BAND-LIMITED STOCHASTIC PROCESSES IN DISCRETE AND CONTINUOUS TIME

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In the theory of stochastic differential equations, it is commonly assumed that the forcing function is a Wiener process. Such a process has an infinite bandwidth in the frequency domain. In practice, however, all stochastic processes have a limited bandwidth.

A theory of band-limited linear stochastic processes is described that reflects this reality, and it is shown how the corresponding ARMA models can be estimated. By ignoring the limitation on the frequencies of the forcing function, in the process of fitting a conventional ARMA model, one is liable derive estimates that are severely biased.

The estimation biases can be avoided by sampling the continuous process at a rate corresponding to the maximum frequency of the forcing function. Then, there is a direct correspondence between the parameters of the band-limited ARMA model and those of an equivalent continuous-time process.

Keywords: Stochastic Differential Equations, Band-Limited Stochastic Processes, Aliasing and Interference

1. Introduction

It is common to assume that the differential equations that are used for modelling stochastic processes in continuous time are driven by a continuous stream of infinitesimal impulses. These impulses, which constitute the increments of a Wiener process, are composed of a non denumerable infinity of sinusoidal components of all frequencies in the interval $[0, \infty)$.

Whereas a Wiener process is a fruitful mathematical abstraction that has many important applications, it is of doubtful relevance to macroeconomic modelling. An inspection of the periodograms of macroeconomic data sequences reveals that their various components tend to reside in strictly limited frequency bands; and it is improbable that they should have arisen from the filtering of Wiener processes.

It is more realistic to assume the such components originate in the filtering of continuous white-noise processes that are band limited in the same manner as the components themselves. However, in adopting this point of view, we are challenged to produce a model of such a process. To achieve this is one the purposes of this paper. A further purpose is to examine the effects of using ordinary Autoregressive Moving Average (ARMA) models to derive parametric representations of band-limited processes.

Conventional ARMA processes are driven by discrete-time white-noise processes, which have spectral density functions that are uniform across a frequency range running from zero to π radians per period, which is the limiting frequency that is observable in sampled data. When the driving process is band limited to a subset of the interval $[0, \pi]$, there are liable to be severe biases in the estimated parameters, unless some account is taken of this fact.

2. Evidence of Band-Limited Processes

We should begin by presenting some evidence to support the assertion that macroeconomic data sequences are commonly composed of components that fall within limited frequency bands.

Figure 1 displays a sequence of the logarithms of the quarterly series of U.K. Gross Domestic Product (GDP) over the period from 1955 to 1994, which comprises a total of 160 observations. Interpolated through this sequence is a quadratic trend, which can be taken to represent the growth path of the economy.

The deviations from this growth path are a combination of a low-frequency business cycle with some high-frequency fluctuations that are due to the seasonal nature of economic activity. These deviations are represented in Figure 2, which also shows an interpolated continuous function that is designed to represent the business cycle. Figure 3 shows that the effect of taking the first differences of the logarithmic data is to emphasise the seasonal fluctuations at the expense of the business cycle.

The periodograms of Figures 4–6 clearly reflect the features of the corresponding data sequences. The periodogram is a discrete periodic function, which is a function of the Fourier transform of the periodic extension of the data. A single cycle of the periodogram occurs in the interval $[-\pi, \pi]$ or, equivalently, in the interval $[0, 2\pi]$. However, since the data is real-valued, the periodograms are symmetric about the zero frequency and, therefore, they are characterised completely by graphs over the interval $[0, \pi]$

Figure 4 is the periodogram of the a saw tooth function that corresponds to the periodic extension of the trended data. It owes its most prominent features to the radical disjunctions that occur at the points where the end of one replication of the sample is joined to the beginning of the next replication. The spike in the vicinity of the zero frequency is so dominant in this periodogram that the remaining features, which are on a much smaller scale, are almost invisible.

The periodogram of the differenced data, which is in Figure 5, shows that, in eliminating the trend, the differencing strongly suppresses the low-frequency components of the data, which include the business cycles.

The periodogram of the deviations from the quadratic trend, which is in Figure 6, gives a clear representation of the spectral effects both of the business cycle and of the seasonal fluctuations, and it show that they reside separate frequency bands.

The spectral structure extending from zero frequency up to $\pi/8$ belongs to the business cycle. The prominent spikes located at the frequency $\pi/2$ and at the limiting Nyquist frequency of π are the property of the seasonal fluctuations.

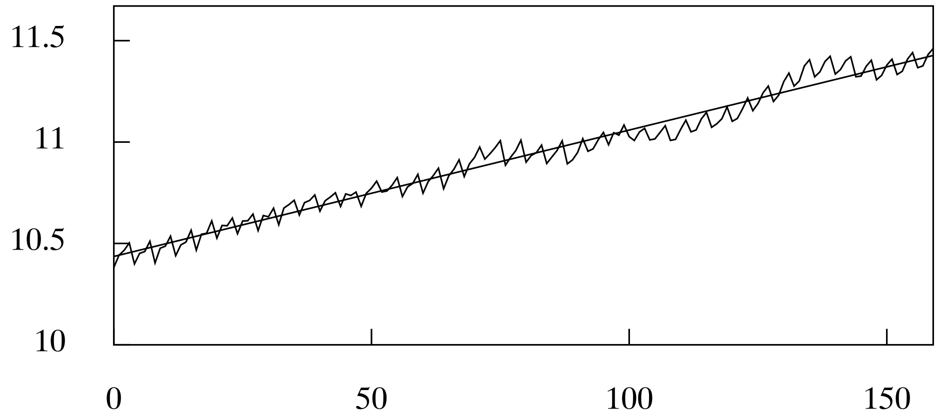

Figure 1. The quarterly series of the logarithms of consumption in the U.K., for the years 1955 to 1994, together with a quadratic trend interpolated by least-squares regression.

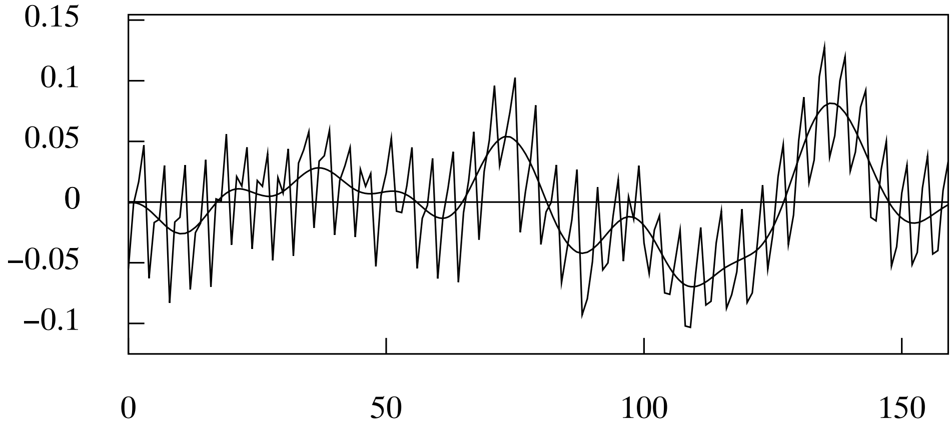

Figure 2. The residual sequence from fitting a quadratic trend to the logarithmic consumption data. The interpolated line, which represents the business cycle, has been synthesised from the Fourier ordinates in the frequency interval $[0, \pi/8]$.

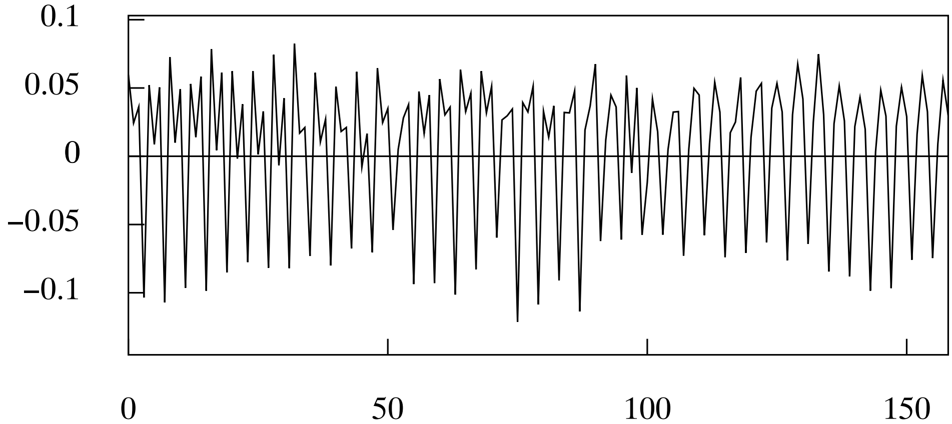

Figure 3. The differences of the logarithmic consumption data.

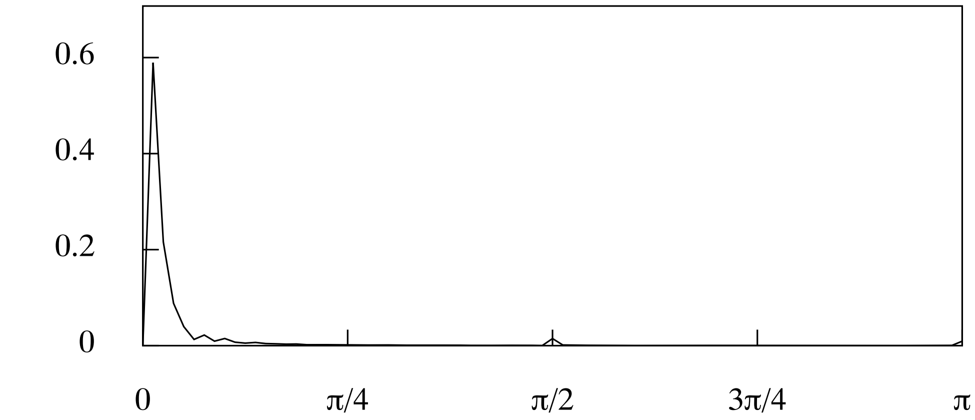

Figure 4. The periodogram of the logarithms of consumption in the U.K., for the years 1955 to 1994.

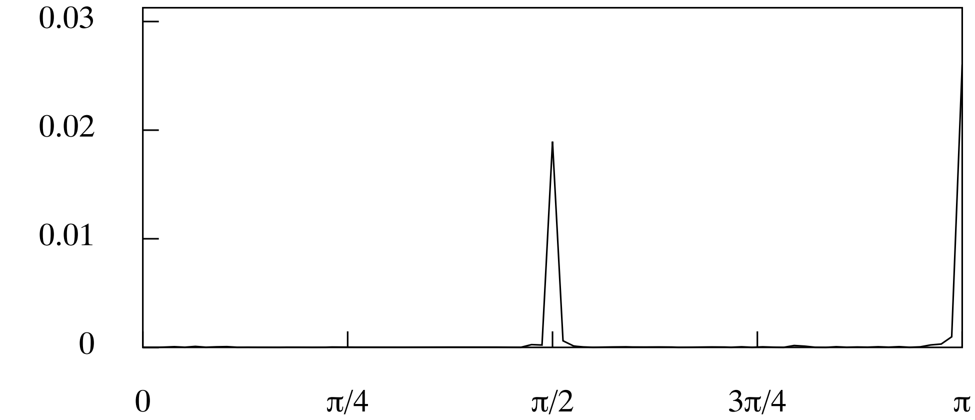

Figure 5. The periodogram of the first differences of the the logarithmic consumption data.

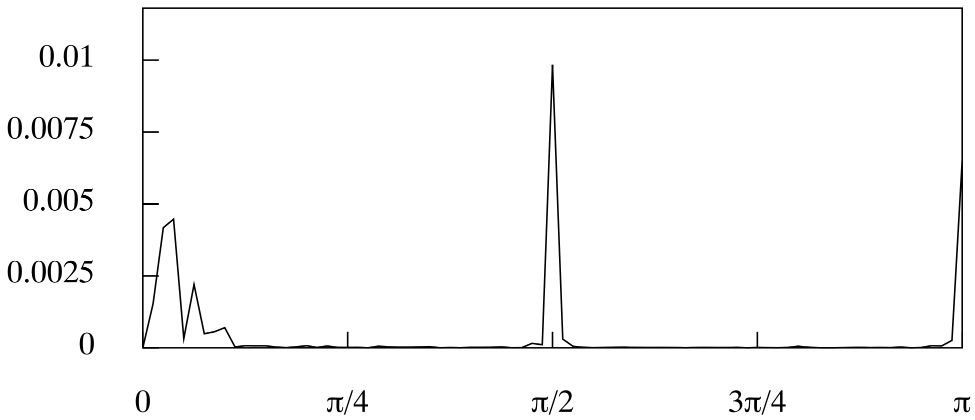

Figure 6. The periodogram of the residual sequence obtained from the linear detrending of the logarithmic consumption data.

Elsewhere in the periodogram, there are wide dead spaces, which are punctuated by the spectral traces of minor elements of noise.

The slowly varying continuous function $z(t)$ interpolated through the deviations of Figure 2 has been created by combining a set of sine and cosine functions of increasing frequencies which are regularly spaced and extend no further than the limiting frequency of the business cycle, which is $\pi/8$. Thus,

$$\begin{aligned} z(t) &= \sum_{j=0}^{d} \{\alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t)\} \\ &= \sum_{j=-d}^{d} \xi_j e^{\mathrm{i}\omega_j t} \quad \text{with} \quad \omega_d \simeq \pi/8, \end{aligned} \tag{1}$$

where $\xi_j = (\alpha_j + \mathrm{i}\beta_j)/2$ and $\xi_{-j} = \xi_j^* = (\alpha_i - \mathrm{i}\beta_j)/2$.

Some justification ought to be given for characterising the spectral structure of a trended sequence in terms of the periodogram of its deviations from an interpolated polynomial trend. If y is the vector of the data, then the vector of the deviations is given by the formula

$$e = Q(Q'Q)^{-1}Q'y, \tag{2}$$

wherein Q' is a submatrix of the pth-order matrix difference operator $\nabla_T^p = (I_T - L_T)^p$, which is obtained by deleting the first p rows. The difference operator $I_T - L_T$ is formed from the identity matrix $I_T = [e_0, e_1, \ldots, e_{T-1}]$ and from the matrix $L_T = [e_1, \ldots, e_{T-1}, 0]$, obtained by deleting the leading vector from I_T and appending column of zeros to the end of the array.

Thus, for example, in the case of a second-order difference operator, which is appropriate to linear detrending, there is

$$\nabla_6^2 = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right] = \begin{bmatrix} Q'_* \\ Q' \end{bmatrix}. \tag{3}$$

It is apparent from equation (2) that the residual vector contains exactly the same information as the vector $Q'y$ of differences.

The periodogram of the polynomial residuals allows us to discern the spectral structure across the entire frequency range. The effect of increasing the degree of the polynomial, which generates a more flexible trend, is to attenuate the low-frequency components relative to the high-frequency components, but the bandwidths of the components in question remain the same. Thus, we have a device for accurately determining the domains of the various spectral structures that correspond to the components of the data sequence.

Our objective is to characterise the dynamics of the business cycle via the parameters of a fitted ARMA model. Such a model is liable to be applied to a seasonally adjusted version of the data, of which the periodogram will lack the spectral spikes at the seasonal frequency of $\pi/2$ and at the harmonic frequency of π. An second-order autoregressive AR(2) model with complex roots is the simplest of the models that might be appropriate to the purpose. The modulus of its roots should reveal the damping characteristics of the cycles, and their argument should indicate the angular velocity or, equivalently, the length, of the cycles.

The parametric spectrum of an AR(2) model is supported on the entire frequency range $[0, \pi]$, which is to say that it is everywhere nonzero within this interval. However, the band-limited process, to which the observed business cycle appears to belong, has a zero-valued spectral density everywhere in the interval $(\pi/8, \pi]$.

The consequence of this disparity is that an AR(2) model that is fitted directly to the data is liable to deliver highly misleading estimates. Thus, it has been widely reported that, when it is fitted to deseasonalised quarterly data, the model will invariably deliver estimates that imply real-valued roots, which fail adequately to represent the dynamics of the business cycle. (See Pagan 1997, for example.)

In order to estimate the parameters successfully, it is necessary to map the low-frequency spectral structure of the business cycle into the interval $[0, \pi]$. This involves creating a new data sequence at a lower sampling rate. It is also necessary to ensure that nothing is carried into the interval $[0, \pi]$ that does not belong to the low-frequency structure. This is achieved by applying an anti-aliasing filter prior to resampling the data at the lesser rate

In the process of describing the simple technique of resampling, we shall provide a model for the continuous-time band-limited process that generates the business cycle component. The parameters of this process are also the parameters of a discrete-time ARMA process that describes the resampled data.

3. The Sampling Process

A sequence that has been sampled at the integer time points from a continuous aperiodic function that is square-integrable will have a transform that is a periodic function. This outcome is manifest in the discrete-time Fourier transform, where it can be seen that the period in question has the length of 2π radians.

To understand this result, consider the Fourier representation of a real–valued square-integrable function $x(t)$ defined over the real line. The following are the corresponding expressions for the function and its Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\omega t}\xi(\omega)d\omega \longleftrightarrow \xi(\omega) = \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega t}x(t)dt. \tag{4}$$

By sampling $x(t)$ at the integer time points, a sequence $\{x_t; t = 0, \pm 1, \pm 2, \ldots\}$ is generated of which the transform is a periodic function. In that case,

$$x_t = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}\omega t}\xi_S(\omega)d\omega \longleftrightarrow \xi_S(\omega) = \sum_{k=-\infty}^{\infty} x_k e^{-\mathrm{i}k\omega}. \tag{5}$$

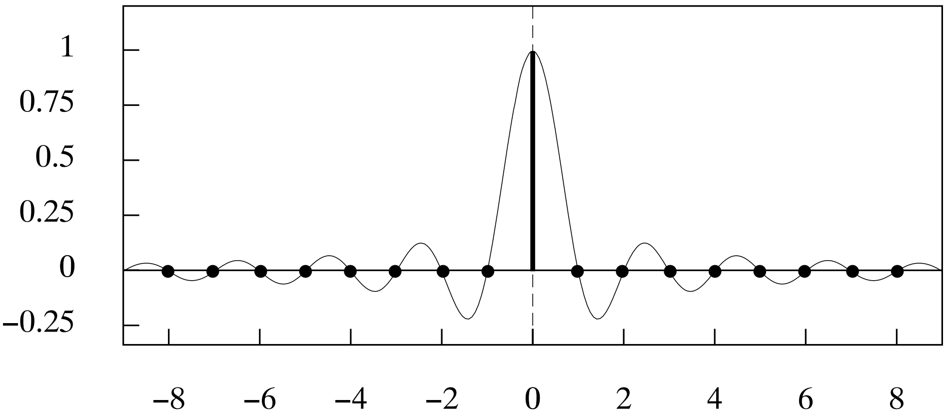

Figure 7. The sinc function wave-packet $\phi_{(0)}(t) = \sin(\pi t)/\pi t$ comprising frequencies in the interval $[0, \pi]$.

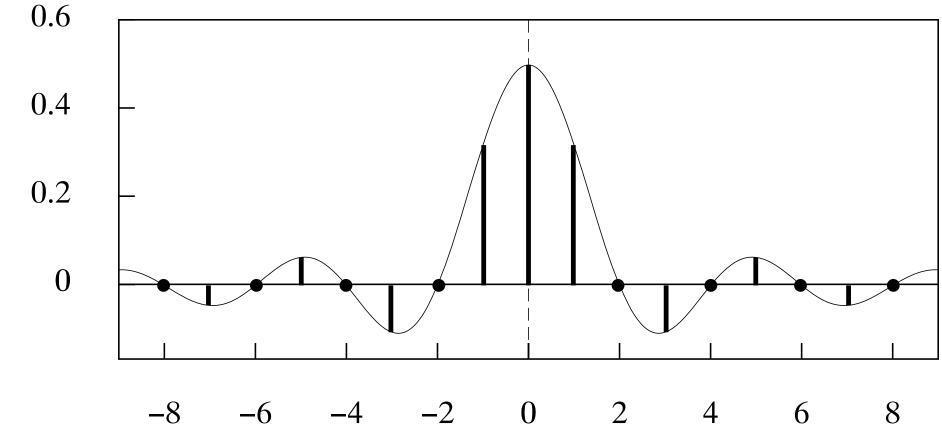

Figure 8. The sinc function wave-packet $\phi_{(1)}(t) = \sin(\pi t/2)/\pi t$ comprising frequencies in the interval $[0, \pi/2]$.

Therefore, at the sampled point x_t, to which the expressions under (4) and (5) both relate, there is,

$$x_t = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\omega t}\xi(\omega)d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}\omega t}\xi_S(\omega)d\omega. \tag{6}$$

The equality of the two integrals implies that

$$\xi_S(\omega) = \sum_{j=-\infty}^{\infty} \xi(\omega + 2j\pi). \tag{7}$$

Thus, the function $\xi_S(\omega)$ is obtained by wrapping $\xi(\omega)$ around a circle of circumference of 2π and adding the coincident ordinates. The two functions will coincide at all frequencies in the interval $[-\pi, \pi]$ if $\xi(\omega) = 0$ for all $|\omega| \geq \pi$. Otherwise, $\xi_S(\omega)$ will be subject to a process of aliasing, whereby elements of the continuous function that are at frequencies in excess of π are confounded with elements at frequencies less than π. Thus, the so-called Nyquist frequency of π radians per period of observation represents the limit of what is directly observable in sampled data.

If the condition is fulfilled that $\xi(\omega) = 0$ for all $|\omega| \geq \pi$, then it should be possible to reconstitute the continuous function $x(t)$ from its sampled ordinates. This is the burden of the famous Nyquist–Shannon sampling theorem—see Shannon (1949, 1998)—which was foreshadowed in the work of Whittaker (1935).

Since $\xi(\omega) = \xi_S(\omega)$ is a continuous function defined of the interval $[-\pi, \pi]$, it may be regarded as a periodic function of a period of 2π. Putting the RHS of (5) into the LHS of (4), and taking the integral over $[-\pi, \pi]$ in consequence of the band-limited nature of the function $x(t)$, gives

$$x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \sum_{k=-\infty}^{\infty} x_k e^{-\mathrm{i}k\omega} \right\} e^{\mathrm{i}\omega t} d\omega = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} x_k \int_{-\pi}^{\pi} e^{\mathrm{i}\omega(t-k)} d\omega. \tag{8}$$

The integral on the RHS is evaluated as

$$\int_{-\pi}^{\pi} e^{\mathrm{i}\omega(t-k)} d\omega = 2\frac{\sin\{\pi(t-k)\}}{t-k}. \tag{9}$$

Putting this into the RHS of (8) gives

$$x(t) = \sum_{k=-\infty}^{\infty} x_k \frac{\sin\{\pi(t-k)\}}{\pi(t-k)} = \sum_{k=-\infty}^{\infty} x_k \phi_0(t-k), \tag{10}$$

where

$$\phi_0(t-k) = \frac{\sin\{\pi(t-k)\}}{\pi(t-k)} \tag{11}$$

is the so-called sinc function, which is the Fourier transform of the following frequency function:

$$\phi_0(\omega) = \begin{cases} 1, & \text{if } |\omega| \in (0, \pi); \\ 1/2, & \text{if } \omega = \pm\pi, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

In the case of a stationary stochastic process, the sampled sequence is not square summable and, therefore, in a strict sense, this proof of the interpolation via the Nyquist–Shannon Theory does not apply. However, the convergence of the interpolation formula of (10), when $x(t) = \{x_t; t = 0, \pm 1, \pm 2, \ldots\}$ is a stationary sequence, can confirmed by considering a sum with $k \in [-N, N]$ for some finite integer N. The variance of the sum of discarded terms can be made arbitrarily small by increasing the value of N.

To investigate the nature of the interpolation, one may consider the value interpolated at an arbitrary point $\tau \in (0, 1)$ lying between the sampled points x_0 and x_1. This is representative of any point lying between sampled values.

The interpolated ordinate x_τ is formed as a weighted average of all the sampled ordinates. The sum of the weights is

$$\sum_{k=-\infty}^{\infty} \frac{\sin\{\pi(\tau-k)\}}{\pi(\tau-k)} = \frac{\sin(\pi\tau)}{\pi\tau} + \sum_{k=1}^{\infty}\left\{\frac{\sin\{\pi(\tau-k)\}}{\pi(\tau-k)} + \frac{\sin\{\pi(\tau+k)\}}{\pi(\tau+k)}\right\}. \quad (13)$$

From the identity $\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$, it follows that

$$\begin{aligned}\sin\{\pi(\tau-k)\} &= \sin(\pi\tau)\cos(\pi k) - \cos(\pi\tau)\sin(\pi k)\\ &= (-1)^k \sin(\pi\tau),\end{aligned} \quad (14)$$

for, with k as an integer, there is $\sin(\pi k) = 0$ and $\cos(\pi k) = (-1)^k$. Likewise, $\sin\{\pi(\tau-k)\} = (-1)^k \sin(\pi\tau)$. Therefore,

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \frac{\sin\{\pi(\tau-k)\}}{\pi(t-k)} &= \frac{\sin(\pi\tau)}{\pi\tau} + \sum_{k=1}^{\infty}(-1)^k\left\{\frac{\sin(\pi\tau)}{\pi(\tau-k)} + \frac{\sin(\pi\tau)}{\pi(\tau+k)}\right\}\\ &= \frac{\sin(\pi\tau)}{\pi}\left\{\frac{1}{\tau} + \sum_{k=1}^{\infty}\frac{(-1)^k 2\tau}{\tau^2-k^2}\right\}\\ &= \frac{\sin(\pi\tau)}{\pi}\left\{\frac{1}{\tau} + \sum_{k=1}^{\infty}(-1)^k\left[\frac{1}{\tau+k} + \frac{1}{\tau-k}\right]\right\}.\end{aligned} \quad (15)$$

The value of this sum is unity. Thus, for example, a constant function, which is not square integrable, will be reconstituted exactly from its sampled values. Also, the weights are absolutely summable, which means that the BIBO (bounded input bounded output) condition is satisfied. This condition is sufficient to ensure that a filtered stationary sequence retains its stationarity.

The sequence of sinc functions $\phi_0(t-k); k \in \mathcal{I} = \{0, \pm1, \pm2, \ldots\}$ constitutes an orthogonal basis for the set of all functions band limited to the frequency interval $[0, \pi]$. To show this, let $\phi_0(\omega)$ be the transform of $\phi_0(t)$ and consider the following autoconvolution:

$$\begin{aligned}\int_t \phi_0(t)\phi_0(\tau-t)dt &= \int_t \phi_0(t)\left\{\frac{1}{2\pi}\int_\omega \phi_0(\omega)e^{\mathrm{i}\omega(\tau-t)}d\omega\right\}dt\\ &= \frac{1}{2\pi}\int_\omega \phi_0(\omega)\left\{\int_t \phi_0(t)e^{-\mathrm{i}\omega t}dt\right\}e^{\mathrm{i}\omega\tau}d\omega\\ &= \frac{1}{2\pi}\int_\omega \phi_0(\omega)\phi_0(\omega)e^{\mathrm{i}\omega\tau}d\omega.\end{aligned} \quad (16)$$

The symmetry of $\phi_0(t)$ allows us to write $\phi_0(\tau-t) = \phi_0(t+\tau)$, whereas the idempotency of $\phi_0(\omega)$ gives $\phi_0^2(\omega) = \phi_0(\omega)$. Together, these two conditions indicate that $\phi_0(t)$ is its own autocorrelation function. Therefore, the condition

$$\phi_0(t-k) = 0 \quad \text{for} \quad k \in \{\pm1, \pm2, \ldots\} \quad (17)$$

indicates that sinc functions separated by integer distances are mutually orthogonal.

The sinc function $\phi_0(t)$ is represented in Figure 5. Here, it is manifest that the values of the ordinates of the function at the nonzero integer points $t \in \{\pm 1, \pm 2, \ldots\}$ are zeros. Also, when the set of sinc functions $\{\phi_0(t-k); k \in \mathcal{I}\}$ at unit displacements are sampled at the integer values of t, the result is nothing but the set of unit impulses at the integer points. This constitutes a basis for the set of all sequences defined over the set of integers.

An important property of the basis sinc functions is their lack of mutual interference at the integer points. Any band-limited function that is composed of a weighted sum of the basis functions will assume values at the integer points that are equal to the amplitudes of the sinc functions centred on those points. There will be no contributions to these values from the other basis functions.

If a function is band limited to the frequency interval $[0, \pi]$, then sampling at the integer points constitutes a critical rate of sampling that is just sufficient to detect an element at the Nyquist frequency of π radians per sampling interval. If the function is band limited in frequency to a subinterval of $[0, \pi]$, then the current rate of sampling is over-rapid, and it may be reduced. In particular, if the function is band limited to the interval $[0, \pi/n]$, where n is an integer, then the continuous function can be reconstituted from a sub sample comprising every nth point.

Figure 8 depicts the sinc function $\phi_1(t) = \sin(\pi t/2)/\pi t$, which is the Fourier transform of a rectangle on the frequency interval $[-\pi/2, \pi/2]$. The set of functions $\{\phi_1(t-2k); t \in \mathcal{R}, k \in \mathcal{I}\}$, which are at displacements of 2 sample points, constitutes a basis of all continuous functions band limited to the interval $[0, \pi/2]$.

Likewise, the 2-point displacements of the sampled version of $\phi_1(t)$, with $t \in \mathcal{I}$, provide a basis for the corresponding band-limited sequences obtained by sampling continuous functions band-limited to $[0, \pi/2]$. Discarding alternate points from such a sampled sequence, and reindexing accordingly, creates a sequence with a frequency content extending over the interval $[0, \pi]$, from which a continuous function can be recreated in the manner of equation (10).

The reconstruction or interpolation of a function in the manner suggested by the sampling theorem is not possible in practice, because it requires summing an infinite number sinc functions, each of which is supported on the entire real line. Nevertheless, a continuous band-limited periodic function, defined on a finite interval, can be reconstituted from a finite number of wrapped or periodic sinc functions, which are Dirichlet kernels by another name. The Dirichlet kernel is obtained by sampling the sinc-function rectangle in the frequency domain.

Consider

$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\omega t}\phi(\omega)d\omega \longleftrightarrow \phi(\omega) = \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega t}\phi(t)dt. \tag{18}$$

Sampling $\psi(\omega)$ at the frequency points $\omega_j = 2\pi j/T; j \in \{0, \pm 1, \pm 2, \ldots\}$ gives rise to a classical Fourier series:

$$\phi^\circ(t) = \sum_{j=-\infty}^{\infty} e^{\mathrm{i}\omega_j t}\phi_j \longleftrightarrow \phi_j = \frac{1}{T}\int_{-T/2}^{T/2} e^{-\mathrm{i}\omega_j t}\phi^\circ(t)dt, \tag{19}$$

where $\phi_j = \phi(\omega_j)$. If the integral on the RHS of (19) is to be reconciled with that of (18), then it must be the case that

$$\phi^{\circ}(t) = T \sum_{q=-\infty}^{\infty} \phi(t + qT), \tag{20}$$

which is to say that the periodic sinc function $\phi^{\circ}(t)$ is formed by wrapping the function $\phi(t)$ around a circle of circumference T and adding the overlying ordinates.

The circular function $\phi^{\circ}(t)$ can be found by transforming into the time domain the ordinates sampled from the frequency-domain rectangle of the sinc function. Thus, it can be shown that if

$$\phi_j = \begin{cases} 1, & \text{if } j \in \{0, \pm 1, \ldots, \pm d\}, \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

then

$$\phi^{\circ}(t) = \frac{\sin([d + 1/2]\omega_1 t)}{\sin(\omega_1 t/2)}, \quad \text{where} \quad \omega_1 = \frac{2\pi}{T}. \tag{22}$$

The functions $\{\phi^{\circ}(t - k); k = 0, 1, \ldots, T - 1\}$ are appropriate for reconstituting a continuous periodic function $x(t)$ defined on the interval $[0, T)$ from its sampled ordinates $x_0, x_1, \ldots, x_{T-1}$. However, the function can also be reconstituted, in the manner of equation (1), from it Fourier ordinates as

$$x(t) = \sum_{j=-d}^{d} \xi_j e^{\mathrm{i}\omega_j t} = \sum_{j=0}^{d} \{\alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t)\}, \tag{23}$$

where $\xi_j = (\alpha_j + \mathrm{i}\beta_j)/2$ and $\xi_{-j} = (\alpha_j - \mathrm{i}\beta_j)/2$.

For economic data, the sampling interval is typically a month, a quarter or a year. For financial data, it may be less. Whatever the case, it is unlikely that this interval will correspond to an integer multiple of the critical rate of sampling of whatever band-limited process might underlie the data. Therefore, it may not be possible to achieve the critical rate by subsampling the data. Nevertheless, if the Nyquist frequency exceeds the maximum frequency of the process, then it will always be possible to resample the data at the critical rate. This can be achieved once the continuous function has been reconstituted, via the formula of (23), from the relevant Fourier ordinates of the available data.

The sinc function is widely dispersed in time. Wave packets of a more compact nature can be formed from alternative band-limited spectral energy functions. The wide dispersion of the sinc function is due to the sharp edges of the rectangular energy function. Energy functions with smoother profiles will generate more compact wave packets that might constitute better models of the elements of a wave train sequence underlying a continuous band-limited stochastic function.

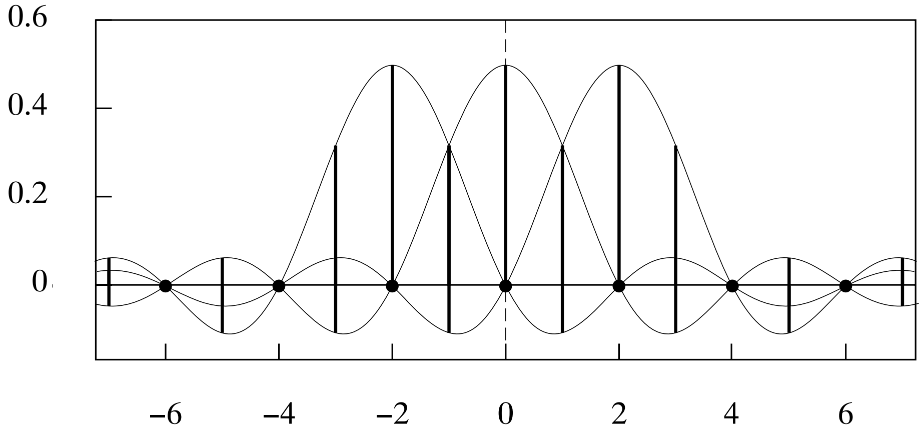

Figure 9. The wave packets $\phi_{(2)}(t)$ and $\phi_{(2)}(t-k)$ suffer no interference when $k \in \{2, 4, 6, \ldots\}$.

4. The Processes Underlying the Data

A clear message that comes from the example of Figure 6, and from others like it, is that the component structures of many macro econometric sequences are distinctly band-limited in frequency. Since these structures are quite unlike those that belong to conventional ARMA or ARIMA processes, it is appropriate to enquire into the nature of the processes that do underlie the discretely sampled data.

Our model of a process underlying discretely sampled data is of a sequence of band-limited wave packets of random amplitudes, which may also have varying profiles. These may arrive at random intervals in the manner of a Poisson process. Since the wave packets are band limited, it will always be possible to resolve their sum into a sequence of sinc functions at regular displacements along the time axis, having amplitudes that can be described by a discrete stochastic process.

In a continuous band-limited white-noise process, the amplitudes of successive regularly-spaced sinc functions constitute a sequence of independently and identically distributed random variables, which corresponds to a discrete-time white-noise process. In a continuous ARMA process driven by a continuous band-limited white-noise, the amplitudes of the sinc functions are described by an ordinary discrete-time ARMA process.

When the interval between successive functions is $\Delta = 1/n$, the generic sinc function is $\sin\{n(t-k)\}/\pi(t-k)$. In the limit, as $\Delta \to 0$, this becomes $\delta(t-k) = \lim(n \to \infty)\sin\{n(t-k)\}/\pi(t-k)$, which is Dirac's delta function, also described as the unit impulse. The integral of this function is zero for all $t < k$ and unity for all $t > k$. The limiting white-noise process $\zeta(t)$, for which

$$E\{\zeta(t)\} = 0 \quad \text{and} \quad E\{\zeta(t)\zeta(t+\tau)\} = \sigma^2\delta(\tau), \tag{24}$$

is commonly described as continuous white noise. It consists of the orthogonal increments of Wiener process, which is the continuous-time version of the random walk.

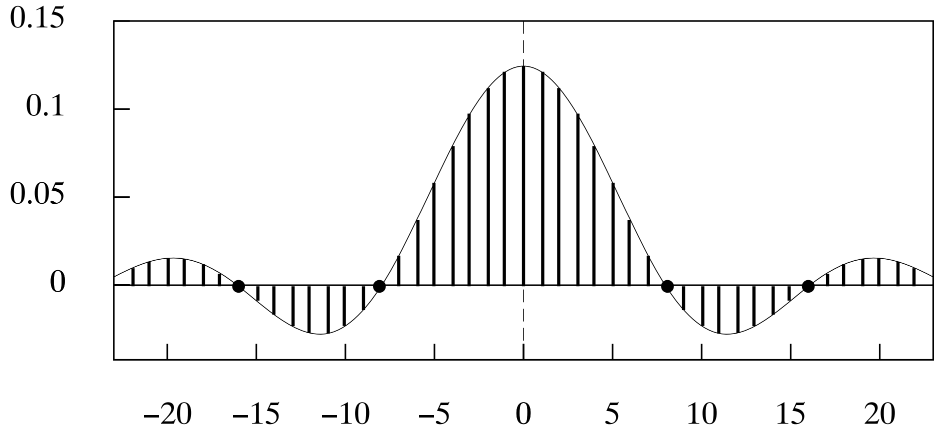

Figure 10. The sinc function wave-packet $\phi_3(t) = \sin(\pi t/8)/\pi t$ comprising frequencies in the interval $[0, \pi/8]$.

An impulse in continuous time has a uniform power spectrum that is distributed over the entire frequency range, with infinitesimal power in any finite interval. A Wiener process is the product of the cumulation of a steam of infinitesimal impulses which are the stationarity and independent increments of the process.

Whereas an ideal impulse is familiar within the conceptual realm of physics, it has no counterpart in the real physical world, where we can expect there to be distinct limits to the frequencies of oscillatory motions. On the other hand, there is a common understanding that, in the real world, time flows continuously. Therefore, there is a tendency to regard discrete-time models, such as ARMA models, as approximations to their more realistic continuous-time counterparts, which are stochastic differential equations.

It is assumed, almost without exception, that the driving forces of such differential equations are provided by the increments of Wiener processes. The sampling of a Wiener process at regular intervals entails the phenomenon of aliasing whereby the cumulated increments gives rise to a uniform spectrum of finite power over the interval $[-\pi, \pi]$. Therefore, according to the usual understanding, the correspondence the parameters of an ARMA model, fitted to the discretely sampled data, and those of the related differential equation is complicated by the problem of aliasing.

In the case of a differential equation driven by a process limited to frequency less that the Nyquist frequency of π, there will be no problem of aliasing to contend with. However, another problem of a different nature may arise, which can be described as interference.

Consider sampling a band-limited process at a rate in excess of the critical Nyquist rate. According to the Shannon interpolation formula, the value of the process at an arbitrary point in time is liable to be expressed as a sum of values sampled from an infinite number of wave packets. The sum will include values sampled from the tails of receding wave packets as well as values that represent the onset of the waves that lie ahead. Since a discrete ARMA looks in one direction only, it appears, at first sight, to be incapable of modelling such a process.

This problem in ARMA modelling that arises from interference, in common with the problem of aliasing, can be overcome by sampling the data at the critical rate that corresponds to the highest frequency present in the underlying process. In that case, there will be no interference, either from the waves that lie ahead of the current sample point or from the waves that have passed by, and the relationship of the current value to the previous sampled values can be described by a discrete ARMA model.

This outcome is evident in Figure 9, which depicts three adjacent elements of a sinc function basis which is appropriate to a continuous stochastic function band limited to the interval $[0, \pi/2]$. If observations are taken at successive integer points, then those for $t \in \{\pm 1, \pm 3, \pm 5, \ldots\}$ will be comprise samples from all of the basis functions. If observations are sampled at two-point intervals, as is the case when $t \in \{0, \pm 2, \pm 4, \ldots\}$, then the sampled values will correspond to the amplitudes of individual sinc functions; and there will be no interference, at the sampled points, from adjacent sinc functions.

In the case the business cycle function of Figure 2, which, on the evidence of Figure 6, is band-limited to the frequency interval $[0, \pi/8]$, only one in eight of the points sampled from this function at unit time intervals needs to be retained in order to convey all of the relevant information.

The periodogram of the resulting subsampled sequence, which has the frequency range of $[0, \pi]$, will have a profile identical to that of the spectral of structure that occupies the interval $[0, \pi/8]$ in Figure 6. The sinc function $\psi_3 = \sin(\pi t/8)\pi t$, of which the 8-point displacements $\{\psi_3(t-8k); k \in \mathcal{I}\}$ provide a basis for functions in the frequency band $[0, \pi/8]$, is illustrated in Figure 10.

5. The Whittle Maximum-likelihood Estimator

We shall now consider methods for estimating the parameters of a band-limited ARMA process. We shall adopt the Whittle's (1951, 1962) version of the maximum-likelihood estimator, which can be modified to take account of only a subset of the frequency range of the data.

Consider a vector $x = [x_0, x_1, \ldots, x_{T-1}]'$ of T observations of which the z-transform is $x(z) = x_0 z + x_1 z + \cdots + x_{T-1} z^{T-1}$. Let $I_T = [e_0, e_1, \ldots, e_{T-1}]$ denote the identity matrix of order T and let $K_T = [e_1, \ldots, e_{T-1}, e_0]$ denote the circulant operator, which is formed by displacing the leading vector of the identity matrix to the end of the array. Then, the circulant data matrix $X = x(K_T)$ is formed by replacing z within $x(z)$ by the matrix K_T. (An account of the algebra of circulant matrices has been provided by Pollock 2002. See, also, Gray 2002.)

The circulant matrix K_T may be factorised as

$$K_T = \bar{U} D U = U \bar{D} \bar{U}, \tag{25}$$

where

$$\begin{aligned} U &= T^{-1/2}[W^{jt}; t, j = 0, \ldots, T-1] \quad \text{and} \\ D &= \mathrm{diag}\{1, W, W^2, \ldots, W^{T-1}\}, \quad \text{with} \\ W &= \exp\{-\mathrm{i}2\pi/T\}, \end{aligned} \tag{26}$$

whilst $\bar{U} = T^{-1/2}[W^{-jt}; t, j = 0, \ldots, T-1]$ is the conjugate of the unitary symmetric matrix U, such that $\bar{U}U = U\bar{U} = I$.

The second equality of (25) follows from the fact that K_T, which is real-valued, must equal its own complex conjugate. The matrix $T^{-1}W = T^{-1/2}U$ is that of the discrete Fourier transform (DFT), whereas $W^{-1} = \bar{W} = T^{1/2}\bar{U}$ is the matrix of the inverse DFT.

Consider factorising the circulant data matrix thus:

$$x(K_T) = X = \bar{U}x(D)U = Ux(\bar{D})\bar{U}. \tag{27}$$

Since $X' = Ux(D)\bar{U} = \bar{U}x(\bar{D})U$, the matrix of empirical circular autocovariances is

$$\begin{aligned} C &= T^{-1}X'X = T^{-1}\bar{U}c(D)U, \quad \text{where} \\ c(D) &= \mathrm{diag}\{g_0, g_1, \ldots, g_{T-1}\} \\ &= T\mathrm{diag}\{|\xi_0|^2, |\xi_1|^2, \ldots, |\xi_{T-1}|^2\}. \end{aligned} \tag{28}$$

The diagonal elements of $c(D)$ are the ordinates of the periodogram. The expected value of C is

$$D(x) = \Gamma = \bar{U}\gamma(D)U, \tag{29}$$

where $\gamma(D)$ contains ordinates of the spectral density function.

On the assumption that it has a normal distribution, the logarithm of the density function of the sample is

$$\log L = \frac{-T}{2}\log(2\pi) - \frac{1}{2}\log|\Gamma| - \frac{1}{2}x'\Gamma^{-1}(x)x. \tag{30}$$

For a circular process, there is $|\Gamma| = \prod_{j=0}^{T-1}\gamma_j$ and

$$x'\Gamma^{-1}(x)x = x'\bar{U}\gamma^{-1}(D)Ux = \sum_{j=0}^{T-1} g_j/\gamma_j. \tag{31}$$

Therefore,

$$\log|D(x)| + x'D^{-1}(x)x = \sum_{j=0}^{T-1}\left\{\log(\gamma_j) + \frac{g_j}{\gamma_j}\right\}. \tag{32}$$

This is the Whittle's criterion function of which the minimisation provides the maximum-likelihood estimates of the parameters of the stochastic process that has generated the data.

A generalised form of the Whittle function is

$$L = \sum_{j=0}^{T-1} w_j\left\{\log(\gamma_j) + \frac{g_j}{\gamma_j}\right\}. \tag{33}$$

Here, w_j is a nonnegative factor, which can be used to vary the weight that is attributed to the jth ordinate of the periodogram. This criterion function has been investigated by Thomson (1986), and it has been employed in various applications by Robinson (1995), by Haywood and Tunnicliffe Wilson (1997) and by Proietti (2007), amongst others.

If x is generated by an ordinary ARMA process, then its z-transform satisfies the equation

$$\alpha(z)x(z) = \mu(z)\varepsilon(z), \tag{34}$$

wherein $\varepsilon(z)$ is the z-transform of a sequence $\varepsilon(t) = \{\varepsilon_t; t = 0, \pm 1, \pm 2, \ldots\}$ of independently and identically distributed mean-zero random disturbances, described as white noise.

The assumptions affecting an ordinary white-noise sequence can be expressed as

$$E\{\varepsilon(z)\} = 0 \quad \text{and} \quad E\{\varepsilon(z)\varepsilon(z^{-1})\} = \gamma_\varepsilon(z) = \sigma^2. \tag{35}$$

Setting $z = \exp\{-i\omega\}$ in $\gamma_\varepsilon(z)$ gives a spectral density function that is uniform over the range $[-\pi, \pi]$ of the frequency variable ω.

In the case of a band-limited white-noise process $\zeta(t)$, for which the spectral density is non zero and uniform over the interval $(-d, d) \in [-\pi, \pi]$ and zero elsewhere, the spectral density function is given by $\gamma_\zeta(\omega) = \sigma^2\phi(\omega)$, where

$$\phi(\omega) = \begin{cases} 1, & \text{if } \omega \in (-d, d), \\ 1/2, & \text{if } \omega = \pm d, \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

When the ARMA transfer function $\mu(z)/\alpha(z)$ is applied to the band-limited noise, a process is derived that has the following autocovariance generating function:

$$\gamma(z) = \sigma^2 \frac{\mu(z)\phi(z)\mu(z^{-1})}{\alpha(z)\alpha(z^{-1})}, \tag{37}$$

where $\phi(z)$ is the z-transform of the sequence of ordinates sampled at the integer time points from the sinc function

$$\phi(t) = \frac{\sin(dt)}{\pi t} = \frac{1}{2\pi}\int_{-d}^{d} e^{\mathrm{i}\omega t}. \tag{38}$$

Setting $z = K_T$ gives $\Gamma = \gamma(K_T)$, which is the matrix of circular autocovariances of (29). Setting $z = \exp\{\mathrm{i}2\pi j/T\} = W^j$, and letting $j = 0, 1, \ldots, T-1$, generates the spectral ordinates $\gamma_0, \gamma_1, \ldots \gamma_{T-1}$, which are the diagonal element of the matrix $\gamma(D)$.

6. Alternative Estimates of an AR(2) Process

We can proceed to use the Whittle criterion to obtain some alternative estimates of the band-limited AR(2) process that appears to be a reasonable model for describing the business cycles that are depicted in Figure 2. We shall take three alternative approaches, which produce highly contrasting results; and it is the process that entails a resampling of the data combined with anti-alias filtering that is the only appropriate one.

For the first approach, we need to work with seasonally adjusted data. To obtain such data, we shall apply a filter that mimics the processes of seasonal adjustment that occur within national central statistical offices.

The filter is derived from a model that combines a white-noise component $\eta(t)$ with a seasonal component obtained by passing an independent white noise $\nu(t)$ through a rational filter with unit poles at the seasonal frequencies and with corresponding zeros close to unity. The z-transform of the output sequence gives

$$\begin{aligned} y(z) &= \eta(z) + \frac{R(z)}{S(z)}\nu(z) \quad \text{or} \\ S(z)y(z) &= S(z)\eta(z) + R(z)\nu(z), \end{aligned} \tag{39}$$

where

$$\begin{aligned} R(z) &= 1 + \rho z + \rho^2 z^2 + \cdots + \rho^{s-1}z^{s-1} \\ &= \prod_{j=1}^{s-1}(1 - \rho e^{2\pi j/s}), \end{aligned} \tag{40}$$

with $z = \exp\{-\mathrm{i}2\pi/\mathrm{T}\}$ and $\rho < 1$, and

$$\begin{aligned} S(z) &= 1 + z + z^2 + \cdots + z^{s-1} \\ &= \prod_{j=1}^{s-1}(1 - e^{2\pi j/s}). \end{aligned} \tag{41}$$

The transfer function of the seasonal-adjustment filter is

$$\beta(z) = \frac{\sigma_\eta^2 S(z)S(z^{-1})}{S(z)S(z^{-1})\sigma_\eta^2 + \sigma_\nu^2 R(z)R(z^{-1})}. \tag{42}$$

Setting $z = \exp\{\mathrm{i}\omega\}$ and letting ω run from 0 to π generates the frequency response of the filter, of which the squared modulus, or squared gain, is plotted in Figure 11 for the case where $\rho = 0.9$ and $\lambda = \sigma_\eta^2/\sigma_\nu^2 = 0.125$.

To derive a version of the filter that is applicable to a finite sample of length T, we may replace z within the component polynomials by the circulant operator K_T. This produces the matrices $Q_S = S(K_T)$ and $Q_R = R(K_T)$. If y is the original data vector, then the seasonally-adjusted data is

$$h = Q_S(Q_S'Q_S + \lambda^{-1}Q_R'Q_R)^{-1}Q_S'y. \tag{43}$$

Similar filters have been derived and illustrated by Pollock (2006, 2007).

Figure 12 shows the seasonally-adjusted version of the data sequence of Figure 2, and Figure 13 shows the extracted seasonal component. The remarkable regularity of the seasonal component, which is not unusual among such estimates, is an artefact of the filter $1 - \beta(z)$, which is complementary to the seasonal-adjustment filter. This filter extracts from the data the elements at the seasonal frequencies $\pi/2$ and π, together with a small proportion of what lies at the adjacent frequencies.

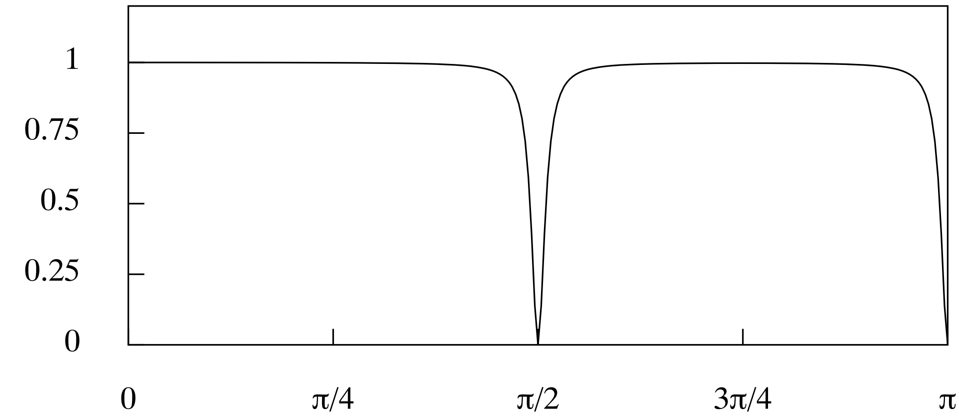

Figure 11. The squared gain of a seasonal adjustment filter to be applied to the quarterly detrended logarithmic consumption data.

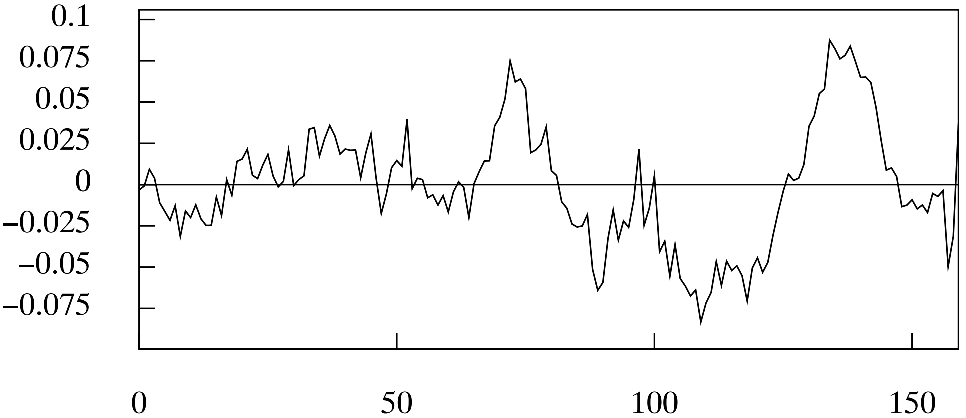

Figure 12. The seasonally adjusted detrended logarithmic consumption data.

The periodogram of the seasonally adjusted data is show in Figure 14. Beyond the upper limit of the spectrum of the low-frequency business cycle component, there are minor traces of a contaminating noise, which serves to roughen the profile of the seasonally-adjusted data. The smooth profile of the business cycle, from which this noise is absent, is shown in Figure 2.

When it is applied to the seasonally-adjusted data, the ordinary Whittle estimator delivers an AR(2) model of which the parametric spectrum is show in Figure 15. The roots of the estimated autoregressive operator, which are real-valued, are show of the left side Figure 17. This outcome is due, in large measure, to the presence of the noise contamination in the seasonally-adjusted data; and its partial removal leads to very different estimates.

The effect of the noise contamination can be reduced by adopting the generalised Whittle estimator, which allows differential weights to be attached to the ordinates of the periodogram. Figure 16 shows the parametric spectrum of an AR(2) model that has been fitted using the generalised Whittle criterion of band-spectral estimation, which is superimposed upon the periodogram of the

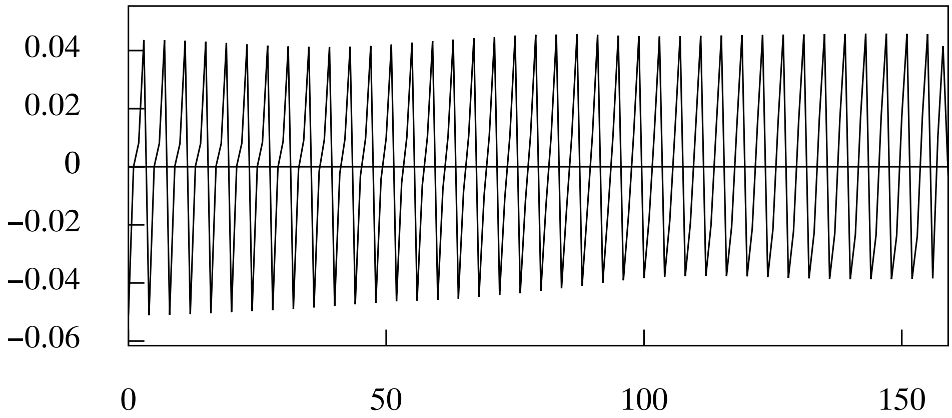

Figure 13. The seasonal component extracted from the detrended logarithmic consumption data in the process of seasonal adjustment.

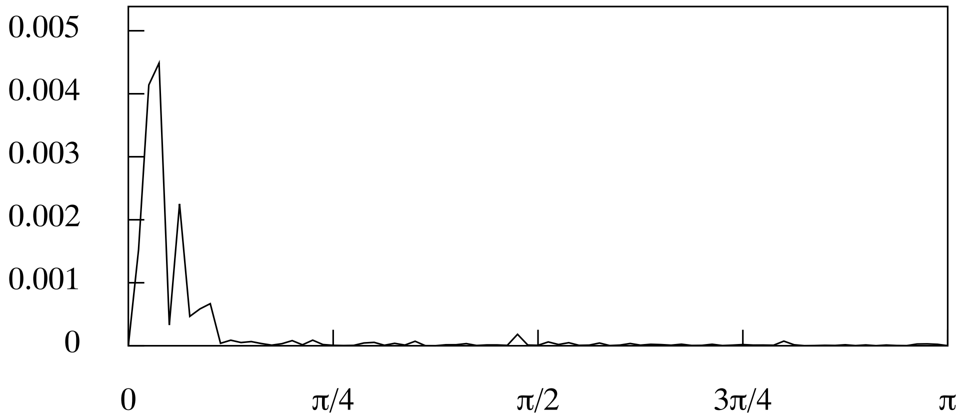

Figure 14. The periodogram of the seasonally adjusted data.

data. In the estimation, the selected periodogram ordinates that lie in the frequency interval $[0, \pi/4]$ have been associated with unit weights $w_j = 1$, and the remainder have been associated with zero weights. The highlighted region of the diagram indicates the range of the selected ordinates.

The effect of the frequency selection upon the parametric spectrum of the estimated model is remarkable. Figure 16 shows a spectrum with a prominent spike at a frequency that corresponds to that of the business cycle. The complex roots of the autoregressive operator are shown on the right side of Figure 17, where it can be seen that they are close to the boundary of the unit circle.

Despite having seemingly appropriate characteristics, the estimates provided by the band-limited method are of doubtful validity. Already, the over-prominent spike of the parametric spectrum appears to be misrepresenting the underlying periodogram. Moreover, in view of the fact that the spectral structure of the business cycle is limited to the interval $[0, \pi/8]$, it would seem appropriate to limit the spectral band of the estimator to the same interval, instead of the wider interval of $[0, \pi/4]$. The effect of this further limitation is to increase

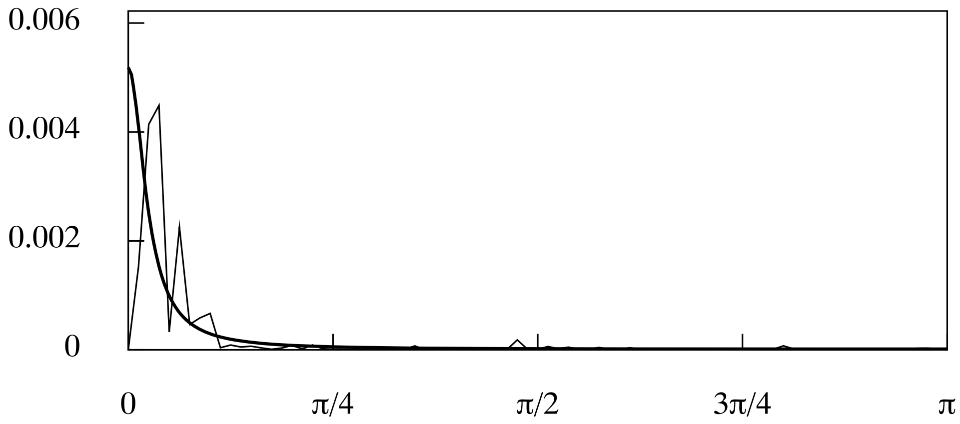

Figure 15. The spectrum of an AR(2) model fitted to the detrended, seasonally adjusted logarithmic income data, superimposed on the periodogram.

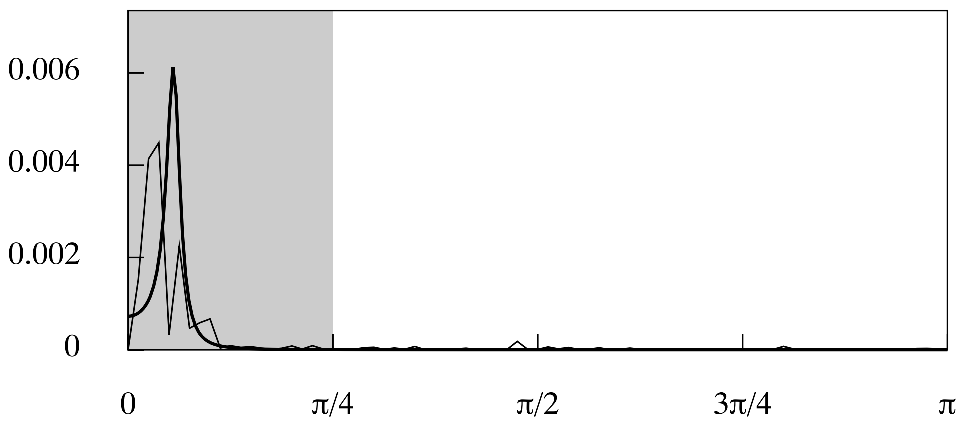

Figure 16. The spectrum of an AR(2) model fitted to the detrended, deseasonalised logarithmic income data via a band-limited autoregressive estimation, superimposed on the periodogram.

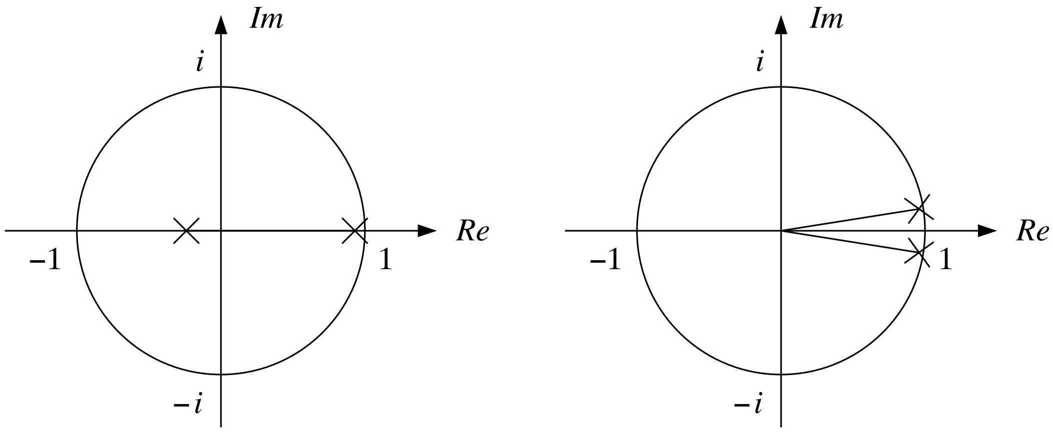

Figure 17. The poles of the AR(2) models fitted to the detrended logarithmic income data. (a) is from unrestricted estimator and (b) is from the band-limited estimator.

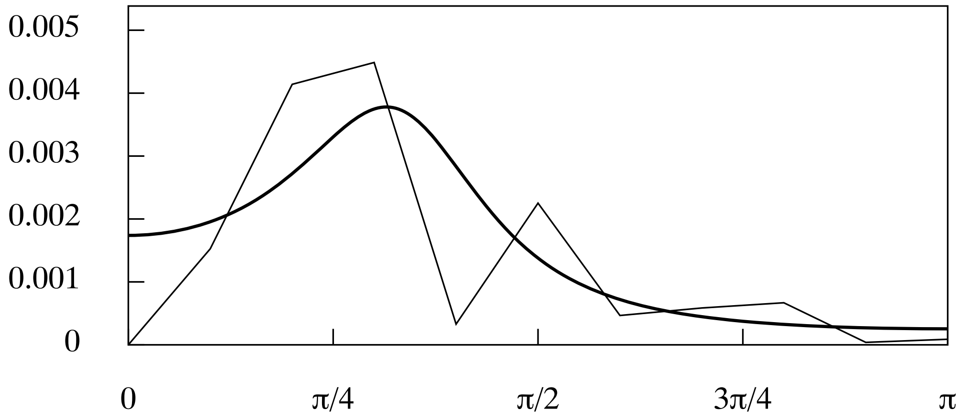

Figure 18. The periodogram of the sub sampled anti-aliased data with the parametric spectrum of an estimated AR(2) model superimposed.

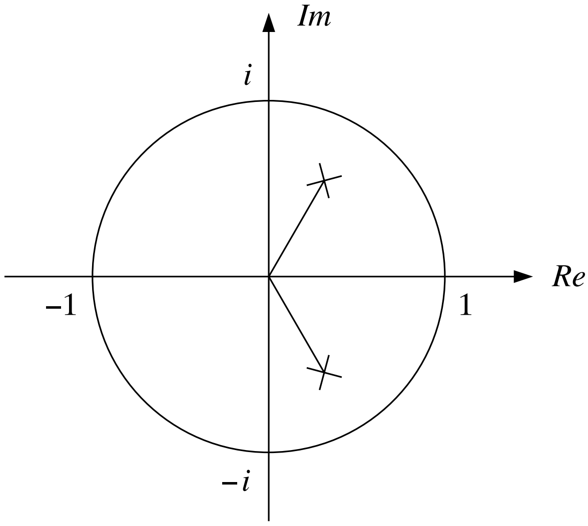

Figure 19. The poles of the AR(2) model fitted to 20 points subsampled at the rate of 1 in 8 from data that has been subjected to an anti-aliasing lowpass filter with cut off at $\pi/8$ radians.

radically the prominence of the spectral spike.

The conclusion of these experiments is that the outcome of the band spectral estimation is highly dependent upon the extent of the frequency band. Extending the band to include a small part of the noise contamination that is evident in the periodogram of the seasonally-adjusted data has a major effect upon the estimated autoregressive parameters

The appropriate estimator of the business cycle parameters is one that takes account only of the information within the corresponding spectral structure and which maps this structure into the interval $[0, \pi]$, which is the domain of an ordinary ARMA model. To achieve this outcome, the spectral elements that fall outside the frequency range of the business cycle must first be removed. This operation, which constitutes an anti-alias filtering, may be carried out either in the time domain or in the frequency domain.

Given the availability of the spectral ordinates of the data, it is straightforward to operate in the frequency domain by setting the rejected ordinates to zeros. Then, a continuous low-frequency function can be synthesised from the selected ordinates. An example is provided by the interpolated function in Figure 2. The synthesised function can be resampled at a rate that corresponds to the maximum frequency within the spectral structure of the business cycle.

Since we have deemed that the maximum frequency within the business cycle is at $\pi/8$, the data is to be resampled at 1/8th of the original rate of observation. This simple fractional rate is a convenient one, since it implies taking one in every eight of the anti-aliased data points. In that case, there is no need synthesise a continuous function for the purpose of resampling the data.

The periodogram of the subsampled anti-aliased data is show in Figure 18 with the parametric spectrum of an estimated AR(2) model superimposed. The periodogram represents a rescaled version of the part of the periodogram of Figure 6, pertaining to the original data, that occupies the frequency range $[0, \pi/8]$; and it appears to be well represented by the parametric spectrum. Also, the roots of the autoregressive operator, which are displayed in Figure 19, appear to give an appropriate representation of the dynamic properties of the business cycle, both as regards its frequency and its damping characteristics.

7. Conclusions and Remarks

In this paper, we have presented a model for a band-limited stochastic process in continuous time and we have demonstrated that, if the rate of sampling corresponds to the maximum frequency within the data, then a sequence of sampled ordinates can be described by an ordinary discrete-time ARMA model. In these circumstances, it may be said that the sampling is at the critical rate.

The model is all-inclusive. It provides the continuous-time counterpart of the conventional ARMA model, where the forcing function is a discrete white-noise process with a uniform spectral density function defined on the frequency interval $[-\pi, \pi]$. It also encompasses the case of continuos-time autoregressive model driven by the increments of a Wiener process, which are unbounded in frequency.

We have concentrated on the case where the sampling is over-rapid and where we can afford to reduce it, either by a process of sub sampling or by resampling a reconstituted version of the continuous signal. There are also cases to be considered where the sampling rate is less than the maximum frequency of the data and where there is an inevitable problem of aliasing. Such circumstances are already accounted for in the existing literature under the rubric of temporal aggregation.

The available theory indicates that, if the data can be described, at the critical rate of sampling, by an AR(p) autoregressive model of order p, then the sub sampled data will be described by an ARMA($p, p-1$) model. The only proviso here is that, in both cases, the sampling rate should be sufficiently rapid to accommodate the dynamics implied by the autoregressive operator. Thus, the maximum value ω_m of the arguments associated with the roots of the operator must be less than the Nyquist frequency of π radians per sample period.

Telser (1967) derived this result in the context of what he described as the skip sampling of a discrete-time autoregressive process, whereas Phadke and Wu (1974) and Pandit and Wu (1975) did so by considering the relations between continuous-time linear stochastic processes driven by the increments of a Wiener process and the models that can be fitted to the sampled data.

Our example of a band-limited process is complicated by the fact that it is a component of a composite process which includes a trend. In this case, we have been able to represent the trend by a quadratic function that is virtually linear. However, in general, we can expect that the orderly development of the data will be disrupted, on occasion, by structural breaks and outliers.

Outliers and level shifts constitute what we may call impact events. Their effect is to add to the periodogram of the quadratically detrended data some components that have diffuse spectra that extend in frequency well beyond the maximum frequency of the business cycle. An outlier on a single point contributes a uniform increment to the ordinates of the periodogram. An abrupt shift of level has the same effect upon the periodogram as do the disjunctions that are liable to occur in the periodic extension of the data where the end of one replication of the sample joins the beginning of the next.

If we wish to preserve the band-limited model in the face of such disturbances, then we should need to remove the outliers and to accommodate any level shifts within the trend. The usual detrending devices are commonly based on models that describe data components that are generated by homogeneous time invariant processes. They are not appropriate to our purpose, and they need to be adapted.

An adaptive device that has proved to be very serviceable is the Reinsch (1976) smoothing spline with a smoothing parameter that is amenable to local variations. A sufficient reduction in the value of the smoothing parameter in the vicinity of a level shift will allow the spline to incorporate the shift.

Two or three such accommodations might be acceptable in a data sequence of a length similar to that of our example. When the disturbances within the data become more numerous, there begin to be doubts over whether it can sustain a constant-coefficient parametric model of the sort that we have attributed to the business cycle. Then, a non-parametric method of analysis is in order that entails wave packets within a hierarchy of frequency bands. This is a multiresolution wavelet analysis in other words.

References

Gray, R.M., (2002), *Toeplitz and Circulant Matrices: A Review,* Information Systems Laboratory, Department of Electrical Engineering, Stanford University, California, (`http://ee.stanford.edu/gray/~toeplitz.pdf`).

Haywood, J., and G. Tunnicliffe Wilson, (1997), Fitting Time Series Models by Minimising Multistep-Ahead Errors: A Frequency Domain Approach, *Journal of the Royal Statistical Society, Series B (Methodological),* Vol. 59, No. 1. (1997), pp. 237–254.

Pandit, S.M., and S.M. Wu, (1975), Unique Estimates of the Parameters of a Continuous Stationary Stochastic Process, *Biometrika,* 62, 497–501.

Pagan, A., (1997), Towards an Understanding of Some Business Cycle Characteristics, *The Australian Economic Review,* 30, 1–15.

Phadke, M.S., and S.M. Wu, (1974), Modelling of Continuous Stochastic Processes from Discrete Observations with Applications to Sunspot Data, *Journal of the American Statistical Association,* 69, 325–329.

Pollock, D.S.G., (2002),Circulant Matrices and Time-Series Analysis, *The International Journal of Mathematical Education in Science and Technology,* 2002, 33, 213–230.

Pollock, D.S.G., (2006), Econometric Methods of Signal Extraction, *Computational Statistics and Data Analysis,* 50, 2268–2292

Pollock, D.S.G., (2007), Wiener–Kolmogorov Filtering, Frequency-Selective Filtering and Polynomial Regression, *Econometric Theory*, 23, 71–88.

Proietti, T., (2007), Band Spectral Estimation and Signal Extraction, Forthcoming in *Economic Modelling.*

Reinsch, C.H., (1976), Smoothing by Spline Functions, *Numerische Mathematik,* 10, 177–183.

Robinson, P. M., (1995), Gaussian Semiparametric Estimation of Long Range Dependence, *Annals of Statistics,* 23, 1630–1661.

Shannon, C.E., (1949), Communication in the Presence of Noise, *Proceedings of the Institute of Radio Engineers,* 37, 10–21. Reprinted in 1998, *Proceedings of the IEEE,* 86, 447–457.

Telser, L.G., (1967), Discrete Samples and Moving Sums in Stationary Stochastic Processes, *Journal of the American Statistical Association,* 62, 318, 484–499.

Thomson, P.J., (1986), Band-Limited Spectral Estimation of Autoregressive-Moving Average Processes, *Journal of Applied Probability*, 23A 'Essays in Time Series and Allied Processes: Papers in honour of E.J. Hannan' (edited by J. Gani and M.B. Priestley), 143–155.

Whittaker, J.M.,(1935), *Interpolatory Function Theory,* Cambridge Tracts in Mathematics and Mathematical Physics, no. 33. Cambridge, U.K., Cambridge University Press.

Whittle, P., (1951), *Hypothesis Testing in Time Series Analysis,* Almqvist and Wiksells Boktryckeri, Uppsala.

Whittle, P., (1962), Gaussian Estimation in Stationary Time Series, *Bulletin of the International Statistical Institute,* 39, 105–129.