Mathematics **2013**, *1*, 65-75; doi:10.3390/math1020065

OPEN ACCESS

mathematics

ISSN 2227-7390

www.mdpi.com/journal/mathematics

Article

On the Class of Dominant and Subordinate Products

Alexander Berkovich * and Keith Grizzell

Department of Mathematics, University of Florida, Gainesville, FL 32611-8105, USA;
E-Mail: grizzell@ufl.edu

* Author to whom correspondence should be addressed; E-Mail: alexb@ufl.edu

Received: 25 March 2013; in revised form: 12 April 2013 / Accepted: 22 April 2013 / Published: 15 May 2013

Abstract: In this paper we provide proofs of two new theorems that provide a broad class of partition inequalities and that illustrate a naïve version of Andrews' anti-telescoping technique quite well. These new theorems also put to rest any notion that including parts of size 1 is somehow necessary in order to have a valid irreducible partition inequality. In addition, we prove (as a lemma to one of the theorems) a rather nontrivial class of rational functions of three variables has entirely nonnegative power series coefficients.

Keywords: q-series; generating functions; partition inequalities; anti-telescoping; rational functions with nonnegative coefficients

Classification: PACS [2010] Primary 11P82; Secondary 11P81; 11P83; 11P84; 05A17; 05A20

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1. Introduction

When examining two q-products Π_1 and Π_2 and their corresponding q-series, it sometimes happens that the coefficients in the q-series for Π_1 are never less than the coefficients in the q-series for Π_2. When that happens, we say that Π_1 is *dominant* (in this pair of products) and that Π_2 is *subordinate*, and we express this relationship with the more succinct notation $\Pi_1 \succcurlyeq \Pi_2$. (Note that $\succcurlyeq$ yields a partial ordering on the set of q-products if we identify products that produce the same q-series; then, any given product may be dominant when paired with some products, subordinate when paired with others, neither when paired with still other products, and both dominant and subordinate only when paired with "itself".) Immediately from this definition it follows that if $\Pi_1 \succcurlyeq \Pi_2$, then the q-series determined by $\Pi_1 - \Pi_2$

must have nonnegative coefficients, *i.e.*, $\Pi_1 - \Pi_2 \succcurlyeq 0$. Thus, determining whether a given pair of products is a dominant/subordinate pair solves an equivalent positivity problem.

Using the standard notations [1]

$$(a;q)_L = \begin{cases} 1 & \text{if } L = 0 \\ \prod_{j=0}^{L-1}(1-aq^j) & \text{if } L > 0 \end{cases} \tag{1}$$

$$(a_1, a_2, \ldots, a_n; q)_L = (a_1;q)_L (a_2;q)_L \cdots (a_n;q)_L \tag{2}$$

and

$$(a;q)_\infty = \lim_{L\to\infty} (a;q)_L \tag{3}$$

we may say that, for example, in the Rogers–Ramanujan difference

$$\frac{1}{(q, q^4; q^5)_\infty} - \frac{1}{(q^2, q^3; q^5)_\infty} \succcurlyeq 0 \tag{4}$$

the first product is dominant and the second product is subordinate. At the 1987 A.M.S. Institute on Theta Functions, Ehrenpreis asked if one can prove this dominance without resorting to the Rogers–Ramanujan identities. In 1999, Kadell [2] provided an affirmative answer to this question. In 2005, Berkovich and Garvan [3] proved a class of finite versions of such inequalities (from which the infinite versions are easily recovered), namely that

$$\frac{1}{(q, q^{m-1}; q^m)_L} \succcurlyeq \frac{1}{(q^r, q^{m-r}; q^m)_L} \tag{5}$$

if and only if $r \nmid (m-r)$ and $(m-r) \nmid r$. Note that this last inequality provides the finite version of Equation (4):

$$\frac{1}{(q, q^4; q^5)_L} \succcurlyeq \frac{1}{(q^2, q^3; q^5)_L} \tag{6}$$

In 2011, Andrews [4] proved the finite little Göllnitz inequality

$$\frac{1}{(q, q^5, q^6; q^8)_L} \succcurlyeq \frac{1}{(q^2, q^3, q^7; q^8)_L} \tag{7}$$

which (in 2012) Berkovich and Grizzell [5] generalized to

$$\frac{1}{(q, q^{y+2}, q^{2y}; q^{2y+2})_L} \succcurlyeq \frac{1}{(q^2, q^y, q^{2y+1}; q^{2y+2})_L} \tag{8}$$

where y is any odd integer greater than 1.

For Equations (4), (5), and (8), the proofs in each case relied solely on the construction of a suitable injection. For Equation (7), however, Andrews relied primarily on his anti-telescoping technique. A naïve version of Andrews' anti-telescoping technique begins with two sequences of products, $\{P(i)\}_{i=1}^{\infty}$ and $\{Q(i)\}_{i=1}^{\infty}$, and the desire to show that, for every $L \geq 1$,

$$\frac{1}{P(L)} \succcurlyeq \frac{1}{Q(L)}$$

One then simply writes (letting $P(0) = Q(0) = 1$)

$$\frac{1}{P(L)} - \frac{1}{Q(L)} = \sum_{i=1}^{L} \frac{Q(i-1)}{P(i)Q(L)} \left(\frac{Q(i)}{Q(i-1)} - \frac{P(i)}{P(i-1)} \right) \tag{9}$$

$$= \sum_{i=1}^{L} \frac{\frac{Q(i)}{Q(i-1)} - \frac{P(i)}{P(i-1)}}{P(i) \cdot \frac{Q(L)}{Q(i-1)}} \tag{10}$$

and if one is lucky enough that each addend in Equation (10) is $\succcurlyeq 0$, then that is all one needs to show in order to prove the desired inequality. This bit of serendipity is by no means trivial; for example, this naïve anti-telescoping fails to help show Equation (6) since, among numerous other terms, the coefficient of q^8 is -1 in the second ($i = 2$) addend of the naïve anti-telescoping of Equation (6) for every $L > 1$. A less naïve approach might sometimes be more beneficial, but for our purposes in this paper the naïve approach outlined above is sufficient.

Now clearly we could multiply every exponent in any inequality akin to Equations (4)–(8) by some common factor to obtain an inequality without $(1-q)$ as the leading factor in the denominator on the left; when looking at the partition-theoretic interpretation, this creates "reducible" examples (but examples nonetheless) where parts of size 1 are not needed to "fill in the gaps". In 2012, at the Ramanujan 125 Conference in Gainesville, Florida, Hamza Yesilyurt asked if the inclusion of the factor $(1 - q)$ was necessary in all irreducible inequalities. We are pleased to answer in the negative, as stated in the following new theorem.

Theorem 1.1 *For any sextuple of positive integers* (L, m, x, y, r, s),

$$\frac{1}{(q^x, q^y, q^{rx+sy}; q^m)_L} \succcurlyeq \frac{1}{(q^{rx}, q^{sy}, q^{x+y}; q^m)_L}$$

Clearly Theorem 1.1 yields infinitely many irreducible examples. More astounding, however, is that the modulus m can be *arbitrary*. Even more amazing still is the relative ease with which the proof can be written using naïve anti-telescoping!

It is also possible, albeit more difficult, to use naïve anti-telescoping to yield the following new theorem.

Theorem 1.2 *For any octuple of positive integers* (L, m, x, y, z, r, s, u),

$$\frac{1}{(q^x, q^y, q^z, q^{rx+sy+uz}; q^m)_L} \succcurlyeq \frac{1}{(q^{rx}, q^{sy}, q^{uz}, q^{x+y+z}; q^m)_L}$$

The extra difficulty in proving Theorem 1.2 comes from the fact that it seems to be impossible to re-write the addends in a natural way that makes it obvious that each addend only contributes nonnegative coefficients to the q-series. Consequently, en route to proving Theorem 1.2, we will require the following unobvious result, which is worthwhile in its own right and is not found anywhere else. (Most notably, we do not find anything of this form in [6], which contains a compendium of rational functions with nonnegative coefficients.)

Lemma 1.3 *Let* r *and* s *be positive integers. Then the multivariate rational function*

$$f(x, y, t) := \frac{(1 - xy)(1 - tx^r)(1 - ty^s) + (1 - t^2)(x - x^r)(y - y^s)}{(1 - tx^r)(1 - ty^s)(1 - x)(1 - y)(1 - tx)(1 - ty)}$$

with $|x| < 1$, $|y| < 1$, *and* $|t| < 1$, *has nonnegative coefficients when written as a power series centered at* $(0, 0, 0)$.

In Section 2, we provide a proof of Theorem 1.1 using a simple rational function identity together with naïve anti-telescoping, followed by a discussion of a partition theoretic interpretation of the difference

$$\frac{1}{(q^x, q^y, q^{rx+sy}; q^m)_L} - \frac{1}{(q^{rx}, q^{sy}, q^{x+y}; q^m)_L}$$

In Section 3 we give a proof of Lemma 1.3, which will be used in the proof of Theorem 1.2 in Sectionn 4. We then conclude in Section 5 with a brief discussion of a more general inequality.

2. Proof of Theorem 1.1

Let $P(i) := (q^x, q^y, q^{rx+sy}; q^m)_i$ and $Q(i) := (q^{rx}, q^{sy}, q^{x+y}; q^m)_i$. We observe that since the identity

$$\begin{aligned}&(1-t\alpha)(1-t\beta)(1-txy) - (1-tx)(1-ty)(1-t\alpha\beta)\\ &\quad = t(x-\alpha)(1-\beta)(1-ty) + t(y-\beta)(1-t\alpha)(1-x)\end{aligned}$$

is true, substituting q^x, q^y, q^{rx}, and q^{sy} for x, y, α, and β, respectively, we can conclude that

$$(1-tq^{rx})(1-tq^{sy})(1-tq^{x+y}) - (1-tq^x)(1-tq^y)(1-tq^{rx+sy}) \tag{11}$$

and

$$tq^x(1-q^{(r-1)x})(1-q^{sy})(1-tq^y) + tq^y(1-q^{(s-1)y})(1-tq^{rx})(1-q^x) \tag{12}$$

are identically equal. Letting $t = q^{(i-1)m}$, we may use the equality of Equations (11) and (12) to write

$$\frac{Q(i-1)}{P(i)Q(L)}\left(\frac{Q(i)}{Q(i-1)} - \frac{P(i)}{P(i-1)}\right) = V(i) + W(i)$$

where

$$V(i) := \frac{q^{m(i-1)+y}(1-q^{(s-1)y})(1-q^x)(1-q^{m(i-1)+rx})}{P(i)\cdot Q(L)/Q(i-1)}$$

and

$$W(i) := \frac{q^{m(i-1)+x}(1-q^{(r-1)x})(1-q^{sy})(1-q^{m(i-1)+y})}{P(i)\cdot Q(L)/Q(i-1)}$$

We note that since $(1-q^x)$ and $(1-q^y)$ are factors of the product $P(i)$ and since $(1-q^{m(i-1)+rx})$ is a factor of the product $Q(L)/Q(i-1)$, we have $V(i) \succcurlyeq 0$ for $1 \le i \le L$. To see that $W(i) \succcurlyeq 0$, we consider the following two cases.

1. Suppose $i = 1$; then $(1-q^x)$ and $(1-q^y) = (1-q^{m(i-1)+y})$ are factors of $P(i) = P(1)$ and $(1-q^{sy})$ is a factor of $Q(L)/Q(i-1) = Q(L)$. Thus, $W(1) \succcurlyeq 0$.
2. Suppose $i > 1$; then $(1-q^x)$, $(1-q^y)$, and $(1-q^{m(i-1)+y})$ are all independent factors of $P(i)$. Thus, $W(i) \succcurlyeq 0$.

Finally, applying the anti-telescoping Equation (9), we have

$$\frac{1}{P(L)} - \frac{1}{Q(L)} = \sum_{i=1}^{L} (V(i) + W(i)) \tag{13}$$

which then suffices to prove the theorem.

It would be nice to have a combinatorial proof of Equation (13), but such has not been discovered by the time this paper was written. We note, however, that a partition interpretation of the right-hand side of Equation (13) is possible. Given a partition π, we let p_j denote the part that is equal to $p + (j-1)m$, and we let $\nu(p_j, \pi)$ represent the number of occurrences of the part p_j in the partition π. Then, for a fixed L we define

$$\mathfrak{M}(p, \pi) := \max\left(\{j : \nu(p_j, \pi) > 0\} \cup \{0\}\right)$$

and

$$\mathfrak{m}(p, \pi) := \min\left(\{j : \nu(p_j, \pi) > 0\} \cup \{L+1\}\right)$$

We may consider $\sum_{i=1}^{L} V(i)$ and $\sum_{i=1}^{L} W(i)$, from Equation (13), as two separate generating functions for partitions into parts congruent to (for $1 \leq i \leq L$) x_i, y_i, $(x+y)_i$, $(rx)_i$, $(ry)_i$, or $(rx+ry)_i$, subject to certain restrictions. (Note: in the cases where a particular part could arise in multiple ways, for example if $x_3 = y_1$ or $rx = y$, then it would be necessary to treat the parts that arise in different ways as distinct, perhaps by assigning them unique colors based on what the base part is; since the base part is always one of x, y, $(x+y)$, rx, sy, and $(rx+sy)$, no more than six colors should be required.) We may take the restrictions as follows.

Restrictions for $\sum_{i=1}^{L} V(i)$:		Restrictions for $\sum_{i=1}^{L} W(i)$:	
V1:	$\mathfrak{M}(y, \pi) \geq \max(\{1, \mathfrak{M}(x, \pi)\})$	W1:	$\mathfrak{M}(x, \pi) > \mathfrak{M}(y, \pi)$
V2:	$\mathfrak{M}(y, \pi) \geq \mathfrak{M}(rx + sy, \pi)$	W2:	$\mathfrak{M}(x, \pi) \geq \mathfrak{M}(rx + sy, \pi)$
V3:	$\mathfrak{m}(rx, \pi) > \mathfrak{M}(y, \pi)$	W3:	$\mathfrak{m}(rx, \pi) \geq \mathfrak{M}(x, \pi)$
V4:	$\mathfrak{m}(sy, \pi) \geq \mathfrak{M}(y, \pi)$	W4:	$\mathfrak{m}(sy, \pi) \geq \max(\{2, \mathfrak{M}(x, \pi)\})$
V5:	$\mathfrak{m}(x + y, \pi) \geq \mathfrak{M}(y, \pi)$	W5:	$\mathfrak{m}(x + y, \pi) \geq \mathfrak{M}(x, \pi)$
V6:	$\nu(x_1, \pi) = 0$	W6:	$\nu(x_1, \pi) < r - 1$
V7:	$\nu(y_1, \pi) < s - 1$	W7:	$\nu(y_1, \pi) < s$

Since the restrictions V1 and W1 are mutually exclusive, we may consider the right-hand side of Equation (13) as the generating function for partitions into parts congruent to (for $1 \leq i \leq L$) x_i, y_i, $(x+y)_i$, $(rx)_i$, $(sy)_i$, or $(rx+sy)_i$ such that the partition satisfies either V1–V7 or W1–W7.

3. Proof of Lemma 1.3

Let $[t^n]F(t)$ denote the coefficient of t^n extracted from $F(t)$ (when written as a Maclaurin series). Direct calculations yield

$$\begin{aligned}[t^n]f(x,y,t) = \quad & \frac{(1-xy)(x^{n+1}-y^{n+1})}{(1-x)(1-y)(x-y)} \\ & + \frac{(-x^{n+r}(1-x^2)+x^{nr+1}(1-x^{2r}))(y-y^s)}{(1-x)(1-y)(x-y)(x^r-y^s)} \\ & + \frac{(-y^{n+s}(1-y^2)+y^{ns+1}(1-y^{2s}))(x-x^r)}{(1-x)(1-y)(x-y)(x^r-y^s)} \\ & + \frac{(x^{(n-1)r}(1-x^{2r})-y^{n-1}(1-y^2))yx^r(x-x^r)(y-y^s)}{(1-x)(1-y)(x-y)(x^r-y^s)(x^r-y)} \\ & + \frac{(y^{(n-1)s}(1-y^{2s})-x^{n-1}(1-x^2))xy^s(x-x^r)(y-y^s)}{(1-x)(1-y)(x-y)(x^r-y^s)(y^s-x)}\end{aligned} \tag{14}$$

Claim:

$$\begin{aligned}[t^n]f(x,y,t) = \quad & \frac{x^n(1-y^{n+1})}{(1-y)(1-x)} + \frac{(y^{n+1}-y^{(n+1)s})(x^n-x^r)}{(1-y)(1-x)} \\ & + \frac{(y^n-y^{ns})(x^2-x^{2r})}{(1-y)(1-x)} + \frac{x(y^n-y^{(n+1)s})}{1-y} \\ & + \sum_{j=1}^{n-1} \frac{x^{(n-j)r}(y^j-y^{js})(1-x^{2r})}{(1-y)(1-x)} \\ & + \sum_{j=0}^{(n-2-\delta(n))/2} \frac{x^{n-2j-1}y^{s(2j+1)}(1+x)}{1-y} \\ & + \sum_{j=1}^{(n-2+\delta(n))/2} \frac{x^{n-2j}y^{2js}(1-y^{s(n+1-2j)})(1+x)}{1-y} \\ & + \frac{y^n}{1-y} + \frac{\delta(n)xy^{(n+1)s}}{1-y}\end{aligned} \tag{15}$$

where $\delta(n) = 0$ if n is even and $\delta(n) = 1$ if n is odd. To verify Equation (15), one first eliminates the sums in Equation (15) to obtain

$$\begin{aligned}[t^n]f(x,y,t) = \quad & \frac{x^n(1-y^{n+1})}{(1-y)(1-x)} + \frac{(y^{n+1}-y^{(n+1)s})(x^n-x^r)}{(1-y)(1-x)} \\ & + \frac{(y^n-y^{ns})(x^2-x^{2r})}{(1-y)(1-x)} + \frac{x(y^n-y^{(n+1)s})}{1-y} \\ & + \frac{y^n}{1-y} + \frac{(1+x)xy^s(x^{n-1}-y^{(n-1)s})}{(1-y)(x-y^s)} \\ & + \frac{yx^r(x^{(n-1)r}-y^{n-1})(1-x^{2r})}{(1-y)(1-x)(x^r-y)} - \frac{y^{s(n+1)}(1+x)(x^2-x^n)}{(1-y)(1-x^2)} \\ & - \frac{y^sx^r(x^{(n-1)r}-y^{s(n-1)})(1-x^{2r})}{(1-y)(1-x)(x^r-y^s)}\end{aligned} \tag{16}$$

Then, one can either verify by hand or use any number of symbolic manipulation programs to verify that the right-hand sides of Equations (16) and (14) are equal by simplifying their difference and getting 0. (The authors used Maple.)

We now observe that Equation (15) implies that $[t^n]f(x,y,t)$ has nonnegative coefficients, provided $r \geq n$. Moreover, the only possible negative coefficients are

$$[x^j y^k t^n] f(x,y,t) \quad \text{with} \quad 1 < r < n \quad \text{and} \quad r \leq j < n < k < (n+1)s$$

since all terms of Equation (15) yield manifestly nonnegative coefficients except for the second term when $r < n$, where we have

$$\frac{(y^{n+1} - y^{(n+1)s})(x^n - x^r)}{(1-y)(1-x)} = -(y^{n+1} + \cdots + y^{(n+1)s-1})(x^r + \cdots + x^{n-1})$$

Now suppose that the coefficient of $x^j y^k t^n$ in the power series for $f(x,y,t)$, centered at $(0,0,0)$, were negative; *i.e.*, $[x^j y^k t^n] f(x,y,t) < 0$. Then, we must have both $1 < r < n$ and $r \leq j < n < k < s(n+1)$. Further, by the symmetry of $f(x,y,t)$ (with respect to the simultaneous swapping of x and r with y and s, respectively) we would know that $[x^k y^j t^n] f(x,y,t) < 0$ as well, and hence $1 < s < n$ and $s \leq k < n < j < r(n+1)$. However, we then have a contradiction since we would have both $j < k$ and $k < j$. Thus, $[x^j y^k t^n] f(x,y,t) \geq 0$, and the lemma is proved.

4. Proof of Theorem 1.2

Let $P(i) := (q^x, q^y, q^z, q^{rx+sy+uz}; q^m)_i$ and $Q(i) := (q^{rx}, q^{sy}, q^{uz}, q^{x+y+z}; q^m)_i$. Our goal will be to show that each addend in the sum on the right-hand side of Equation (10) has nonnegative coefficients. We will do this by considering two cases based on the index of summation i in Equation (10): $i = 1$ and $2 \leq i \leq L$. First, though, we observe that

$$(1-t\alpha)(1-t\beta)(1-t\gamma)(1-txyz) - (1-tx)(1-ty)(1-tz)(1-t\alpha\beta\gamma) \tag{17}$$

is identically equal to

$$\begin{aligned} &\tfrac{1}{2}t(x-\alpha)\left[(1-t\beta)(1-t\gamma)(1-yz) + (1-ty)(1-tz)(1-\beta\gamma)\right] \\ &+ \tfrac{1}{2}t(y-\beta)\left[(1-t\gamma)(1-t\alpha)(1-zx) + (1-tz)(1-tx)(1-\gamma\alpha)\right] \\ &+ \tfrac{1}{2}t(z-\gamma)(1-tx)(1-ty)(1-\alpha\beta) \\ &+ \tfrac{1}{2}t(z-\gamma)\left[(1-t\alpha)(1-t\beta)(1-xy) + (1-t^2)(x-\alpha)(y-\beta)\right] \end{aligned} \tag{18}$$

Substituting q^x, q^y, q^z, q^{rx}, q^{sy}, and q^{uz} for x, y, z, α, β, and γ, respectively, we may then conclude that

$$\begin{aligned} &(1-tq^{rx})(1-tq^{sy})(1-tq^{uz})(1-tq^{x+y+z}) \\ &-(1-tq^x)(1-tq^y)(1-tq^z)(1-tq^{rx+sy+uz}) \end{aligned} \tag{19}$$

is identically equal to

$$\begin{aligned}
&\tfrac{1}{2}tq^x(1-q^{(r-1)x})\left[(1-tq^{sy})(1-tq^{uz})(1-q^{y+z})\right.\\
&\qquad\left.+\,(1-tq^{y})(1-tq^{z})(1-q^{sy+uz})\right]\\
&+\tfrac{1}{2}tq^y(1-q^{(s-1)y})\left[(1-tq^{uz})(1-tq^{rx})(1-q^{z+x})\right.\\
&\qquad\left.+\,(1-tq^{z})(1-tq^{x})(1-q^{uz+rx})\right]\\
&+\tfrac{1}{2}tq^z(1-q^{(u-1)z})(1-tq^{x})(1-tq^{y})(1-q^{rx+sy})\\
&+\tfrac{1}{2}tq^z(1-q^{(u-1)z})\left[(1-tq^{rx})(1-tq^{sy})(1-q^{x+y})\right.\\
&\qquad\left.+\,(1-t^2)(q^{x}-q^{rx})(q^{y}-q^{sy})\right]
\end{aligned}\tag{20}$$

Let $t := q^{(i-1)m}$. Then, the numerator of the ith addend in Equation (10), namely

$$\frac{Q(i)}{Q(i-1)} - \frac{P(i)}{P(i-1)}$$

is given precisely by Equation (20). Now turning to the denominator of Equation (10), we may write

$$\frac{Q(L)}{Q(i-1)} = \frac{(q^{rx},q^{sy},q^{uz},q^{x+y+z};q^m)_L}{(q^{rx},q^{sy},q^{uz},q^{x+y+z};q^m)_{i-1}} = (tq^{rx},tq^{sy},tq^{uz},tq^{x+y+z};q^m)_{L-i+1}$$

and so we have that

$$(1-tq^{rx})(1-tq^{sy})(1-tq^{uz}) \text{ divides } \frac{Q(L)}{Q(i-1)} \text{ whenever } 1 \le i \le L \tag{21}$$

Similarly, from the definition of $P(i)$ we may deduce that

$$(1-q^x)(1-q^y)(1-q^z) \text{ divides } P(1) \tag{22}$$

and whenever $i > 1$ that

$$(1-q^x)(1-q^y)(1-q^z)(1-tq^x)(1-tq^y)(1-tq^z) \text{ divides } P(i) \tag{23}$$

When $i = 1$ we have $t = 1$, and hence the numerator of the first addend in Equation (10) simplifies to

$$\begin{aligned}
Q(1)-P(1) = &\tfrac{1}{2}q^x(1-q^{(r-1)x})\left[(1-q^{sy})(1-q^{uz})(1-q^{y+z})\right.\\
&\qquad\left.+\,(1-q^{y})(1-q^{z})(1-q^{sy+uz})\right]\\
&+\tfrac{1}{2}q^y(1-q^{(s-1)y})\left[(1-q^{uz})(1-q^{rx})(1-q^{z+x})\right.\\
&\qquad\left.+\,(1-q^{z})(1-q^{x})(1-q^{uz+rx})\right]\\
&+\tfrac{1}{2}q^z(1-q^{(u-1)z})\left[(1-q^{x})(1-q^{y})(1-q^{rx+sy})\right.\\
&\qquad\left.+\,(1-q^{rx})(1-q^{sy})(1-q^{x+y})\right]
\end{aligned}\tag{24}$$

Meanwhile, the denominator of the first addend in Equation (10) contains all of the factors indicated in Equation (21): $(1-q^{rx})$, $(1-q^{sy})$, $(1-q^{uz})$. The denominator also contains all of the factors indicated by Equation (22): $(1-q^x)$, $(1-q^y)$, $(1-q^z)$. These factors, together with the "trick" of re-writing, for example,

$$(1-q^{x+y}) = (1-q^x) + q^x(1-q^y) \tag{25}$$

is enough to see that the first addend in Equation (10) only has nonnegative coefficients.

When $2 \leq i \leq L$, we have $t = q^{(i-1)m} \neq 1$, and hence the numerator of the ith addend in Equation (10) is precisely Equation (20). From Equations (21) and (23) we have the following factors in the denominator: $(1 - tq^{rx})$, $(1 - tq^{sy})$, $(1 - tq^{uz})$, $(1 - q^x)$, $(1 - q^y)$, $(1 - q^z)$, $(1 - tq^x)$, $(1 - tq^y)$, $(1 - tq^z)$. Again employing the "trick" Equation (25) as necessary, we can handle most of the ith addend similar to before, except for the last term of Equation (20), which contains the factor

$$\left[(1 - q^{x+y})(1 - tq^{rx})(1 - tq^{sy}) + (1 - t^2)(q^x - q^{rx})(q^y - q^{sy})\right] \tag{26}$$

This factor is potentially problematic due to the presence of the factor $(1 - t^2)$ in the second term.

If we let f be given as in Lemma 1.3, then Equation (26) becomes

$$f(q^x, q^y, t)(1 - tq^{rx})(1 - tq^{sy})(1 - q^x)(1 - q^y)(1 - tq^x)(1 - tq^y)$$

The last term of Equation (20), when divided by the nine factors listed above, then becomes

$$\frac{\frac{1}{2}tq^z(1 - q^{(u-1)z})f(q^x, q^y, t)}{(1 - tq^{uz})(1 - q^z)(1 - tq^z)}$$

which, in light of Lemma 1.3, clearly now has no negative coefficients. Thus, having shown that all addends in Equation (10) admit only nonnegative coefficients, Theorem 1.2 is proved.

5. Concluding Remarks

It seems to always be possible to find a suitable "splitting" to handle the $L = 1$ case, no matter how many variables are used. For example, if we increase from three to four main variables ($x_1, \ldots, x_4$, with corresponding $r_1, \ldots, r_4$), for $L = 1$ we have

$$\begin{aligned}
&\frac{1}{(1-q^{x_1})(1-q^{x_2})(1-q^{x_3})(1-q^{x_4})(1-q^{r_1x_1+r_2x_2+r_3x_3+r_4x_4)})} \\
&- \frac{1}{(1-q^{r_1x_1})(1-q^{r_2x_2})(1-q^{r_3x_3})(1-q^{r_4x_4})(1-q^{x_1+x_2+x_3+x_4})} \\
&= \frac{h(x_1,x_2,x_3,r_1,r_2,r_3)+h(x_1,x_2,x_4,r_1,r_2,r_4)+h(x_1,x_3,x_4,r_1,r_3,r_4)+h(x_2,x_3,x_4,r_2,r_3,r_4)}{(1-q^{x_1+x_2+x_3+x_4})(1-q^{r_1x_1+r_2x_2+r_3x_3+r_4x_4})}
\end{aligned}$$

where $h(x_1, x_2, x_3, r_1, r_2, r_3) :=$

$$\begin{aligned}
&\frac{q^{x_1}(1-q^{(r_1-1)x_1})}{(1-q^{x_1})(1-q^{r_1x_1})} \cdot \frac{q^{x_2}(1-q^{(r_2-1)x_2})}{(1-q^{x_2})(1-q^{r_2x_2})} \cdot \frac{q^{x_3}(1-q^{(r_3-1)x_3})}{(1-q^{x_3})(1-q^{r_3x_3})} \\
&+ \frac{1}{2} \cdot \frac{q^{x_1}(1-q^{(r_1-1)x_1})}{(1-q^{x_1})(1-q^{r_1x_1})} \cdot \frac{q^{x_2}(1-q^{(r_2-1)x_2})}{(1-q^{x_2})(1-q^{r_2x_2})} + \frac{1}{2} \cdot \frac{q^{x_2}(1-q^{(r_2-1)x_2})}{(1-q^{x_2})(1-q^{r_2x_2})} \cdot \frac{q^{x_3}(1-q^{(r_3-1)x_3})}{(1-q^{x_3})(1-q^{r_3x_3})} \\
&+ \frac{1}{2} \cdot \frac{q^{x_1}(1-q^{(r_1-1)x_1})}{(1-q^{x_1})(1-q^{r_1x_1})} \cdot \frac{q^{x_3}(1-q^{(r_3-1)x_3})}{(1-q^{x_3})(1-q^{r_3x_3})} + \frac{1}{2} \cdot \frac{q^{x_1}(1-q^{(r_1-1)x_1})}{1-q^{x_1}} \cdot \frac{q^{r_2x_2}}{(1-q^{r_1x_1})(1-q^{r_2x_2})} \\
&+ \frac{1}{2} \cdot \frac{q^{x_1}(1-q^{(r_1-1)x_1})}{1-q^{x_1}} \cdot \frac{q^{r_3x_3}}{(1-q^{r_1x_1})(1-q^{r_3x_3})} + \frac{1}{2} \cdot \frac{q^{x_2}(1-q^{(r_2-1)x_2})}{(1-q^{x_2})(1-q^{r_2x_2})} \cdot \frac{q^{r_3x_3}}{1-q^{r_3x_3}} \\
&+ \frac{1}{2} \cdot \frac{q^{x_2}(1-q^{(r_2-1)x_2})}{(1-q^{x_2})(1-q^{r_2x_2})} \cdot \frac{q^{r_1x_1}}{1-q^{r_1x_1}} + \frac{1}{2} \cdot \frac{q^{x_3}(1-q^{(r_3-1)x_3})}{(1-q^{x_3})(1-q^{r_3x_3})} \cdot \frac{q^{r_2x_2}}{1-q^{r_2x_2}} \\
&+ \frac{1}{2} \cdot \frac{q^{x_3}(1-q^{(r_3-1)x_3})}{(1-q^{x_3})(1-q^{r_3x_3})} \cdot \frac{q^{r_1x_1}}{1-q^{r_1x_1}} + \frac{1}{3} \cdot \frac{q^{x_1}(1-q^{(r_1-1)x_1})}{(1-q^{x_1})(1-q^{r_1x_1})} + \frac{1}{3} \cdot \frac{q^{x_2}(1-q^{(r_2-1)x_2})}{(1-q^{x_2})(1-q^{r_2x_2})} \\
&+ \frac{1}{3} \cdot \frac{q^{x_3}(1-q^{(r_3-1)x_3})}{(1-q^{x_3})(1-q^{r_3x_3})} + \frac{q^{x_1}(1-q^{(r_1-1)x_1})}{(1-q^{x_1})(1-q^{r_1x_1})} \cdot \frac{q^{x_2}(1-q^{(r_2-1)x_2})}{(1-q^{x_2})(1-q^{r_2x_2})} \cdot \frac{q^{r_3x_3}}{1-q^{r_3x_3}} \\
&+ \frac{q^{x_1}(1-q^{(r_1-1)x_1})}{(1-q^{x_1})(1-q^{r_1x_1})} \cdot \frac{q^{x_3}(1-q^{(r_3-1)x_3})}{(1-q^{x_3})(1-q^{r_3x_3})} \cdot \frac{q^{r_2x_2}}{1-q^{r_2x_2}} \\
&+ \frac{q^{x_2}(1-q^{(r_2-1)x_2})}{(1-q^{x_2})(1-q^{r_2x_2})} \cdot \frac{q^{x_3}(1-q^{(r_3-1)x_3})}{(1-q^{x_3})(1-q^{r_3x_3})} \cdot \frac{q^{r_1x_1}}{1-q^{r_1x_1}} \\
&+ \frac{q^{x_1}(1-q^{(r_1-1)x_1})}{(1-q^{x_1})(1-q^{r_1x_1})} \cdot \frac{q^{r_2x_2}}{1-q^{r_2x_2}} \cdot \frac{q^{r_3x_3}}{1-q^{r_3x_3}} + \frac{q^{x_2}(1-q^{(r_2-1)x_2})}{(1-q^{x_2})(1-q^{r_2x_2})} \cdot \frac{q^{r_1x_1}}{1-q^{r_1x_1}} \cdot \frac{q^{r_3x_3}}{1-q^{r_3x_3}} \\
&+ \frac{q^{x_3}(1-q^{(r_3-1)x_3})}{(1-q^{x_3})(1-q^{r_3x_3})} \cdot \frac{q^{r_2x_2}}{1-q^{r_2x_2}} \cdot \frac{q^{r_1x_1}}{1-q^{r_1x_1}}
\end{aligned}$$

satisfies $h(x_1, x_2, x_3, r_1, r_2, r_3) \succcurlyeq 0$. Finding a suitable "splitting" with $t := q^{(i-1)m}$ inserted into opportune locations, as we did in the proofs of the Theorems 1.1 and 1.2, is a much more difficult task here. (We think of this as inserting the t's since we wish to recover the $L = 1$ case when we let $t = 1$.) The authors of this manuscript do not currently possess such a "splitting" for this case. Nonetheless, the authors are fairly confident in the veracity of the following proposal.

Proposal 5.1 *For any* $(2n+2)$*-tuple* $(L, m, x_1, \ldots, x_n, r_1, \ldots, r_n)$ *of positive integers,*

$$\frac{1}{(q^{x_1}, \ldots, q^{x_n}, q^{\Sigma}; q^m)_L} \succcurlyeq \frac{1}{(q^{r_1 x_1}, \ldots, q^{r_n x_n}, q^{\sigma}; q^m)_L} \tag{27}$$

where $\Sigma := r_1 x_1 + \cdots + r_n x_n$ *and* $\sigma := x_1 + \cdots + x_n$.

We note that Proposal 5.1 is true for $L = 1$ since the right-hand side of Equation (27) could be interpreted as the generating function for partitions into parts from the set $S := \{x_1, \ldots, x_n, \Sigma\}$ (parts with the same numeric value but distinct origins having different colors, thus ensuring $|S| = n + 1$) such that for any such partition π, there is an integer A with the property that

$$\begin{aligned} A &\equiv \nu(x_1, \pi) \pmod{r_1} \\ A &\equiv \nu(x_2, \pi) \pmod{r_2} \\ &\vdots \\ A &\equiv \nu(x_n, \pi) \pmod{r_n} \end{aligned}$$

where $\nu(p, \pi)$ is the number of occurrences of the part p in the partition π. This set of partitions is a subset of the set of all partitions into parts from the set S, which is what the left-hand side of Equation (27) would count. To see this clearly, we let π' be a partition with parts from the set $S' := \{r_1 x_1, \ldots, r_n x_n, \sigma\}$ and let $\mu' := \min(\{\nu(r_i x_i, \pi') : 1 \leq i \leq n\})$. Then we can explicitly define an injection (for $L = 1$) mapping $\pi' \mapsto \pi$ as follows:

$$\begin{aligned} \nu(\Sigma, \pi) &:= \mu' \\ \nu(x_i, \pi) &:= r_i \cdot (\nu(r_i x_i, \pi') - \mu') + \nu(\sigma, \pi') \end{aligned}$$

Clearly we can then choose $A = \nu(\sigma, \pi')$. Now this mapping is invertible since if we let $\mu := \min(\{\nu(x_i, \pi) : 1 \leq i \leq n\})$ we have

$$\begin{aligned} \nu(\sigma, \pi') &= \mu \\ \nu(r_i x_i, \pi') &= \frac{\nu(x_i, \pi) - \mu}{r_i} + \nu(\Sigma, \pi) \end{aligned}$$

Thus, the proposal is proved for $L = 1$.

Finally, we intend to explore possible connections with the recent work "A q-rious positivity" by S. Ole Warnaar and Wadim Zudilin (see [7]). In particular, we are $q$uite q-rious as to how the validity of inequalities, like those in this paper, for $L = 1$ might imply the validity for all positive L, a sentiment that seems echoed by the authors of [7].

Acknowledgements

We would like to thank George Andrews and Wadim Zudilin for their interest and helpful discussions.

References

1. Andrews, G.E. *The Theory of Partitions*; Cambridge Mathematical Library, Cambridge University Press: Cambridge, UK, 1998; Reprint of the 1976 original.
2. Kadell, K.W.J. An injection for the Ehrenpreis Rogers-Ramanujan problem. *J. Combin. Theory Ser. A* **1999**, *86*, 390–394.
3. Berkovich, A.; Garvan, F.G. Dissecting the Stanley partition function. *J. Combin. Theory Ser. A* **2005**, *112*, 277–291.
4. Andrews, G.E. Differences of partition functions: The anti-telescoping method. *Dev. Math.* **2013**, *28*, 1–20.
5. Berkovich, A.; Grizzell, K. Races among products. *J. Combin. Theory Ser. A* **2012**, *119*, 1789–1797.
6. Gessel, I.M. Integer quotients of factorials and algebraic generating functions, MIT Combinatorics Seminar, 30 September 2011, as transmitted via the world-wide web. Available online: http://people.brandeis.edu/ gessel/homepage/slides/int-quot.pdf (accessed on 14 May 2013).
7. Warnaar, S.O.; Zudilin, W. A q-rious positivity. *Aequationes Math.* **2011**, *81*, 177–183.