mathematics

MDPI

Article

Robust Nonsmooth Interval-Valued Optimization Problems Involving Uncertainty Constraints

Rekha R. Jaichander [1,2], Izhar Ahmad [3,4,*], Krishna Kummari [1] and Suliman Al-Homidan [3,5]

[1] Department of Mathematics, School of Science, GITAM-Hyderabad Campus, Hyderabad 502329, India; rjrekhasat@gmail.com (R.R.J.); krishna.maths@gmail.com (K.K.)
[2] Department of Mathematics, St. Francis College for Women-Begumpet, Hyderabad 500006, India
[3] Department of Mathematics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia; homidan@kfupm.edu.sa
[4] Center for Intelligent Secure Systems, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
[5] Center for Smart Mobility and Logistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
* Correspondence: drizhar@kfupm.edu.sa

Abstract: In this paper, Karush-Kuhn-Tucker type robust necessary optimality conditions for a robust nonsmooth interval-valued optimization problem (UCIVOP) are formulated using the concept of LU-optimal solution and the generalized robust Slater constraint qualification (GRSCQ). These Karush-Kuhn-Tucker type robust necessary conditions are shown to be sufficient optimality conditions under generalized convexity. The Wolfe and Mond-Weir type robust dual problems are formulated over cones using generalized convexity assumptions, and usual duality results are established. The presented results are illustrated by non-trivial examples.

Keywords: Generalized convexity; robust nonsmooth interval-valued optimization problem; $\mathcal{LU}$-optimal solution; optimality; duality

MSC: 90C17; 90C26; 90C46

check for updates

Citation: Jaichander, R.R.; Ahmad, I.; Kummari, K.; Al-Homidan, S. Robust Nonsmooth Interval-Valued Optimization Problems Involving Uncertainty Constraints. *Mathematics* **2022**, *10*, 1787. https://doi.org/10.3390/math10111787

Academic Editors: Chuangyin Dang, Siyang Gao and Zhiping Chen

Received: 10 April 2022
Accepted: 18 May 2022
Published: 24 May 2022

1. Introduction

Robust optimization has become a prominent predetermined framework for investigating multi-objective optimization problems with data uncertainty. Robust optimization is a relatively new field of research that allows academicians to solve a wide range of optimization problems, particularly when challenged with real-life situations where the data input for a multi-objective linear semi-infinite program is frequently noisy or uncertain due to prediction or measurement inaccuracies, as well as in industrial settings. Under this framework, the objective and constraint functions are only considered to belong to "uncertainty sets" in function space. For single-objective optimization problems, Soyster [1] was the first researcher to study robust optimization problem. Quality products are necessary in sectors as the engineering environment grows increasingly competitive. Variations in various engineering procedures generate unexpected deviations from the function that a designer intended for. The goal of robust design is to avoid such occurrences. In industrial engineering, robust design has been developed to improve product quality and reliability. "Robustness is defined as the capacity of a technology, product, or process to work with little sensitivity to elements that cause unpredictability (in the manufacturing or user environment) and ageing while maintaining the lowest unit manufacturing cost." explained Taguchi [2], the pioneer of robust design. For more recent developments on robust optimization problems, the readers are advised to refer to [3–8]. Lee et al. [9] and Oksanen et al. [10] used robustness for transforming data to prognostic information and open platform communications technology for agricultural machinery telemetry respectively. Lie et al. [11] used robustness in electrical machines which greatly increased motor performance and diminished computational cost. Further, robust optimization has several real life applications, readers are advised to refer to [12–15].

An interval-valued optimization problem is based on interval coefficients with closed intervals. Interval-valued optimization problems can contribute a more useful alternative for evaluating the uncertainty therein. In recent years, interval-valued optimization has become a major topic in applied mathematics. This is due to the fact that, in many circumstances, the theory regarding the parameters of a physical world system is unknown. Hence, these parameters cannot be accurately evaluated. Many developments in the theory of interval-valued optimization problems have been carefully investigated, readers are advised to refer [16–20]. Interval-based models have a wide range of real-world applications such as inventory [21], genetic algorithm [22] and engineering applications [23].

There are numerous mathematical models that are employed in applied mathematics, economics, engineering, stochastics management and decision sciences for which convexity is no longer sufficient. Various expansions of convex functions have been proposed in the literature. Many of these functions provide more than one property resulting in models that are better adaptable to real-world conditions than convex models. Beginning with the pioneering work of Arrow and Enthoven [24], efforts were formed to cripple the convexity assumption and thus investigate the application of optimality conditions. In this attempt, Hanson [25] introduced a new category of functions which are applicable to optimization theory, which was termed as the category of invex functions by Craven [26]. Some of the recent advances related to generalized convexity with applications to group dynamic problems, portfolio and location theory were analysed in detail, readers are advised to refer to [27–29].

The basic purpose of multi-objective optimization research is to identify the best possible objective values by finding the global Pareto efficient solution. In practice, users may be less interested in discovering so-called global best solutions, especially if they are extremely sensitive to variable perturbations, which are unavoidable in practice. Practitioners are interested in building robust solutions that are less vulnerable to minor alterations in these situations. Hence, in this paper we emphasis the robustness for a nonsmooth interval-valued optimization problem.

The general problem dealing with minimizing (or maximizing) functions that are generally not differentiable at their minimizers (or maximizers) refers to nonsmooth optimization. Nonsmooth calculus, an extension of differential calculus, has recently become a key advancement in mathematical sciences, particularly in the fields of mathematics, operations research and engineering. Suneja et al. [30] used Clarke's generalized gradients to develop generalized convexity and optimality conditions related to vector optimization problem along with duality results in Mond-Weir type problems. Chen et al. [31] employed a modified objective function approach to examine the optimality conditions which are applicable to multi-objective fractional programming problems and a family of nonsmooth multi-objective optimization problems with cone constraints. In [32], Lee and Son explored the necessary optimality theorem for a nonsmooth optimization problem in the presence of data uncertainty. In the face of data uncertainty, Lee and Lee [33] have interpreted nonsmooth optimality theorems for weakly and properly robust efficient solutions to a nonsmooth multi-objective problem with more than two locally Lipschitz objective and constraint functions. In [34], Chuong studied optimality conditions for robust (weakly) pareto-optimal solutions which are in terms of limiting subdifferentials and multipliers. The primal and its robust dual problem (strictly) with generalized convexity assumptions were investigated further for weak/strong duality relations. There are no results on robust $\mathcal{LU}$-optimal solution of nonsmooth/nonconvex uncertain constrained interval-valued optimization problem (UCIVOP), that we are aware of.

Guided by the above works, this paper uses a robust methodology to analyse a nonsmooth/nonconvex uncertain constrained interval-valued optimization problem (UCIVOP).

The details of the manuscript are given as follows. Section 2 recalls some preliminary definitions and basic results. Section 3 establishes robust optimality conditions for (UCIVOP), based on the assumptions of generalized convexity. Sections 4 and 5 are concerned with the formulation of Wolfe and Mond-Weir type robust dual problems over cones involving generalized convexity assumptions, followed by the development of duality results. Section 6 deals with the conclusion.

2. Preliminaries

Let $\mathcal{R}^n$ and $\mathcal{R}^n_+$ denote respectively n-dimensional Euclidean space and its nonnegative orthant. Let $\mathbb{I}$ be the set of all closed bounded intervals in $\mathcal{R}$. Suppose $I_1 = [\iota^{\mathcal{L}}, \iota^{\mathcal{U}}], I_2 = [\varsigma^{\mathcal{L}}, \varsigma^{\mathcal{U}}] \in \mathbb{I}$, then

(i) $I_1 + I_2 = \{\iota + \varsigma : \iota \in I_1 \text{ and } \varsigma \in I_2\} = [\iota^{\mathcal{L}} + \varsigma^{\mathcal{L}}, \iota^{\mathcal{U}} + \varsigma^{\mathcal{U}}]$,

(ii) $I_1 = \{-\iota : \iota \in I_1\} = [-\iota^{\mathcal{U}}, -\iota^{\mathcal{L}}]$,

(iii) $I_1 - I_2 = I_1 + (-I_2) = [\iota^{\mathcal{L}} - \varsigma^{\mathcal{U}}, \iota^{\mathcal{U}} - \varsigma^{\mathcal{L}}]$,

(iv) $c + I_1 = \{c + \iota : \iota \in I_1\} = [c + \iota^{\mathcal{L}}, c + \iota^{\mathcal{U}}]$,

(v) $cI_1 = \{c\iota : \iota \in I_1\} = \begin{cases} [c\iota^{\mathcal{L}}, c\iota^{\mathcal{U}}], \text{ if } c \geq 0, \\ [c\iota^{\mathcal{U}}, c\iota^{\mathcal{L}}], \text{ if } c < 0, \end{cases}$

where $c \in \mathcal{R}$.

For $I_1 = [\iota^{\mathcal{L}}, \iota^{\mathcal{U}}]$ and $I_2 = [\varsigma^{\mathcal{L}}, \varsigma^{\mathcal{U}}]$, the partial ordering $\leq_{\mathcal{LU}}$ on $\mathbb{I}$ is defined as $I_1 \leq_{\mathcal{LU}} I_2$ if and only if $\iota^{\mathcal{L}} \leq \varsigma^{\mathcal{L}}$ and $\iota^{\mathcal{U}} \leq \varsigma^{\mathcal{U}}$. Moreover, we represent $I_1 <_{\mathcal{LU}} I_2$ if and only if $I_1 \leq_{\mathcal{LU}} I_2$ along with $I_1 \neq I_2$. In other words, $I_1 <_{\mathcal{LU}} I_2$ if and only if

$$\begin{array}{lll} & \iota^{\mathcal{L}} < \varsigma^{\mathcal{L}}, & \iota^{\mathcal{U}} < \varsigma^{\mathcal{U}}, \\ \text{or} & \iota^{\mathcal{L}} \leq \varsigma^{\mathcal{L}}, & \iota^{\mathcal{U}} < \varsigma^{\mathcal{U}}, \\ \text{or} & \iota^{\mathcal{L}} < \varsigma^{\mathcal{L}}, & \iota^{\mathcal{U}} \leq \varsigma^{\mathcal{U}}. \end{array}$$

A nonempty subset L of $\mathcal{R}^n$ is a convex cone if $L + L \subseteq L$ and $lL \subseteq L$ for all $l > 0$. A proper, closed and convex cone with nonempty interior is denoted by $\mathcal{C} \subset \mathcal{R}$. Let Φ_{j_0} be a nonempty, convex and compact subset of $\mathcal{R}^{n_{j_0}}$, for $j_0 = 1, 2, \ldots, m_0$ and $\sum_{j_0=1}^{m_0} n_{j_0} = p$. Let $\Psi = [\Psi^L, \Psi^U] : \mathcal{R}^n \to \mathbb{I}$ and $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{m_0})^T : \mathcal{R}^n \times \mathcal{R}^p \to \mathcal{R}^{m_0}$ be an interval-valued and vector-valued mappings and $\varphi_{j_0} : \mathcal{R}^n \times \mathcal{R}^{n_{j_0}} \to \mathcal{R}$ for $j_0 = 1, 2, \ldots, m_0$, where the transpose T is the superscript. $Int\mathcal{C}$ stands for the interior of $\mathcal{C}$. $\mathcal{C}^* = \{\eta \in \mathcal{R}_+ : \eta^T \vartheta_0 \geq 0, \forall \vartheta_0 \in \mathcal{C}\} = \mathcal{C}$ is the dual cone of $\mathcal{C}$.

In this article, we consider the subsequent uncertain constrained interval-valued optimization problem (UCIVOP):

$$\text{(UCIVOP)} \qquad \min_{\vartheta_0 \in \mathcal{R}^n} \ \Psi(\vartheta_0) = [\Psi^{\mathcal{L}}(\vartheta_0), \Psi^{\mathcal{U}}(\vartheta_0)]$$
$$\text{subject to}$$
$$-\varphi(\vartheta_0, \varrho) \in \mathcal{S},$$

where $\mathcal{S} \subset \mathcal{R}^{m_0}$ is a proper, closed and convex cone, $\varrho = (\varrho_1, \varrho_2, ..., \varrho_{m_0})^T \in \mathcal{R}^p$ is the vector of uncertain parameter with $\varrho_{j_0} \in \Phi_{j_0}$, $j_0 = 1, 2, ..., m_0$ and $\vartheta_0 \in \mathcal{R}^n$ is the vector of decision variable. The uncertainty set-valued function $\Phi : J \rightrightarrows \mathcal{R}^p$ is given by $\Phi(j_0) = \varrho_{j_0}$, $\forall\ j_0 \in J$. For the sake of convenience, we set $\Phi = \prod_{j_0=1}^{m_0} \Phi_{j_0}$. As Φ_{j_0} are nonempty, convex and compact sets for all $j_0 = 1, 2, ..., m_0$, Φ is a nonempty, convex and compact subset of $\mathcal{R}^p$. We assume $\mathcal{S} = \mathcal{R}^{m_0}_+$ and $\mathcal{C} = \mathcal{R}_+$ without losing generality throughout this study.

In this study, we use a robust methodology to explore (UCIVOP). The robust counterpart of (UCIVOP) is as follows:

$$\text{(RIVOP)} \qquad \min_{\vartheta_0 \in \mathcal{R}^n} \quad \Psi(\vartheta_0) = [\Psi^{\mathcal{L}}(\vartheta_0), \Psi^{\mathcal{U}}(\vartheta_0)]$$
$$\text{subject to}$$
$$-\varphi(\vartheta_0, \varrho) \in \mathcal{S}, \forall \varrho \in \Phi.$$

A vector ϑ_0 is a feasible solution of (RIVOP), it is said to be a robust feasible solution of (UCIVOP). The collection of all robust feasible solutions of (UCIVOP) is denoted by $\mathbb{H}$ where $\mathbb{H} = \{\vartheta_0 \in \mathcal{R}^n : -\varphi(\vartheta_0, \varrho) \in \mathcal{S}, \forall \varrho \in \Phi\}$.

Definition 1. *The robust feasible point $\bar{\vartheta}_0$ is refered to as a robust $\mathcal{LU}$-optimal solution of (UCIVOP), if there is no robust feasible solution ϑ_0 of (UCIVOP) such that $\Psi(\vartheta_0) <_{\mathcal{LU}} \Psi(\bar{\vartheta}_0)$.*

We offer numerous specific scenarios to emphasise the generality of our interval-valued robust optimization problems (IVROPs).

Case (i). If $\Psi^{\mathcal{L}}(\vartheta_0) = \Psi^{\mathcal{U}}(\vartheta_0)$ for all $\vartheta_0 \in \mathcal{R}^n$, then (UCIVOP) reduces to the subsequent problem:

$$\text{(ROP)} \qquad \min_{\vartheta_0 \in \mathcal{R}^n} \quad \Psi(\vartheta_0)$$
$$\text{subject to}$$
$$-\varphi(\vartheta_0, \varrho) \in \mathcal{S}, \forall \varrho \in \Phi.$$

which is the robust optimization problem (ROP). Many researchers [35–37] have examined the Karush-Kuhn-Tucker type necessary optimality conditions for (ROP).

Case (ii). If the constraint functions are independent of the uncertainty parameter ϱ_{j_0}, for each $j_0 = 1, 2, \ldots, m_0$ then (UCIVOP) reduces to the subsequent interval-valued optimization problem:

$$\text{(IVOP)} \qquad \min_{\vartheta_0 \in \mathcal{R}^n} \quad \Psi(\vartheta_0) = [\Psi^{\mathcal{L}}(\vartheta_0), \Psi^{\mathcal{U}}(\vartheta_0)]$$
$$\text{subject to}$$
$$-\varphi(\vartheta_0) \in \mathcal{S}.$$

Ishibuchi and Tanaka [38], Inuiguchi and Kume [39] and Wu [40,41] investigated the Karush-Kuhn-Tucker necessary optimality conditions of the interval-valued optimization problems based on the assumption that each of the constraints are convex and continuously differentiable.

Definition 2. *(See, Chen et al. [42]) A real-valued function $h : \mathcal{R}^n \to \mathcal{R}$ is a locally Lipschitz if and only if, for any $v \in \mathcal{R}^n$, there exist a positive constant τ and a neighborhood $\mathcal{V}$ of v such that, for any $\vartheta_0, \varpi_0 \in \mathcal{V}$,*

$$|h(\vartheta_0) - h(\varpi_0)| \leq \tau ||\vartheta_0 - \varpi_0||,$$

where $||.||$ stands for any norm in $\mathcal{R}^n$.

The Clarke's generalized subgradient (See, Clarke [43]) of h at v is $\partial h(v) = \{\zeta \in \mathcal{R}^n : h^0(v; z) \geq \zeta^T z, \ \forall z \in \mathcal{R}^n\}$, where

$$h^0(v; z) = \limsup_{\substack{\varpi_0 \to v \\ t \downarrow 0}} \frac{h(\varpi_0 + tz) - h(\varpi_0)}{t},$$

clearly,

$$h^0(v;z) = \max\{\langle \zeta, z\rangle : \zeta \in \partial h(v)\}.$$

It is generally known that if $h : \mathcal{R}^n \to \mathcal{R}$ is a locally Lipschitz function, then the Clarke's generalized subgradient $\partial h : \mathcal{R}^n \rightrightarrows \mathcal{R}^n$ is nonempty and compact-valued function as well as upper semicontinuous on $\mathcal{R}^n$ i.e., for any sequences (ϑ_n) and (ϖ_n) of $\mathcal{R}^n$ with $\vartheta_n \to \vartheta_0 \in \mathcal{R}^n$ and $\varpi_n \in \partial h(\vartheta_n), \forall n \in N$, there exists a subsequence $(\varpi_{n_k}) \to \varpi_0 \in \partial h(\vartheta_0)$. An interval-valued function $\Psi = [\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}] : \mathcal{R}^n \to \mathbb{I}$ is said to be locally Lipschitz on $\mathcal{R}^n$ if and only if $\Psi^{\mathcal{L}}$, $\Psi^{\mathcal{U}}$ are locally Lipschitz on $\mathcal{R}^n$. Throughout this paper, we will always presume that $\Psi^{\mathcal{L}}$, $\Psi^{\mathcal{U}}$ are locally Lipschitz functions on $\mathcal{R}^n$ and that φ is locally Lipschitz function on $\mathcal{R}^n$ with respect to the first argument and its components are upper semicontinuous with respect to the second argument.

Definition 3. *Let $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ be type II $(\mathcal{C}, \mathcal{C}, \mathcal{R}^{m_0}_+)$-generalized convex at $\bar{\vartheta}_0 \in \mathcal{R}^n$. If for each $\vartheta_0 \in \mathbb{H}$ and $A^{\mathcal{L}} \in \partial\Psi^{\mathcal{L}}(\bar{\vartheta}_0)$, $A^{\mathcal{U}} \in \partial\Psi^{\mathcal{U}}(\bar{\vartheta}_0)$, $B \in \partial_{\vartheta_0}\varphi(\bar{\vartheta}_0, \varrho)$, $\varrho \in \Phi$, there exists $\xi \in \mathcal{R}$ such that*

$$\begin{aligned} \Psi^L(\vartheta_0) - \Psi^L(\bar{\vartheta}_0) - A^L\xi &\in \mathcal{C}, \\ \Psi^U(\vartheta_0) - \Psi^U(\bar{\vartheta}_0) - A^U\xi &\in \mathcal{C}, \\ -\varphi(\bar{\vartheta}_0, \varrho) - B\xi &\in \mathcal{R}^{m_0}_+. \end{aligned}$$

Definition 4. (i). *$f : \mathcal{R}^n \to \mathbb{I}$ is pseudo convex at $\bar{\vartheta}_0 \in \mathcal{R}^n$, if for any $\vartheta_0 \in \mathcal{R}^n$ and $\mu \in \mathcal{C}^* \setminus \{0\}$ the following holds:*

$$\mu^T f(\vartheta_0) < \mu^T f(\bar{\vartheta}_0) \Rightarrow A^T(\vartheta_0 - \bar{\vartheta}_0) < 0, \; \forall A \in \partial(\mu^T f)(\bar{\vartheta}_0).$$

(ii). *$f : \mathcal{R}^n \to \mathbb{I}$ is strictly pseudo convex at $\bar{\vartheta}_0 \in \mathcal{R}^n$, if for any $\vartheta_0 \in \mathcal{R}^n \setminus \{\bar{\vartheta}_0\}$ and $\mu \in \mathcal{C}^* \setminus \{0\}$ the subsequent equation holds:*

$$\mu^T f(\vartheta_0) \le \mu^T f(\bar{\vartheta}_0) \Rightarrow A^T(\vartheta_0 - \bar{\vartheta}_0) < 0, \; \forall A \in \partial(\mu^T f)(\bar{\vartheta}_0).$$

(iii). *$\mathcal{G} : \mathcal{R}^n \times \mathcal{R}^p \to \mathcal{R}^{m_0}$ is generalized quasi convex at $\bar{\vartheta}_0 \in \mathcal{R}^n$, if for any $\vartheta_0 \in \mathcal{R}^n$ and $\varrho \in \Phi$ the following holds:*

$$\mathcal{G}(\vartheta_0, \varrho) \le \mathcal{G}(\bar{\vartheta}_0, \varrho) \Rightarrow B^T(\vartheta_0 - \bar{\vartheta}_0) \le 0, \; \forall B \in \partial_{\vartheta_0}\mathcal{G}(\bar{\vartheta}_0, \varrho).$$

Definition 5. *$(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type I $(\mathcal{C}, \mathcal{C}, \mathcal{R}^{m_0}_+)$-pseudo convex at $\bar{\vartheta}_0 \in \mathcal{R}^n$, if $\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}$ are pseudo convex functions and φ is generalized quasi convex at $\bar{\vartheta}_0 \in \mathcal{R}^n$.*

Definition 6. *$(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type II $(\mathcal{C}, \mathcal{C}, \mathcal{R}^{m_0}_+)$-pseudo convex at $\bar{\vartheta}_0 \in \mathcal{R}^n$, if $\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}$ are strictly pseudo convex functions and φ is generalized quasi convex at $\bar{\vartheta}_0 \in \mathcal{R}^n$.*

Remark 1. *It is noted that the Definitions 4–6 are properly wider than convex functions (see [44], Example 2.2 and [34], Example 3.10).*

Definition 7. *(See, Chen et al. [42]) Let X_0 be a nonempty subset of $\mathcal{R}^n$. $f : X_0 \to \mathcal{R}$ is said to be $\mathcal{C}$-convexlike if the set $f(X_0) + \mathcal{C}$ is convex.*

Definition 8. *(See, Chen et al. [42]) The generalized robust Slater constraint qualification (GRSCQ) is satisfied if there exists $z_0 \in \mathcal{R}^n$ such that $-\varphi(z_0, \varrho) \in int\mathcal{R}^{m_0}_+, \forall \varrho \in \Phi$.*

Lemma 1. *(See, Chen et al. [42]) Let $\mathcal{C} \subseteq \mathcal{R}$ be a closed convex cone with $int\mathcal{C} \neq \emptyset$. Then, $\varpi_0 \in int\mathcal{C} \Leftrightarrow u^T\varpi_0 > 0, \forall u \in \mathcal{C}^* \setminus \{0\}$.*

The following is always denoted by Assumption $\mathbb{D}$ (See, Chen et al. [42]) in the rest of this work.

(B1): With respect to the first argument, φ is locally Lipschitz and uniformly on Φ with respect to the second argument i.e., for each $\vartheta_0 \in \mathcal{R}^n$, there is a positive constant K and an open neighborhood $\mathcal{V}$ of ϑ_0 such that

$$||\varphi(\varpi_0, \varrho) - \varphi(\vartheta_0, \varrho)|| \leq K||\varpi_0 - \vartheta_0||, \ \forall \varpi_0, \vartheta_0 \in \mathcal{V}, \ \varrho \in \Phi.$$

(B2): For each $j_0 \in \{1, 2, \ldots, m_0\}$, the function $\varrho_{j_0} \longmapsto \varphi_{j_0}(., \varrho_{j_0})$ is concave on Φ_{j_0}. We define a family of real-valued functions ϕ, $\phi_{j_0} : \mathcal{R}^n \to \mathcal{R}$, for each $j_0 \in \{1, 2, \ldots, m_0\}$ as follows:

$$\phi_{j_0}(\vartheta_0) = \max_{\varrho_{j_0} \in \Phi_{j_0}} \varphi_{j_0}(\vartheta_0, \varrho_{j_0}) \tag{1}$$

$$\phi(\vartheta_0) = \max_{j_0 \in \{1,2,\ldots,m_0\}} \phi_{j_0}(\vartheta_0) \tag{2}$$

Since φ_{j_0} is upper semicontinuous and Φ_{j_0} is nonempty, convex and compact for each $j_0 \in \{1, 2, \ldots, m_0\}$, ϕ_{j_0} is clearly defined. By the auxiliary function (1), the following is an equivalent description of the set $\mathbb{H}$ of robust feasible solutions.

$$\mathbb{H} = \{\vartheta_0 \in \mathcal{R}^n : \phi_{j_0}(\vartheta_0) \leq 0, j_0 = 1, 2, \ldots, m_0\} = \{\vartheta_0 \in \mathcal{R}^n : \phi(\vartheta_0) \leq 0\}.$$

3. Karush-Kuhn-Tucker Robust $\mathcal{LU}$-Optimality Conditions

Chen et al. [42] established the Karush-Kuhn-Tucker robust necessary optimality conditions for weakly robust efficient solution for a robust non-smooth multi-objective optimization problem. In perspective of Chen et al. [42], if we take into account $k = 2$ then we arrive at the subsequent Karush-Kuhn-Tucker robust necessary optimality conditions for robust $\mathcal{LU}$ optimal solution.

Theorem 1 (Kuhn-Tucker-type robust necessary $\mathcal{LU}$-optimality conditions)**.** *Let φ satisfy the Assumption $\mathbb{D}$, ϕ is R_+-convexlike, $\Psi^{\mathcal{L}}$, $\Psi^{\mathcal{U}}$ be $\mathfrak{C}$-convexlike and (GRSCQ) holds at $\bar{\vartheta}_0$. If $\bar{\vartheta}_0$ is a robust $\mathcal{LU}$-optimal solution, then there exist $\bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}} \in \mathfrak{C}^* \setminus \{0\}$, $\bar{\gamma} \in \mathcal{R}_+^{m_0}$ and $\bar{\varrho} \in \Phi$ such that*

$$\begin{aligned} &0 \in \left\{ \partial \Psi^{\mathcal{L}}(\bar{\vartheta}_0)^T \bar{\mu}^{\mathcal{L}} + \partial \Psi^{\mathcal{U}}(\bar{\vartheta}_0)^T \bar{\mu}^{\mathcal{U}} + \partial_{\vartheta_0} \varphi(\bar{\vartheta}_0, \bar{\varrho})^T \bar{\gamma} \right\}, \\ &\bar{\gamma}^T \varphi(\bar{\vartheta}_0, \bar{\varrho}) = 0. \end{aligned} \tag{3}$$

Now, we establish Karush-Kuhn-Tucker-type robust sufficient $\mathcal{LU}$-optimality conditions for (UCIVOP).

Theorem 2 (Sufficient $\mathcal{LU}$-optimality conditions)**.** *Let $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ be type II $(\mathfrak{C}, \mathfrak{C}, \mathcal{R}_+^{m_0})$-generalized convex at $\bar{\vartheta}_0 \in \mathbb{H}$. Assume that there exist $\bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}} \in \mathfrak{C}^* \setminus \{0\}$, $\bar{\gamma} \in \mathcal{R}_+^{m_0}$ and $\bar{\varrho} \in \Phi$ such that*

$$0 \in \left\{ \partial \Psi^{\mathcal{L}}(\bar{\vartheta}_0)^T \bar{\mu}^{\mathcal{L}} + \partial \Psi^{\mathcal{U}}(\bar{\vartheta}_0)^T \bar{\mu}^{\mathcal{U}} + \partial_{\vartheta_0} \varphi(\bar{\vartheta}_0, \bar{\varrho})^T \bar{\gamma} \right\}, \tag{4}$$

$$\bar{\gamma}^T \varphi(\bar{\vartheta}_0, \bar{\varrho}) = 0. \tag{5}$$

Then $\bar{\vartheta}_0$ is a robust $\mathcal{LU}$-optimal solution.

Proof. Suppose $\bar{\vartheta}_0$ is not a robust $\mathcal{LU}$-optimal solution of (UCIVOP), then there exists $\hat{\vartheta}_0 \in \mathbb{H}$ such that

$$\Psi(\hat{\vartheta}_0) <_{\mathcal{LU}} \Psi(\bar{\vartheta}_0).$$

That is,

$$\Psi^{\mathcal{L}}(\hat{\vartheta}_0) < \Psi^{\mathcal{L}}(\bar{\vartheta}_0)$$
$$\Psi^{\mathcal{U}}(\hat{\vartheta}_0) < \Psi^{\mathcal{U}}(\bar{\vartheta}_0),$$

or

$$\Psi^{\mathcal{L}}(\hat{\vartheta}_0) \leq \Psi^{\mathcal{L}}(\bar{\vartheta}_0)$$
$$\Psi^{\mathcal{U}}(\hat{\vartheta}_0) < \Psi^{\mathcal{U}}(\bar{\vartheta}_0),$$

or

$$\Psi^{\mathcal{L}}(\hat{\vartheta}_0) < \Psi^{\mathcal{L}}(\bar{\vartheta}_0)$$
$$\Psi^{\mathcal{U}}(\hat{\vartheta}_0) \leq \Psi^{\mathcal{U}}(\bar{\vartheta}_0).$$

Since $\bar{\mu}^{\mathcal{L}} \geq 0$, $\bar{\mu}^{\mathcal{U}} \geq 0$, then the preceeding inequalities together yield

$$\bar{\mu}^{\mathcal{L}}[\Psi^{\mathcal{L}}(\hat{\vartheta}_0) - \Psi^{\mathcal{L}}(\bar{\vartheta}_0)] + \bar{\mu}^{\mathcal{U}}[\Psi^{\mathcal{U}}(\hat{\vartheta}_0) - \Psi^{\mathcal{U}}(\bar{\vartheta}_0)] < 0. \tag{6}$$

Since $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type II $(\mathcal{C}, \mathcal{C}, \mathcal{R}^{m_0}_{+})$-generalized convex at $\bar{\vartheta}_0$, then there exists $\xi \in \mathcal{R}$ such that

$$\begin{aligned}\Psi^{\mathcal{L}}(\hat{\vartheta}_0) - \Psi^{\mathcal{L}}(\bar{\vartheta}_0) - A^{\mathcal{L}}\xi \in \mathcal{C},\ \forall A^{\mathcal{L}} \in \partial\Psi^{\mathcal{L}}(\bar{\vartheta}_0),\\ \Psi^{\mathcal{U}}(\hat{\vartheta}_0) - \Psi^{\mathcal{U}}(\bar{\vartheta}_0) - A^{\mathcal{U}}\xi \in \mathcal{C},\ \forall A^{U} \in \partial\Psi^{U}(\bar{\vartheta}_0),\\ -\varphi(\bar{\vartheta}_0, \bar{\varrho}) - B\xi \in \mathcal{R}^{m_0}_{+},\ \forall B \in \partial_{\vartheta_0}\varphi(\bar{\vartheta}_0, \bar{\varrho}).\end{aligned}$$

From the above inequalities, one has

$$\begin{aligned}\bar{\mu}^{\mathcal{L}}[\Psi^{\mathcal{L}}(\hat{\vartheta}_0) - \Psi^{\mathcal{L}}(\bar{\vartheta}_0) - A^{\mathcal{L}}\xi] \geq 0,\ \forall A^{\mathcal{L}} \in \partial\Psi^{\mathcal{L}}(\bar{\vartheta}_0),\\ \bar{\mu}^{\mathcal{U}}[\Psi^{\mathcal{U}}(\hat{\vartheta}_0) - \Psi^{\mathcal{U}}(\bar{\vartheta}_0) - A^{\mathcal{U}}\xi] \geq 0,\ \forall A^{\mathcal{U}} \in \partial\Psi^{\mathcal{U}}(\bar{\vartheta}_0),\\ \bar{\gamma}^{T}[-\varphi(\bar{\vartheta}_0, \bar{\varrho}) - B\xi] \geq 0,\ \forall B \in \partial_{\vartheta_0}\varphi(\bar{\vartheta}_0, \bar{\varrho}).\end{aligned}$$

Combining the above inequalities, we get

$$\begin{aligned}&[\bar{\mu}^{\mathcal{L}}A^{\mathcal{L}} + \bar{\mu}^{\mathcal{U}}A^{\mathcal{U}} + \bar{\gamma}^{T}B]\xi\\ &\leq \bar{\mu}^{\mathcal{L}}[\Psi^{\mathcal{L}}(\hat{\vartheta}_0) - \Psi^{\mathcal{L}}(\bar{\vartheta}_0)] + \bar{\mu}^{\mathcal{U}}[\Psi^{\mathcal{U}}(\hat{\vartheta}_0) - \Psi^{\mathcal{U}}(\bar{\vartheta}_0)] - \bar{\gamma}^{T}\varphi(\bar{\vartheta}_0, \bar{\varrho}).\end{aligned} \tag{7}$$

By using (5), the inequality (7) gives

$$[\bar{\mu}^{\mathcal{L}}A^{\mathcal{L}} + \bar{\mu}^{\mathcal{U}}A^{\mathcal{U}} + \bar{\gamma}^{T}B]\xi \leq \bar{\mu}^{\mathcal{L}}[\Psi^{\mathcal{L}}(\hat{\vartheta}_0) - \Psi^{\mathcal{L}}(\bar{\vartheta}_0)] + \bar{\mu}^{\mathcal{U}}[\Psi^{\mathcal{U}}(\hat{\vartheta}_0) - \Psi^{\mathcal{U}}(\bar{\vartheta}_0)]. \tag{8}$$

By using (6), the inequality (8) implies

$$[\bar{\mu}^{\mathcal{L}}A^{\mathcal{L}} + \bar{\mu}^{\mathcal{U}}A^{\mathcal{U}} + \bar{\gamma}^{T}B]\xi < 0,$$

which contradicts (4) and hence the theorem. □

The subsequent example demonstrates Theorem 2.

Example 1. *Now we examine the uncertain constrained interval-valued optimization problem.*

$$\begin{aligned}\text{(UCIVOP-1)} \qquad &\min_{\vartheta_1, \vartheta_2 \in \mathcal{R}} \quad \Psi(\vartheta_1, \vartheta_2) = [\Psi^{\mathcal{L}}(\vartheta_1, \vartheta_2), \Psi^{\mathcal{U}}(\vartheta_1, \vartheta_2)]\\ &\text{subject to}\\ &-\varphi(\vartheta_1, \vartheta_2, \varrho) \in \mathcal{R}^{2}_{+}, \text{where } \varrho \in \Phi.\end{aligned}$$

The robust counterpart of (UCIVOP-1) is defined as given below:

$$\text{(RIVOP-1)} \qquad \min_{\vartheta_1,\vartheta_2 \in \mathcal{R}} \quad \Psi(\vartheta_1,\vartheta_2) = ([1,1]\vartheta_1 + [1,1]\vartheta_2^2 + [4,5])$$

subject to

$$\left\{ -\vartheta_1{}^2 + 2\vartheta_1 + 3 - ln(1+\varrho_1), \vartheta_2 + |\varrho_2| \right\}^T \in R^2_+, \forall \varrho \in \Phi,$$

where $\varrho = (\varrho_1,\varrho_2) \in \Phi = \Phi_1 \times \Phi_2$.

Let $\Phi_1 = [-0.5, 0]$, $\Phi_2 = [-1,1]$ *and* $\mathcal{C} = \mathcal{R}_+$. *Let* $\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}} : \mathcal{R}^2 \to \mathcal{R}$ *and* $\varphi : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}^2$. *One can validate that the robust feasible set is* $\mathbb{H} = [-1,1] \times \mathcal{R}_+$. *Then,* $\partial\psi^{\mathcal{L}}(\vartheta_1,\vartheta_2)^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \partial\psi^{\mathcal{U}}(\vartheta_1,\vartheta_2)^T$ *and* $\partial_{\vartheta_0}\varphi(\bar{\vartheta}_1,\bar{\vartheta}_2,\bar{\varrho}) = \begin{pmatrix} -4 & 0 \\ 0 & -1 \end{pmatrix}$. *One can easily verify that* $(\Psi^{\mathcal{L}},\Psi^{\mathcal{U}},\varphi)$ *is type II* $(C, C, R^{m_0}_+)$*-generalized convex at* $\bar{\vartheta}_0 = (\bar{\vartheta}_1,\bar{\vartheta}_2) = (-1,0)$. *Moreover, there exists* $\bar{\mu}^{\mathcal{L}} = 1/2 = \bar{\mu}^{\mathcal{U}}$, *where* $\bar{\mu}^{\mathcal{L}},\bar{\mu}^{\mathcal{U}} \in C^* \setminus \{0\}$ *and* $\bar{\gamma} = (1/4,1)^T \in R^2_+$, $\bar{\varrho} = 0 \in \Phi$. *It can be easily verified that*

$$0 \in \left\{ \partial\Psi^{\mathcal{L}}(\bar{\vartheta}_1,\bar{\vartheta}_2)^T\bar{\mu}^{\mathcal{L}} + \partial\Psi^{\mathcal{U}}(\bar{\vartheta}_1,\bar{\vartheta}_2)^T\bar{\mu}^{\mathcal{U}} + \partial_{\vartheta_0}\varphi(\bar{\vartheta}_1,\bar{\vartheta}_2,\bar{\varrho})^T\bar{\gamma} \right\},$$

$$\bar{\gamma}^T\varphi(\bar{\vartheta}_1,\bar{\vartheta}_2,\bar{\varrho}) = 0.$$

Then $\bar{\vartheta} = (-1,0) \in \mathbb{H}$ *is a robust* $\mathcal{LU}$ *-optimal solution of (UCIVOP-1) (See Figure 1) .*

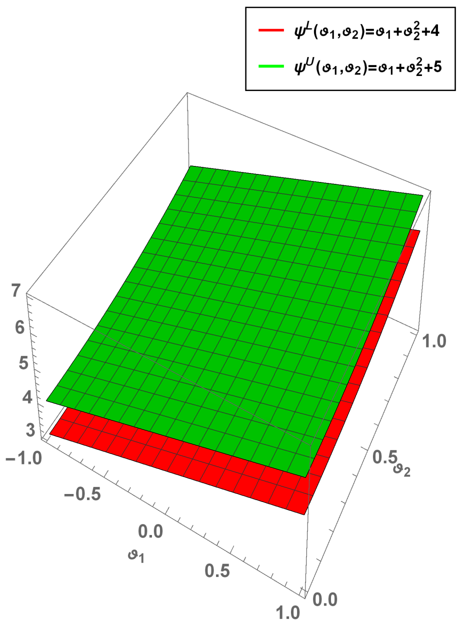

Figure 1. Graphical view of the objective functions $\Psi^{\mathcal{L}}(\vartheta_1,\vartheta_2)$ and $\Psi^{\mathcal{U}}(\vartheta_1,\vartheta_2)$ of the problem (UCIVOP-1).

Theorem 3 (Sufficient $\mathcal{LU}$-optimality conditions)**.** *Assume that there exist* $\bar{\mu}^{\mathcal{L}},\bar{\mu}^{\mathcal{U}} \in \mathcal{C}^* \setminus \{0\}$, $\bar{\gamma} \in \mathcal{R}^{m_0}_+$ *and* $\bar{\varrho} \in \Phi$ *such that*

$$0 \in \left\{ \partial\Psi^{\mathcal{L}}(\bar{\vartheta}_0)^T\bar{\mu}^{\mathcal{L}} + \partial\Psi^{\mathcal{U}}(\bar{\vartheta}_0)^T\bar{\mu}^{\mathcal{U}} + \partial_{\vartheta_0}\varphi(\bar{\vartheta}_0,\bar{\varrho})^T\bar{\gamma} \right\}, \tag{9}$$

$$\bar{\gamma}^T\varphi(\bar{\vartheta}_0,\bar{\varrho}) = 0. \tag{10}$$

(i). If $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type I $(\mathcal{C}, \mathcal{C}, \mathcal{R}^{m_0}_{+})$-pseudo convex at $\bar{\vartheta}_0 \in \mathbb{H}$, then $\bar{\vartheta}_0$ is a robust $\mathcal{LU}$-optimal solution.
(ii). If $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type II $(\mathcal{C}, \mathcal{C}, \mathcal{R}^{m_0}_{+})$-pseudo convex at $\bar{\vartheta}_0 \in \mathbb{H}$, then $\bar{\vartheta}_0$ is a robust $\mathcal{LU}$-optimal solution.

Proof. Firstly we validate (i).

Suppose $\bar{\vartheta}_0$ is not a robust $\mathcal{LU}$-optimal solution of (UCIVOP), then there exists $\hat{\vartheta}_0 \in \mathbb{H}$ such that

$$\Psi(\hat{\vartheta}_0) <_{\mathcal{LU}} \Psi(\bar{\vartheta}_0).$$

That is,

$$\begin{aligned} \Psi^{\mathcal{L}}(\hat{\vartheta}_0) &< \Psi^{\mathcal{L}}(\bar{\vartheta}_0) \\ \Psi^{\mathcal{U}}(\hat{\vartheta}_0) &< \Psi^{\mathcal{U}}(\bar{\vartheta}_0), \end{aligned}$$

or

$$\begin{aligned} \Psi^{\mathcal{L}}(\hat{\vartheta}_0) &\leq \Psi^{\mathcal{L}}(\bar{\vartheta}_0) \\ \Psi^{\mathcal{U}}(\hat{\vartheta}_0) &< \Psi^{\mathcal{U}}(\bar{\vartheta}_0), \end{aligned}$$

or

$$\begin{aligned} \Psi^{\mathcal{L}}(\hat{\vartheta}_0) &< \Psi^{\mathcal{L}}(\bar{\vartheta}_0) \\ \Psi^{\mathcal{U}}(\hat{\vartheta}_0) &\leq \Psi^{\mathcal{U}}(\bar{\vartheta}_0). \end{aligned}$$

Since $\bar{\mu}^{\mathcal{L}} \geq 0$, $\bar{\mu}^{\mathcal{U}} \geq 0$, then the preceeding inequalities together yield

$$\bar{\mu}^{\mathcal{L}}[\Psi^{\mathcal{L}}(\hat{\vartheta}_0) - \Psi^{\mathcal{L}}(\bar{\vartheta}_0)] + \bar{\mu}^{\mathcal{U}}[\Psi^{\mathcal{U}}(\hat{\vartheta}_0) - \Psi^{\mathcal{U}}(\bar{\vartheta}_0)] < 0. \tag{11}$$

By virtue of (9), there exist $\vartheta_1 \in \partial(\bar{\mu}^{\mathcal{L}}\Psi^{\mathcal{L}})(\bar{\vartheta}_0)$, $\vartheta_2 \in \partial(\bar{\mu}^{\mathcal{U}}\Psi^{\mathcal{U}})(\bar{\vartheta}_0)$, $\varpi_1 \in \partial_{\vartheta_0}\varphi(\bar{\vartheta}_0, \bar{\varrho})^T\bar{\gamma}$ such that

$$\vartheta_1 + \vartheta_2 + \varpi_1 = 0. \tag{12}$$

Since $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type I $(\mathcal{C}, \mathcal{C}, \mathcal{R}^{m_0}_{+})$-pseudo convex at $\bar{\vartheta}_0$, for any $\hat{\vartheta}_0$ and $\bar{\mu}^{\mathcal{L}}$, $\bar{\mu}^{\mathcal{U}} \in \mathcal{C}^* \setminus \{0\}$ the following hold:

$$\bar{\mu}^{\mathcal{L}}\Psi^{L}(\hat{\vartheta}_0) < \bar{\mu}^{\mathcal{L}}\Psi^{\mathcal{L}}(\bar{\vartheta}_0) \Rightarrow A^{\mathcal{L}}(\hat{\vartheta}_0 - \bar{\vartheta}_0) < 0, \ \forall A^{L} \in \partial(\bar{\mu}^{\mathcal{L}}\Psi^{\mathcal{L}})(\bar{\vartheta}_0). \tag{13}$$

$$\bar{\mu}^{\mathcal{U}}\Psi^{\mathcal{U}}(\hat{\vartheta}_0) < \bar{\mu}^{\mathcal{U}}\Psi^{\mathcal{U}}(\bar{\vartheta}_0) \Rightarrow A^{\mathcal{U}}(\hat{\vartheta}_0 - \bar{\vartheta}_0) < 0, \ \forall A^{\mathcal{U}} \in \partial(\bar{\mu}^{\mathcal{U}}\Psi^{\mathcal{U}})(\bar{\vartheta}_0). \tag{14}$$

$$\varphi(\hat{\vartheta}_0, \bar{\varrho}) \leq \varphi(\bar{\vartheta}_0, \bar{\varrho}) \Rightarrow B^T(\hat{\vartheta}_0 - \bar{\vartheta}_0) \leq 0, \ \forall B \in \partial_{\vartheta_0}\varphi(\bar{\vartheta}_0, \bar{\varrho}). \tag{15}$$

By using the inequalities (11), (13) and (14), we get

$$[A^{\mathcal{L}} + A^{\mathcal{U}}](\hat{\vartheta}_0 - \bar{\vartheta}_0) < 0. \tag{16}$$

The inequality (16) along with (12) gives

$$-\varpi_1(\hat{\vartheta}_0 - \bar{\vartheta}_0) < 0. \tag{17}$$

Since $\varpi_1 \in \partial_{\vartheta_0}\varphi(\bar{\vartheta}_0, \bar{\varrho})^T\bar{\gamma}$, we get $\varpi_1 = B_0^T\bar{\gamma}$, for some $B_0 \in \partial_{\vartheta_0}\varphi(\bar{\vartheta}_0, \bar{\varrho})$. Thus,

$$-B_0^T\bar{\gamma}(\hat{\vartheta}_0 - \bar{\vartheta}_0) < 0, \text{ for some } B_0 \in \partial_{\vartheta_0}\varphi(\bar{\vartheta}_0, \bar{\varrho}). \tag{18}$$

Since $\bar{\vartheta}_0 \in \mathbb{H}$ and using (10) in the inequality (15) with $\bar{\gamma} \in \mathcal{R}^{m_0}_{+}$, we obtain

$$B^T\bar{\gamma}(\hat{\vartheta}_0 - \bar{\vartheta}_0) \leq 0, \ \forall B \in \partial_{\vartheta_0}\varphi(\bar{\vartheta}_0, \bar{\varrho}).$$

This is a contradiction to (18).

Assertion (ii) is proved similar to part (i) by using the type II pseudo convexity of $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ at $\bar{\vartheta}_0$. $\square$

4. Wolfe Type Robust Dual Problem

Let us examine the subsequent Wolfe type robust dual problem for (UCIVOP).

$$\text{(WIRD)} \qquad \max_{(\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma)} \Big(\Psi(\varpi_0) + \gamma^T \varphi(\varpi_0, \varrho)\Big) = \Big\{[\Psi^{\mathcal{L}}(\varpi_0), \Psi^{\mathcal{U}}(\varpi_0)] + \gamma^T \varphi(\varpi_0, \varrho)\Big\}$$

subject to

$$0 \in \Big\{\partial \Psi^{\mathcal{L}}(\varpi_0)^T \mu^{\mathcal{L}} + \partial \Psi^{\mathcal{U}}(\varpi_0)^T \mu^{\mathcal{U}} + \partial_{\vartheta_0} \varphi(\varpi_0, \varrho)^T \gamma\Big\},$$

$$\varrho \in \Phi,\ \mu^{\mathcal{L}} + \mu^{\mathcal{U}} = 1,\ \mu^{\mathcal{L}}, \mu^{\mathcal{U}} \in \mathcal{C}^* \setminus \{0\},\ \gamma \in \mathcal{R}^{m_0}_+.$$

The robust feasible set of (WIRD) is represented as $\mathbb{H}^W_D$, which is the set of all points of the form $(\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma) \in \mathcal{R}^n \times \Phi \times \mathcal{C}^* \setminus \{0\} \times \mathcal{C}^* \setminus \{0\} \times \mathcal{R}^{(m_0)}_+$ that satisfies the constraints of (WIRD).

Remark 2. *(i). If $\Psi^{\mathcal{L}}(\varpi_0) = \Psi^{\mathcal{U}}(\varpi_0)$ for all $\varpi_0 \in \mathcal{R}^n$, then (WIRD) model reduces to Wolfe type dual model (WRD) of Chen et al. [42].*
(ii). In the absence of uncertain parameter ρ in the constraints, the (WIRD) model reduces to (WD) model of Singh et al. [45].

Definition 9. *The robust feasible point $(\bar{\varpi}_0, \bar{\varrho}, \bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}}, \bar{\gamma}) \in \mathbb{H}^W_D$ is called a robust $\mathcal{LU}$-optimal solution of (WIRD), if there does not exist a robust feasible solution $(\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma)$ of (WIRD) such that $\Psi(\bar{\varpi}_0) + \bar{\gamma}^T \varphi(\bar{\varpi}_0, \bar{\varrho}) <_{\mathcal{LU}} \Psi(\varpi_0) + \gamma^T \varphi(\varpi_0, \varrho)$.*

The following section describes the duality results between (UCIVOP) and (WIRD).

Theorem 4 (Weak Duality)**.** *Let $\vartheta_0 \in \mathbb{H}$ and $(\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma) \in \mathbb{H}^W_D$. If $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type II$(\mathcal{C}, \mathcal{C}, \mathcal{R}^{m_0}_+)$-generalized convex at ϖ_0, then the subsequent inequality cannot hold :*

$$\Psi(\vartheta_0) <_{\mathcal{LU}} \Psi(\varpi_0) + \gamma^T \varphi(\varpi_0, \varrho).$$

Proof. Suppose, if possible

$$\Psi(\vartheta_0) <_{\mathcal{LU}} \Psi(\varpi_0) + \gamma^T \varphi(\varpi_0, \varrho). \tag{19}$$

That is,

$$\Psi^{\mathcal{L}}(\vartheta_0) < \Psi^{\mathcal{L}}(\varpi_0) + \gamma^T \varphi(\varpi_0, \varrho)$$
$$\Psi^{\mathcal{U}}(\vartheta_0) < \Psi^{\mathcal{U}}(\varpi_0) + \gamma^T \varphi(\varpi_0, \varrho),$$

or

$$\Psi^{\mathcal{L}}(\vartheta_0) \leq \Psi^{\mathcal{L}}(\varpi_0) + \gamma^T \varphi(\varpi_0, \varrho)$$
$$\Psi^{\mathcal{U}}(\vartheta_0) < \Psi^{\mathcal{U}}(\varpi_0) + \gamma^T \varphi(\varpi_0, \varrho),$$

or

$$\Psi^{\mathcal{L}}(\vartheta_0) < \Psi^{\mathcal{L}}(\varpi_0) + \gamma^T \varphi(\varpi_0, \varrho)$$
$$\Psi^{\mathcal{U}}(\vartheta_0) \leq \Psi^{\mathcal{U}}(\varpi_0) + \gamma^T \varphi(\varpi_0, \varrho).$$

Since $\mu^{\mathcal{L}} \geq 0$, $\mu^{\mathcal{U}} \geq 0$, then the preceeding inequalities together yield

$$\mu^{\mathcal{L}}[\Psi^{\mathcal{L}}(\vartheta_0) - \Psi^{\mathcal{L}}(\varpi_0)] + \mu^{\mathcal{U}}[\Psi^{\mathcal{U}}(\vartheta_0) - \Psi^{\mathcal{U}}(\varpi_0)] - \gamma^T \varphi(\varpi_0, \varrho)[\mu^{\mathcal{L}} + \mu^{\mathcal{U}}] < 0. \tag{20}$$

Since $(\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma) \in \mathbb{H}_D^W$, we obtain $\mu^{\mathcal{L}}, \mu^{\mathcal{U}} \in \mathcal{C}^* \setminus \{0\}$, $\gamma \in \mathcal{R}_+^{m_0}$, $\varrho \in \Phi$,

$$\mu^{\mathcal{L}} + \mu^{\mathcal{U}} = 1, \tag{21}$$

$$0 \in \left\{ \partial \Psi^{\mathcal{L}}(\varpi_0)^T \mu^{\mathcal{L}} + \partial \Psi^{\mathcal{U}}(\varpi_0)^T \mu^{\mathcal{U}} + \partial_{\vartheta_0} \varphi(\varpi_0, \varrho)^T \gamma \right\}. \tag{22}$$

By using (21), the inequality (20) gives

$$\mu^{\mathcal{L}}[\Psi^{\mathcal{L}}(\vartheta_0) - \Psi^{\mathcal{L}}(\varpi_0)] + \mu^{\mathcal{U}}[\Psi^{U}(\vartheta_0) - \Psi^{\mathcal{U}}(\varpi_0)] - \gamma^T \varphi(\varpi_0, \varrho) < 0. \tag{23}$$

Since $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type II $(\mathcal{C}, \mathcal{C}, \mathcal{R}_+^{m_0})$-generalized convex at ϖ_0, then there exists $\xi \in \mathcal{R}$ such that

$$\begin{aligned}
\Psi^{\mathcal{L}}(\vartheta_0) - \Psi^{\mathcal{L}}(\varpi_0) - A^{\mathcal{L}}\xi \in \mathcal{C},\ \forall A^{L} \in \partial \Psi^{\mathcal{L}}(\varpi_0), \\
\Psi^{\mathcal{U}}(\vartheta_0) - \Psi^{\mathcal{U}}(\varpi_0) - A^{\mathcal{U}}\xi \in \mathcal{C},\ \forall A^{\mathcal{U}} \in \partial \Psi^{\mathcal{U}}(\varpi_0), \\
-\varphi(\varpi_0, \varrho) - B\xi \in \mathcal{R}_+^{m_0},\ \forall B \in \partial_{\vartheta_0} \varphi(\varpi_0, \varrho).
\end{aligned}$$

From the above inequalities, one has

$$\begin{aligned}
\mu^{\mathcal{L}}[\Psi^{\mathcal{L}}(\vartheta_0) - \Psi^{\mathcal{L}}(\varpi_0) - A^{\mathcal{L}}\xi] \geq 0,\ \forall A^{\mathcal{L}} \in \partial \Psi^{\mathcal{L}}(\varpi_0), \\
\mu^{\mathcal{U}}[\Psi^{\mathcal{U}}(\vartheta_0) - \Psi^{\mathcal{U}}(\varpi_0) - A^{\mathcal{U}}\xi] \geq 0,\ \forall A^{\mathcal{U}} \in \partial \Psi^{\mathcal{U}}(\varpi_0), \\
\gamma^T[-\varphi(\varpi_0, \varrho) - B\xi] \geq 0,\ \forall B \in \partial_{\vartheta_0} \varphi(\varpi_0, \varrho).
\end{aligned}$$

Combining the above inequalities, we get

$$\begin{aligned}
&[\mu^{\mathcal{L}} A^{\mathcal{L}} + \mu^{\mathcal{U}} A^{\mathcal{U}} + \gamma^T B]\xi \\
&\leq \mu^{\mathcal{L}}[\Psi^{\mathcal{L}}(\vartheta_0) - \Psi^{\mathcal{L}}(\varpi_0)] + \mu^{\mathcal{U}}[\Psi^{\mathcal{U}}(\vartheta_0) - \Psi^{\mathcal{U}}(\varpi_0)] - \gamma^T \varphi(\varpi_0, \varrho).
\end{aligned} \tag{24}$$

By using (23), the inequality(24) implies

$$[\mu^{L} A^{\mathcal{L}} + \mu^{\mathcal{U}} A^{\mathcal{U}} + \gamma^T B]\xi < 0,\ \forall A^{\mathcal{L}} \in \partial \Psi^{\mathcal{L}}(\varpi_0),\ \forall A^{\mathcal{U}} \in \partial \Psi^{\mathcal{U}}(\varpi_0),\ \forall B \in \partial_{\vartheta_0} \varphi(\varpi_0, \varrho),$$

which contradicts (22) and hence the theorem. □

We now re-explore Example 1 to demonstrate Theorem 4.

Example 2. *Let us examine the uncertain constrained interval-valued optimization problem.*

$$\begin{aligned}
\text{(UCIVOP-1)} \qquad & \min_{\vartheta_1, \vartheta_2 \in \mathcal{R}} \ \Psi(\vartheta_1, \vartheta_2) = [\Psi^{\mathcal{L}}(\vartheta_1, \vartheta_2), \Psi^{\mathcal{U}}(\vartheta_1, \vartheta_2)] \\
& \text{subject to} \\
& -\varphi(\vartheta_1, \vartheta_2, \varrho) \in \mathcal{R}_+^2, \text{where } \varrho \in \Phi.
\end{aligned}$$

Let $\mathcal{R}^n, \mathcal{R}, \mathcal{R}^{m_0}, \mathcal{R}^p, \mathcal{C}, \Phi_1, \Phi_2, \Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}$ and φ be same as Example 1. The robust counterpart of (UCIVOP-1) is defined as follows:

$$\text{(RIVOP-1)} \qquad \min_{\vartheta_1,\vartheta_2\in\mathcal{R}} \quad \Psi(\vartheta_1,\vartheta_2) = ([1,1]\vartheta_1 + [1,1]\vartheta_2^2 + [4,5])$$

subject to

$$\left\{ -\vartheta_1{}^2 + 2\vartheta_1 + 3 - ln(1+\varrho_1), \vartheta_2 + |\varrho_2| \right\}^T \in \mathcal{R}^2_+, \forall \varrho \in \Phi,$$

where $\varrho = (\varrho_1,\varrho_2) \in \Phi = \Phi_1 \times \Phi_2$. Recall that the robust feasible set of (RIVOP-1) is $\mathbb{H} = [-1,1] \times \mathcal{R}_+$.

We consider a Wolfe type robust dual problem (WIRD-1) for (RIVOP-1) as follows:

$$\text{(WIRD-1)} \qquad \max_{(\varpi_1,\varpi_2,\varrho,\mu^{\mathcal{L}},\mu^{\mathcal{U}},\gamma)} \Bigg(([1,1]\varpi_1 + [1,1]\varpi_2^2 + [4,5]) + \gamma^T \left\{ \varpi_1^2 - 2\varpi_1 - 3 + ln(1+\varrho_1), -\varpi_2 - |\varrho_2| \right\}^T \Bigg)$$

subject to

$$0 \in \left\{ \partial\Psi^{\mathcal{L}}(\bar{\varpi}_1,\bar{\varpi}_2)^T\mu^{\mathcal{L}} + \partial\Psi^{\mathcal{U}}(\bar{\varpi}_1,\bar{\varpi}_2)^T\mu^{\mathcal{U}} + \partial_{\vartheta_0}\varphi(\bar{\varpi}_1,\bar{\varpi}_2,\bar{\varrho})^T\gamma \right\},$$

where $\mu^{\mathcal{L}} + \mu^{\mathcal{U}} = 1$, $\mu^{\mathcal{L}},\mu^{\mathcal{U}} \in \mathcal{C}^* \setminus \{0\}$ and $\gamma \in \mathcal{R}^{m_0}_+$. Again from Example 1, we have $(\vartheta_1,\vartheta_2,\varrho,\mu^{\mathcal{L}},\mu^{\mathcal{U}},\gamma) \in \mathbb{H}^W_D$ where $\mu^{\mathcal{L}} = 1/2 = \mu^{\mathcal{U}}$, $\gamma = (1/4,1)^T \in \mathcal{R}^2_+$, $\varrho = 0 \in \Phi$. For any $(\varpi_1,\varpi_2,\varrho,\mu^{\mathcal{L}},\mu^{\mathcal{U}},\gamma) \in \mathbb{H}^W_D$, we conclude that $\mu^{\mathcal{L}},\mu^{\mathcal{U}} \in \mathcal{C}^* \setminus \{0\}$, $\gamma = (\gamma_1,\gamma_2)^T \in \mathcal{R}^2_+$. Hence, it follows

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2\varpi_2 \end{pmatrix}\mu^{\mathcal{L}} + \begin{pmatrix} 1 \\ 2\varpi_2 \end{pmatrix}\mu^{\mathcal{U}} + \begin{pmatrix} 2\varpi_1 - 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$$

From the above equation, we get $\mu^{\mathcal{L}} + \mu^{\mathcal{U}} + 2(\varpi_1 - 1)\gamma_1 = 0$ and $2\varpi_2(\mu^{\mathcal{L}} + \mu^{\mathcal{U}}) - \gamma_2 = 0$. Combining these equations along with $\mu^{\mathcal{L}},\mu^{\mathcal{U}},\gamma_1,\gamma_2 \geq 0$ and $\mu^{\mathcal{L}} + \mu^{\mathcal{U}} = 1$, we obtain $\varpi_2 \geq 0$, $\varpi_1 \leq 1$. Taking into account $\varpi_2 \geq 0$, $\varpi_1 \leq 1$ and $\forall(\varpi_1,\varpi_2,\varrho,\mu^{\mathcal{L}},\mu^{\mathcal{U}},\gamma) \in \mathbb{H}^W_D$, we have,

$$\begin{aligned} &\mu^{\mathcal{L}}[\Psi^{\mathcal{L}}(\vartheta_1,\vartheta_2) - \Psi^{\mathcal{L}}(\varpi_1,\varpi_2)] + \mu^{\mathcal{U}}[\Psi^{\mathcal{U}}(\vartheta_1,\vartheta_2) - \Psi^{\mathcal{U}}(\varpi_1,\varpi_2)] \\ &\qquad - (\mu^{\mathcal{L}} + \mu^{\mathcal{U}})\gamma^T\varphi(\varpi_1,\varpi_2,\varrho) \\ &= -1/4 - 3\varpi_1/2 - \varpi_1^2/4 - \varpi_2^2 + \varpi_2 - ln(1+\varrho_1)/4 + |w_2| \nless 0. \end{aligned}$$

Therefore, weak duality theorem of (WIRD-1) holds.

Theorem 5 (Strong Duality)**.** *Let $\bar{\vartheta}_0$ be a robust $\mathcal{LU}$-optimal solution of (UCIVOP) and (GRSCQ) hold. Assume that φ satisfy the Assumption $\mathbb{D}$, ϕ is $\mathcal{R}_+$-convexlike and $\Psi^{\mathcal{L}}$, $\Psi^{\mathcal{U}}$ be $\mathcal{C}$-convexlike. Then, there exist $\bar{\varrho} \in \Phi$, $\bar{\mu}^{\mathcal{L}},\bar{\mu}^{\mathcal{U}} \in \mathcal{C}^* \setminus \{0\}$, $\bar{\gamma} \in R^{m_0}_+$ such that $(\bar{\vartheta}_0,\bar{\varrho},\bar{\mu}^{\mathcal{L}},\bar{\mu}^{\mathcal{U}},\bar{\gamma}) \in \mathbb{H}^W_D$. Furthermore, if $(\Psi^{\mathcal{L}},\Psi^{\mathcal{U}},\varphi)$ is type II $(\mathcal{C},\mathcal{C},\mathcal{R}^{m_0}_+)$-generalized convex at ϖ_0, where $(\varpi_0,\varrho,\mu^{\mathcal{L}},\mu^{\mathcal{U}},\gamma) \in \mathbb{H}^W_D$ then $(\bar{\vartheta}_0,\bar{\varrho},\bar{\mu}^{\mathcal{L}},\bar{\mu}^{\mathcal{U}},\bar{\gamma})$ is a robust $\mathcal{LU}$-optimal solution of (WIRD).*

Proof. As a result of Theorem 1, there exist $\bar{\varrho} \in \Phi$, $\bar{\mu}^{\mathcal{L}},\bar{\mu}^{\mathcal{U}} \in \mathcal{C}^* \setminus \{0\}$, $\bar{\gamma} \in \mathcal{R}^{m_0}_+$ such that $(\bar{\vartheta}_0,\bar{\varrho},\bar{\mu}^{\mathcal{L}},\bar{\mu}^{\mathcal{U}},\bar{\gamma}) \in \mathbb{H}^W_D$ and

$$\bar{\gamma}^T\varphi(\bar{\vartheta}_0,\bar{\varrho}) = 0. \tag{25}$$

By Theorem 4, the subsequent inequality does not hold:

$$\Psi(\bar{\vartheta}_0) <_{\mathcal{LU}} \Psi(\varpi_0) + \gamma^T \varphi(\varpi_0, \varrho),\ \forall (\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma) \in \mathbb{H}_D^W.$$

This implies $(\bar{\vartheta}_0, \bar{\varrho}, \bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}}, \bar{\gamma})$ is a robust $\mathcal{LU}$-optimal solution of (WIRD). □

Theorem 6 (Converse Duality). *Let $(\varpi_0, \varrho, \mu^{\mathcal{L}}, \bar{\mu}^{L}, \bar{\mu}^{\mathcal{U}}, \bar{\gamma})$ be a robust $\mathcal{LU}$-optimal solution of (WIRD) with $\bar{\varpi}_0 \in \mathbb{H}$. If $(\Psi^L, \Psi^U, \varphi)$ is type II $(\mathfrak{C}, \mathfrak{C}, \mathcal{R}_+^{m_0})$-generalized convex at $\bar{\varpi}_0$, then $\bar{\varpi}_0$ is a robust $\mathcal{LU}$-optimal solution of (UCIVOP).*

Proof. Since $(\bar{\varpi}_0, \bar{\varrho}, \bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}}, \bar{\gamma})$ is a robust $\mathcal{LU}$-optimal solution of (WIRD) and $\bar{\varpi}_0 \in \mathbb{H}$. Then, it follows from Theorem 4, the subsequent inequality does not hold:

$$\Psi(\vartheta_0) <_{\mathcal{LU}} \Psi(\bar{\varpi}_0) + \bar{\gamma}^T \varphi(\bar{\varpi}_0, \bar{\varrho}),\ \forall\ \vartheta_0 \in \mathbb{H}.$$

This implies, the subsequent inequality does not hold: $\Psi(\vartheta_0) <_{\mathcal{LU}} \Psi(\bar{\varpi}_0)$.
Hence, $\bar{\varpi}_0$ is a robust $\mathcal{LU}$-optimal solution of (UCIVOP). □

5. Mond-Weir Type Robust Dual Problem

Let us examine the subsequent Mond-Weir type robust dual problem for (UCIVOP).

(MWIVRD)
$$\max_{(\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma)} \Psi(\varpi_0) = [\Psi^{\mathcal{L}}(\varpi_0), \Psi^{\mathcal{U}}(\varpi_0)]$$
subject to
$$0 \in \left\{ \partial\Psi^{\mathcal{L}}(\varpi_0)^T \mu^{\mathcal{L}} + \partial\Psi^{\mathcal{U}}(\varpi_0)^T \mu^{\mathcal{U}} + \partial_{\vartheta_0} \varphi(\varpi_0, \varrho)^T \gamma \right\},$$
$$\gamma^T \varphi(\varpi_0, \varrho) \geq 0,\ \varrho \in \Phi,$$
$$\mu^{\mathcal{L}}, \mu^{\mathcal{U}} \in \mathfrak{C}^* \setminus \{0\},\ \gamma \in \mathcal{R}_+^{m_0}.$$

It is worth mentioning that (MWIVRD) is viewed as a likely version to the Mond-Weir type robust dual problem of (UCIVOP).

(MWIVRD)
$$\max_{(\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma)} \Psi(\varpi_0) = [\Psi^{\mathcal{L}}(\varpi_0), \Psi^{\mathcal{U}}(\varpi_0)]$$
subject to
$$0 \in \left\{ \partial\Psi^{\mathcal{L}}(\varpi_0)^T \mu^{\mathcal{L}} + \partial\Psi^{\mathcal{U}}(\varpi_0)^T \mu^{\mathcal{U}} + \partial_{\vartheta_0} \varphi(\varpi_0, \varrho)^T \gamma \right\},$$
$$\gamma^T \varphi(\varpi_0, \varrho) \geq 0,$$
$$\mu^{\mathcal{L}}, \mu^{\mathcal{U}} \in \mathfrak{C}^* \setminus \{0\},\ \gamma \in \mathcal{R}_+^{m_0},$$

where $\varrho \in \Phi$ is an uncertain parameter.

The robust feasible set of (MWIVRD) is denoted by $\mathbb{H}_D^{MW}$, which is the set of all points of the form $(\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma) \in \mathcal{R}^n \times \Phi \times \mathfrak{C}^* \setminus \{0\} \times \mathfrak{C}^* \setminus \{0\} \times \mathcal{R}_+^{(m_0)}$ that satisfies the constraints of (MWIVRD).

Remark 3. *(i). If $\Psi^{\mathcal{L}}(\varpi_0) = \Psi^{\mathcal{U}}(\varpi_0)$ for all $\varpi_0 \in \mathcal{R}^n$, then (MWIVRD) reduces to Mond-Wier type dual model (MWRD) of Chen et al. [42].*
(ii). In the absence of uncertain parameter ρ in the constraints, the (MWIVRD) reduces to (MWD) model of Singh et al. [45].

Definition 10. *The robust feasible point* $(\bar{\varpi}_0, \bar{\varrho}, \bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}}, \bar{\gamma}) \in \mathbb{H}_D^{MW}$ *is called a robust* $\mathcal{LU}$*-optimal solution of (MWIVRD), if there does not exist a robust feasible solution* $(\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma)$ *of (MWIVRD) such that* $\Psi(\bar{\varpi}_0) <_{\mathcal{LU}} \Psi(\varpi_0)$.

The following section describes the duality results between (UCIVOP) and (MWIVRD).

Theorem 7 (Weak Duality)**.** *Let* $\vartheta_0 \in \mathbb{H}$ *and* $(\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma) \in \mathbb{H}_D^{MW}$.
(i). If $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ *is type I* $(\mathcal{C}, \mathcal{C}, \mathcal{R}_+^{m_0})$*-pseudo convex at* ϖ_0*, then the subsequent inequality cannot hold :*

$$\Psi(\vartheta_0) <_{\mathcal{LU}} \Psi(\varpi_0).$$

(ii). If $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ *is type II* $(\mathcal{C}, \mathcal{C}, \mathcal{R}_+^{m_0})$*-pseudo convex at* ϖ_0*, then the subsequent inequality cannot hold:*

$$\Psi(\vartheta_0) <_{\mathcal{LU}} \Psi(\varpi_0).$$

Proof. Firstly we validate (i).

Suppose, if possible

$$\Psi(\vartheta_0) <_{\mathcal{LU}} \Psi(\varpi_0).$$

That is,

$$\Psi^{\mathcal{L}}(\vartheta_0) < \Psi^{\mathcal{L}}(\varpi_0)$$
$$\Psi^{\mathcal{U}}(\vartheta_0) < \Psi^{\mathcal{U}}(\varpi_0),$$

or

$$\Psi^{\mathcal{L}}(\vartheta_0) \leq \Psi^{\mathcal{L}}(\varpi_0)$$
$$\Psi^{\mathcal{U}}(\vartheta_0) < \Psi^{\mathcal{U}}(\varpi_0),$$

or

$$\Psi^{\mathcal{L}}(\vartheta_0) < \Psi^{\mathcal{L}}(\varpi_0)$$
$$\Psi^{\mathcal{U}}(\vartheta_0) \leq \Psi^{\mathcal{U}}(\varpi_0).$$

Since $\mu^{\mathcal{L}} \geq 0$, $\mu^{\mathcal{U}} \geq 0$, then the preceeding inequalities together yield

$$\mu^{L}[\Psi^{\mathcal{L}}(\vartheta_0) - \Psi^{\mathcal{L}}(\varpi_0)] + \mu^{\mathcal{U}}[\Psi^{\mathcal{U}}(\vartheta_0) - \Psi^{\mathcal{U}}(\varpi_0)] < 0. \tag{26}$$

Since $(\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma) \in \mathbb{H}_D^{MW}$, there exist $\mu^{\mathcal{L}}, \mu^{\mathcal{U}} \in \mathcal{C}^* \setminus \{0\}$, $\gamma \in \mathcal{R}_+^{m_0}$, $\varrho \in \Phi$ such that

$$\gamma^T \varphi(\varpi_0, \varrho) \geq 0, \tag{27}$$

$$0 \in \left\{ \partial\Psi^{\mathcal{L}}(\varpi_0)^T \mu^{\mathcal{L}} + \partial\Psi^{\mathcal{U}}(\varpi_0)^T \mu^{\mathcal{U}} + \partial_{\vartheta_0}\varphi(\varpi_0, \varrho)^T \gamma \right\}. \tag{28}$$

By virtue of (28), there exist $\vartheta_1 \in \partial(\mu^{\mathcal{L}}\Psi^{\mathcal{L}})(\varpi_0)$, $\vartheta_2 \in \partial(\mu^{\mathcal{U}}\Psi^{\mathcal{U}})(\varpi_0)$, $\varpi_1 \in \partial_{\vartheta_0}\varphi(\varpi_0, \varrho)^T\gamma$ such that

$$\vartheta_1 + \vartheta_2 + \varpi_1 = 0. \tag{29}$$

Since $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type I $(\mathcal{C}, \mathcal{C}, \mathcal{R}_+^{m_0})$-pseudo convex at ϖ_0, for any ϑ_0 and $\mu^{\mathcal{L}}$, $\mu^{\mathcal{U}} \in \mathcal{C}^* \setminus \{0\}$ the following hold:

$$\mu^{\mathcal{L}}\Psi^{\mathcal{L}}(\vartheta_0) < \mu^{\mathcal{L}}\Psi^{\mathcal{L}}(\varpi_0) \Rightarrow A^{\mathcal{L}}(\vartheta_0 - \varpi_0) < 0,\ \forall A^{\mathcal{L}} \in \partial(\mu^{\mathcal{L}}\Psi^{\mathcal{L}})(\varpi_0). \tag{30}$$

$$\mu^{\mathcal{U}}\Psi^{\mathcal{U}}(\vartheta_0) < \mu^{\mathcal{U}}\Psi^{\mathcal{U}}(\varpi_0) \Rightarrow A^{\mathcal{U}}(\vartheta_0 - \varpi_0) < 0,\ \forall A^{\mathcal{U}} \in \partial(\mu^{\mathcal{U}}\Psi^{\mathcal{U}})(\varpi_0). \tag{31}$$

$$\varphi(\vartheta_0,\varrho) \leq \varphi(\varpi_0,\varrho) \Rightarrow B^T(\vartheta_0 - \varpi_0) \leq 0,\ \forall B \in \partial_{\vartheta_0}\varphi(\varpi_0,\varrho). \tag{32}$$

By using the inequalities (26), (30) and (31), we obtain

$$[A^{\mathcal{L}} + A^{\mathcal{U}}](\vartheta_0 - \varpi_0) < 0. \tag{33}$$

The inequality (33) along with (29) gives

$$-\varpi_1(\vartheta_0 - \varpi_0) < 0. \tag{34}$$

Since $\varpi_1 \in \partial_{\vartheta_0}\varphi(\varpi_0,\varrho)^T\gamma$, we get $\varpi_1 = B_0^T\gamma$, for some $B_0 \in \partial_{\vartheta_0}\varphi(\varpi_0,\varrho)$. Thus,

$$-B_0^T\gamma(\vartheta_0 - \varpi_0) < 0,\ \text{for some } B_0 \in \partial_{\vartheta_0}\varphi(\varpi_0,\varrho). \tag{35}$$

Since $\vartheta_0 \in \mathbb{H}$ and using (27) in the inequality (32) with $\gamma \in \mathcal{R}_+^{m_0}$, we have

$$B^T\gamma(\vartheta_0 - \varpi_0) \leq 0,\ \forall B \in \partial_{\vartheta_0}\varphi(\varpi_0,\varrho).$$

This is a contradiction to (35).

Assertion (ii) is proved similar to part (i) by using the type II pseudo convexity of $(\Psi^{\mathcal{L}},\Psi^{\mathcal{U}},\varphi)$ at ϖ_0. □

We demonstrate weak duality theorem by the subsequent illustration.

Example 3. *Consider the subsequent uncertain constrained interval-valued optimization problem.*

$$\text{(UCIVOP-2)}\qquad \min_{\vartheta_1,\vartheta_2\in\mathcal{R}} \Psi(\vartheta_1,\vartheta_2) = [\Psi^{\mathcal{L}}(\vartheta_1,\vartheta_2), \Psi^{\mathcal{U}}(\vartheta_1,\vartheta_2)]$$
$$\text{subject to}$$
$$-\varphi(\vartheta_1,\vartheta_2,\varrho) \in \mathcal{R}_+^2, \text{where } \varrho \in \Phi.$$

The robust counterpart of (UCIVOP-2) is defined as follows:

$$\text{(RIVOP-2)}\qquad \min_{\vartheta_1,\vartheta_2\in\mathcal{R}} \Psi(\vartheta_1,\vartheta_2) = ([1,1]\vartheta_1 + [1,1]\vartheta_2^3 + [4,5])$$
$$\text{subject to}$$
$$\left\{-{\vartheta_1}^2 + 2\vartheta_1 + 3 - ln(1+\varrho_1), \vartheta_2 + |\varrho_2|\right\}^T \in \mathcal{R}_+^2, \forall\varrho \in \Phi,$$

where $\varrho = (\varrho_1,\varrho_2) \in \Phi = \Phi_1 \times \Phi_2$. *Clearly, the robust feasible set of (RIVOP-2) is* $\mathbb{H} = [-1,1] \times \mathcal{R}_+$.

We consider a Mond-Weir type robust dual problem (MWIVRD-2) for (RIVOP-2) as follows:

$$\text{(MWIVRD-2)}\qquad \max_{(\varpi_1,\varpi_2,\varrho,\mu^{\mathcal{L}},\mu^{\mathcal{U}},\gamma)} \Psi(\varpi_1,\varpi_2) = ([1,1]\varpi_1 + [1,1]\varpi_2^3 + [4,5])$$
$$\text{subject to}$$
$$0 \in \left\{\partial\Psi^{\mathcal{L}}(\varpi_1,\varpi_2)^T\mu^{\mathcal{L}} + \partial\Psi^{U}(\varpi_1,\varpi_2)^T\mu^{\mathcal{U}} + \partial_{\vartheta_0}\varphi(\varpi_1,\varpi_2,\varrho)^T\gamma\right\},$$
$$\gamma^T\varphi(\varpi_0,\varrho) \geq 0, \varrho \in \Phi,$$
$$\mu^{\mathcal{L}},\mu^{\mathcal{U}} \in \mathfrak{C}^* \setminus \{0\},\ \gamma \in \mathcal{R}_+^{m_0}.$$

Clearly, $(\Psi^{\mathcal{L}},\Psi^{\mathcal{U}},\varphi)$ *is type I and type II* $(\mathfrak{C},\mathfrak{C},\mathcal{R}_+^{m_0})$*-pseudo convex at* ϖ_0 *(See Figure 2). It follows from Example 1, we have* $(\vartheta_1,\vartheta_2,\varrho,\mu^{\mathcal{L}},\mu^{\mathcal{U}},\gamma) \in \mathbb{H}_D^{MW}$ *where* $\mu^{\mathcal{L}} = 1/2 = \mu^{\mathcal{U}}$,

$\gamma = (1/4, 1)^T \in \mathcal{R}^2_+$, $\varrho = 0 \in \Phi$. *For any* $(\varpi_1, \varpi_2, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma) \in \mathbb{H}^{MW}_D$, $\mu^{\mathcal{L}}, \mu^{\mathcal{U}} \in \mathfrak{C}^* \setminus \{0\}$, $\gamma = (\gamma_1, \gamma_2)^T \in \mathcal{R}^2_+$, *we get*

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3\varpi_2^2 \end{pmatrix} \mu^{\mathcal{L}} + \begin{pmatrix} 1 \\ 3\varpi_2^2 \end{pmatrix} \mu^{\mathcal{U}} + \begin{pmatrix} 2\varpi_1 - 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$$

The above equation yields $\mu^{\mathcal{L}} + \mu^{\mathcal{U}} + 2(\varpi_1 - 1)\gamma_1 = 0$ *and* $3\varpi_2^2(\mu^{\mathcal{L}} + \mu^{\mathcal{U}}) - \gamma_2 = 0$. *Combining these equations along with* $\mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma_1, \gamma_2 \geq 0$, *we obtain* $\varpi_2 \geq 0$. *Since,* $\gamma^T \varphi(\varpi_0, \varrho) \geq 0$, *that is,*

$$\begin{pmatrix} \gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} \varpi_1^2 - 2\varpi_1 - 3 + ln(1 + \varrho_1) \\ -\varpi_2 - |\varrho_2| \end{pmatrix} \geq 0.$$

This implies,

$$\gamma_1[\varpi_1^2 - 2\varpi_1 - 3 + ln(1 + \varrho_1)] + \gamma_1[-\varpi_2 - |\varrho_2|] \geq 0.$$

Since $\gamma_1 \geq 0$ *and* $-\varpi_2 - |\varrho_2| \leq 0$, *we have*

$$\varpi_1^2 - 2\varpi_1 - 3 + ln(1 + \varrho_1) \geq 0.$$

This implies, $\varpi_1 \leq -1$ *for* $\varrho_1 = 0$. *Taking into account* $\varpi_2 \geq 0$, $\varpi_1 \leq -1$ *and* $\forall (\varpi_1, \varpi_2, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma) \in \mathbb{H}^{MW}_D$, *we get,*

$$\mu^{\mathcal{L}}[\Psi^{\mathcal{L}}(\vartheta_1, \vartheta_2) - \Psi^{\mathcal{L}}(\varpi_1, \varpi_2)] + \mu^{\mathcal{U}}[\Psi^{U}(\vartheta_1, \vartheta_2) - \Psi^{\mathcal{U}}(\varpi_1, \varpi_2)] = -1 - \varpi_1 - \varpi_2^3 \not< 0.$$

Therefore, weak duality theorem of (MWIVRD-2) holds.

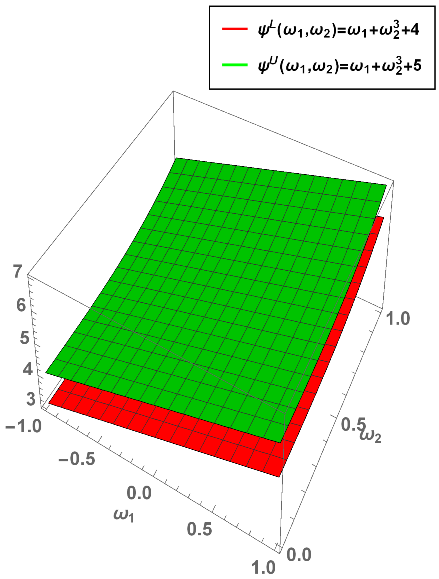

Figure 2. Graphical view of the objective functions $\Psi^{\mathcal{L}}(\varpi_1, \varpi_2)$ and $\Psi^{\mathcal{U}}(\varpi_1, \varpi_2)$ of the problem (MWIVRD-2).

Theorem 8 (Strong Duality)**.** *Let* $\bar{\vartheta}_0$ *be a robust* $\mathcal{LU}$*-optimal solution of (UCIVOP) and (GRSCQ) hold. Assume that* φ *satisfy the Assumption* $\mathbb{D}$, ϕ *is* $\mathcal{R}_+$ *- convexlike and* $\Psi^{\mathcal{L}}$, $\Psi^{\mathcal{U}}$ *be* $\mathfrak{C}$*-convexlike. Then, there exist* $\bar{\varrho} \in \Phi$, $\bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}} \in \mathfrak{C}^* \setminus \{0\}$, $\bar{\gamma} \in \mathcal{R}^{m_0}_+$ *such that* $(\bar{\vartheta}_0, \bar{\varrho}, \bar{\mu}^{\mathcal{L}}, \bar{\mu}^{U}, \bar{\gamma}) \in \mathbb{H}^{MW}_D$.
(i). If $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ *is type I* $(\mathfrak{C}, \mathfrak{C}, \mathcal{R}^{m_0}_+)$*-pseudo convex at* ϖ_0 *where* $(\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma) \in \mathbb{H}^{MW}_D$, *then* $(\bar{\vartheta}_0, \bar{\varrho}, \bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}}, \bar{\gamma})$ *is a robust* $\mathcal{LU}$*-optimal solution of (MWIVRD).*
(ii). If $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ *is type II* $(\mathfrak{C}, \mathfrak{C}, \mathcal{R}^{m_0}_+)$*-pseudo convex at* ϖ_0 *where* $(\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma) \in \mathbb{H}^{MW}_D$, *then* $(\bar{\vartheta}_0, \bar{\varrho}, \bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}}, \bar{\gamma})$ *is a robust* $\mathcal{LU}$*-optimal solution of (MWIVRD).*

Proof. Let $\bar{\vartheta}_0$ be a robust $\mathcal{LU}$-optimal solution of (UCIVOP) and (GRSCQ) hold. Then by Theorem 1, there exist $\bar{\varrho} \in \Phi$, $\bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}} \in \mathcal{C}^* \setminus \{0\}$, $\bar{\gamma} \in \mathcal{R}_+^{m_0}$ such that $(\bar{\vartheta}_0, \bar{\varrho}, \bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}}, \bar{\gamma})$ satisfies

$$0 \in \left\{ \partial \Psi^{\mathcal{L}}(\bar{\vartheta}_0)^T \bar{\mu}^{\mathcal{L}} + \partial \Psi^{\mathcal{U}}(\bar{\vartheta}_0)^T \bar{\mu}^{\mathcal{U}} + \partial_{\vartheta_0} \varphi(\bar{\vartheta}_0, \bar{\varrho})^T \bar{\gamma} \right\},$$
$$\bar{\gamma}^T \varphi(\bar{\vartheta}_0, \bar{\varrho}) = 0.$$

Hence, $(\bar{\vartheta}_0, \bar{\varrho}, \bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}}, \bar{\gamma}) \in \mathbb{H}_D^{MW}$.

To prove (i).

As $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type I pseudo convex at ϖ_0 and utilizing the assumption (i) of Theorem 7, we get that the subsequent inequality does not hold:

$$\Psi(\bar{\vartheta}_0) <_{\mathcal{LU}} \Psi(\varpi_0), \ \forall (\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma) \in \mathbb{H}_D^{MW},$$

which implies that $(\bar{\vartheta}_0, \bar{\varrho}, \bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}}, \bar{\gamma})$ is a robust $\mathcal{LU}$-optimal solution of (MWIVRD).

(ii). As $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type II pseudo convex at any ϖ_0 and utilizing the assumption (ii) of Theorem 7, we get that the subsequent inequality cannot hold:

$$\Psi(\bar{\vartheta}_0) <_{\mathcal{LU}} \Psi(\varpi_0), \ \forall (\varpi_0, \varrho, \mu^{\mathcal{L}}, \mu^{\mathcal{U}}, \gamma) \in \mathbb{H}_D^{MW},$$

which implies that $(\bar{\vartheta}_0, \bar{\varrho}, \bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}}, \bar{\gamma})$ is a robust $\mathcal{LU}$-optimal solution of (MWIVRD). □

Theorem 9 (Converse Duality)**.** *Let $(\bar{\varpi}_0, \bar{\varrho}, \bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}}, \bar{\gamma})$ be a robust $\mathcal{LU}$-optimal solution of (MWIVRD) with $\bar{\varpi}_0 \in \mathbb{H}$.*
(i). If $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type I $(\mathcal{C}, \mathcal{C}, \mathcal{R}_+^{m_0})$-pseudo convex at $\bar{\varpi}_0$, then $\bar{\varpi}_0$ is a robust $\mathcal{LU}$-optimal solution of (UCIVOP).
(ii). If $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type II $(\mathcal{C}, \mathcal{C}, \mathcal{R}_+^{m_0})$-pseudo convex at $\bar{\varpi}_0$, then $\bar{\varpi}_0$ is a robust $\mathcal{LU}$-optimal solution of (UCIVOP).

Proof. To prove (i).

Since $(\bar{\varpi}_0, \bar{\varrho}, \bar{\mu}^{\mathcal{L}}, \bar{\mu}^{\mathcal{U}}, \bar{\gamma})$ is a robust $\mathcal{LU}$-optimal solution of (MWIVRD) with $\bar{\varpi}_0 \in \mathbb{H}$. As $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type I pseudo convex at $\bar{\varpi}_0$ and utilizing the assumption (i) of Theorem 7, the subsequent inequality does not hold:

$$\Psi(\vartheta_0) <_{\mathcal{LU}} \Psi(\bar{\varpi}_0), \ \forall \vartheta_0 \in \mathbb{H}.$$

Hence, $\bar{\varpi}_0$ is a robust $\mathcal{LU}$-optimal solution of (UCIVOP).

(ii) As $(\Psi^{\mathcal{L}}, \Psi^{\mathcal{U}}, \varphi)$ is type II pseudo convex at any $\bar{\varpi}_0$ and utilizing the assumption (ii) of Theorem 7, the subsequent inequality cannot hold:

$$\Psi(\vartheta_0) <_{\mathcal{LU}} \Psi(\bar{\varpi}_0), \ \forall \vartheta_0 \in \mathbb{H}.$$

Hence, $\bar{\varpi}_0$ is a robust $\mathcal{LU}$-optimal solution of (UCIVOP). □

6. Conclusions

This paper uses the $\mathcal{LU}$-optimal solution and the generalized robust Slater constraint qualifications (GRSCQ) to formulate Karush- Kuhn-Tucker type robust necessary optimality conditions for an uncertain constrained interval-valued optimization problem (UCIVOP). These Karush-Kuhn-Tucker type robust necessary conditions are shown to be sufficient optimality conditions under generalized convexity. An illustration is provided to demonstrate the robust sufficient optimality theorem's validity. Further to that, Karush-Kuhn-Tucker robust necessary conditions are used to formulate Wolfe and Mond-Weir type robust dual problems over cones. The validity of Wolfe and Mond-Weir's weak duality theorems is demonstrated. Finally, the usual duality results are demonstrated using the generalized convexity assumptions. It would be interesting to see if these results could be obtained for other types of nonconvex vector optimization problems with multiple interval-valued objective functions, as well as other types of extremum problems. As a result, we are generalizing our current results to multiple interval-valued optimization problems with uncertainty, which we will focus on in our future research.

Author Contributions: Methodology, I.A., K.K. and S.A.-H.; Software, R.R.J.; Supervision, K.K.; Writing—original draft, R.R.J. and I.A.; Writing—review & editing, S.A.-H. All authors have read and agreed to the published version of the manuscript.

Funding: Deanship of Research Oversight and Coordination, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The second and fourth authors would like to thank the King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia to provide the financial support. Also, the authors would like to express their gratitude to the anonymous referees for their insightful recommendations and comments, which dramatically improved this manuscript to its present state.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Soyster, A.L. Convex programming with set-inclusive constraints and applications to inexact linear programming. *Oper. Res.* **1973**, *21*, 1154–1157. [CrossRef]
2. Taguchi, G.; Chowdhury, S.; Taguchi, S. *Robust Engineering*; McGraw-Hill: New York, NY, USA, 2000.
3. Bokrantz, R.; Fredriksson, A. Necessary and sufficient conditions for pareto efficiency in robust multi-objective optimization. *Eur. J. Oper. Res.* **2017**, *262*, 682–692. [CrossRef]
4. Deb, K.; Gupta, H. Introducing robustness in multi-objective optimization. *Evol. Comput.* **2006**, *14*, 463–494. [CrossRef] [PubMed]
5. Ide, J.; Köbis, E. Concepts of efficiency for uncertain multi-objective optimization problems based on set order relations. *Math. Methods Oper. Res.* **2014**, *80*, 99–127. [CrossRef]
6. Ide, J.; Schöbel, A. Robustness for uncertain multi-objective optimization: A survey and analysis of different concepts. *OR Spectr.* **2016**, *38*, 235–271. [CrossRef]
7. Syberfeldt, A.; Gustavsson, P. Increased robustness of product sequencing using multi-objective optimization. *Procedia CIRP* **2014**, *17*, 434–439. [CrossRef]
8. Wiecek, M.M.; Dranichak, G.M. Robust multi-objective optimization for decision making under uncertainty and conflict. In *Optimization Challenges in Complex, Networked and Risky Systems*; INFORMS: Catonsville, MD, USA, 2016; pp. 84–114.
9. Lee, J.; Wu, F.; Zhao, W.; Ghaffari, M.; Liao, L.; Siegel, D. Prognostics and health management design for rotary machinery systems-Reviews, methodology and applications. *Mech. Syst. Signal Process.* **2014**, *42*, 314–334. [CrossRef]
10. Oksanen, T.; Linkolehto, R.; Seilonen, I. Adapting an industrial automation protocol to remote monitoring of mobile agricultural machinery: A combine harvester with IoT. *IFAC-PapersOnLine* **2016**, *49*, 127–131. [CrossRef]
11. Lei, G.; Bramerdorfer, G.; Ma, B.; Guo, Y.; Zhu, J. Robust design optimization of electrical machines: Multi-objective approach. *IEEE Trans. Energy Convers.* **2020**, *36*, 390–401. [CrossRef]
12. Ahmadi-Nezamabad, H.; Zand, M.; Alizadeh, A.; Vosoogh, M.; Nojavan, S. Multi-objective optimization based robust scheduling of electric vehicles aggregator. *Sustain. Cities Soc.* **2019**, *47*, 101494. [CrossRef]
13. Doolittle, E.K.; Kerivin, H.L.M.; Wiecek, M.M. Robust multi-objective optimization problem with application to internet routing. *Ann. Oper. Res.* **2018**, *271*, 487–525. [CrossRef]
14. Krüger, C.; Castellani, F.; Geldermann, J.; Schöbel, A. Peat and pots: An application of robust multi-objective optimization to a mixing problem in agriculture. *Comput. Electron. Agric.* **2018**, *154*, 265–275. [CrossRef]
15. Palma, C.D.; Nelson, J.D. Bi-objective multi-period planning with uncertain weights: A robust optimization approach. *Eur. J. For. Res.* **2010**, *129*, 1081–1091. [CrossRef]
16. Ahmad, I.; Kummari, K.; Al-Homidan, S. Sufficiency and duality for interval-valued optimization problems with vanishing constraints using weak constraint qualifications. *Int. J. Anal. Appl.* **2020**, *18*, 784–798.
17. Kummari, K.; Ahmad, I. Sufficient optimality conditions and duality for nonsmooth interval-valued optimization problems via L-invex-infine functions. *Politehn. Univ. Buchar. Sci. Bull. Ser. A Appl. Math. Phys.* **2020**, *82*, 45–54.
18. Singh, D.; Dar, B.; Goyal, A. KKT optimality conditions for interval-valued optimization problems. *J. Nonlinear Anal. Optim. Theory Appl.* **2014**, *5*, 91–103.

19. Zhang, J.; Zheng, Q.; Zhou, C.; Ma, X.; Li, L. On interval-valued pseudo-linear functions and interval-valued pseudo-linear optimization problems. *J. Funct. Spaces* **2015**, *2015*, 610848.
20. Zhao, J.; Bin, M. Karush-Kuhn-Tucker optimality conditions for a class of robust optimization problems with an interval-valued objective function. *Open Math.* **2020**, *18*, 781–793. [CrossRef]
21. Shaikh, A.A.; Das, S.C.; Bhunia, A.K.; Panda, G.C.; Khan, M.A.A. A two-warehouse EOQ model with interval-valued inventory cost and advance payment for deteriorating item under particle swarm optimization. *Soft Comput.* **2019**, *23*, 13531–13546. [CrossRef]
22. Wang, W.; Xiong, J.; Xie, M. A study of interval analysis for cold-standby system reliability optimization under parameter uncertainty. *Comput. Ind. Eng.* **2016**, *97*, 93–100. [CrossRef]
23. Zhang, Z.; Wang, X.; Lu, J. Multi-objective immune genetic algorithm solving nonlinear interval-valued programming. *Eng. Appl. Artif. Intell.* **2018**, *67*, 235–245. [CrossRef]
24. Arrow, K.J.; Enthoven, A.C. Quasi-concave programming. *Econometrica* **1961**, 779–800. [CrossRef]
25. Hanson, M.A. On sufficiency of the Kuhn-Tucker conditions. *J. Math. Anal. Appl.* **1981**, *80*, 545–550. [CrossRef]
26. Craven, B.D. Invex functions and constrained local minima. *Bull. Aust. Math. Soc.* **1981**, *24*, 357–366. [CrossRef]
27. De Carvalho Bento, G.; Bitar, S.D.B.; da Cruz Neto, J.X.; Soubeyran, A.; de Oliveira Souza, J.C. A proximal point method for difference of convex functions in multi-objective optimization with application to group dynamic problems. *Comput. Optim. Appl.* **2020**, *75*, 263–290. [CrossRef]
28. Fakhar, M.; Mahyarinia, M.R.; Zafarani, J. On nonsmooth robust multi-objective optimization under generalized convexity with applications to portfolio optimization. *Eur. J. Oper. Res.* **2018**, *265*, 39–48. [CrossRef]
29. Günther, C. *On Generalized Convex Constrained Multi-Objective Optimization and Application in Location Theory*; University and State Library of Saxony-Anhalt: Halle, Germany, 2018.
30. Suneja, S.K.; Khurana, S.; Bhatia, M. Optimality and duality in vector optimization involving generalized type I functions over cones. *J. Glob. Optim.* **2011**, *49*, 23–35. [CrossRef]
31. Chen, J.W.; Cho, Y.J.; Kim, J.K.; Li, J. Multi-objective optimization problems with modified objective functions and cone constraints and applications. *J. Glob. Optim.* **2011**, *49*, 137–147. [CrossRef]
32. Lee, G.M.; Son, P.T. On nonsmooth optimality theorems for robust optimization problems. *Bull. Korean Math. Soc.* **2014**, *51*, 287–301. [CrossRef]
33. Lee, G.M.; Lee, J.H. On nonsmooth optimality theorems for robust multi-objective optimization problems. *J. Nonlinear Convex Anal.* **2015**, *16*, 2039–2052.
34. Chuong, T.D. Optimality and duality for robust multi-objective optimization problems. *Nonlinear Anal.* **2016**, *134*, 127–143. [CrossRef]
35. Ben-Tal, A.; Ghaoui, L.E.l.; Nemirovski, A. *Robust Optimization*; Princeton Series in Applied Mathematics; Princeton University Press: Princeton, NJ, USA, 2009.
36. Jeyakumar, V.; Li, G.Y. Strong duality in robust convex programming: Complete characterizations. *SIAM J. Optim.* **2010**, *20*, 3384–3407. [CrossRef]
37. Mangasarian, O.L. *Nonlinear Programming*; SIAM: Philadelphia, PA, USA, 1994.
38. Ishibuchi, H.; Tanaka, H. Multi-objective programming in optimization of the interval objective function. *Eur. J. Oper. Res.* **1990**, *48*, 219–225. [CrossRef]
39. Inuiguchi, M.; Kume, Y. Goal programming problems with interval coefficients and target intervals. *Eur. J. Oper. Res.* **1991**, *52*, 345–360. [CrossRef]
40. Wu, H.C. The Karush-Kuhn-Tucker optimality conditions in an optimization problem with interval-valued objective function. *Eur. J. Oper. Res.* **2007**, *176*, 46–59. [CrossRef]
41. Wu, H.C. The Karush-Kuhn-Tucker optimality conditions in multi-objective programming problems with interval-valued objective functions. *Eur. J. Oper. Res.* **2009**, *196*, 49–60. [CrossRef]
42. Chen, J.; Köbis, E.; Yao, J.C. Optimality conditions and duality for robust nonsmooth multi-objective optimization problems with constraints. *J. Optim. Theory Appl.* **2019**, *181*, 411–436. [CrossRef]
43. Clarke, F.H. *Optimization and Nonsmooth Analysis*; Society for Industrial and Applied Mathematics: Philadelphia, PA, USA, 1990.
44. Saadati, M.; Oveisiha, M. Optimality conditions for robust nonsmooth multi-objective optimization problems in Asplund spaces. *arXiv* **2021**, arXiv:2105.14366.
45. Singh, D.; Dar, B.A.; Kim, D.S. Sufficiency and duality in non-smooth interval valued programming problems. *J. Ind. Manag. Optim.* **2019**, *15*, 647–665. [CrossRef]