function $T_A(x)$, falsity-membership function $F_A(x)$ and indeterminacy-membership function $I_A(x)$. These membership functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$, i.e., $T_A(x) : X \to ]0^-, 1^+[$, $I_A(x) : X \to ]0^-, 1^+[$ and $F_A(x) : X \to ]0^-, 1^+[$, where $0^- = 0 - \varepsilon$ and $1^+ = 1 + \varepsilon$, while $\varepsilon$ is a number greater than 0. We mention that NSs have no any restriction on the sum of the three membership functions $T_A(x)$, $I_A(x)$ and $F_A(x)$, and so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.*

To cope with the challenges regarding the application of NSs to real problems in technical and scientific way, Wang et al. [12] introduced the idea of SVNSs as follows.

**Definition 2 [12].** *Let $X$ be the space of objects with basic elements denoted by $x \in X$ A SVNS $B$ in $X$ is characterized by a truth-membership function $T_B(x)$, falsity-membership function $F_B(x)$ and indeterminacy-membership function $I_B(x)$, where the three membership functions $T_B(x)$, $I_B(x)$ and $F_B(x)$ are all belonging to the interval $[0, 1]$ for each point $x$ in $X$, i.e., $T_B(x) \in [0, 1]$, $I_B(x) \in [0, 1]$ and $F_B(x) \in [0, 1]$ with a constraint $0 \leq T_B(x) + I_B(x) + F_B(x) \leq 3$.*

**Definition 3 [12].** *Let $A$ and $B$ are two SVNSs in $X$. Then, $A$ is said to be contained in $B$ (i.e., $A \subseteq B$) if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$ for all $x \in X$.*

**Definition 4 [12].** *Two SVNSs $A$ and $B$ are said to be equal (i.e., $A = B$) if and only if $A \subseteq B$ and $B \subseteq A$.*

For a given SVNS $A$ in $X$, a triplet $\langle T_A(x), I_A(x), F_A(x) \rangle$ for $x \in X$ is termed as a singled-valued neutrosophic number (SVNN), which is an element of the SVNS $A$ in $X$.

For simplicity, we can just write a SVNN $c = \langle T_c, I_c, F_c \rangle$ or $e = \langle T_e, I_e, F_e \rangle$ as an element of the SVNS $A$ in $X$.

**Definition 5 [36].** *For a SVNS $A$ in $X$ with its two SVNNs $c = \langle T_c, I_c, F_c \rangle$ and $e = \langle T_e, I_e, F_e \rangle$, the normalized hamming distance between $c$ and $e$ is defined as*

$$d(c,e) = \frac{1}{3}\{|T_c - T_e| + |I_c - I_e| + |F_c - F_e|\}$$

**Definition 6 [21].** *Let $c = \langle T_c, I_c, F_c \rangle$ and $e = \langle T_e, I_e, F_e \rangle$ be two SVNNs in the SVNS $A$ in $X$. The Hausdorff distance between these two SVNNs $c$ and $e$ is defined as*

$$d_{Xu}(c,e) = \max\{|T_c - T_e|, |I_c - I_e|, |F_c - F_e|\}$$

We mention that, Xu et al. [21] proposed the so-called Hausdroff distance $d_{Xu}(c,e)$ only for two SVNNs, not for two SVNSs in which they used $d_{Xu}(c,e)$ to measure the distance between two numbers (points), not between two sets. However, the original Hausdroff metric is used to define a distance between two sets. That is, there not yet has any Hausdroff distance for SVNSs in the literature. We next give a review of Hausdorff metric between two sets.

**Definition 7 (Hausdorff metric [19,37]).** *Let $Y$ and $Z$ be two non-empty compact, bounded and closed subsets in a matric space $S$ where $d(y,z)$ is a metric for $S$. Then, the directed Hausdorff forward and backward distances can be defined, respectively, as*

$$d_F(Y,Z) = \max_{y \in Y}\left\{\min_{z \in Z} d(y,z)\right\} \text{and} d_B(Y,Z) = \max_{z \in Z}\left\{\min_{y \in Y} d(y,z)\right\}.$$

*Then, the maximum $H(Y,Z)$ of these two direct distances with*

$$H(Y,Z) = Max\{d_F(Y,Z), d_B(Y,Z)\}$$

*is called Hausdorff metric between the two sets $Y$ and $Z$.*

It is noted that these direct distances are not symmetric, and so both of the Hausdorff direct distances are not equal in most cases, i.e., $d_F(Y,Z) \neq d_B(Y,Z)$, and so the maximum of these two direct distances is taken as Hausdorff metric. For example, suppose $\widehat{U} = [\widehat{u}_1, \widehat{u}_2]$ and $\widehat{V} = [\widehat{v}_1, \widehat{v}_2]$ are two intervals in a real space $\Re$. Then the Hausdorff distance between these two intervals is $H(\widehat{U}, \widehat{V}) = Max\left\{\left|\widehat{u}_1 - \widehat{v}_1\right|, \left|\widehat{u}_2 - \widehat{v}_2\right|\right\}$. It is obvious that the so-called Hausdorff distance $d_{Xu}(c,e)$ for SVNNs in Definition 6 by Xu et al. [21] is not a Hausdorff distance for SVNSs. Thus, we next propose a novel Hausdorff distance with its corresponding similarity measures for SVNSs completely based on Definition 7 of Hausdorff metric in next section.

## 4. Hausdorff Distance and Similarity Measures for SVNSs Based on Hausdorff Metric

In this section, we propose a novel scheme to calculate Hausdorff distance between two SVNSs with Definition 7 of Hausdorff metric between two sets. We then use our proposed Hausdorff distance between SVNSs to construct new similarity measures for SVNSs using some algebraic functions. In order to have more clear descriptions, we give a tabular representation of notations and variables used in this paper, as shown in Table 2.

**Table 2.** Notations and variables.

| Notations | Explanation |
|---|---|
| $\mathrm{d}(A,B)$ | Hamming distance between SVNSs $A$ and $B$ |
| $h_f(A,B)$ | Hausdorff forward direct distance between $A$ and $B$ |
| $h_b(A,B)$ | Hausdorff backward direct distance between $A$ and $B$ |
| $HD(A,B)$ | Hausdorff distance between $A$ and $B$ |
| $h_f^*(A,B)$ | Average based Hausdorff forward direct distance between $A$ and $B$ |
| $h_b^*(A,B)$ | Average based Hausdorff backward direct distance between $A$ and $B$ |
| $d^*(A,B)$ | Minimum distance between $A$ and $B$ |
| $HD^*(A,B)$ | Average based Hausdorff distance between $A$ and $B$ |
| $\psi$ | A monotonically decreasing function |
| $SM(A,B)$ | Similarity measure between $A$ and $B$ |
| $SM_L(A,B)$ | Similarity measure between $A$ and $B$ based on simple linear function |
| $SM_R(A,B)$ | Similarity measure between $A$ and $B$ based on rational function |
| $SM_E(A,B)$ | Similarity measure between $A$ and $B$ based on exponential function |
| $O_i$ | ith alternative |
| $q_j$ | $j$th criteria/attribute |
| $M_{ij}$ | The evaluation value of $i$th alternative on the basis of the $j$th criteria |
| $(M_{ij})_{k\times n}$ | Single-valued neutrosophic decision matrix (SVNDM) |
| $O^*$ | Ideal alternative (A theoretical standard for comparison) |
| ${T_j}^*$ | The maximum value of Truth membership degree for $j$th criteria among all of the alternatives |
| ${I_j}^*$ | The minimum value of indeterminacy degree for $j$th criteria among all of the alternatives |
| ${F_j}^*$ | The minimum value of falsity degree for $j$th criteria among all of the alternatives |
| $SM^*$ | Best alternative/option |
| $N$ | Set of nominees |
| $N^*$ | Ideal Nominee (A theoretical standard) |
| $B$ | Set of service brands |
| $B^*$ | Ideal service brand (A theoretical standard) |

### *4.1. Hausdorff Distance for SVNSs*

In this subsection, we propose new Hausdorff distances for SVNSs.

**Definition 8.** *Let $A$ and $B$ be two SVNSs on a finite universal set $X = \{x_1, x_2, x_3, \ldots, x_n\}$. Then, we define the Hausdorff forward direct distance between $A$* and *$B$ as follows:*

$$h_f(A,B) = \max_{i=1,2,\ldots,n}\left\{\min_{j=1,2,\ldots,n} \mathrm{d}(A_i, B_j)\right\} \tag{1}$$

*where $\mathrm{d}(A_i, B_j)$ is the hamming distance between the ith point of $A$ in $X$ and the jth point of $B$ in $X$ with $A_i = \langle T_A(x_i), I_A(x_i), F_A(x_i)\rangle$, $B_j = \langle T_B(x_j), I_B(x_j), F_B(x_j)\rangle$, and $d(A_i, B_j) = \frac{1}{3}\left\{\left|T_A(x_i) - T_B(x_j)\right| + \left|I_A(x_i) - I_B(x_j)\right| + \left|F_A(x_i) - F_B(x_j)\right|\right\}$. Similarly, the Hausdorff backward direct distance between $A$ and $B$ is defined as:*

$$h_b(A,B) = \max_{j=1,2,\ldots,n}\left\{\min_{i=1,2,\ldots,n} \mathrm{d}(B_j, A_i)\right\} \tag{2}$$

*where* $\mathrm{d}(B_j, A_i)$ *is the hamming distance between the jth point of B in X and the ith point of A in X with* $d(B_j, A_i) = \frac{1}{3}\{|T_B(x_j) - T_A(x_i)| + |I_B(x_j) - I_A(x_i)| + |F_B(x_j) - F_A(x_i)|\}$ *Hence, our proposed Hausdorff distance is mathematically expressed as:*

$$HD(A, B) = \max\left\{h_f(A, B), h_b(A, B)\right\} \tag{3}$$

**Theorem 1.** *Let* $X = \{x_1, \ldots, x_n\}$ *be a finite universe of discourses. For any three SVNSs* $A$, $B$ *and C in X , the proposed Hausdorff distance* $HD(A, B)$ *satisfies the following properties (a)–(d).*

*(a)* $0 \leq HD(A, B) \leq 1$;
*(b)* $HD(A, B) = 0$ iff $\mathrm{A} = \mathrm{B}$;
*(c)* $HD(A, B) = HD(B, A)$;
*(d)* $If\ A \subseteq B \subseteq C$, *then* $HD(A, B) \leq HD(A, C)$ *and* $HD(B, C) \leq HD(A, C)$.

**Proof.** First, the three properties (a) to (c) are straightly forward and obvious, and so we only prove the property (d). Since $A \subseteq B \subseteq C$, so $T_A(x_i) \leq T_B(x_i) \leq T_C(x_i)$, $I_A(x_i) \geq I_B(x_i) \geq I_C(x_i)$ and $F_A(x_i) \geq F_B(x_i) \geq F_C(x_i)$. Hence we have $|T_A(x_i) - T_B(x_i)| \leq |T_A(x_i) - T_C(x_i)|$, $|I_A(x_i) - I_B(x_i)| \leq |I_A(x_i) - I_C(x_i)|$, and $|F_A(x_i) - F_B(x_i)| \leq |F_A(x_i) - F_C(x_i)|$. Similarly, $|T_B(x_i) - T_C(x_i)| \leq |T_A(x_i) - T_C(x_i)|$, $|I_B(x_i) - I_C(x_i)| \leq |I_A(x_i) - I_C(x_i)|$ and $|F_B(x_i) - F_C(x_i)| \leq |F_A(x_i) - F_C(x_i)|$ Let $l$, $m$ and $n$ be three non-negative numbers with $l \leq m \leq n$. Then, according to the characteristics of a distance measure $D$, we know that $D(l, m) \leq D(l, n)$. So we have $\min_j\{\mathrm{d}(A_i, B_j)\} \leq \min_j\{\mathrm{d}(A_i, C_j)\}\ \forall i$. After applying maximum over $i$ on both sides of the inequality, we can have $\max_i\left\{\min_j\{\mathrm{d}(A_i, B_j)\right\}\right\} \leq \max_i\left\{\min_j\{\mathrm{d}(A_i, C_j)\right\}\right\}$. Using our proposed hausdorff forward direct distance, we can say $h_f(A, B) \leq h_f(A, C)$. In a similar way, we take maximum over $j$ on both sides of the inequality $\min_i\{\mathrm{d}(B_j, A_i) \leq \min_i\{\mathrm{d}(C_j, A_i)\right\}$. We find an expression for backward hausdorff distance as $\max_j\left\{\min_i\{\mathrm{d}(B_j, A_i)\right\}\right\} \leq \max_j\left\{\min_i\{\mathrm{d}(C_j, A_i)\right\}\right\}$. Thus, we can write $h_b(A, B) \leq h_b(A, C)$. After combining both hausdorff direct distances and taking their maxima on both sides, the resulting expression is of the form $\max\left\{h_f(A, B), h_b(A, B)\right\} \leq \max\left\{h_f(A, C), h_b(A, C)\right\}$. Thus, $HD(A, B) \leq HD(A, C)$. In a same way, we can also demonstrate $HD(B, C) \leq HD(A, C)$. This completes the proof of the property (d). □

To calculate the distance of overall elements of a set from the nearest point of another set, we next propose an average-based Hausdorff distance for SVNSs as follows.

**Definition 9.** *Let A and B be two SVNSs on a finite universal set* $X = \{x_1, x_2, x_3, \ldots, x_n\}$. *We define an average-based Hausdorff forward and backward direct distances as follows. The average-based Hausdorff forward direct distance between the two SVNSs A and B is defined as*

$$h_f^*(A, B) = \frac{1}{n}\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{d}^*(A_i, B_j) \tag{4}$$

*where* $d^*(A_i, B_j) = \min_{j=1,2,\ldots,n} \mathrm{d}(A_i, B_j)$ *for* $i = 1, 2, \ldots, n$, *and* $\mathrm{d}(A_i, B_j)$ *is defined the same as in Definition 8. Similarly, we also define the average-based Hausdorff backward direct distance as*

$$h_b^*(A, B) = \frac{1}{n}\sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{d}^*(B_j, A_i) \tag{5}$$

*where* $d^*(B_j, A_i) = \min_{i=1,2,\ldots,n} \mathrm{d}(B_j, A_i)$ *for* $j = 1, 2, \ldots, n$. *Thus, the average-based Hausdorff distance between two SVNSs A and B is defined as*

$$HD^*(A,B) = \max\left\{h_f^*(A,B), h_b^*(A,B)\right\} \tag{6}$$

**Theorem 2.** *Let* $X = \{x_1, x_2, x_3, \ldots, x_n\}$ *be a universe of discourses and let E, F and G be three SVNSs. Then, the proposed average-based distance measure* $HD^*(E,F)$ *satisfies the following properties (a)–(d).*

*(a)* $0 \le HD^*(E,F) \le 1$;
*(b)* $HD^*(E,F) = 0$ iff $\mathrm{E} = \mathrm{F}$;
*(c)* $HD^*(E,F) = HD^*(F,E)$;
*(d)* $If\ \mathrm{E} \subseteq F \subseteq G$, *then* $HD^*(E,F) \le HD^*(E,G)$ *and* $HD^*(F,G) \le HD^*(E,G)$;

**Proof.** Proof of Theorem 2 is the same as that of Theorem 1. □

### *4.2. Similarity Measures for SVNSs*

Using the duality principle of distance and similarity, we develop some similarity measures (SMs) for SVNSs by using our proposed Hausdorff distances for SVNSs. Let us consider a monotonically decreasing function $\psi$ which can be used to establish a foundation for developing a linkage between the proposed distance and similarity measures. According to Theorems 1 and 2, we have $0 \le HD(A,B) \le 1$ and $0 \le HD^*(A,B) \le 1$. Hence, we formulate an analogical characteristics for its corresponding similarity measure as $\psi(1) \le \psi(HD(A,B)) \le \psi(0)$, and then we have $0 \le \psi(HD(A,B)) - \psi(1) \le \psi(0) - \psi(1)$, and so $0 \le \frac{\psi(HD(A,B)) - \psi(1)}{\psi(0) - \psi(1)} \le 1$. Similarly, based on the average-based Hausdorff distance for SVNSs, we have $\psi(1) \le \psi(HD^*(A,B)) \le \psi(0)$, and $0 \le \psi(HD^*(A,B)) - \psi(1) \le \psi(0) - \psi(1)$. Thus, we obatin $0 \le \frac{\psi(HD^*(A,B)) - \psi(1)}{\psi(0) - \psi(1)} \le 1$.

This is the non-negativity property of the defined SMs based on our proposed Hausdorff distance for SVNSs. Although both of $\frac{\psi(HD(A,B)) - \psi(1)}{\psi(0) - \psi(1)}$ and $\frac{\psi(HD^*(A,B)) - \psi(1)}{\psi(0) - \psi(1)}$ are reliable SMs based on our proposed Hausdorff distances for SVNSs, we only define the SMs using our proposed average-based Hausdorff distance for SVNSs in the following. That can be formally defined as follows.

**Definition 10.** *For two SVNSs A and B defined in a finite universal set* $X = \{x_1, x_2, x_3, \ldots, x_n\}$, *we define a similarity measure* $SM(A,B)$ *on the basis of the average-based Hausdorff distance* $HD^*(A,B)$ *using a monotonically decreasing function* $\psi$ *as follows:*

$$SM(A,B) = \frac{\psi(HD^*(A,B)) - \psi(1)}{\psi(0) - \psi(1)} \tag{7}$$

*According to Definition 10, we can construct varioussimilarity measures for SVNSs through a usefully monotonically decreasing function* $\psi$. *For example, we use the simple linear function* $\psi(x) = 1 - x$, *and then we give a new SM between two SVNSs A and B as*

$$SM_L(A,B) = 1 - HD^*(A,B) \tag{8}$$

*In a similar way, we can also use a simple rational function* $\psi(x) = \frac{1}{1+x}$ *to construct the following SM for two SVNSs A and B as*

$$SM_R(A,B) = \frac{1 - HD^*(A,B)}{1 + HD^*(A,B)} \tag{9}$$

*Moreover, we consider another useful and well-known exponential function $\psi(x) = e^{-x}$, and then we formulate another SM in an exponential form as*

$$SM_E(A, B) = \frac{e^{-HD^*(A,B)} - e^{-1}}{1 - e^{-1}} \tag{10}$$

### *4.3. Numerical Analysis and Illustration*

In this subsection, we give a synthetic example generated on the basis of intuititive configuration which is coherent and convincing to demonstrate the calculation of the largest distance between two SVNSs from the nearest points of each other as well as the measurement mechanism of our proposed methods to compute the Hausdorff distances between those two SVNSs in accordance with the decorum of genuine notion of Hausdorff metric presented by [26,30].

**Example 1.** *Let $A$ and $B$ be two SVNSs defined over $X = \{x_1, x_2, x_3, x_4\}$ with*

$$A = \{\langle x_1, 0.8, 0.4, 0.5\rangle, \langle x_2, 0.7, 0.3, 0.4\rangle, \langle x_3, 0.6, 0.2, 0.3\rangle, \langle x_4, 0.5, 0.1, 0.1\rangle\}$$

$$B = \{\langle x_1, 0.8, 0.5, 0.3\rangle, \langle x_2, 0.9, 0.7, 0.6\rangle, \langle x_3, 0.7, 0.4, 0.2\rangle, \langle x_4, 0.5, 0.1, 0.1\rangle\}.$$

*Intuitively, it can be seen that the 4th point of both SVNSs A and B in X are the same. So, the 4th point of the SVNS B in X is the nearest point to the SVNS A in X and vice versa. Hence, we would calculate the distances from the 4th point of the SVNS B in X and to identify the farthest distance of the SVNS A from it. Therefore, we present a graphical representation of these distances using Hamming distance, as shown in Figure 1.*

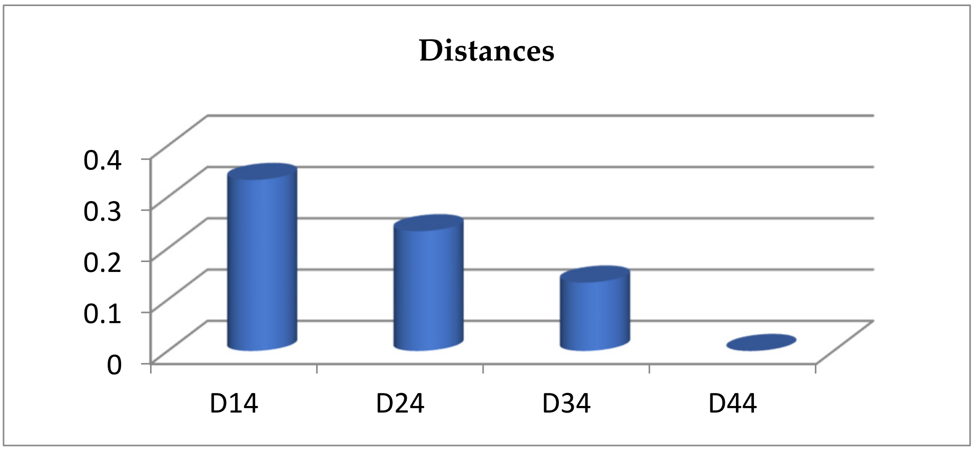


**Figure 1.** Distances from the 4th point of the SVNS $B$.

From Figure 1, it can be seen that the 1st point of the SVNS $A$ in $X$ is has a largest distance from the 4th point of the SVNS $B$ in $X$. So, this distance is the maximum distance of the SVNS $A$ in $X$ from the nearest point of the SVNS $B$ in $X$ with the value of 0.333. Likewise, it can be also seen that the 4th point of the SVNSs $A$ in $X$ is the nearest point to the SVNS $B$ in $X$. Thus, we would compute the distances from the 4th point of the SVNS $A$ in $X$ and to spot the farthest distance of the SVNS $B$ from it. Hence, the graphical representation of these distances using Hamming distance is shown in Figure 2.

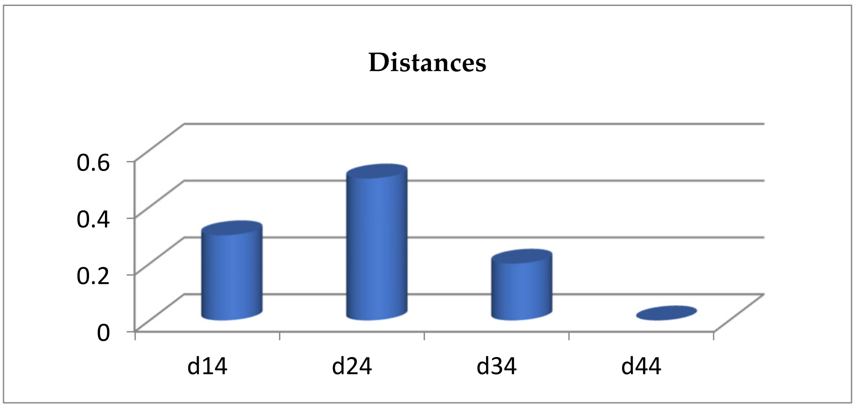


**Figure 2.** Distances from the 4th point of the SVNS $A$.

It can be seen from Figure 2 that the 2nd point of the SVNS $B$ in $X$ is at a maximum distance from the 4th point of the SVNS $A$ in $X$. And so, this is the largest distance of the SVNS $B$ in $X$ from the nearest point of the SVNS $A$ in $X$ with the value of 0.5. Thus, the maximum one of these two farthest distances would be the largest distance between the SVNSs $A$ and $B$ in $X$ from the nearest points of each other. That is, $\max\{0.333, 0.5\} = 0.5$. This can be graphically expressed in Figure 3.

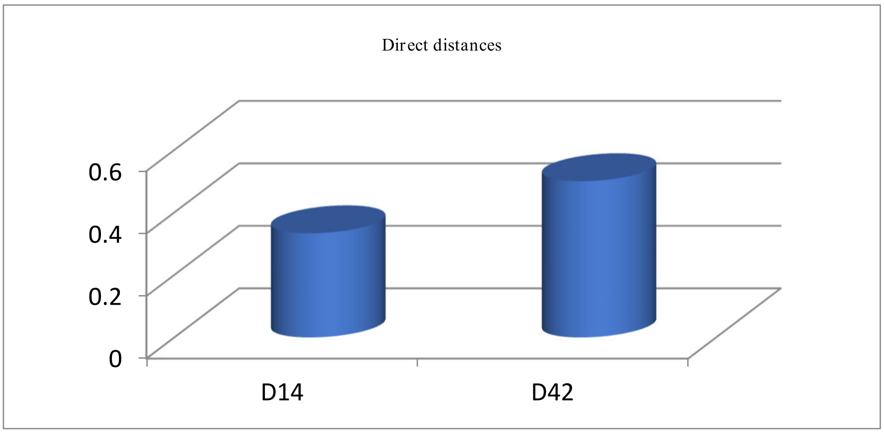


**Figure 3.** Forward and backward mximum distances.

From Figure 3, it can be seen that the distance of the SVNS $B$ in $X$ from the 4th point of the SVNS $A$ in $X$ is the largest distance. After implementing our proposed Hausdorff distances, we have the following results. The Hausdorff forward and backward direct distances between two SVNSs $A$ and $B$ in $X$ are $h_f(A, B) = 0.333$ and $h_b(A, B) = 0.50$, respectively. Thus, the Hausdorff distances between the SVNSs $A$ and $B$ in $X$ is $HD(A, B) = \max\left\langle h_f(A, B), h_b(A, B)\right\rangle = 0.50$. By comparing the intuition based on these calculation results and those of our proposed Hausdorff distance, we can say that our proposed methods are well suited, logically reasonable and reliable to calculate Hausdorff distance between two SVNSs as per the spirit of Hausdorff metrics between two sets.

## 5. Application of the Proposed Methods in Multi-Criteria Decision Making

In this section, we demonstrate the application and practicality of our proposed methods in multi-criteria decision making (MCDM) under the single-valued neutrosophic environment.We initially present the theoretical concept of implementing our proposed methods to solve MCDM problems. Suppose we have a set of $k$ options/alternatives $O = \{O_1, O_2, O_3, \ldots, O_k\}$ to be evaluated by decision makers with respect to a set of $n$ quality criterion $Q = \{q_1, q_2, q_3, \ldots, q_n\}$. The evaluation value of the $i$th alternative $O_i$ on the basis of the $j$th criteria $q_j$ can be represented mathematically as $M_{ij} = \langle q_j, T_{O_i}(q_j), I_{O_i}(q_j), F_{O_i}(q_j) \rangle$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n$, which is a SVNN. Hence, we can represent the single-valued neutrosophic decision matrix (SVNDM) in a matrix notation as $M = (M_{ij})_{k\times n}$. That is, the value of $M_{ij}$ is used to calculate the alternatives $O_i$ on the basis of the quality criterion $q_j$ in the set $Q$.

It needs a theoretical standard to compare with, and to be in a position to rank the set of alternatives based on their strengths/characteristics and consequently to recognize the best alternative. In order to construct the theoretical standard as an ideal alternative, we extended the idea of positive ideal solution for Pythagorean fuzzy TOPSIS used by Hussian and Yang [38] to be the notion of ideal single-valued neutrosophic value (SVNV) used by Ye [27] for the construction of single valued neutrosophic MCDM method based on correlation coefficient, which can be presented as

$$O^* = \{\langle q_j, T_j{}^*, I_j{}^*, F_j{}^* \rangle / q_j \in Q\},\ j = 1, 2, \ldots, n \tag{11}$$

where $T_j{}^* = \max_{i=1,2,\ldots k} \langle T_i(q_j) \rangle$, $I_j{}^* = \min_{i=1,2,\ldots k} \langle I_i(q_j) \rangle$, and $F_j{}^* = \min_{i=1,2,\ldots k} \langle F_i(q_j) \rangle$ for $j = 1, 2, \ldots, n$.

We then utilize our proposed methods to compute the similarity between ideal alternative and all of the options under consideration. In this connection, an alternative is considered to be the best option which has a highest value of similarity with ideal alternative by using the principle of maximum similarity. That can be mathematically expressed as

$$SM^* = \max_{i=1,2,\ldots k} \langle SM(O^*, O_i) \rangle \tag{12}$$

That is, the $i$th alternative is the best option to choose, as it has the maximum similarity with the ideal alternative, as compared to the other options (symbolically).

We next implement our proposed methods to solve two MCDM problems of a private sector higher secondary school to show its practicality, and have been elaborated in the following.

**Example 2.** *It is a well known fact that the most precious feelings is the one that we truly matters, we meaningfully add value to the entire system and we are recognizing for that. In teaching learning, a positive feedback and recognition of the student's efforts is a great stimulus to inspire and motivate the students for their continued progress and a good conduct. Every institution encourages its students to be more productive in their learning through awards and rewards to develop a feeling of pride and sense of achievement. Student of the year award is one of the most worthy awards at a private sector higher secondary school to recognize the efforts of its students, who shows an outstanding performance in all of the three major domains (Curricular, ex-curricular and behavior/conduct). A committee of its field specialists, academicians and management had formulated the following set of criteria to evaluate the performance of the deserving students and to nominate the most deserving student for this outstanding award. (i) $y_1$: Annual result score, (ii) $y_2$: Participation in class, research projects and assignments, (iii) $y_3$: Obedience, loyalty and community service, (iv) $y_4$: Conduct, punctuality and attendance, (v) $y_5$: Club performance and leadership skills, and (vi) $y_6$: Relation with peers, admin staff and teachers. After a thorough evaluation of the students, group of top three outstanding performers/students have been nominated and the characteristics of these nominees $N = \{N_1, N_2, N_3\}$ have been expressed with the help of SVNNs, keeping in view the above designed set of criterion; $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. The summarized form of their evaluation values have been recorded and presented in the following single-valued neutrosophic decision matrix (SVNDM) of Table 3, in which all of the criterions are considered to be equally weighted.*

**Table 3.** Single-valued neutrosophic decision matrix (SVNDM).

| Students | Criterion (Y) | | | | | |
|---|---|---|---|---|---|---|
| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |
| $N_1$ | (0.93, 0, 0.07) | (0.9, 0.2, 0.3) | (0.88, 0.3, 0.1) | (0.83, 0.2, 0.4) | (0.9, 0.3, 0.1) | (0.87, 0.22, 0.34) |
| $N_2$ | (0.92, 0, 0.08) | (0.8, 0.3, 0.4) | (0.9, 0.3, 0.4) | (0.8, 0.35, 0.2) | (0.84, 0.4, 0.3) | (0.86, 0.3, 0.4) |
| $N_3$ | (0.91, 0, 0.09) | (0.85, 0.4, 0.2) | (0.8, 0.4, 0.2) | (0.82, 0.3, 0.3) | (0.9, 0.3, 0.4) | (0.88, 0.4, 0.28) |

Firstly, we determine the ideal alternative with the help of Equation (11) as

$$N^* = \{\langle y_1, 0.93, 0, 0.7\rangle, \langle y_2, 0.9, 0.2, 0.2\rangle, \langle y_3, 0.9, 0.3, 0.1\rangle, \langle y_4, 0.83, 0.2, 0.2\rangle, \langle y_5, 0.9, 0.3, 0.1\rangle, \langle y_6, 0.88, 0.22, 0.28\rangle\}$$

In order to identify the best candidate among these top three nominees, we compute their similarity with the ideal alternative, using our proposed similarity measures. Furthermore, keeping in view the principle of maximum similarity, we choose the student having highest value of similarity for the award of best student. After applying our proposed methods, the results of similarity measure(s) between ideal alternative and the top three nominees have been furnished in the following Table 4.

**Table 4.** Similarity measures between ideal alternative $N^*$ and nominees $N_i (i = 1, 2, 3)$.

| SMs | $N_1$ | $N_2$ | $N_3$ |
|---|---|---|---|
| $SM_L(N_i, N*)$ | **0.933889** | 0.875977 | 0.898796 |
| $SM_R(N_i, N*)$ | **0.903611** | 0.82417 | 0.854633 |
| $SM_E(N_i, N*)$ | **0.909167** | 0.833461 | 0.862637 |

From Table 4, it is clear that the student $N_1$ has the highest value of similarity with the ideal alternative as compared to the other two candidates. Hence we can say that the student $N_1$ can be chosen as the most excellent option for best student award. The overall rankings of the nominees as per their value of similarity with ideal alternative can be presented as $N_1 \succ N_3 \succ N_2$.

**Example 3.** *A safe and secure campus environment is very important for all of the family members of the institution including students, faculty and staff to have a healthy learning and to deliver their best. Every institution is putting efforts to make their campuses safer place for the students to enable them to concentrate on learning the knowledge, skills and attitude needed for a meaningfully successful educational and professional career. Keeping in view the importance of campus safety and security, the top management of the private sector higher secondary school hired a team of field specialists to constitute a committee including representatives from its administration and technical staff to evaluate the available options of CCTV camera brands. With mutual consensus, the members of committee had devised the following set of equally weighted criterion to assess the technicalities and quality of the available alternatives under consideration and to reach at a wise decision to purchase the best brand among them to strengthen the security system of the school. (i) C1:image sensor and motion detection, (ii) C2: Night vision and long distance real time transmission, (iii) C3: Wide dynamic range and quality of video, (iv) C4: Lens and weather resistance, (v) C5: On screen display and cloud storage, and (vi) C6: Power, auto gain control and outside control. After a comprehensive assessment of the received quotations/services, the committee had selected top four brands for further evaluation and to select the best alternative. The unanimous judgment/opinion of the committee based on the designed criterion and characteristics information of the top four CCTV cameras have been articulated in the form of SVNNs. These four alternatives (Service Brands) and the set of criterion can be presented in the form of sets respectively as* $B = \{B_1, B_2, B_3, B_4\}$ *and* $C = \{C_1, C_2, C_3, C_4, C_5, C_6\}$*. The overall summary of their evaluation and characteristic information have been documented in the form of single-valued neutrosophic decision matrix (SVNDM) and demonstrated in the following Table 5, where* $B^*$ *in the Table 5 is the ideal alternative, computed using Equation (11). We apply our similarity measures to calculate the similarity of ideal alternative with all of the four options of CCTV cameras to choose the best one. We select the CCTV camera that has a highest similarity with the ideal alternative, using principle of maximum similarity.*

**Table 5.** Single-valued neutrosophic decision matrix (SVNDM).

| Services Brands | Criterion (C) | | | | | |
|---|---|---|---|---|---|---|
| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ |
| $B_1$ | (0.85, 0.24, 0.2) | (0.8, 0.28, 0.2) | (0.84, 0.4, 0.3) | (0.8, 0.2, 0.4) | (0.82, 0.2, 0.4) | (0.8, 0.2, 0.3) |
| $B_2$ | (0.8, 0.3, 0.2) | (0.78, 0.1, 0.3) | (0.82, 0.3, 0.15) | (0.7, 0.3, 0.4) | (0.8, 0.3, 0.1) | (0.82, 0.24, 0.4) |
| $B_3$ | (0.9, 0.1, 0.2) | (0.7, 0.3, 0.4) | (0.8, 0.2, 0.2) | (0.75, 0.2, 0.3) | (0.8, 0.4, 0.2) | (0.78, 0.3, 0.2) |
| $B_4$ | (0.88, 0.2, 0.22) | (0.84, 0.18, 0.28) | (0.9, 0.2, 0.3) | (0.8, 0.3, 0.2) | (0.85, 0.2, 0.2) | (0.9, 0.3, 0.2) |
| $B^*$ | (0.89, 0.1, 0.2) | (0.84, 0.1, 0.2) | (0.9, 0.2, 0.15) | (0.8, 0.2, 0.2) | (0.85, 0.2, 0.1) | (0.9, 0.2, 0.2) |

In Table 6, the results of similarity values of four alternatives with ideal option using our proposed similarity measures are presented.

**Table 6.** Similarity values of $B^*$ and $B_i(i = 1, 2, 3, 4)$.

| SMs | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| $SM_L(B_i, B*)$ | 0.917222 | 0.873889 | 0.896111 | **0.932778** |
| $SM_R(B_i, B*)$ | 0.847101 | 0.776024 | 0.811777 | **0.874024** |
| $SM_E(B_i, B*)$ | 0.874321 | 0.812562 | 0.843899 | **0.897152** |

Table 6 shows that the alternative $B_4$ has the highest value of similarity with the ideal option/alternative among all of the four available alternatives. It is therefore, using the principle of maximum similarity, $B_4$ can be chosen as the best CCTV camera among the options under consideration for installation in the aforementioned private sector higher secondary school to fortify its security system. In addition, the overall alternatives can be ranked on the basis their similarity values with ideal alternative as $B_4 \succ B_1 \succ B_3 \succ B_2$. So these real examples and applications can justify the practicality and suitability of our proposed method to deal the MCDM problems in a simple and novel way.

## 6. Managerial Insights and Advantages

This section presents some of the managerial insights and advantages of our proposed research study. In this paper, we demonstrate the design and structure of our proposed SVN-MCDM method in detail to reinforce the understanding and theoretical foundation of the managerial personnel and decision makers. We have discussed two real life MCDM problems including the selection of best CCTV camera among the top four brands available in the market to strengthen the campus safety of a private sector higher secondary school and also the selection process of the best student of the year for award with detailed critical to quality attributes. We further implement our SVN-MCDM method to solve these two real life problems of MCDM under SVNE with step-wise explanations of the scheme as well as the illustration of characteristics of the alternatives which are very important to comprehend and to reach at a better decision. On the basis of this detailed demonstration and articulation of the implementation procedure of our proposed SVN-MCDM method in both of the MCDM problems, the managerial personnel and decision makers can easily accomplish a trustworthy and unbiased evaluation of the alternatives and identify a best choice among the alternatives under consideration. Furthermore, in our proposed Hausdorff distance method, there are multiple comparisons between the elements of two SVNSs to identify the nearest point/element of one set to another, and vice versa, for covering all of the aspects of distinction and eventually similarity prospects in its corresponding SMs. In addition, SVNSs are more powerful tools to express vague, imprecise and indeterminate information in a very good way. So it provides more space and flexibility for decision makers to express their judgment and evaluation values about the alternatives on the basis of some desirable features. Thus, the integration of Hausdorff distance and SVNSs can better provide a flexible room for decision makers and facilitate the MCDM process in a more healthy way. Our proposed SVN-MCDM method is very simple and easy to execute with logically strong grounds and appealing mathematical foundations. So the managers can easily handle any situation during the evaluation process of the available options under consideration in the set of desired quality criterion with good understanding to execute MCDM process and to choose the deserving best alternative. A graphical representation of the SVN-MCDM process is furnished in Figure 4.

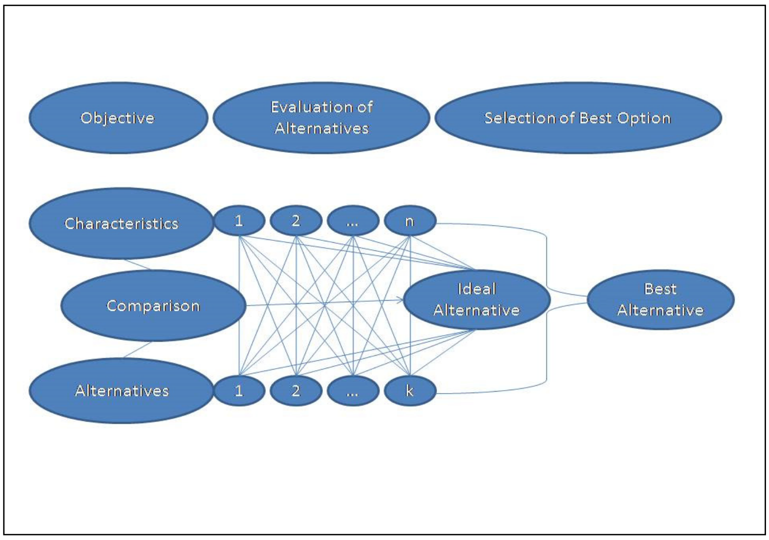


**Figure 4.** Graphical representation of SVN-MCDM process.

## 7. Conclusions

SVNS is a powerful tool to represent uncertain, imprecise, incomplete, and inconsistent information in a very comprehensive way to solve the real world problems of the engineering and scientific domains. In this paper, we proposed two novel Hausdorff distances for SVNSs which were based on the original definition of Hausdorff metric. We then developed some similarity measures (SMs) using our proposed Hausdorff distance for SVNSs. More theoretical properties of the proposed Hausdorff distance and SMs for SVNSs were presented. We also use an example to demonstrate that our proposed Hausdorff distance for SVNSs is logically reasonable and suitability as per the legitimate notion and idea of Hausdorff distance between two sets. Furthermore, we constructed a single-valued neutrosophic MCDM (SVN-MCDM) method with the help of our proposed SMs based on the proposed Hausdorff distances for SVNSs. In order to demonstrate its practicality and effectiveness, we apply our SVN-MCDM to solve two real world MCDM problems under single-valued neutrosophic environment to identify and select the best alternative as well as to rank overall alternatives. The computational results showed that our proposed method is feasible, valid and has a great potential in dealing real world MCDM problems. In our future works, we will extend our proposed SMs and Hausdorff distance for SVNSs to the complex neutrosophic systems and bi-polar neutrosophic environments to address real MCDM problems in various fields, such as medical diagnosis, industrial engineering, renewable energy management and risk management.

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