

Article

Cosmological Linear Perturbations in the Models of Dark Energy and Modified Gravity

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Abstract: The quasi-static solutions of the matter density perturbation in various dark energy models and modified gravity models have been investigated in numerous papers. However, the oscillating solutions in those models have not been investigated enough so far. In this paper, we review the behavior of the oscillating solutions, which have a possibility to unveil the difference between the models of the late-time accelerated expansion of the Universe, by using appropriate approximations.

Keywords: dark energy; modified gravity; cosmological perturbation

1. Introduction

It is clarified by the observations of Type Ia supernovae in 1990s [1,2] that the current universe is acceleratedly expanding if the Universe is almost homogeneous. We need to introduce some energy which have negative pressure to explain the accelerated expansion when we utilize the Friedmann equations which describe dynamics of the isotropic homogeneous universe. The energy introduced in this way is called dark energy. There are candidates of dark energy, e.g., introducing the cosmological constant into the Friedmann equations, assuming the existence of the classical scalar field spreading over the whole universe, and so on. Whereas, there are modified gravity theories which can cause the accelerated expansion of the Universe not by introducing dark energy but by modifying the geometry of space-time or the gravitational constant. It is known that the Λ Cold Dark Matter (Λ CDM) model, where Λ means cosmological constant, is almost consistent with the observations of cosmic microwave background radiation, baryon acoustic oscillation, and type Ia supernovae. The Λ CDM model is regarded as the standard model of cosmology because it is simple besides is consistent with the observations.

However, the other models of dark energy and modified gravity can realize an almost same expansion history of the Universe compared to that of the Λ CDM model. Therefore, we cannot determine the true model which describes the real universe only from the growth history of the Universe. In this paper, it is mentioned whether or not differences between the models are always appeared by considering the evolution of the matter density perturbation as a perturbation from the background space-time of the Universe. The Λ CDM model, k -essence model [3–5] and $F(R)$ gravity model [6–10] will be considered as the typical models of dark energy and modified gravity.

The cosmological perturbation theory is often used under the sub-horizon approximation, which consists of the two approximations in the small scale $a^2/k \ll 1/H^2$ and in the Hubble scale evolution $1/dt \sim H$, so that the perturbation should be consistent with the Newton gravity. However, the sub-horizon approximation is merely an approximation and is not always correct. In the following section, we will see what kinds of behaviors of the solutions are appeared when we do not adopt the quasi-static approximation $1/dt \sim H$. We use units of $k_B = c = \hbar = 1$ and denote the gravitational constant $8\pi G$ by κ^2 in the following.

2. Evolutions of the Matter Density Perturbation in Each Model of Dark Energy and Modified Gravity

The evolution equation of the matter density perturbation in the Λ CDM model is often expressed as follows:

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0 \quad (1)$$

where $\delta \equiv \delta\rho/\rho$, ρ is the energy density of the matter, H is the Hubble rate defined by $\dot{a}(t)/a(t)$, and Ω_m is the matter fraction of the energy density of the Universe. Equation (1) is derived by using the sub-horizon approximation, whereas, if we do not use the sub-horizon approximation, then we obtain [11]

$$\begin{aligned} \frac{d^2\delta}{dN^2} + \left[1 + \frac{3}{2}(1+w)\Omega_m + 3(c_s^2 - w) \right. \\ \left. - \frac{d}{dN} \ln \left| -\frac{2k^4}{3a^4 H^2 \kappa^2 \rho} + 3(1+w) \left(1 + \frac{k^2}{3a^2 H^2} \right) \right| \right] \frac{d\delta}{dN} \\ - \left\{ \frac{k^2}{3a^2 H^2} (2 + 3w - 3c_s^2 + 3w_{\text{eff}}) + 3(w - c_s^2) - \frac{9}{2}(1+w)(w_{\text{eff}} - w)\Omega_m \right. \\ \left. - \left[\frac{k^2}{3a^2 H^2} + 3(w - c_s^2) - \frac{3}{2}(1+w)\Omega_m \right] \frac{d}{dN} \ln \left| -\frac{2k^4}{3a^4 H^2 \kappa^2 \rho} \right. \right. \\ \left. \left. + 3(1+w) \left(1 + \frac{k^2}{3a^2 H^2} \right) \right| \right\} \delta = 0 \quad (2) \end{aligned}$$

where w is the equation of state parameter of the matter $w \equiv p/\rho$, c_s is the sound speed $c_s^2 \equiv \delta p/\delta\rho$, k is the wave number, a is a scale factor, and $N \equiv \ln a(t)$. w_{eff} is the effective equation of state

parameter expressed as $w_{\text{eff}} \equiv -2\dot{H}/3H^2 - 1$. By expanding Equation (2) under the approximation $a/k \ll 1/H$ gives

$$\begin{aligned} \frac{d^2\delta}{dN^2} + \left\{ \frac{1}{2} - 6w + 3c_s^2 - \frac{3}{2}w_{\text{eff}} + O\left(\left(\frac{k^2}{a^2H^2}\right)^{-1}\right) \right\} \frac{d\delta}{dN} + \left\{ \frac{c_s^2k^2}{a^2H^2} \right. \\ \left. + 3(w - c_s^2)(1 + 3w + 3w_{\text{eff}}) - \frac{3}{2}(1 + 3w)(1 + w_{\text{eff}}) + O\left(\left(\frac{k^2}{a^2H^2}\right)^{-1}\right) \right\} \delta = 0 \end{aligned} \quad (3)$$

It is found from Equation (3) that there are the wave number dependence of the matter density perturbation in the Λ CDM model, though it is sometimes said that the wave number dependence of the matter density perturbation is the peculiar property of $F(R)$ gravity model. As we have just seen, to evaluate the matter density perturbation without using the sub-horizon approximation can unveil some properties we have never known. In particular, the difference between the case the sub-horizon approximation is used and the case the sub-horizon approximation is not used is conspicuously appeared in k -essence model and $F(R)$ gravity model. In the following, we treat the equation of state parameter and the sound speed as $w = c_s = 0$ by focusing on from the matter dominant era onwards.

k -essence model is one of dark energy models, and its action is described by

$$S = \int d^4x \sqrt{-g} \left\{ \frac{R}{2\kappa^2} - K(\phi, X) + L_{\text{matter}} \right\}, \quad X \equiv -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi \quad (4)$$

Here, ϕ is a scalar field and L_{matter} expresses the Lagrangian density of the matter. In k -essence model, the evolution of the matter density perturbation is described not by a two dimensional equation but by a four dimensional equation [12] because the number of the parameters in the Einstein equation are increased by the existence of the scalar field, *i.e.*, $\delta\phi$ and its derivatives are appeared in the linearized equations. We can decompose the four dimensional equation into the following two dimensional Equation (5) and the solution Equation (6) by considering that the scale of the density fluctuation we can observe is much less than the horizon scale of the Universe $a/k \ll 1/H$. The equation is given as

$$c_\phi^2 \left\{ \frac{d^2\delta}{dN^2} + \left(\frac{1}{2} - \frac{3}{2}w_{\text{eff}} \right) \frac{d\delta}{dN} - \frac{3}{2}\Omega_m\delta \right\} = 0 \quad (5)$$

where c_ϕ is the sound speed in k -essence model defined by $c_\phi^2 \equiv (p_\phi)_{,X}/(\rho_\phi)_{,X} = K_{,X}/(K_{,X} + \dot{\phi}^2 K_{,XX})$ [4]. Here, ρ_ϕ and p_ϕ are the energy density and the pressure of the scalar field, respectively. The subscript $_{,X}$ means derivative with respect to X . The solution is expressed by

$$\delta_{\text{oci}}(N) = C(N) \cos \left[\int^N dN' \frac{c_\phi k}{aH} \right] + r_1 C(N) \sin \left[\int^N dN' \frac{c_\phi k}{aH} \right] \quad (6)$$

$$\begin{aligned} \frac{d}{dN} \ln |C(N)| &= \frac{d}{dN} \ln |\dot{\phi}| - \frac{3}{4} \frac{d}{dN} \ln |K_{,X}| \\ &+ \frac{1}{4} \frac{d}{dN} \ln |K_{,X} + \dot{\phi}^2 K_{,XX}| + \frac{d}{dN} \ln |4K_{,X} + \dot{\phi}^2 K_{,XX}| \end{aligned} \quad (7)$$

where r_1 is an arbitrary real constant. Equation (5) is equivalent to Equation (3) in the leading terms when c_ϕ is not vanished. Therefore, the quasi-static solution of the matter density perturbation in k -essence model is almost identical to that of the Λ CDM model if the background evolution of the Universe is tuned to satisfy observations. On the other hand, the oscillating solution represented by

Equation (6), which cannot be realized in the Λ CDM model, is peculiarity of k -essence model. If we use the subhorizon approximation, the oscillating solution Equation (7) is neglected and we only obtain the quasi-static Equation (6). The behavior of the oscillating solution is depending on the form of the function $K(\phi, X)$ and it can be decaying or growing. Therefore, we should evaluate the behavior of the solution by calculating the effective growth factor represented by Equation (7) in each model. While, only the oscillating solution of the matter density perturbation is influenced by the sound speed of the scalar field. Effects of the sound speed on large scale structure of the Universe is numerically studied in Reference [13].

Next, we consider the following action as $F(R)$ gravity model,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} [R + f(R)] + S_{\text{matter}} \quad (8)$$

where f is an arbitrary function of the scalar curvature R , and $f(R)$ represents the deviation from the Einstein gravity. When we use the spatially flat Friedmann-Lemaître-Robertson-Walker metric, $ds^2 = a^2(\eta)(d\eta^2 - \sum_{i=1}^3 dx^i dx^i)$, the Friedmann-Lemaître equations are written by

$$\frac{3\mathcal{H}'}{a^2}(1 + f_R) - \frac{1}{2}(R + f) - \frac{3\mathcal{H}}{a^2}f'_R = -\kappa^2\rho \quad (9)$$

$$\frac{1}{a^2}(\mathcal{H}' + 2\mathcal{H}^2)(1 + f_R) - \frac{1}{2}(R + f) - \frac{1}{a^2}(\mathcal{H}f'_R + f''_R) = \kappa^2 w\rho \quad (10)$$

where $R = 6a^{-2}(\mathcal{H}' + \mathcal{H}^2)$, $f_R \equiv df(R)/dR$, and the prime represents the differentiation with respect to conformal time η . ρ is the energy density of the matter coming from the variation of S_{matter} and w is the equation of state parameter expressed by $w = p/\rho$. The Hubble rate with respect to conformal time \mathcal{H} is defined by $\mathcal{H} \equiv a'/a$. It is known that $F(R)$ gravity model is conformally equivalent to the scalar field model, which has a non-minimal coupling between the scalar field and the matter. Therefore, the evolution equation of the matter density perturbation is expected to be four dimensional same as in k -essence model. In fact, it is shown in Reference [14] that the evolution equation is four dimensional in $F(R)$ gravity model. The coefficients of the equation are, however, too complicated to be definitely written down, so we need to expand the coefficients by applying the approximations $|f_R| \equiv |df(R)/dR| \ll 1$ and $a^2/k^2 \ll 1/H^2$. Then, it is necessary to be careful which approximations we should give priority to. In the following, we consider the case that the approximations $|f_R|, |Rf_{RR}|, |R^2f_{RRR}| \ll 1$, where subscripts R means derivative with respect to R , take priority over $a^2/k^2 \ll 1/H^2$ to describe the expansion history of the Universe similar to that of the Λ CDM model. The four dimensional equation is, then, expressed as follows [15]:

$$\begin{aligned} \delta'''' + \left\{ \frac{12\mathcal{H}^2(-2 + \mathcal{H}''/\mathcal{H}^3)f_{RRR}}{a^2 f_{RR}} + \frac{1 - \mathcal{H}'/\mathcal{H}^2}{-2 + \mathcal{H}''/\mathcal{H}^3} + O(\mathcal{H}^2/\chi^2) \right\} \mathcal{H}\delta''' \\ + \chi^2 \left\{ (1 + O(\mathcal{H}^2/\chi^2)) \delta'' + \mathcal{H} (1 + O(\mathcal{H}^2/\chi^2)) \delta' \right. \\ \left. + \mathcal{H}^2 \left(2\frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{\mathcal{H}''}{\mathcal{H}^3} + O(\mathcal{H}^2/\chi^2) \right) \delta \right\} = 0 \end{aligned} \quad (11)$$

$$\chi \equiv \sqrt{\frac{a^2}{3f_{RR}} \frac{1 - \mathcal{H}'/\mathcal{H}^2}{2 - \mathcal{H}''/\mathcal{H}^3}} \quad (12)$$

Noting to the terms proportional to χ^2 , we obtain

$$\frac{d^2\delta}{dN^2} + \left(\frac{1}{2} - \frac{3}{2}w_{\text{eff}}\right) \frac{d\delta}{dN} + \left(2\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3}\right) \delta = 0 \quad (13)$$

Equation (13) is equivalent to Equation (3) when the absolute values of the derivatives of $f(R)$ with respect to R are little. On the other hand, if we use the WKB approximation under the condition $|\chi/(a\mathcal{H})| \gg 1$ then we have

$$\delta(\eta) = C_1 e^{\int f_{\text{eff}} dN + i \int \chi d\eta} + C_2 e^{\int f_{\text{eff}} dN - i \int \chi d\eta} \quad (14)$$

where C_1 and C_2 are arbitrary constants, and the effective growth factor f_{eff} is defined as

$$f_{\text{eff}} = 1 - \frac{5}{2} \frac{d}{dN} \ln |\chi| - 2 \frac{d}{dN} \ln |f_{RR}| + \frac{1 - \mathcal{H}'/\mathcal{H}^2}{2 - \mathcal{H}''/\mathcal{H}^3} \quad (15)$$

Considering the Friedmann Equations (9) and (10), and the condition $|f_R|, |Rf_{RR}|, |R^2 f_{RRR}| \ll 1$, we can simplify Equation (15) into

$$f_{\text{eff}} \simeq -\frac{1}{2} + \frac{9}{2} \left(2 - \frac{\mathcal{H}''}{\mathcal{H}^3}\right) \frac{\mathcal{H}^2 f_{RRR}}{a^2 f_{RR}} \quad (16)$$

Here, $\mathcal{H}''/\mathcal{H}^3 \simeq 1/2$ is held in the matter dominant era. Whereas, viable models of $F(R)$ gravity are generally satisfies the condition $f_{RR} > 0$ imposed from the quantum stability. Therefore, the behavior of the oscillating solution is determined by the sign of f_{RRR} . If the form of f_{RR} is described by negative power law of R or $\exp(-\alpha R)$, $\alpha > 0$, then f_{RRR} and f_{eff} are negative. That is to say, the behavior of the matter density perturbation is determined by the quasi-static solution because the other solution Equation (14) is decaying oscillating solution. In this case, it is difficult to find the difference between $F(R)$ gravity model and the Λ CDM model from the matter density perturbation. In fact, famous viable models of $F(R)$ gravity have such a behavior, so we can make a model which cannot be distinguished from the Λ CDM model by the observations concerned with the background and the linear perturbative evolution of the Universe. While, we can also make a model which reproduces the background evolution of the Universe in the Λ CDM model but realizes the different evolution of the matter density perturbation from the Λ CDM model if $f_{RRR} > 0$. In this case, the difference could be observed in the large scale structure of the Universe because there is the oscillatory behavior depending on the redshift in the evolution of the matter density perturbation.

We considered the case that approximations $|f_R|, |Rf_{RR}|, |R^2 f_{RRR}| \ll 1$ take priority over $a^2/k^2 \ll 1/H^2$, however, the other cases are also interesting. For example, if we give priority $a/k \ll 1/H$ over $|Rf_{RR}| \ll 1$, the quasi-static solution of the matter density perturbation grows faster than the Λ CDM model as it is well known. However, we should note that the background evolution of the Universe is modified by the term proportional to f_{RR} in this case. The oscillating behavior of the oscillating solution is decaying when $a^2/k^2 \ll |Rf_{RR}/H^2|$, so it is enough to consider only the quasi-static solution.

3. Summary

The following behaviors of the matter density perturbation in the models of dark energy and modified gravity are unveiled by considering them without applying the subhorizon approximation. In the Λ CDM model, the wave number dependence of the matter density perturbation is appeared in sub-leading terms. There is not only the quasi-static solution but also the oscillating solution which can give unignorable contributions in k -essence model. This oscillating solution is a peculiar property in k -essence model, and its behavior depends on the sound speed of the scalar field, a time derivative of the scalar field and X derivatives of the action $K(\phi, X)$. Although there is the oscillating solution in $F(R)$ gravity, viable $F(R)$ gravity models cannot be distinguished from the Λ CDM model by evaluating the growth rate of the structure formation when we fit their background evolution to the observational results. Because the oscillating solutions in those models are decaying solutions. However, we can also make a model which reproduces the background evolution of the Universe in the Λ CDM model but realizes the different evolution of the matter density perturbation from the Λ CDM model if $f_{RRR} > 0$. Thus, a careful investigation of the nonlinear effect would be important.

$F(R)$ gravity model is conformally equivalent to the scalar field model, which has a non-minimal coupling between the scalar field and the matter. The clear differences between k -essence model and $F(R)$ gravity model in the matter density perturbation are whether there is an influence to the quasi-static evolution from $\delta\phi$ or $f(R)$ and the sound speed dependence of the oscillating solution.

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Conflicts of Interest

The author declares no conflict of interest.

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