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On the Definition of Standard Parallels in Map Projections

Miljenko Lapaine

Faculty of Geodesy, University of Zagreb, Kačićeva 26, 10000 Zagreb, Croatia; mlapaine@geof.hr

Abstract: The article belongs to the field of theoretical research on map projections. It is observed that there is no unique and generally accepted definition of standard parallels in the cartographic literature. For some authors, a standard line is a line along which there is no distortion, and for others, it is a line along which there is no distortion of length. At the same time, it is forgotten that the length distortions at any point generally change and depend on the direction. The main goal of this article is very simple: the sentence "linear deformation is zero in all directions" is expressed using a mathematical formula. Besides that, the paper introduces equidistance in a broader sense. This is a novelty in the theory of map projections. Equidistance is defined at a point, along a line and in an area, especially in the direction of the parallels and especially in the direction of the meridian. This enables an unambiguous definition of standard parallels. Theoretical considerations are illustrated with examples of cylindrical projections. The practical value of the proposed approach is manifested in the possibility of a better understanding of the distribution of distortions in any map projection used.

Keywords: map projection; parallel equidistant in the direction of the parallel; parallel equidistant in the direction of the meridian; standard parallel

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1. Introduction

Map projection is the mapping of a curved surface—for example, the Earth's sphere or an ellipsoid—into a plane. The changes that occur during such mapping are called distortions. We distinguish distortions of lengths, areas and angles. If there is no distortion at every point of a line/curve, we say that it is a line with zero distortion or a standard line. If this is true for a parallel, we say it is a standard parallel. At first glance, these are well-known, generally accepted definitions, but in this paper, we will show that this is not the case and provide a correct, mathematically based approach.

Lapaine and Menezes [1] wrote about nomenclature problems in the theory of map projections. This paper discusses the same topic in even more detail and comprehensiveness.

The principal (linear) scale (PS) of a map is the ratio of the length in the projection plane to its original on the surface (sphere or ellipsoid) that is being projected/mapped. The PS is usually written on the map because it determines the general degree of reduction in the lengths shown on the map. For most maps, it is usually referred to and written as the scale of the map, e.g., 1:25,000.

It should be well known that the scale changes from point to point, and at a particular point, it usually depends on the direction as well. The local linear scale factor c is the ratio of the differential of the arc of the curve in the projection plane to the differential of the corresponding arc of the original curve on the ellipsoidal or spherical surface (more details in Section 2).

The local scale (LS) is the product of the principal scale (PS) and the local linear scale factor c:

$$\mathrm{LS} = \mathrm{PS} \times c. \tag{1}$$

Since the local linear scale also depends on the direction, instead of (1), it would be better to write

$$\mathrm{LS}(\alpha) = \mathrm{PS} \times c(\alpha), \tag{2}$$

where we marked the observed direction with α.

The linear distortion is the difference between 1 and the local linear scale factor. It is expressed as a number, so if $c(\alpha) = 1$, then instead of saying "no distortion", it is better to say "distortion equal to zero".

Hinks [2] says that in conic projections there is a parallel, and sometimes two, which has the true length. Such a parallel is called standard. "One parallel, and sometimes a second, is made of the true length; that is to say, if a map is to be on the scale of one-millionth, the length of the complete parallel on the map will be one-millionth of the corresponding terrestrial parallel. This is called a Standard parallel."

Kavrayskiy [3] says that parallels that preserve length are called *standard* or, in the case of cylindrical projections, *main*. "Параллели сохраяющие длину, називаются также стандартними, а в случае цилиндрических проекций главными."

The same author [4] has this definition: parallels on which the main scale is preserved are called standard or main parallels. "Параллели по каторым сохраняется главный масштаб називают иногда стандартними или главными параллелами."

Maling [5] says that if one line with zero distortion is replaced by two, they are known as standard parallels in the normal aspect of the conical and cylindrical projections. "If the single line of zero distortion is replaced by two, which are known as *standard parallels* in the normal aspect of both Conical and Cylindrical projections, ..."

Vakhrameeva et al. [6] have this definition: depending on the size of the mapping area and the method of determining the parameters, conic projections can have one or two parallels whose lengths do not deform; such *parallels* are called *main*. "В зависимости от размеров изображаемой территории и способа определения параметров в конической проекции есть одна или две параллели, длины которых не искажуютя; такие параллели називаются главными."

At the beginning of his handbook on map projections, Snyder [7] has in his list the designations φ_1, φ_2 ... for standard parallels of projections with two standard parallels: "φ_1, φ_2 ... standard parallels of latitude for projections with two standard parallels. These are true to scale and free of angular distortion."

Snyder and Voxland [8] in the dictionary, in their album of map projections, write: "Standard parallel—In the normal aspect of a projection, a parallel of latitude along which the scale is as stated for that map. There are one or two standard parallels on most cylindrical and conic map projections and one on many polar stereographic projections." Why stereographic projections?

In *Enzyklopädischer Wörterbuch Kartographie in 25 Sprachen* [9] on p. 76, we can read this:

> 37.1 *Berührungslinie* = Linie auf der Kugel oder auf dem Ellipsoid [ex: Breitenkreis, Meridian etc.] in der eine für die kartographische Abbildung benutzte Hilfsabbildungsfläche diese (a) berührt [ex: (1) *Berührungsbreitenkreis*, (2) *Berührungsmeridian*] oder (b) schneidet [ex: (3) *Schnittbreitenkreis*, (4) *Schnittmeridian*].
>
> NB: Berührt die Hilfsabbildugsfläche einen Punkt, so spricht man vom (5) *Berührungs-punkt*.
>
> E: *standard line*. A line on a Map Projection along which the Principal Scale obtains: (1) *tangent parallel*; (2) *tangent meridian*; (3) *secant parallel*; (4) *secant meridian*.

The terms in the other 23 languages follow.

In the previous definition in German, *Berührungslinie* literally means line of contact, and the other names are parallel of contact, meridian of contact, parallel of intersection and meridian of intersection. The auxiliary surface (*Hilfsabbildugsfläche*) plays a decisive role, while distortion is not mentioned at all!

In the English definition in the same dictionary, a standard line is "a line in a map projection along which the scale is equal to the main scale". The auxiliary surface is not explicitly mentioned, but the names tangential parallel, tangential meridian, intersecting parallel and intersecting meridian appear. In French, it is "a line of contact or intersection with an auxiliary projection surface", and in Russian it is "zero distortion line, i.e., a line on a map where the main scale is preserved at every point".

Kennedy and Koop [10] and ESRI's online dictionary [11] have this definition: "Standard line: A line on a sphere or spheroid that has no length compression or expansion after being projected. Commonly, a standard parallel or central meridian". So, only what happens along the line is important, not around it, although it should be known that the local linear scale factor and scale at each point depend on the direction (Tissot's indicatrix).

The same authors Kennedy and Koop [10] do not define a standard parallel as a parallel that is a standard line, but as a parallel where the projection surface touches the surface (?!). "The line of latitude where the projection surface touches the surface." And they continue: "A tangent conic or cylindrical projection has one standard parallel, while a secant conic or cylindrical projection has two. A standard parallel has no distortion."

Furuti [12] wrote the following: "Lines (straight or not) in the map with constant scale and length proportional to corresponding lines on Earth are called standard lines." Based on such a definition, Furuti concludes that in the sinusoidal (Sanson–Flamsteed) projection, all parallels are standard (?!). According to him, the same applies to Werner's projection.

Kessler and Battersby [13] define a standard line as a line without distortion, which in many cases coincides with a specific line of latitude and is called a standard parallel: "The third parameter controls the location of the line of no distortion (called a standard line) which, in many cases, coincides with a specific line of latitude and is referred to as a standard parallel".

In the new cartographic dictionary [14], a standard parallel is a parallel along which there is no deformation of lengths, areas, or angles. It is a parallel along which the main scale is preserved at all points in all directions.

Based on the above, we can conclude that there is terminological confusion. The definition of a standard line and a standard parallel is not universally accepted. For some authors, a standard line is a line along which there are no distortions, i.e., there are no length distortions at every point and in every direction. In other words, there is no distortion of areas or angles [4,7,13–16]. For others, it is a line along which there is only no distortion of lengths. It is viewed only in the direction of the line and not in other directions. It is forgotten that length distortions at any point generally change and depend on direction [2,3,6,8,10–12].

In this article, we will propose a solution to the observed problem, i.e., unambiguously define standard lines and standard parallels.

Before that, let us ask ourselves: what is the role of standard parallels in the theory and practice of map projections?

Hinks [2] says that a conic projection with two standard parallels is much better than one with only one standard parallel, for all maps over a considerable range of latitude. In the same book, Hinks deals with the choice of two standard parallels under different conditions. In the last century, almost all authors of textbooks on map projections wrote about the choice of standard parallels. This topic is still current [17].

Maling [5] considers that an important modification of any conic projection is the replacement of one standard parallel by two. According to him, this is equivalent to the geometric concept of a cone intersecting a sphere, which has the effect of redistributing local scales because the main scale is preserved along two parallels. He concludes that a greater range of latitudes can be mapped without excessive distortions.

Let us mention at the end of this introductory part that so far no one has considered the distribution of distortions in projections with three or more standard parallels.

2. Distortions When Mapping Points, Lines and Areas

We will say that the distortion is equal to zero at some point if the local linear scale factor in all directions is equal to 1, i.e., if

$$c(\alpha) = 1, \text{ for each } \alpha \in [0, 2\pi], \tag{3}$$

where $c(\alpha)$ is defined in (4)

$$c^2(\alpha) = E\cos^2\alpha + \frac{F}{\cos\varphi}\sin 2\alpha + \frac{G}{\cos^2\varphi}\sin^2\alpha \tag{4}$$

where E, F and G are coefficients of the first differential form of the map projection. We have deliberately, without limitation of generality, used a sphere of a radius of 1 to facilitate understanding and to shorten the derivations.

Requirement (3) is equivalent to the condition

$$c_{min} = c_{max} = 1 \tag{5}$$

This tells us that Tissot's indicatrix (ellipse) has been transformed into a unit circle. Let us emphasize that the attribute "for each" is important in (3) and that it is not enough that the specified property be valid for only one α. In other words, for an ellipse to be a circle, it is necessary that all its radii be equal, or that its semi-axes be of equal length. It is not enough for one semi-axis of an ellipse to be equal to 1 for the ellipse to be a unit circle.

Let us introduce the following for the sake of writing:

$$a = c_{max},\ b = c_{min}. \tag{6}$$

First, let us note that $a > 0$ and $b > 0$ are always true.

Table 1 shows all possible cases when $a = 1$, $b = 1$ or $a = b = 1$. It is very simple but new in the theory of map projections—see Table 1 [1]:

Table 1. All possibilities of standard points and lines vs. equidistant points and lines in map projections. If $a = b = 1$, we have a standard point or line. If $a = 1$ or $b = 1$, we have an equidistant point, line or area. The case (3,3) is not possible.

	a = 1	b = 1	$a = b$ =1
At a point	(1,1)	(1,2)	**(1,3)**
Along a line	(2,1)	(2,2)	**(2,3)**
In an area	**(3,1)**	**(3,2)**	**(3,3)**

(1,1)

If at some point $a = 1$, we can say that this point is locally *equidistant* (distortion is equal to zero) in the direction of the maximum local linear scale factor.

(1,2)

If at some point $b = 1$, we can say that this point is locally *equidistant* (distortion is equal to zero) in the direction of the minimal local linear scale factor.

(1,3)

If at some point $a = b = 1$, we can say that this is locally *equidistant in all directions*, the point is *with zero distortion*, a *zero-distortion point* or a *standard point*. At that point, the distortion of areas and angles is also equal to zero.

(2,1)

If at all points of a line $a = 1$, then it is not generally a line with zero distortion, but we can say that this line is *equidistant* in the direction of the maximum local linear scale factor.

(2,2)

If at all points of a line $b = 1$, then it is not generally a line with zero distortion, but we can say that this line is *equidistant* in the direction of the minimal local linear scale factor.

(2,3)

If at all points of a line $a = b = 1$, then it is a *line with zero distortion*, a *zero-distortion line* or a *standard line*. At all points of that line, the distortion of areas and angles is also equal to zero.

(3,1)

If at all points of an area $a = 1$, then we can say that this area is *equidistant* in the direction of the maximum local linear scale factor.

(3,2)

If at all points of an area $b = 1$, then we can say that this area is *equidistant* in the direction of the minimal local linear scale factor.

(3,3)

The expression $a = b = 1$ cannot be true at all points of a two-dimensional area on the map, as this would mean a map projection without distortion. Leonhard Euler first proved that this was not possible [18].

Instead of a and b, we usually use these three functions:

$$h = \sqrt{E},\ k = \frac{\sqrt{G}}{\cos\varphi},\ \cos\beta = \frac{F}{\sqrt{EG}} \tag{7}$$

h is the local linear scale factor in the meridian direction

k is the local linear scale factor in the parallel direction

β is the angle between the image of the meridian and the parallel in the projection plane.

The formulas of Apollonius that connect the semi-axes of an ellipse and its two conjugate diameters are known [8]

$$a = \frac{\sqrt{h^2 + k^2 + 2hk\sin\beta} + \sqrt{h^2 + k^2 - 2hk\sin\beta}}{2},$$

$$b = \frac{\sqrt{h^2 + k^2 + 2hk\sin\beta} - \sqrt{h^2 + k^2 - 2hk\sin\beta}}{2}.$$

Table 2 shows all possible cases when $h = 1$, $k = 1$ or $h = k = 1$, compared with Table 1.

Table 2. The same as Table 1 but adapted to the normal aspect projection.

	***h* = 1**	***k* = 1**	***h* = *k* =1**
At a point	(1,1)	(1,2)	**(1,3)**
Along a line	(2,1)	(2,2)	**(2,3)**
In an area	**(3,1)**	**(3,2)**	**(3,3)**

If at some point $h = 1$, then at that point in the direction of the meridian, the length distortion is equal to zero (Figure 1).

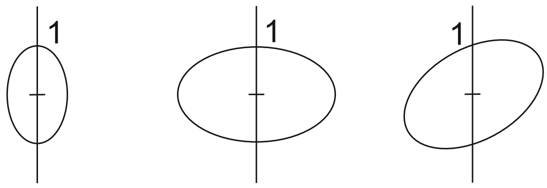

Figure 1. Tissot's indicatrix when $h = 1$ at a point. At that point, the linear distortion is zero in the direction of the meridian. The direction of the meridian image is drawn with a part of a vertical line.

If for all points of a meridian $h = 1$, it is a meridian along which or in the direction of which the linear distortion is equal to zero—the meridian is equidistantly mapped in the direction of the meridian.

If for all points of a parallel $h = 1$, it is a parallel equidistantly mapped in the direction of the meridian.

If in all points of the mapped area $h = 1$, it is common to say that the map projection is equidistant along the meridians. For example, the Postel projection is an azimuthal projection equidistant along the meridians. A simple cylindrical projection is also equidistant along the meridians.

If at some point $k = 1$, then at that point in the direction of the parallel, the length distortion is equal to zero (Figure 2).

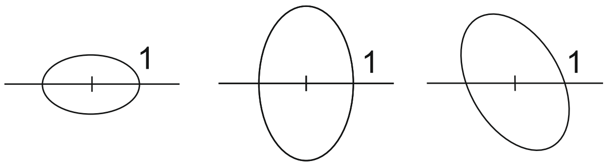

Figure 2. Tissot's indicatrix when $k = 1$ at a point. At that point, the linear distortion is zero in the direction of the parallel. The direction of the parallel image is drawn with a part of a horizontal line.

If in all points of a parallel $k = 1$, it is a parallel along which or in the direction of which the linear distortion is equal to zero—it is a parallel equidistantly mapped in the direction of the parallel.

If in all points of a meridian $k = 1$, it is a meridian equidistantly mapped in the direction of the parallel.

If for all points of an area $k = 1$, it is customary to say that the map projection is equidistant along parallels. For example, a normal aspect orthographic projection is equidistant along parallels. The sinusoidal (Sanson) projection is also equidistant along parallels.

Let us consider some special cases where the images of meridians and parallels intersect at right angles ($F = 0$).

If at all points of a meridian $h = k = 1$ is valid, it is a meridian with a linear distortion equal to zero in all directions, i.e., it is a *standard meridian*. (Figure 3)

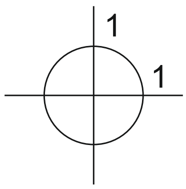

Figure 3. Tissot's indicatrix is a unit circle when $h = k = 1$ at a point. We say that if a point was mapped equidistantly in all directions, the point is *standard*. At that point, the linear distortion is equal to zero in all directions.

If at all points of a parallel $h = k = 1$ is valid, it is a parallel with a linear distortion equal to zero in all directions, i.e., it is a *standard parallel*. (Figure 3)

Some Consequences

If the projection is in normal aspect and conformal, then at every point $h = k$. Therefore, if $h = 1$ or $k = 1$ applies in a conformal projection along a parallel, that parallel is standard. Examples of such projections are in Mercator normal aspect or Lambert conformal conic.

If the projection is in normal aspect and equal area, then at each point $hk = 1$. Therefore, if $h = 1$ or $k = 1$ holds true along a parallel, that parallel is standard. Examples of such projections are Lambert equal-area cylindrical or Albers conic. This applies only to conic and cylindrical projections in which meridians and parallels are the main directions. The Bonne projection is a normal aspect equal-area projection in which $k = 1$ holds on all parallels, but only one parallel is standard.

The reason for the different understanding of standard parallels is probably the fact that in conformal and equal-area conic and cylindrical projections, standard and equidistant parallels do not differ.

In the theory of map projections, we distinguish between projections that are equidistant along meridians and those that are equidistant along parallels. If the projection is normal aspect and equidistant along the meridians, then at each point $h = 1$. If in such a projection along a parallel $k = 1$, that parallel is standard. An example can be the Postel projection (azimuthal equidistant along meridians).

Analogously, if the projection is normal aspect and equidistant along parallels, then at each point $k = 1$. If $h = 1$ in such a projection along a meridian, then that meridian is a standard one. For example, the central meridian in the sinusoidal (Sanson) projection is the standard meridian.

3. Equidistant and Standard Parallels in Normal Aspect Cylindrical Projections

Normal aspect cylindrical projections of the unit sphere are mappings defined using formulas

$$x = n(\lambda - \lambda_0),\ y = y(\varphi), \tag{8}$$

where $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$ are geographic coordinates, constants are $n \geq 0$ and $\lambda_0 \in [-\pi, \pi]$, and function $y = y(\varphi)$ is continuous and monotony increasing. As with any map projection, x and y are the coordinates of a point in a rectangular (mathematical, right-oriented) coordinate system in the plane. Although the attribute *cylindrical* is usually interpreted via mapping onto a cylinder and then developing that surface into a plane, this is generally not true. In fact, the opposite is true. A map made in cylindrical projection can be folded into a cylinder. Anyone can see that Equation (8) describes mapping to a plane and not to the surface of a cylinder. For such mapping, we have

$$E = \left(\frac{dy}{d\varphi}\right)^2,\ F = 0,\ G = n^2 \tag{9}$$

and the first differential form reads

$$Ed\varphi^2 + 2Fd\varphi d\lambda + Gd\lambda^2 = \left(\frac{dy}{d\varphi}\right)^2 d\varphi^2 + n^2 d\lambda^2. \tag{10}$$

The square of the local linear scale factor for mapping a sphere of a radius of 1 in the normal aspect of the cylindrical projection (8) is

$$c^2 = \frac{\left(\frac{dy}{d\varphi}\right)^2 d\varphi^2 + n^2 d\lambda^2}{d\varphi^2 + \cos^2\varphi d\lambda^2}. \tag{11}$$

The factors of the local linear scale along the meridian, or the parallel, respectively, are defined generally in (7), and for normal aspect cylindrical projections they will be

$$h = h(\varphi) = \frac{dy}{d\varphi}, \tag{12}$$

$$k = k(\varphi) = \frac{n}{\cos\varphi}. \tag{13}$$

From (12) and (13), we see that both factors of local linear scales depend on the latitude φ.

We will differentiate equidistant parallels in the direction of the parallel ($k = 1$) and in the direction of the meridian ($h = 1$).

If in (12), $h = 1$ for some φ, then the corresponding parallel is mapped equidistantly in the direction of the meridian, i.e., in the direction perpendicular to that parallel. Tissot's indicatrices on that parallel look like those in Figure 1 on the left and in the middle.

If in (13), $k = 1$ for some φ, then the corresponding parallel is mapped equidistantly in the direction of that parallel. Tissot's indicatrices on that parallel look like those in Figure 2 on the left and in the middle.

The normal aspect of the cylindrical projection for a parallel that is mapped equidistantly in the direction of that parallel, and to which the latitude $\varphi = \varphi_1$ corresponds

$$k(\varphi_1) = 1, \tag{14}$$

must hold or consider (13)

$$n = \cos\varphi_1. \tag{15}$$

From formula (15) and the properties of the cosine trigonometric function, we can conclude that if a parallel of the latitude φ_1 is equidistantly mapped in the direction of that parallel, then this also applies to $-\varphi_1$.

What can we say about the number of parallels *equidistantly mapped in the direction of the parallels*? From formula (15), we can conclude that there can be one ($n = 1,\ \varphi_1 = 0$), two ($0 < n < 1$) or none ($n > 1$) such parallels.

The normal aspect of the cylindrical projection for the parallel that is mapped equidistantly in the direction of the meridian, and to which the latitude $\varphi = \varphi_2$ corresponds

$$h(\varphi_2) = 1, \tag{16}$$

must be or consider (12)

$$\frac{dy}{d\varphi}(\varphi_2) = 1. \tag{17}$$

What can we say about the number of parallels *equidistantly mapped in the direction of the meridian*? From formula (17), we can conclude that there can be several such parallels, which we will illustrate with examples.

For the normal aspect of cylindrical projection along the standard parallel corresponding to the latitude $\varphi = \varphi_0$,

$$k(\varphi_0) = h(\varphi_0) = 1 \tag{18}$$

It must be fulfilled, i.e.,

$$n = \cos\varphi_0 \text{ and } \frac{dy}{d\varphi}(\varphi_0) = 1 \tag{19}$$

We can conclude about the number of *standard parallels* in the normal aspect of cylindrical projections as follows. Since a standard parallel must be mapped equidistantly in the direction of that parallel, and there can be a maximum of two such parallels, the same applies to standard parallels.

The following are examples of cylindrical projections with different properties with respect to equidistantly mapped and standard parallels.

3.1. Example 1

Let the normal aspect of the cylindrical projection be defined as follows:

$$x = \cos\varphi_i \cdot \lambda,\ y = \frac{\pi\varphi + \varphi^2}{\pi + 2\varphi_j}, \tag{20}$$

where $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$, φ_i and φ_j are two given latitudes, and $\varphi_j \neq -\frac{\pi}{2}$. From (20), we can obtain

$$n = \cos\varphi_i, \tag{21}$$

$$\frac{dy}{d\varphi}(\varphi) = \frac{\pi + 2\varphi}{\pi + 2\varphi_j}. \tag{22}$$

From (21), we conclude that for the chosen latitude $\varphi_i \neq 0$, there are two parallels that are equidistantly mapped in the direction of these parallels and to which the latitudes $\pm\varphi_i$ correspond. From (22), we conclude that $\frac{dy}{d\varphi}(\varphi) = 1$ if $\varphi = \varphi_j$.

If we take $\varphi_j = \varphi_i$ or $\varphi_j = -\varphi_i$, we will have two parallels mapped equidistantly in the direction of these parallels, but only one of them will be standard.

3.2. Example 1a

If $\varphi_i = \frac{\pi}{6}$, and $\varphi_j = -\frac{\pi}{6}$, we have the projection equations

$$x = \frac{\sqrt{3}}{2} \cdot \lambda,\ y = \frac{3\left(\pi\varphi + \varphi^2\right)}{2\pi}. \tag{23}$$

It is for that projection that

$$n = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}, \tag{24}$$

This means that the parallels $\varphi_i = \pm\frac{\pi}{6}$ are mapped with zero distortion in the direction of these parallels; thus, these are equidistant parallels in the direction of the parallels. Furthermore,

$$\frac{dy}{d\varphi} = \frac{3(\pi + 2\varphi)}{2\pi}, \tag{25}$$

So, the parallel that is mapped with zero distortion in the direction of the meridian will correspond to the latitude $\varphi = \varphi_j = -\frac{\pi}{6}$. Since this parallel with zero distortion is mapped both along the parallel and along the meridian, we conclude that in this projection, we have one standard parallel with latitude $-\frac{\pi}{6}$ (Figures 4–6).

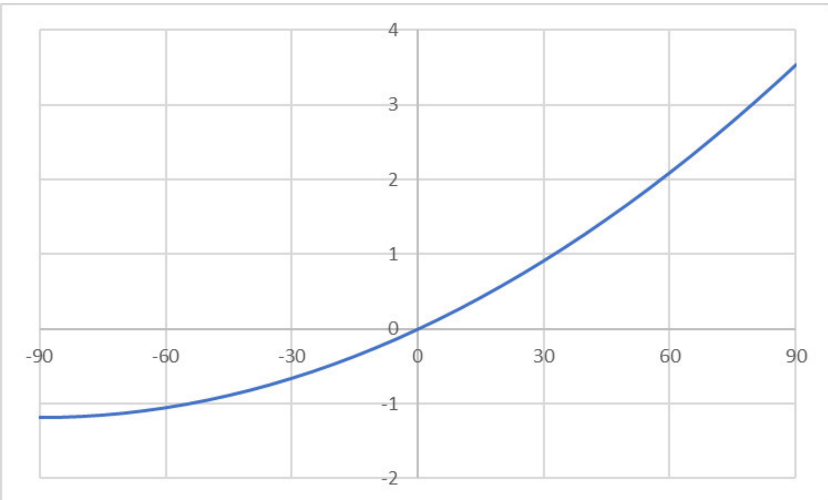

Figure 4. $y = \frac{3\left(\pi\varphi+\varphi^2\right)}{2\pi}$. Horizontal axis in degrees.

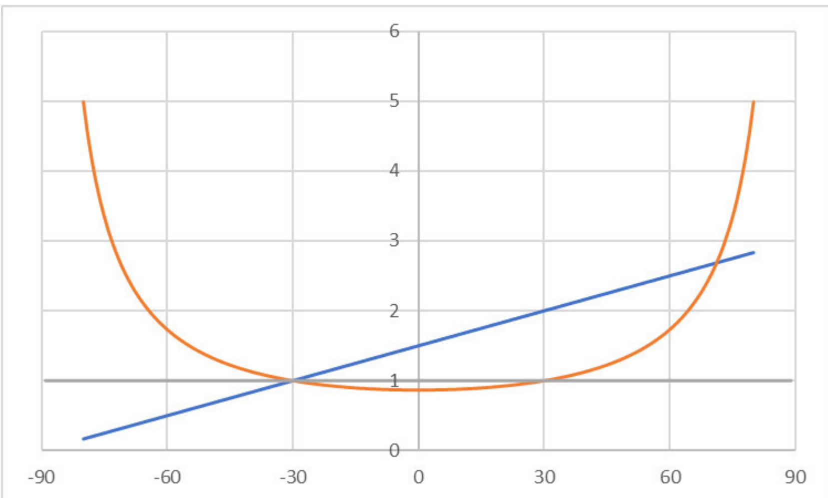

Figure 5. $h = \frac{dy}{d\varphi} = \frac{3(\pi+2\varphi)}{2\pi}$ (blue) and $k = \frac{\sqrt{3}}{2\cos\varphi}$ (red). Horizontal axis in degrees.

Figure 6. World map in cylindrical projection $x = \frac{\sqrt{3}}{2} \cdot \lambda$, $y = \frac{3(\pi\varphi + \varphi^2)}{2\pi}$. Equidistantly mapped parallels $-30°$ and $30°$. Standard parallel $-30°$.

3.3. Example 1b

If we take $\varphi_i = \frac{\pi}{6}$ and $\varphi_j = \frac{\pi}{3}$, the projection Equation (20) reads

$$x = \frac{\sqrt{3}}{2} \cdot \lambda, \ y = \frac{3(\pi\varphi + \varphi^2)}{5\pi}. \tag{26}$$

For that projection, $n = \frac{\sqrt{3}}{2} = \cos\varphi_i$, which means that the parallels $\varphi_i = \pm\frac{\pi}{6}$ are mapped equidistantly in the direction of these parallels. Furthermore,

$$\frac{dy}{d\varphi} = \frac{3(\pi + 2\varphi)}{5\pi}, \tag{27}$$

The parallel that is mapped with zero distortion in the direction of the meridian will correspond to the latitude $\varphi = \varphi_j = \frac{\pi}{3}$. So, in that projection we have two parallels mapped with zero distortion along them $\left(\pm\frac{\pi}{6}\right)$, one parallel mapped with zero distortion in the direction of the meridian $\left(\frac{\pi}{3}\right)$, and therefore no standard parallel (Figures 7–9).

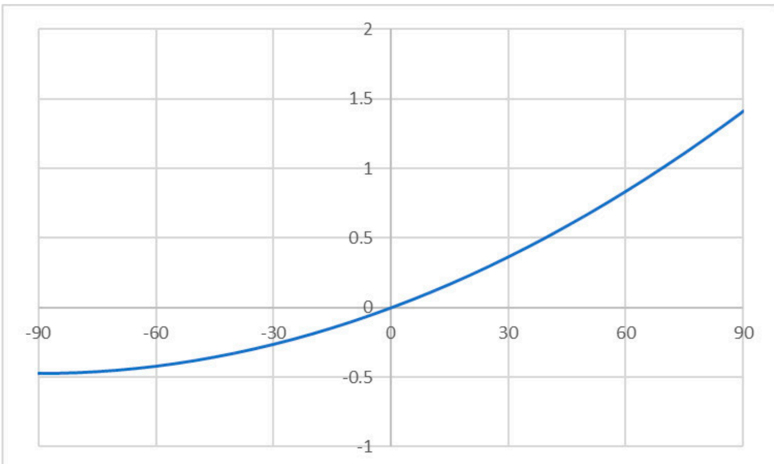

Figure 7. $y = \frac{3(\pi\varphi + \varphi^2)}{5\pi}$. Horizontal axis in degrees.

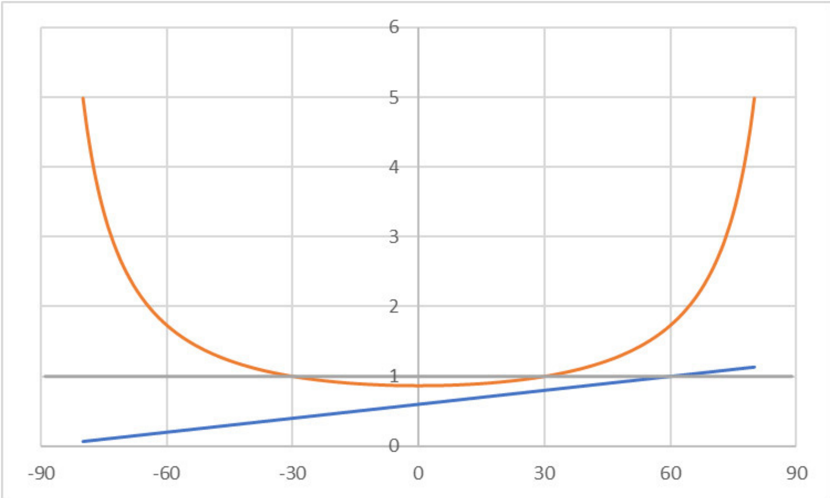

Figure 8. $h = \frac{dy}{d\varphi} = \frac{3(\pi+2\varphi)}{5\pi}$ (blue) and $k = \frac{\sqrt{3}}{2\cos\varphi}$ (red). Horizontal axis in degrees.

Figure 9. World map in cylindrical projection $x = \frac{\sqrt{3}}{2}\cdot\lambda$, $y = \frac{3(\pi\varphi+\varphi^2)}{5\pi}$. Equidistantly mapped parallels $-30°$ and $30°$. No standard parallels.

3.4. Example 2

Let the normal aspect cylindrical projection equations be given as follows:

$$x = \frac{\sqrt{3}}{2}\cdot\lambda,\ y = \frac{a}{5}\varphi^5 + \frac{c}{3}\varphi^3 + e\varphi, \tag{28}$$

where

$$a = \frac{\varepsilon}{\varphi_1^2\varphi_2^2},\ \ c = -\varepsilon\left(\frac{1}{\varphi_1^2} + \frac{1}{\varphi_2^2}\right),\ e = 1+\varepsilon, \tag{29}$$

φ_1 and φ_2 are two given latitudes, and $\varepsilon > 0$ is a given real number. For that projection, we have

$$n = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}, \tag{30}$$

which means that the parallels $\varphi = \pm\frac{\pi}{6}$ are mapped with zero distortion in the direction of those parallels. Furthermore, it is for that projection that

$$\frac{dy}{d\varphi} = a\varphi^4 + c\varphi^2 + e. \tag{31}$$

It is not hard to see that

$$\frac{dy}{d\varphi}(-\varphi_1) = \frac{dy}{d\varphi}(\varphi_1) \text{ and } \frac{dy}{d\varphi}(-\varphi_2) = \frac{dy}{d\varphi}(\varphi_2). \tag{32}$$

3.5. Example 2a

Let us take $\varphi_1 = \frac{\pi}{6}$, $\varphi_2 = \frac{\pi}{3}$ and $\varepsilon = 0.5$. From (29), it is easy to calculate

$$a = \frac{162}{\pi^4}, \;\; c = -\frac{45}{2\pi^2}, \; e = 1.5. \tag{33}$$

Furthermore,

$$\frac{dy}{d\varphi}\left(\pm\frac{\pi}{6}\right) = \frac{dy}{d\varphi}\left(\pm\frac{\pi}{3}\right) = 1, \tag{34}$$

and this means that the four parallels $\varphi_1 = \pm\frac{\pi}{6}$ and $\varphi_2 = \pm\frac{\pi}{3}$ are mapped with zero distortion in the direction perpendicular to them. Of these four parallels, two are mapped with zero distortion in the direction of these parallels (32), so they are also standard parallels of the cylindrical projection (30) considering (31) and (35) (Figures 10–12).

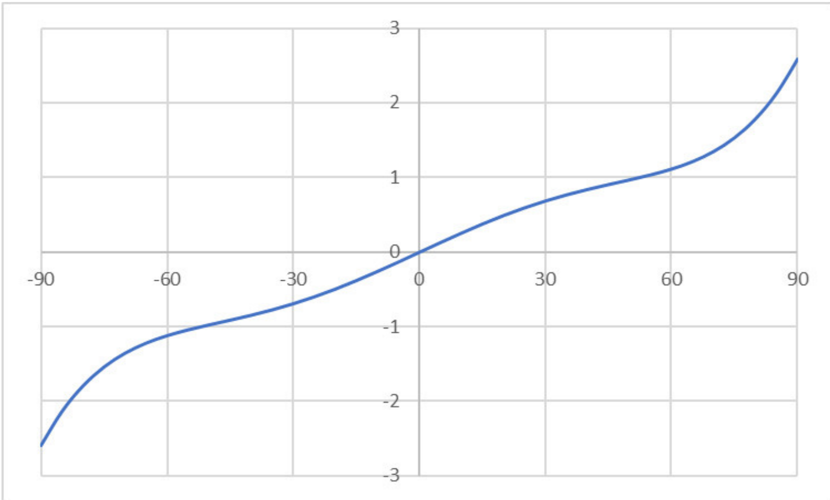

Figure 10. $y = \frac{a}{5}\varphi^5 + \frac{c}{3}\varphi^3 + e\varphi$; a, c and e from (35).

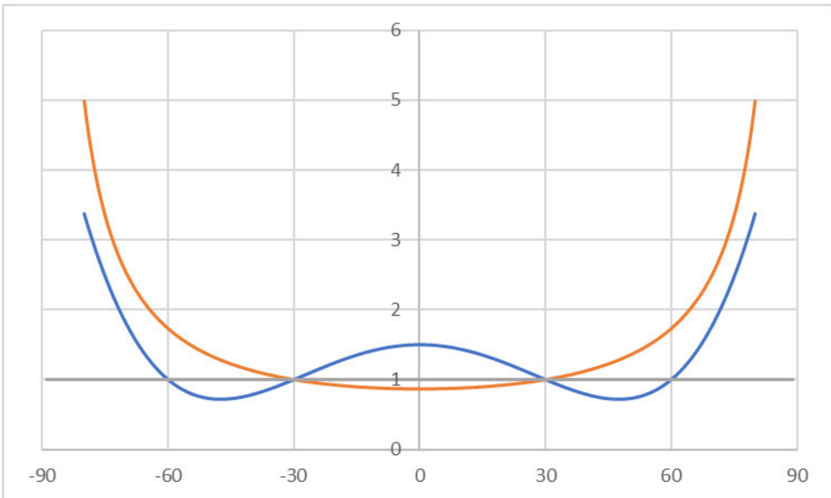

Figure 11. $h = \frac{dy}{d\varphi} = a\varphi^4 + c\varphi^2 + e$ (blue) and $k = \frac{\sqrt{3}}{2\cos\varphi}$ (red); a, c and e from (35).

Figure 12. Map of the world in cylindrical projection $x = \frac{\sqrt{3}}{2} \cdot \lambda$, $y = \frac{a}{5}\varphi^5 + \frac{c}{3}\varphi^3 + e\varphi$; a, c and e from (35). Two standard parallels ($\pm 30^\circ$).

3.6. Example 2b

Let the normal aspect cylindrical projection be defined as in (30) and (31). Let us take $\varphi_1 = \varphi_2 = \frac{\pi}{4}$ and $\varepsilon = 0.5$. From (31), we calculate

$$a = \frac{128}{\pi^4}, \ \ c = -\frac{16}{\pi^2}, \ e = 1.5. \tag{35}$$

For that projection according to (33), the parallels $\varphi = \pm\frac{\pi}{6}$ are mapped with zero distortion in the direction of those parallels. Furthermore, for that projection

$$h\left(\pm\frac{\pi}{4}\right) = \frac{dy}{d\varphi}\left(\pm\frac{\pi}{4}\right) = a\left(\pm\frac{\pi}{4}\right)^4 + b\left(\pm\frac{\pi}{4}\right)^2 + e = 1, \tag{36}$$

and this means that the two parallels $\varphi_1 = \varphi_2 = \pm\frac{\pi}{4}$ are mapped equidistantly in the direction perpendicular to them. So, for that projection, two parallels were mapped equidistantly in the direction of those parallels, and the other two equidistantly in the direction of the meridian. There are no standard parallels in this projection (Figures 13–15).

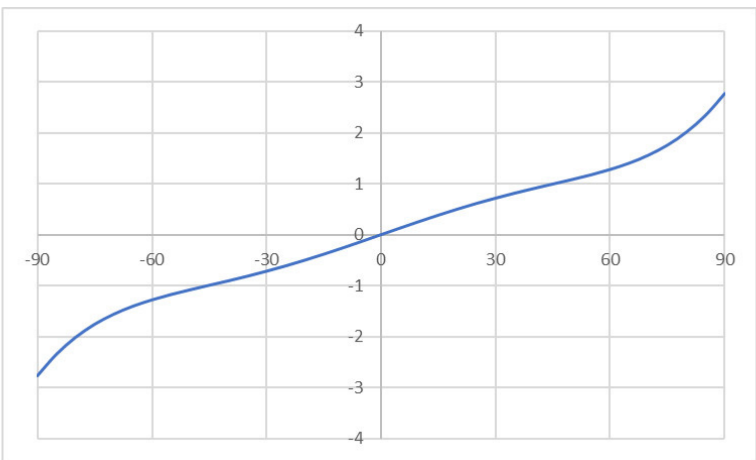

Figure 13. $y = \frac{a}{5}\varphi^5 + \frac{c}{3}\varphi^3 + e\varphi$; a, c and e from (36).

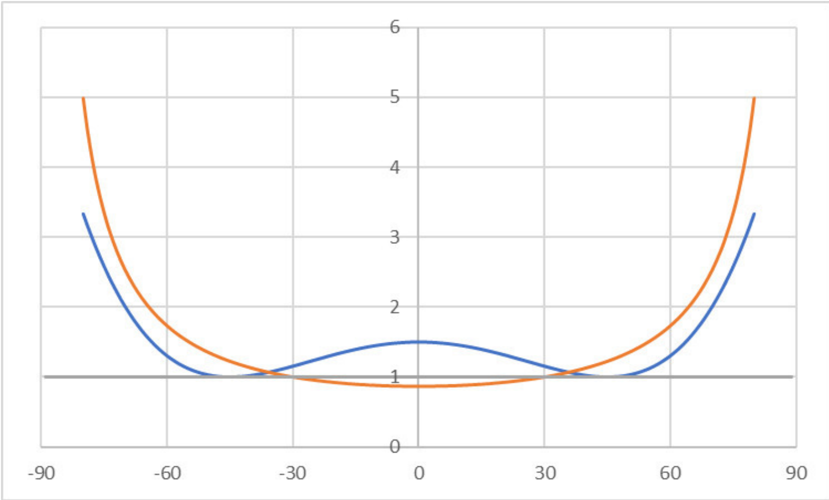

Figure 14. $h = \frac{dy}{d\varphi} = a\varphi^4 + c\varphi^2 + e$ (blue) and $k = \frac{\sqrt{3}}{2\cos\varphi}$ (red); a, c and e from (36).

Figure 15. The world map in cylindrical projection $x = \frac{\sqrt{3}}{2} \cdot \lambda$, $y = \frac{a}{5}\varphi^5 + \frac{c}{3}\varphi^3 + e\varphi$; a, c and e from (36). Two parallels are mapped equidistantly in the direction of parallels and the other two equidistantly in the direction of meridians. But there are no standard parallels.

3.7. Example 3. Patterson Projection

One of the newer cylindrical projections is the Patterson projection. The Miller 1 projection was introduced by Osborn Maitland Miller [19]. The Patterson projection was derived from the Miller 1 projection and modified graphically. The equations of the Patterson cylindrical projection are as follows [20]:

$$x = \lambda,\ y = \varphi\Big(c_1 + \varphi^2\varphi^2\Big(c_2 + \varphi^2\Big(c_3 + \varphi^2 c_4\Big)\Big)\Big), \tag{37}$$

where

$$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ \lambda \in [-\pi, \pi],\ c_1 = 1.0148,\ c_2 = 0.23185,\ c_3 = -0.14499,\ c_4 = 0.02406.$$

For that projection, we have

$$h = \frac{dy}{d\varphi} = c_1 + \varphi^2\varphi^2\Big(5c_2 + \varphi^2\Big(7c_3 + 9\varphi^2 c_4\Big)\Big),\ k = \frac{1}{\cos\varphi}. \tag{38}$$

It is easy to see that $k(0) = 1$, i.e., that the equator is the only equidistantly mapped parallel. Furthermore, $h(0) = c_1 > 1$, from which it follows that the equator is not a standard parallel. Thus, Patterson's cylindrical projection does not have a single standard parallel (Figures 16 and 17 and Section 3.7).

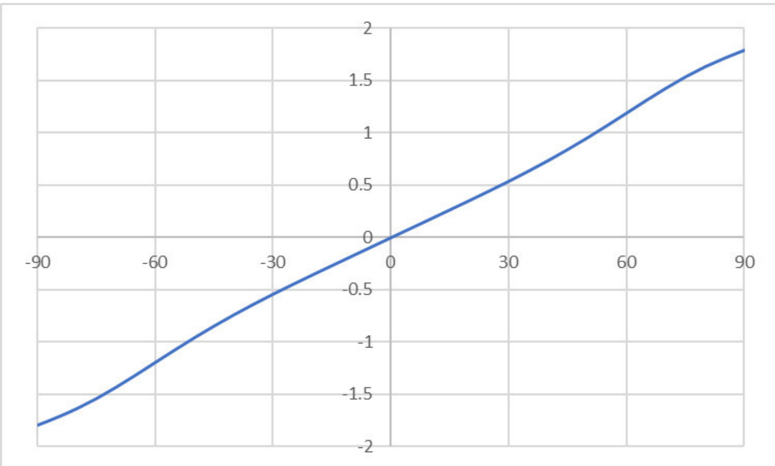

Figure 16. $y = \varphi(c_1 + \varphi^2\varphi^2(c_2 + \varphi^2(c_3 + \varphi^2 c_4)))$ for the Patterson projection.

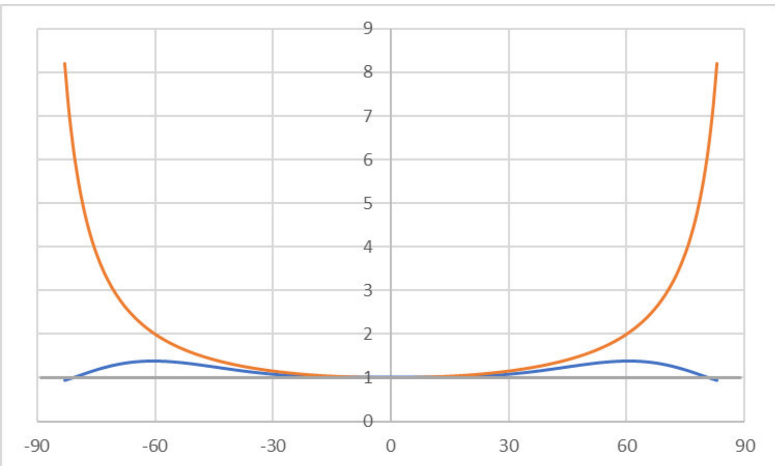

Figure 17. $h = \frac{dy}{d\varphi} = c_1 + \varphi^2\varphi^2(5c_2 + \varphi^2(7c_3 + 9\varphi^2 c_4))$ (blue) and $k = \frac{1}{\cos\varphi}$ (red) for the Patterson projection. There are no standard parallels because $h(0) = c_1 > 1$.

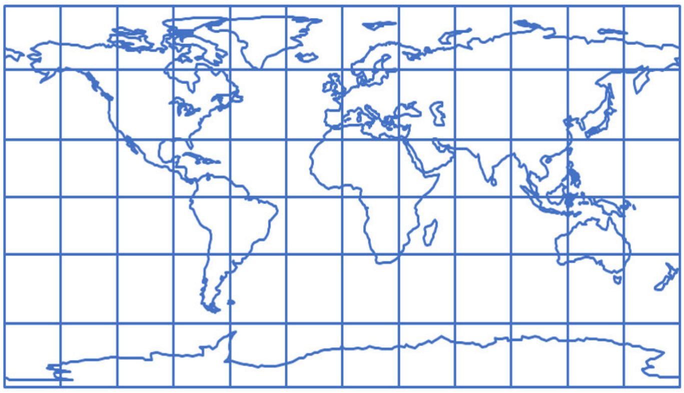

Figure 18. The map of the world in the Patterson projection: $x = \lambda, y = \varphi\left(c_1 + \varphi^2\varphi^2\left(c_2 + \varphi^2\left(c_3 + \varphi^2 c_4\right)\right)\right)$. There are no standard parallels.

3.8. Example 4. Equidistant Cylindrical Projection

The Patterson cylindrical projection is quite similar to the equidistant cylindrical projection (Figures 19–21), which has the equations:

$$x = \lambda,\ y = \varphi, \tag{39}$$

where $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$. For that projection, we have

$$h = \frac{dy}{d\varphi} = 1,\ k = \frac{1}{\cos\varphi}. \tag{40}$$

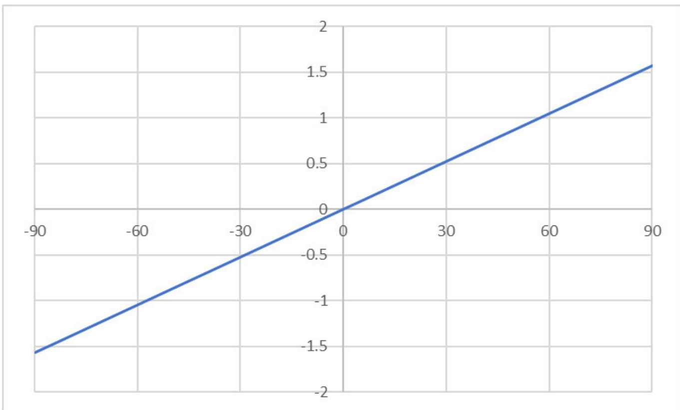

Figure 19. $y = \varphi$ for the equidistant cylindrical projection.

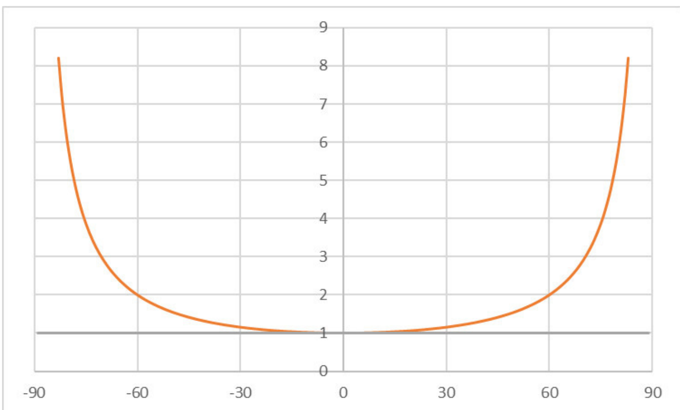

Figure 20. $h = \frac{dy}{d\varphi} = 1$ and $k = \frac{1}{\cos\varphi}$ (red) for the equidistant cylindrical projection. The equator is the standard parallel.

Figure 21. World map in the equidistant cylindrical projection $x = \lambda$, $y = \varphi$. The standard parallel is the equator.

The equidistant cylindrical projection given by Equation (39) has one standard parallel —the equator (Figures 19–21). It is not clear what would be the advantage of the Patterson projection over the equidistant cylindrical one.

4. Conclusions

Equidistant projections are known in the theory of map projections, but this paper introduces equidistance in a broader sense. Equidistance is defined at a point, along a line and in an area, especially in the direction of the parallel and especially in the direction of the meridian. This is a novelty in the theory of map projections.

It was observed that for normal aspect cylindrical map projections, one should distinguish zero distortion in the direction of the parallels and perpendicular to them, i.e., in the direction of the meridian. Distortion equal to zero in the parallel direction is actually equidistance in the parallel direction. Analogously, a distortion equal to zero in the meridian direction is equidistance in the meridian direction. Distortion equal to zero in all directions means that it is a standard point. A line that consists of standard points is a standard line. Theoretical considerations are illustrated with appropriate examples.

If we want to visualize it, Tissot's indicatrix has become a unit circle in standard points. Often, in textbooks or books on map projections, we can find maps of the world with Tissot's distortion ellipses drawn. If some of these ellipses are unit circles, we have standard points. If in some direction the radius of the ellipse is equal to 1, we have an equidistantly mapped point in that direction. And then, by expanding the approach just described to parallels and meridians, we immediately obtain standard and equidistantly mapped parallels or meridians.

The paper describes a theoretical approach that would not have a purpose by itself. The practical value of this approach is manifested in the possibility of a better understanding of the distribution of distortions in any map projection used. To prevent misunderstandings of standard parallels in the theory of map projections and their teaching, we recommend a clear distinction between equidistant parallels in a certain direction and standard parallels.

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References

1. Lapaine, M.; Menezes, P. Standard, secant and equidistant parallels/Standardne, presječne i ekvidistantne paralele. *Kartogr. I Geoinformacije* **2020**, *19*, 34, 40–62. [CrossRef]
2. Hinks, A.R. *Map Projections*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1912; p. 1921.
3. Kavrayskiy, V.V. *Izbrannye trudy, Tom II, Matematicheskaja kartografija, Vypusk 1*; Izdanie Upravlenija nachal'nika Gidrograficheskoy Sluzhby VMF: USSR, 1958.

4. Kavrayskiy, V.V. *Izbrannye trudy, Tom II, Matematicheskaja kartografija, Vypusk 2*; Izdanie Upravlenija nachal'nika Gidrograficheskoy Sluzhby VMF: USSR, 1959.
5. Maling, D.H. *Coordinate Systems and Map Projections*; Reprinted 1980; George Philip: London, UK, 1973; 255p.
6. Vakhrameeva, L.A.; Bugaevskiy, L.M.; Kazakova, Z.L. *Matematicheskaja Kartografija*; Nedra: Moscow, Russia, 1986.
7. Snyder, J.P. *Map Projections—A Working Manual*; Prof. Paper 1395, Reprinted 1989 and 1994 with corrections; U.S. Geological Survey: Reston, VA, USA, 1987; 383p.
8. Snyder, J.P.; Voxland, M.P. *Album of Map Projections*; Professional Paper. 1453; U. S. Geological Survey: Reston, VA, USA, 1989.
9. Neumann, J. *Enzyklopädischer Wörterbuch Kartographie in 25 Sprachen*; 2. erweiterte Ausgabe, K G Saur: München, Germany, 1997.
10. Kennedy, M.; Koop, S. *Understanding Map Projections*; ESRI: Redlands, CA, USA, 1994; Available online: http://downloads2.esri.com/support/documentation/ao_/710Understanding_Map_Projections.pdf (accessed on 21 April 2021).
11. Standard Line. ESRI. Available online: http://support.esri.com/en/other-resources/gis-dictionary/search/standard%2520line (accessed on 10 March 2019).
12. Furuti, C.A. Map Projections. Available online: http://www.progonos.com/furuti/MapProj/ (accessed on 18 June 2017).
13. Kessler, F.C.; Battersby, S.E. *Working with Map Projections*; CRC Press: Boca Raton, FL, USA, 2019.
14. Frančula, N.; Lapaine, M.; Jazbec, I.-P. *Kartografski Rječnik*; Hrvatsko Kartografsko Društvo and Naklada Dominović: Zagreb, Croatia, 2020.
15. Frančula, N.; Lapaine, M. *Geodetsko-Geoinformatički Rječnik*; Državna Geodetska Uprava: Zagreb, Croatia, 2008; ISBN 978-953-6971-12-1.
16. Borčić, B.; Kreiziger, I.; Lovrić, P.; Frančula, N. *Višejezični Kartografski Rječnik*; Geodetski Fakultet Sveučilišta u Zagrebu, Zbornik Radova, no. 15: Zagreb, Croatia, 1977.
17. Šavrič, B.; Jenny, B. Automating the selection of standard parallels for conic map projections. *Comput. Geosci.* **2016**, *90*, 202–212. [CrossRef]
18. Euler, L. De repraesentatione superficiei sphaericae super plano. In *Acta Academiae Scientiarum Imperialis Petropolitanae*; Imperatorskaja akademija nauk: Petrpolis, Russia, 1777.
19. Miller, O.M. Notes on cylindrical world map projections. *Geogr. Rev.* **1942**, *32*, 424–430. [CrossRef]
20. Patterson, T.; Šavrić, B.; Jenny, B. Introducing the Patterson Cylindrical Projection. *Cartogr. Perspect.* **2014**, *78*, 77–81. [CrossRef]