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Momentum Halo in The Rayleigh Scattering by a Bose–Einstein Condensate

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Abstract: A ring of radius $\hbar k_0$ in the momentum distribution of a Bose–Einstein condensate is visible when the atoms scatter a single photon. Here, we describe an approximated theory of this effect, leading to an analytic expression of the isotropic momentum scattering rate.

Keywords: Rayleigh scattering; cold atoms

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1. Introduction

Superradiant Rayleigh scattering in a Bose–Einstein condensate provides a striking example of collective enhancement in the interaction of light and matter in ultracold atomic samples [1–4]. In this regime, an elongated BEC is exposed to an off-resonant laser pulse pump beam directed along the condensate symmetry axis. The laser is far detuned from the atomic resonance, so that resonant absorption is suppressed and the only scattering mechanism present is Rayleigh scattering. The atoms, after a transient where they initially scatter in random directions, start to back-scatter photons along the main axis of the condensate. Then, they interfere with the atoms in the original momentum state, creating a matter–wave grating with the right periodicity to further scatter the laser photons in the same mode, gaining a recoil momentum of $2\hbar k_0$, where k_0 is the wave number of the pump photon.

Figure 1 shows two typical results of experiments with a Bose–Einstein condensate exposed to a single far-off detuned laser beam. The left image shows a ring pattern with radius $\hbar k_0$ (image taken from Ref. [5]), whereas the right image shows a superradiant Rayleigh scattering experiment (image taken from Ref. [6]): both figures show the absorption image in which the left peak is the condensate in its original momentum state around $\mathbf{p} = 0$, whereas the right peak in the right image is formed by atoms recoiling after the superradiant Rayleigh scattering at $\mathbf{p} = 2\hbar\mathbf{k}_0$. The ring observed in the left picture and the spherical halo centered between the two density peaks in the right picture are due to nonenhanced spontaneous processes—i.e., random isotropic emission following the absorption of a single laser photon [5]. In Figure 1, the ring appears filled with atoms since the absorption image reports the 'column-integrated' three-dimensional momentum distribution.

Although the origin of this momentum halo is rather clear from a physical point of view, it has not received much attention theoretically. However, although it is formed by atoms recoiling by spontaneous emission, the coherent nature of a Bose–Einstein condensate makes it appear more as a cooperative process rather than a single-particle random process. In this sense, the effect can be interpreted as a cooperative light scattering from an ensemble of weakly excited atoms [7–9]. On these lines, a description of the momentum halo has been proposed in ref. [10] in terms of Mie scattering from ultracold atoms. Mie scattering acts as a seed of the superradiant Rayleigh scattering, weakly populating the momentum state $\mathbf{p} = 2\hbar\mathbf{k}_0$, further enhanced by the superradiant coherent process. However, the Mie scattering approach remains rather difficult, and is limited to spherical, sharp-edged atomic distributions, the solution of which can be obtained only numerically [11]. Conversely, in this work we propose a simplified analytic approach, from which we obtain an expression of the cooperative momentum scattering rate, $\Gamma(\mathbf{p})$, showing the momentum halo due to isotropic Rayleigh scattering.

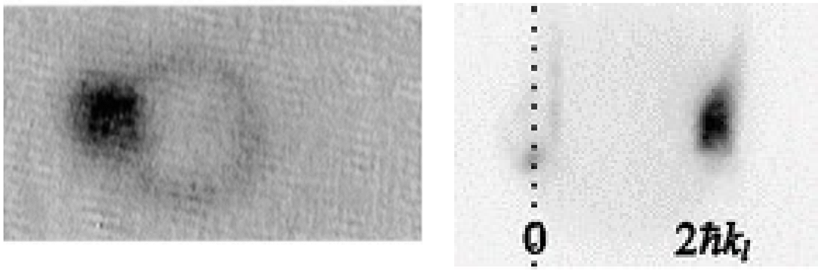

Figure 1. Two examples of momentum halos observed in the experiments with ultracold atoms after exposition to a single far-off detuned laser beam incident from left to right. (**Left**): Absorption image taken after 5.6 ms time-of-flight, showing a ring pattern with radius $\hbar k_0$, resulting from dipole emission of atoms illuminated by a single traveling wave (Adapted with permission from Ref. [5]. Copyright 1999, The American Physical Society). (**Right**): Superradiant Rayleigh scattering experiment, where the absorption image, taken after 45 ms time-of-flight, shows the atoms transferred in the momentum state with $p = 2\hbar k_0$ surrounded by an halo centered at $p = \hbar k_0$ (Adapted with permission from Ref. [6]. Copyright 2008, The American Physical Society).

2. Quantum Model

Consider the atomic system as a bosonic ensemble of N two-level atoms described by the field operator

$$\hat{\Psi}(\mathbf{r},t) = \hat{\Psi}_g(\mathbf{r},t) + \hat{\Psi}_e(\mathbf{r},t) \tag{1}$$

for the ground g and excited e internal states, obeying to bosonic equal-time commutation rules $[\hat{\Psi}_\alpha(\mathbf{r},t), \hat{\Psi}^\dagger_\beta(\mathbf{r}',t)] = \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')$. The N-atom Hamiltonian is

$$H = \sum_{j=1}^{N}\left\{\frac{p_j^2}{2m} + \hbar\left(\frac{\Omega_0}{2}\hat{\sigma}_j e^{i\Delta_0 t - i\mathbf{k}_0\cdot\mathbf{r}_j} + h.c.\right) + \hbar\sum_{\mathbf{k}} g_k\left(\hat{\sigma}_j \hat{a}^\dagger_{\mathbf{k}} e^{i\Delta_k t - i\mathbf{k}\cdot\mathbf{r}_j} + h.c.\right)\right\}, \tag{2}$$

where $\Omega_0 = dE_0/\hbar$ is the Rabi frequency of a linearly polarized incident laser field with electric field E_0 and frequency $\omega_0 = ck_0$, d is the electric dipole, $\Delta_0 = \omega_0 - \omega_a$ is the detuning between the laser and the atomic transition frequencies and $g_k = \sqrt{d^2\omega_k/2\hbar\epsilon_0 V_{ph}}$, where V_{ph} is the photon mode volume; $\hat{\sigma}_j = |g\rangle_j\langle e|_j$ and $\hat{a}_{\mathbf{k}}$ are the lowering operator for the jth atom and the photon annihilation operator in the mode $\mathbf{k}$, respectively. We assume $\Delta_0 \gg \Gamma$ where $\Gamma = d^2\omega_a^3/2\pi\hbar\epsilon_0 c^3$ is the spontaneous decay rate. In the second quantization, the Hamiltonian operator is

$$\begin{aligned}
\hat{H}(t) &= \int d\mathbf{r}\left\{\hat{\Psi}^\dagger_g(\mathbf{r},t)\left[-\hbar^2\frac{\nabla^2}{2m}\right]\hat{\Psi}_g(\mathbf{r},t) + \hat{\Psi}^\dagger_e(\mathbf{r},t)\left[-\hbar^2\frac{\nabla^2}{2m}\right]\hat{\Psi}_e(\mathbf{r},t)\right\} \\
&+ \frac{\hbar\Omega_0}{2}\int d\mathbf{r}\left\{\hat{\Psi}^\dagger_e(\mathbf{r},t)\hat{\Psi}_g(\mathbf{r},t)e^{-i\Delta_0 t + i\mathbf{k}_0\cdot\mathbf{r}} + \hat{\Psi}^\dagger_g(\mathbf{r},t)\hat{\Psi}_e(\mathbf{r},t)e^{i\Delta_0 t - i\mathbf{k}_0\cdot\mathbf{r}}\right\} \\
&+ \hbar\sum_{\mathbf{k}} g_{\mathbf{k}}\int d\mathbf{r}\left\{\hat{\Psi}^\dagger_e(\mathbf{r},t)\hat{\Psi}_g(\mathbf{r},t)\hat{a}_{\mathbf{k}}e^{-i\Delta_k t + i\mathbf{k}\cdot\mathbf{r}} + \hat{\Psi}^\dagger_g(\mathbf{r},t)\hat{\Psi}_e(\mathbf{r},t)\hat{a}^\dagger_{\mathbf{k}}e^{i\Delta_k t - i\mathbf{k}\cdot\mathbf{r}}\right\}.
\end{aligned} \tag{3}$$

The first and second terms of the second and third lines describe absorption and emission of a pump (Ω_0) or vacuum mode ($\hat{a}_{\mathbf{k}}$) photon, respectively. We write the Heisenberg equations for the field operators $\hat{\Psi}_g$, $\hat{\Psi}'_e = \hat{\Psi}_e\exp(i\Delta_0 t)$ and $\hat{a}_{\mathbf{k}}$. For large atom numbers and far detuning from the atomic transition frequency, one can neglect quantum fluctuations and treat these operators as c-numbers, $\hat{\Psi}_{g,e}(\mathbf{r},t) \rightarrow \psi_{g,e}(\mathbf{r},t)$, $\hat{a}_{\mathbf{k}}(t) \rightarrow a_{\mathbf{k}}(t)$. Their dynamical equations are

$$\frac{\partial \psi_g}{\partial t} = i\frac{\hbar}{2m}\nabla^2\psi_g - i\frac{\Omega_0}{2}\psi_e' e^{-i\mathbf{k}_0\cdot\mathbf{r}} - i\psi_e'\sum_{\mathbf{k}} g_{\mathbf{k}} a_{\mathbf{k}}^* e^{-i(\omega_0-\omega_k)t - i\mathbf{k}\cdot\mathbf{r}}, \tag{4}$$

$$\frac{\partial \psi_e'}{\partial t} = i\Delta_0\psi_e' - i\frac{\hbar}{2m}\nabla^2\psi_e' - i\frac{\Omega_0}{2}\psi_g e^{i\mathbf{k}_0\cdot\mathbf{r}} - i\psi_g\sum_{\mathbf{k}} g_{\mathbf{k}} a_{\mathbf{k}} e^{i(\omega_0-\omega_k)t + i\mathbf{k}\cdot\mathbf{r}} \tag{5}$$

$$\frac{da_{\mathbf{k}}}{dt} = -ig_{\mathbf{k}} e^{-i(\omega_0-\omega_k)t}\int d\mathbf{r}\,\psi_g^*(\mathbf{r},t)\psi_e'(\mathbf{r},t)e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{6}$$

In order to describe the evolution in the momentum space, we introduce the Fourier transforms of ψ_g and ψ_e':

$$c_g(\mathbf{p},t) = \frac{1}{(2\pi\hbar)^3}\int d\mathbf{r}\,\psi_g(\mathbf{r},t)e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar} \tag{7}$$

$$c_e(\mathbf{p},t) = \frac{1}{(2\pi\hbar)^3}\int d\mathbf{r}\,\psi_e'(\mathbf{r},t)e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}. \tag{8}$$

The Equations (4)–(6) transform into:

$$\begin{aligned}\frac{\partial c_g(\mathbf{p},t)}{\partial t} &= -i\frac{p^2}{2m\hbar}c_g(\mathbf{p},t) - i\frac{\Omega_0}{2}c_e(\mathbf{p}+\hbar\mathbf{k}_0,t)\\ &- i\sum_{\mathbf{k}} g_{\mathbf{k}} a_{\mathbf{k}}^* e^{-i(\omega_0-\omega_k)t}c_e(\mathbf{p}+\hbar\mathbf{k},t),\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{\partial c_e(\mathbf{p},t)}{\partial t} &= i\left[\Delta_0 - \frac{p^2}{2m\hbar}\right]c_e(\mathbf{p},t) - i\frac{\Omega_0}{2}c_g(\mathbf{p}-\hbar\mathbf{k}_0,t)\\ &- i\sum_{\mathbf{k}} g_{\mathbf{k}}\hat{a}_{\mathbf{k}} e^{i(\omega_0-\omega_k)t}c_g(\mathbf{p}-\hbar\mathbf{k},t),\end{aligned} \tag{10}$$

$$\frac{da_{\mathbf{k}}}{dt} = -ie^{-i(\omega_0-\omega_k)t}g_{\mathbf{k}}(2\pi\hbar)^3\int d\mathbf{p}\,c_g^*(\mathbf{p},t)c_e(\mathbf{p}+\hbar\mathbf{k},t). \tag{11}$$

We solve Equations (9)–(11) assuming a weak field $\Omega_0 \ll \Delta_0$: At the first order in Ω_0, the atoms are weakly excited (i.e., $|c_e(\mathbf{p},t)| \ll |c_g(\mathbf{p},t)|$) and the population of the ground state does not significantly change, with

$$c_g(\mathbf{p},t) \approx c_g(\mathbf{p},0)e^{-i(p^2/2m\hbar)t}. \tag{12}$$

Defining $c_e(\mathbf{p},t) = \tilde{c}_e(\mathbf{p},t)\exp[-i(p^2/2m\hbar)t]$, Equations (10) and (11) become

$$\begin{aligned}\frac{\partial \tilde{c}_e(\mathbf{p},t)}{\partial t} &= i\Delta_0\tilde{c}_e(\mathbf{p},t) - i\frac{\Omega_0}{2}c_g(\mathbf{p}-\hbar\mathbf{k}_0)e^{i(\mathbf{k}_0\cdot\mathbf{p}/m)t - i\omega_R t}\\ &- i\sum_{\mathbf{k}} g_{\mathbf{k}} a_{\mathbf{k}} e^{i(\omega_0-\omega_k+\mathbf{k}\cdot\mathbf{p}/m-\hbar k^2/2m)t}c_g(\mathbf{p}-\hbar\mathbf{k}),\end{aligned} \tag{13}$$

$$\frac{da_{\mathbf{k}}}{dt} = -ig_{\mathbf{k}}(2\pi\hbar)^3\int d\mathbf{p}\,c_g^*(\mathbf{p})e^{-i(\omega_0-\omega_k+\hbar k^2/2m+\mathbf{k}\cdot\mathbf{p}/m)t}\tilde{c}_e(\mathbf{p}+\hbar\mathbf{k},t), \tag{14}$$

where $\omega_R = \hbar k_0^2/2m$ is the recoil frequency and $c_g(\mathbf{p}) = c_g(\mathbf{p},0)$. We eliminate the field variable by integrating Equation (14) over time and substituting it in Equation (13),

$$\begin{aligned}\frac{\partial \tilde{c}_e(\mathbf{p},t)}{\partial t} &= i\Delta_0\tilde{c}_e(\mathbf{p},t) - i\frac{\Omega_0}{2}c_g(\mathbf{p}-\hbar\mathbf{k}_0)e^{i(\mathbf{k}_0\cdot\mathbf{p}/m)t - i\omega_R t}\\ &- (2\pi\hbar)^3\sum_{\mathbf{k}} g_{\mathbf{k}}^2 c_g(\mathbf{p}-\hbar\mathbf{k})\int d\mathbf{p}'\,c_g^*(\mathbf{p}')e^{i[\mathbf{k}\cdot(\mathbf{p}-\mathbf{p}'-\hbar\mathbf{k})/m]t}\\ &\times \int_0^t dt'\,e^{i[\omega_0-\omega_k+\hbar k^2/2m+\mathbf{k}\cdot\mathbf{p}'/m]t'}\tilde{c}_e(\mathbf{p}'+\hbar\mathbf{k},t-t')\end{aligned} \tag{15}$$

The second term of Equation (15) describes the absorption process of the photon $\hbar\mathbf{k}_0$. The exponential factors in the third term of Equation (15) reflect the energy and momentum conservation of the emission photon process, i.e., $\mathbf{p} = \mathbf{p}' + \hbar\mathbf{k}$ and $p^2/2m = p'^2/2m + \hbar\omega_k$: the atoms in the excited state with momentum $\mathbf{p}'$ at time t' decay to the ground state recoiling with momentum $-\hbar\mathbf{k}$, after a free evolution within the time interval $t - t'$. The free evolution consists of quantum diffusion and drift caused by the momentum of the emitted photon and ends at time t when recombination to the ground state accompanied by emission of a photon of momentum $\mathbf{k}$ takes place.

We can assume the Markov approximation (i.e., $\tilde{c}_e(\mathbf{p}' + \hbar\mathbf{k}, t - t') \approx \tilde{c}_e(\mathbf{p}' + \hbar\mathbf{k}, t)$ in the time integral of Equation (15)) and a continuous distribution for $\mathbf{k}$ (i.e., $\sum_{\mathbf{k}} \approx V_{ph}/(2\pi)^3 \int d\mathbf{k}$), with $g_{\mathbf{k}} \approx g_{k_0}$,

$$\begin{aligned}\frac{\partial \tilde{c}_e(\mathbf{p},t)}{\partial t} &= i\Delta_0 \tilde{c}_e(\mathbf{p},t) - i\frac{\Omega_0}{2} c_g(\mathbf{p} - \hbar\mathbf{k}_0) e^{i(\mathbf{k}_0\cdot\mathbf{p}/m)t - i\omega_R t} \\ &- \pi V_{ph}\hbar^3 g_{k_0}^2 \int d\mathbf{k} c_g(\mathbf{p} - \hbar\mathbf{k}) \int d\mathbf{p}'' c_g^*(\mathbf{p}'' - \hbar\mathbf{k}) e^{i[\mathbf{k}\cdot(\mathbf{p}-\mathbf{p}'')/m]t} \\ &\times \delta(\omega_0 - \omega_k - \hbar k^2/2m + \mathbf{k}\cdot\mathbf{p}''/m)\tilde{c}_e(\mathbf{p}'',t) \end{aligned} \tag{16}$$

where we changed the integration variable, defining $\mathbf{p}'' = \mathbf{p}' + \hbar\mathbf{k}$, and where we have approximated the time integral as a Dirac delta in the limit $t \to \infty$. The approach adopted here is similar to the Weisskopf–Wigner theory for the spontaneous emission of a photon in the vacuum modes for a single excited atom [12]. We outline that with the approximation (12) we limit the description to only the spontaneous decay from the excited state. Instead, the description of the superradiant Rayleigh scattering would require the adiabatic elimination of the excited state from Equation (10), writing $c_e(\mathbf{p}, t)$ as proportional to $c_g(\mathbf{p} - \hbar\mathbf{k}_0)$ and $c_g(\mathbf{p} - \hbar\mathbf{k}) \approx c_g(\mathbf{p} + \hbar\mathbf{k}_0)$, with the backward emission assumption, $\mathbf{k} \approx -\mathbf{k}_0$. Then, inserting $c_e(\mathbf{p})$ into Equation (9) and eliminating the scattered field $a_{\mathbf{k}}$ in the same way as for Equation (15), we obtain the self-interaction of the condensate with the matter–wave grating formed by the interference between $c_g(\mathbf{p})$ and $c_g(\mathbf{p} + 2\hbar\mathbf{k}_0)$ [13].

Returning to Equation (16), let us assume an initial spherical Gaussian profile:

$$\psi_g(\mathbf{r}) = \sqrt{\frac{N}{V}} e^{-r^2/4\sigma^2}, \quad c_g(\mathbf{p}) = \sqrt{\frac{N}{(2\pi\hbar)^3 V_p}} e^{-p^2/4\sigma_p^2} \tag{17}$$

where $V = (2\pi)^{3/2}\sigma^3$, $V_p = (2\pi)^{3/2}\sigma_p^3$ and $\sigma_p = \hbar/2\sigma$. They satisfy the normalization relations, $\int d\mathbf{r}\, |\psi_g(\mathbf{r})|^2 = N$ and $\int d\mathbf{p}\, |c_g(\mathbf{p})|^2 = N/(2\pi\hbar)^3$. Assuming the momentum distribution described by $|c_g(\mathbf{p})|^2$ is sufficiently narrow for large values of $k_0\sigma$, we can approximate $\mathbf{p}'' \approx \mathbf{p}$ in the integral over $\mathbf{p}''$ in Equation (16), obtaining

$$\begin{aligned}\frac{\partial \tilde{c}_e(\mathbf{p},t)}{\partial t} &= i\Delta_0 \tilde{c}_e(\mathbf{p},t) - i\frac{\Omega_0}{2} c_g(\mathbf{p} - \hbar\mathbf{k}_0) e^{i(\mathbf{k}_0\cdot\mathbf{p}/m)t - i\omega_R t} \\ &- \pi V_p V_{ph}\hbar^3 g_{k_0}^2 \int d\mathbf{k} |c_g(\mathbf{p} - \hbar\mathbf{k})|^2 \\ &\times \delta(\omega_0 - \omega_k - \hbar k^2/2m + \mathbf{k}\cdot\mathbf{p}/m)\tilde{c}_e(\mathbf{p},t) \end{aligned} \tag{18}$$

Neglecting the recoil energy $\hbar k^2/2m \sim \omega_R \ll \omega_0$ and the Doppler shift $\mathbf{k}\cdot\mathbf{p}/m \sim k_0(p/m) \ll \omega_0$, and introducing $\Gamma = k_0^2 V_{ph} g_{k_0}^2/\pi c$, we obtain finally

$$\frac{\partial \tilde{c}_e(\mathbf{p},t)}{\partial t} = i\Delta_0 \tilde{c}_e(\mathbf{p},t) - i\frac{\Omega_0}{2} c_g(\mathbf{p} - \hbar\mathbf{k}_0) - \frac{1}{2}\Gamma(\mathbf{p})\tilde{c}_e(\mathbf{p},t) \tag{19}$$

where the decay rate for atomic momentum is

$$\Gamma(\mathbf{p}) = \frac{2\pi^2 V_p \hbar^3}{k_0^2}\Gamma \int d\mathbf{k} |c_g(\mathbf{p} - \hbar\mathbf{k})|^2 \delta(k - k_0). \tag{20}$$

The momentum rate (20) can be evaluated analytically for a Gaussian distribution. In fact, changing the integration variable from $\mathbf{k}$ to $\mathbf{q} = \mathbf{k}_0 - \mathbf{k}$,

$$\begin{aligned}\Gamma(\mathbf{p}) &= \frac{N}{4\pi k_0^2}\Gamma e^{-|\mathbf{p}-\hbar\mathbf{k}_0|^2/2\sigma_p^2} \\ &\times \int d\mathbf{q} e^{-\hbar^2 q^2/2\sigma_p^2 - \hbar\mathbf{q}\cdot(\mathbf{p}-\hbar\mathbf{k}_0)/\sigma_p^2}\delta(k_0 - |\mathbf{k}_0 - \mathbf{q}|).\end{aligned} \tag{21}$$

Taking the vectors $\mathbf{p}$ and $\mathbf{q}$ with their z-axis along the direction of $\mathbf{k}_0$, defining in polar coordinates $\mathbf{p} = p(\sin\theta_p\cos\phi_p, \sin\theta_p\sin\phi_p, \cos\theta_p)$ and $\mathbf{q} = q(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$,

$$\begin{aligned}\Gamma(\mathbf{p}) &= \frac{N}{4\pi k_0^2}\Gamma e^{-|\mathbf{p}-\hbar\mathbf{k}_0|^2/2\sigma_p^2} \\ &\times \int_0^\infty dq q^2 e^{-\hbar^2 q^2/2\sigma_p^2} \\ &\times \int_0^\pi d\theta \sin\theta e^{-(\hbar q/\sigma_p^2)(p\cos\theta_p - \hbar k_0)\cos\theta} \\ &\times \int_0^{2\pi} d\phi\, e^{-(\hbar p q/\sigma_p^2)\sin\theta_p\sin\theta\cos(\phi-\phi_p)}\frac{1}{q}\delta\left(\cos\theta - \frac{q}{2k_0}\right).\end{aligned} \tag{22}$$

Solving the integral over ϕ and θ,

$$\begin{aligned}\Gamma(\mathbf{p}) &= \frac{N}{2k_0^2}\Gamma e^{-|\mathbf{p}-\hbar\mathbf{k}_0|^2/2\sigma_p^2}\int_0^{2k_0} dq q e^{-\hbar^2 q^2/2\sigma_p^2} \\ &\times e^{-(\hbar q^2/2k_0\sigma_p^2)(p\cos\theta_p - \hbar k_0)} \\ &\times I_0\left[(\hbar p q/2k_0\sigma_p^2)\sin\theta_p\sqrt{4k_0^2 - q^2}\right]\end{aligned} \tag{23}$$

where $I_0(x)$ is the modified Bessel function of zero order. Changing integration variable from q to $x = q/2k_0$, defining $\bar{\mathbf{p}} = \mathbf{p}/(\hbar k_0)$ and $\bar{\sigma} = k_0\sigma$,

$$\Gamma(\mathbf{p}) = 2N\Gamma e^{-2\bar{\sigma}^2(\bar{p}^2+1-2\bar{p}\cos\theta_p)}\int_0^1 dx x e^{-8\bar{\sigma}^2 x^2\bar{p}\cos\theta_p} I_0\left[8\bar{\sigma}^2\bar{p}\sin\theta_p x\sqrt{1-x^2}\right]. \tag{24}$$

Using the special integral [14]

$$2\int_0^1 dx x e^{-2ax^2} I_0[2bx\sqrt{1-x^2}] = e^{-a}\frac{\sinh\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}$$

we obtain

$$\Gamma(\mathbf{p}) = N\Gamma e^{-2\bar{\sigma}^2(\bar{p}^2+1)}\frac{\sinh[4\bar{\sigma}^2\bar{p}]}{4\bar{\sigma}^2\bar{p}}. \tag{25}$$

This function for $k_0\sigma \gg 1$ describes a ring centered in $\mathbf{p} = 0$ with radius $\hbar k_0$ and thickness $\hbar/(2\sigma)$. Figure 2 shows $\Gamma(\mathbf{p})/N\Gamma$ vs p_x and p_z (in units of $\hbar k_0$) for $k_0\sigma = 4$ and $p_y = 0$.

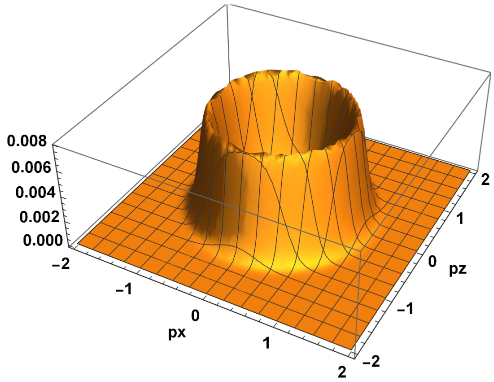

Figure 2. $\Gamma(\mathbf{p})/N\Gamma$ vs p_x and p_z in units of $\hbar k_0$, for $k_0\sigma = 4$ and $p_y = 0$.

This describes the isotropic emission of a photon with momentum $\hbar\mathbf{k}_0$. The atoms, after absorption of a photon with momentum $\hbar\mathbf{k}_0$ along the z-axis and the subsequent isotropic emission, recoil with a momentum distribution proportional to $\Gamma(\mathbf{p} - \hbar\mathbf{k}_0)$. Experimentally, the column-integrated momentum distribution is observed in time-of-flight images. This leads to defining the projected distribution $D^y(p_x, p_z) = \int \Gamma(p_x, p_y, p_z - \hbar k_0)dp_y$, shown in Figure 3 for $k_0\sigma = 4$. The blob on the left represents the initial momentum distribution, in qualitative agreement with the experimental results of Figure 1.

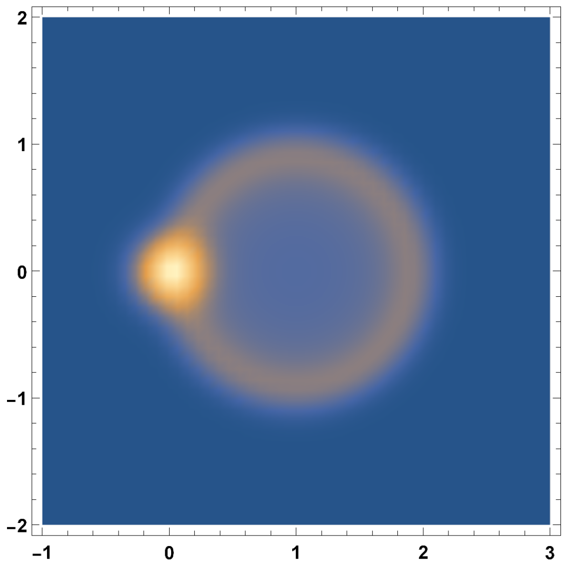

Figure 3. $D^y(p_x, p_z) = \int \Gamma(p_x, p_y, p_z - \hbar k_0)dp_y$, vs p_x and p_z in units of $\hbar k_0$, for $k_0\sigma = 4$.

The expression (25) recalls a similar result of the cooperative decay rate [8], $\Gamma_N = N\Gamma e^{-2\bar{\sigma}^2} \sinh(2\bar{\sigma}^2)/2\bar{\sigma}^2$, obtained by considering N atoms with a spherical Gaussian distribution and excited state approximated by the timed Dicke state $\psi_e(\mathbf{r}) \sim \beta_{TDS}\psi_g(\mathbf{r}) \exp(i\mathbf{k}_0 \cdot \mathbf{r})$ [15]. The analogies between the present result, based on a quantum matter–wave description, and those of Ref. [8], referring to a discrete ensemble of cold classical dipoles without any information about their momentum distribution, may infer that the spontaneous light scattering for a Bose–Einstein condensate can be considered a cooperative effect, with a rate proportional to the resonant optical thickness $b_0 = N/4\bar{\sigma}^2$.

3. Conclusions

From an approximated approach of the equations describing the interaction of a Bose–Einstein condensate with an incident laser beam in the momentum space, we have obtained an expression for the momentum scattering rate, describing the Rayleigh scattering in a form of a ring with radius $\hbar k_0$. The expression has been obtained in the case of a spherical Gaussian distribution, but the general expression (20) could be evaluated for other atomic distributions, for instance with an ellipsoidal shape. This should still result in a ring-like pattern with average radius $\hbar k_0$, but asymmetrical and with more complex features, depending on the Fresnel number $F = k_0\sigma_r^2/\sigma_z$, where σ_r and σ_z are the transverse and longitudinal size, respectively [8]. The result for the spherical Gaussian distribution reproduces the observed halo qualitatively well in the momentum distribution, shown in the left image of Figure 1, due to the isotropic re-emission of the single photon absorbed by the atoms.

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