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Article

A Comparison Study of the Classical and Modern Results of Semi-Local Convergence of Newton–Kantorovich Iterations-II

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Abstract: This article is an independently written continuation of an earlier study with the same title [Mathematics, 2022, 10, 1225] on the Newton Process (NP). This process is applied to solve nonlinear equations. The complementing features are: the smallness of the initial approximation is expressed explicitly in turns of the Lipschitz or Hölder constants and the convergence order $1 + p$ is shown for $p \in (0, 1]$. The first feature becomes attainable by further simplifying proofs of convergence criteria. The second feature is possible by choosing a bit larger upper bound on the smallness of the initial approximation. This way, the completed convergence analysis is finer and can replace the classical one by Kantorovich and others. A two-point boundary value problem (TPBVP) is solved to complement this article.

Keywords: iterative processes; Banach space; semi-local convergence

MSC: 49M15; 65J15; 65G99

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1. Introduction

Let X_1 and X_2 be Banach spaces, and let Ω be a nonempty convex subset of X_1. In addition, $F : \Omega \subset X_1 \longrightarrow X_2$ is a Fréchet differentiable mapping between the Banach spaces X_1 abd X_2. Let also $\mathcal{L}(X_1, X_2)$ denote the space of bounded linear operators from X_1 into X_2. The nonlinear equation

$$F(x) = 0, \tag{1}$$

plays an important role due to the fact that many applications can be brought to look like it. The celebrated Newton Process (NP) in the following form

$$x_0 \in \Omega,\ F'(x_n)s_n = -F(x_n),\ x_{n+1} = x_n + s_n, \tag{2}$$

for $n = 0, 1, 2, \ldots$ is widely used to solve Equation (1) iteratively. This set up is motivated by the solution of corresponding differential equations (see also the Numerical Section 4).

Kantorovich initiated the semi-local convergence (SLC) analysis of (NP) by using the contraction mapping principle due to Banach [1,2]. He presented two different proofs based on majorant functions and recurrent relations [2,3]. The Newton–Kantorovich Theorem contains the (SLC) for (NP). A plethora of researchers utilized this result, in applications, and also as a theoretical tool.

An elementary scalar equation given in [1,2,4–9] shows that convergence criteria may not be satisfied. However, (NP) may converge. That is why these criteria are weakened in [6] without new conditions. However, only linear convergence was obtained for (NP) with the techniques employed in [6].

In the present study, by employing different and more precise techniques, the elusive convergence order $1+p$ is obtained for $p \in (0,1]$. This is achieved by choosing a bit smaller upper bound on $\|F'(x_0)^{-1}F(x_0)\|$. Another new feature involves an explicit upper bound on the smallness of the initial approximation not given in [6]. Notice also that the present study is written completely independently of the corresponding one in [6]. The former reference is only mentioned to stretch the differences and the benefits of the new approach. Consequently, new results can always replace corresponding ones by Kantorovich [2] and others [7–11], as preceding results imply the one in this study but not necessarily vice versa. The method in this study uses smaller Lipschitz or Hölder parameters to achieve these extensions, which are specializations of earlier ones. That is, no additional effort is needed. The generality of this idea allows its application to other processes [3,5–7,9–11]. This will be the topic of future work.

Majorization of (NP) is presented in Section 2. The (SLC) of (NP) can be found in Section 3. Section 4 contains a Boundary Value Problem (BVP) as an application. Conclusions complete this study in Section 5.

2. Majorization

Let K_0, L_0, K, L be positive parameters and $\eta \geq 0$.

The sequence generated for $p \in (0,1]$ and for $\forall n = 0,1,2,\ldots$ by $v_0 = 0, v_1 = \eta$

$$\begin{aligned} v_2 &= \eta + \frac{K\eta^{1+p}}{(1+p)(1-K_0\eta^p)}, \\ v_{n+2} &= v_{n+1} + \frac{L(v_{n+1}-v_n)^{1+p}}{(1+p)(1-L_0 v_{n+1}^p)}, \end{aligned} \tag{3}$$

plays a critical role as a majorizing sequence for (NP) in the Lipschitz case ($p=1$) as well as the Hölder one ($p \in (0,1)$).

Two convergence results for sequence $\{v_n\}$ are developed.

Lemma 1. *Suppose*

$$K_0\eta^p < 1$$

and

$$L_0 v_{n+1}^p < 1$$

$\forall n = 0,1,2,\ldots$. *Then, the following assertions hold*

$$0 \leq v_n \leq v_{n+1}$$

and

$$\lim_{n\longrightarrow\infty} v_n = t^* \leq \min\left\{v, \left(\frac{1}{L_0}\right)^{\frac{1}{p}}\right\},$$

where $v = (\frac{1}{K_0})^{\frac{1}{p}}$.

Proof. The assertions follow by the definition of the sequence $\{v_n\}$ and the conditions of Lemma 1. □

Another convergence result follows.

It is convenient to develop parameters, $d = v_2 - v_1$, $\delta_0 = \frac{Kd^p}{(1+p)(1-K_0\eta^p)}$, $\delta_1 = 1 - \frac{d}{\left(\frac{1}{L_0}\right)^p - \eta}$ and set $S = [0, v)$. Moreover, introduce functions depending on parameter p and defined on the interval S for $n = 1,2,\ldots$ by

$$h_{n,p}(t) = L\eta^p t^{np} + (1+p)L_0 t(\eta + \frac{1-t^{n+1}}{1-t}d)^p - (1+p)t$$

and

$$g_{n,p}(t) = Lt^p - L + (1+p)L_0t[(t^{-n} + \ldots + 1 + t)^p - (t^{-n} + \ldots + 1)^p].$$

Next, some properties for these functions are presented.

Lemma 2. *The following assertions hold*

$$h_{n+1,p}(t) = h_{n,p}(t) + g_{n,p}(t)t^{np}\eta^p,$$

$$\begin{aligned} g_{n+1,p}(t) - g_{n,p}(t) &= (1+p)L_0t[(t^{-n+1} + \ldots + t)^p - (t^{-n+1} + \ldots + t + 1)^p \\ &\quad -((t^{-n} + \ldots + 1 + t)^p - (t^{-n} + \ldots + 1)^p)], \end{aligned}$$

$$g_{n,1}(t) = Lt - L + (1+p)L_0t^{1+p},$$

$$g_{n+1,1}(t) - g_{n,1}(t) = 0 \ \ \forall\, t \in S$$

and

$$g_{n+1,p}(t) - g_{n,p}(t) \leq 0 \ \ \forall\, t \in S,\ p \in (0,1).$$

Define the parameter δ *by*

$$\delta = \begin{cases} \frac{2L}{L+\sqrt{L^2+8L_0L}}, & \text{if } p = 1 \\ \text{the smallest zero in } S - \{0\} \text{ of the function } g_{1,p}, & \text{if } p \in (0,1). \end{cases}$$

Moreover, suppose
(I) $0 \leq \delta_0 \leq \delta \leq \delta_1$, if $p = 1$ *or*
(II) $\eta \leq \frac{1}{2}$, $\eta_2^{-1} := \min\{v, \eta_0, \eta_1\}$, *if* $p \in (0,1)$, *where the parameter* η_0 *is the smallest zero in* $(0, v)$ *of the function*

$$\varphi(t) = K\left[\frac{Kt^{1+p}}{(1+p)(1-K_0t^p)}\right]^p - \delta(1+p)(1-K_0t^p)$$

and

$$\eta_1 = \left(\frac{(1+p)\delta}{L\delta^p + (1+p)L_0\delta(1+\delta+\delta^2)^p}\right)^{\frac{1}{p}}.$$

Then, the sequence $\{v_n\}$ *is such that*

$$0 \leq v_{n+1} - v_n \leq \delta(v_n - v_{n-1}),$$

$$0 \leq v_n \leq v_{n+1} \leq \eta + (1 + \delta_0\frac{1-\delta^n}{1-\delta}d) \leq \eta + \frac{\delta_0 d}{1-\delta} =: t^{**},$$

and

$$\lim_{n\longrightarrow\infty} v_n = t^* \in [0, t^{**}],$$

where $\delta_0 = \frac{L(v_2-v_1)^p}{(1+p)(1-L_0t^p)}$.

Proof. The assertions on functions $h_{n,p}$ and $g_{n,p}$ follow immediately by their definitions. If $p = 1$ by using the quadratic formula the parameter $\delta \in (0,1)$ is obtained. Then, the definition of the function $g_{1,p}$ for $p \in (0,1)$ implies $g_{1,p}(0) < 0$ and $g_{1,p}(1) > 0$. Let δ again stand for the smallest zero of the function $g_{1,p}$ in $S - \{0\}$ assured to exist by the (IVT) (intermediate Value Theorem). Similarly the definition of parameter η_0 is assured by (IVT), since $\varphi(0) = -\delta(1+p) < 0$ and $\varphi(t) \longrightarrow \infty$ as $t \longrightarrow v^-$.

Notice also that under (I)

$$h_{n,1}(t) \leq h_{n+1,1}(t) \leq h_\infty(t) \leq 0 \ \ \forall t \in [0, \delta],$$

whereas under condition (II)

$$g_{1,p}(t) \leq 0 \;\; \forall\; t \in [0, \eta_0],$$

$$h_{1,p}(t) \leq 0 \;\; \forall\; t \in [0, \eta_1]$$

and

$$h_{n+1,p}(t) \leq h_{n,p}(t) \;\; \forall\; t \in [0, \delta],$$

where $h_{\infty,p}(t) = t[L_0(\eta + \frac{d}{1-t})^p - 1]$. Hence, the sequence $\{v_n\}$ is bounded from above by t^{**} and non-decreasing. □

Next, we show that condition (I) can be solved in terms of η as in case (II).

Define the real quadratic polynomials q, q_1, q_2 by

$$q(t) = L_0(K - 2K_0)t^2 + 2L_0 t - 1,$$

$$q_1(t) = (LK + 2\delta L_0(K - 2K_0))t^2 + 4\delta(L_0 + K_0)t - 4\delta,$$

and

$$q_2(t) = L_0(K - 2(1-\delta)K_0)t^2 + 2(1-\delta)(L_0 + K_0)t - 2(1-\delta).$$

The discriminants $\triangle, \triangle_1, \triangle_2$ of these polynomials can be written as

$$\triangle = 4L_0(L_0 + K - 2K_0) > 0,$$

$$\triangle_1 = 16\delta(\delta(L_0 - K_0)^2 + (L + 2\delta L_0)K) > 0$$

and

$$\triangle_2 = 4(1-\delta)((1-\delta)(L_0 - K_0)^2 + 2L_0 K) > 0,$$

respectively. It follows by the definition of δ, q_1 and q_2 that

$$L = \frac{2L_0\delta^2}{1-\delta},\; LK + 2\delta L_0(K - 2K_0) = \frac{2L_0\delta}{1-\delta}(K - 2(1-\delta)K_0),$$

and so

$$q_1(t) = \frac{2L_0\delta}{1-\delta} q_2(t).$$

That is the polynomials q_1 and q_2 have the same roots. Denote by $\frac{1}{2r_1}$ the unique positive root of polynomial q. This root is given by the quadratic formula and can be written as

$$\frac{1}{2r_1} = \frac{1}{L_0 + \sqrt{L^2 + L_0(K - 2K_0)}}.$$

Moreover, denote by $\frac{1}{2r_2}$ the common positive root of the polynomials q_1 and q_2. This root can be written as

$$\frac{1}{2r_2} = \frac{2}{\delta(L_0 + K_0) + \sqrt{(\delta(L_0 + K_0))^2 + \delta(KL + 2\delta L_0(K - 2K_0))}}.$$

Define the parameter η_3 by

$$\eta_3^{-1} = \min\left\{\frac{1}{2r_1}, \frac{1}{2r_2}\right\}.$$

Suppose that the nonnegative number η is such that

$$\eta_3 \eta \leq \frac{1}{2}. \tag{4}$$

It is worth noticing that criterion (4) is written this way to make it looks like the usual Kantorovich criterion for Newton's method in the Lipschitz case [2,10,11].

By the choice of the parameters r_1 and r_2 the polynomials q, q_1, q_2 and the condition (4) we get follows that

$$L_0 v_2 < 1,$$

since $q(\eta) < 0$ and $K_0 \eta < 1$. We infer that

$$q_1(\eta) \leq 0$$

and

$$q_2(\eta) \leq 0.$$

Furthermore, the following estimate holds

$$\delta_0 \leq \delta \leq 1 - \frac{L_0(v_2 - v_1)}{1 - L_0 v_1}. \tag{5}$$

Indeed, the left hand side inequality reduces to showing $q_1(\eta) \leq 0$ and the right hand side to showing $q_2(\eta) \leq 0$. Conditions (4) provides the smallness of η to force convergence of the sequence $\{v_n\}$. By choosing $\frac{1}{2\eta_3}$ to be a little bit larger the convergence $1 + p$ is recovered as follows. Let $\epsilon \geq 0$. Set $b = \frac{L}{1+p}(1 + \epsilon)$ and $c = b^{-\frac{1}{p}}$.

Define function $\psi_{\infty,p}$ on interval S by

$$\psi_{\infty,p}(t) = (1 + \epsilon) L_0 \left(t + \frac{d(t)}{(1 - t)} \right)^p - \epsilon,$$

where

$$d(t) = \frac{K t^p}{(1 + p)(1 - K_0 t^p)}.$$

These definitions imply $\psi_{\infty,p}(0) = -\epsilon < 0$ and $\psi_{\infty,p}(t) \longrightarrow \infty$ as $t \longrightarrow v^-$. Denote by $\frac{1}{\eta_4}$ the smallest zero of the function $\psi_{\infty,p}$ on the interval $(0, v)$. Define

$$\eta_5 = \max \begin{cases} \{\eta_3, \frac{1}{2}c, \frac{1}{2\eta_4}\}, & \text{if } p = 1 \\ \{\eta_2, \frac{1}{2}c, \frac{1}{2\eta_4}\}, & \text{if } p \in (0, 1). \end{cases}$$

Let the sequence $\{v_n\}$ be defined as in the formula (3). Then its convergence is of order $1 + p$.

Lemma 3. *Let $\eta \geq 0$ be such that*

$$\eta_5 \eta < \frac{1}{2}. \tag{6}$$

Then, the following assertions hold

$$0 \leq v_{n+1} - v_n \leq \frac{1}{c} (c\eta)^{(1+p)^n}$$

and

$$t^* - v_n \leq \frac{1}{c(1 - c\eta)} (c\eta)^{(1+p)^n}.$$

The convergence order of the sequence $\{v_n\}$ is $1 + p$.

Proof. Induction is used to show

$$0 \leq \frac{L}{(1 + p)(1 - L_0 v_{n+1}^p)} \leq b, \tag{7}$$

where $b_0^{1+p} = \sup_{n\geq 1} \frac{L^p}{(1+p)^p(1-L_0 v_n^p)^p}$, $v_0 = \eta$ and $b \geq b_0$. Then, this assertion holds for $n = 1$ by the choice of η_0. Then, the assertion (7) holds if using Lemma 2

$$(1+\epsilon)L_0[\eta + (1 + (1 + t + \ldots + t^{n-1})d]^p - \epsilon \leq 0.$$

Define the functions

$$\psi_{n,p}(t) = (1+\epsilon)L_0(\eta + (1 + (1 + t + \ldots + t^{n-1})d)^p - \epsilon \leq 0.$$

It suffices to show

$$\psi_{n,p}(t) \leq 0 \text{ at } t = \delta.$$

The definitions of the functions $\{\psi_{n,p}\}$ yield

$$\begin{aligned} \psi_{n+1,p}(t) - \psi_{n,p}(t) &= (1+\epsilon)L_0\{[\eta + (1 + (1 + t + \ldots + t^n)d]^p \\ &\quad -[\eta + (1 + (1 + t + \ldots + t^{n-1})d]^p\} \geq 0. \end{aligned}$$

Define the function $\psi_{\infty,p}$ on the interval S by

$$\psi_{\infty,p}(t) = \lim_{n\longrightarrow\infty} \psi_{n,p}(t).$$

By the definition of the functions $\psi_{\infty,p}$, it suffices to show $\psi_{\infty,p}(t) \leq 0$, which is true by the choice of η_4. The induction is completed. It follows by the sequence $\{v_n\}$ and Lemma 2

$$\begin{aligned} b^p(v_{n+1} - v_n) &\leq (b(v_n - v_{n-1}))^{1+p} \\ &\leq b^{1+p}(b(v_{n-1} - v_{n-2})^{1+p})^{1+p} \\ &= b^{1+p}b^{1+p}(v_{n-1} - v_{n-2})^{(1+p)^2} \\ &\leq \ldots \\ &\leq b^{(1+p)+(1+p)+\ldots+(1+p)^{n-1}}\eta^{(1+p)^n}, \end{aligned}$$

so

$$\begin{aligned} v_{n+1} - v_n &\leq b^{1+(1+p)+\ldots+(1+p)^{n-1}}\eta^{(1+p)^n} \\ &= b^{\frac{(1+p)^{n-1}}{p}}\eta^{(1+p)^n} \\ &= \frac{1}{c}(c\eta)^{(1+p)^n}, \end{aligned}$$

which shows the first assertion. Moreover, if $k = 1, 2, \ldots$

$$\begin{aligned} v_{n+k} - v_n &\leq v_{n+k} - v_{n+k-1} + \ldots + v_{n+1} - v_n \\ &\leq \frac{1}{c}[(c\eta)^{(1+p)^{n+k-1}} + \ldots + (c\eta)^{(1+p)^n}] \\ &\leq \frac{1}{c}(c\eta)^{(1+p)^n}\frac{1 - (c\eta)^{(1+p)^n}}{1 - c\eta}. \end{aligned}$$

The second assertion follows if $k \longrightarrow \infty$ in the preceding calculation. □

It is worth noticing that Lemma 3 is used to provide weak convergence conditions for (NP). Then, the upper bounds on the iterates v_{n+1} make the proof of Lemma 3 possible.

Next, the Banach lemma on the invertible operators is recalled.

Lemma 4 ([1,2]). *If Q is a bounded linear operator in X_1, Q^{-1} exists if and only if there is a bounded linear operator P in X_1 such that P^{-1} exists and*

$$\|I - PQ\| \leq 1.$$

If Q^{-1} exists, then

$$Q^{-1} = \sum_{n=0}^{\infty} (I - PT)^n P$$

and

$$\|Q^{-1}\| \leq \frac{\|P\|}{1 - \|I - PQ\|}.$$

3. Convergence of (NP)

The notation $U(w, \rho), U[w, \rho]$ means the open and closed balls with radius $\rho > 0$ and center $w \in X_1$, respectively. The parameters $K_0, L_0, K, L,$ and η are connected with the operator F as follows. Consider conditions (A):

Suppose

(A1) There exist $x_0 \in \Omega$, $\eta \geq 0$ such that $F'(x_0)^{-1} \in \mathcal{L}(X_2, X_1), x_1 = x_0 - F'(x_0)^{-1}F(x_0)$

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta,$$

$$\|F'(x_0)^{-1}(F'(x_1) - F'(x_0))\| \leq K_0\|x_1 - x_0\|^p$$

and

$$\|F'(x_0)^{-1}(F'(x_0 + \xi(x_1 - x_0)) - F'(x_0))\| \leq K\|\xi(x_1 - x_0)\|^p.$$

(A2) $\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq L_0\|x - x_0\|^p$ for $\forall x \in \Omega$.

Set $B_1 = U(x_0, \left(\frac{1}{L_0}\right)^{\frac{1}{p}}) \cap \Omega$.

(A3) $\|F'(x_0)^{-1}(F'(x + \xi(y - x)) - F'(x))\| \leq L\|\xi(y - x)\|^p$ for $\forall x, y \in B_1$ and for $\forall \xi \in [0, 1)$.

(A4) The conditions of the preceding Lemma 1 or Lemma 2 or Lemma 3 hold.

(A5) $U[x_0, t^*] \subset \Omega$.

Notice that $K_0 \leq K \leq L_0$.

Next, conditions (A) are applied to show the main convergence result for (NP).

Theorem 1. *Under the conditions in (A) any (NP) sequence $\{x_n\}$ is convergent to a solution $x^* \in U[x_0, t^*]$ of the equation $F(x^*) = 0$. Moreover, upper bounds of the form*

$$\|x^* - x_n\| \leq t^* - v_n \tag{8}$$

hold for all $n = 0, 1, 2, \ldots$.

Proof. The assertions

$$\|x_{i+1} - x_i\| \leq v_{i+1} - v_i, \tag{9}$$

and

$$U[x_{i+1}, t^* - v_{i+1}] \subseteq U[x_i, t^* - v_i], \tag{10}$$

are shown by induction $\forall i = 0, 1, 2, \ldots$. Let $u \in U[x_1, t^* - v_1]$. The following inequalities are consequences of conditions (A1) together with the equality $v_0 = 0$.

$$\|x_1 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \eta = v_1 - v_0,$$

$$\|u - x_0\| \leq \|u - x_1\| + \|x_1 - x_0\| \leq t^* - v_1 + v_1 - v_0 = t^*.$$

So, the vector $u \in U[x_0, t^* - v_0]$. That is assertions (9) and (10) hold for $i = 0$. Assume these assertions hold if $i = 0, 1, \ldots, n$. It follows for each $\xi \in [0, 1]$

$$\|x_i + \xi(x_{i+1} - x_i) - x_0\| \quad \leq \quad v_i + \xi(v_{i+1} - v_i) \leq t^*,$$

and

$$\|x_{i+1}-x_i\| \leq \sum_{j=1}^{i+1}\|x_j-x_{j-1}\| \leq \sum_{j=1}^{i+1}(v_j-v_{j-1}) = v_{i+1}.$$

By the induction hypotheses, by Lemmas 1–3 and the conditions (A1), (A2), and (A4), it follows that

$$\begin{aligned}\|F'(x_0)^{-1}(F'(x_{i+1})-F'(x_0))\| &\leq \bar{K}\|x_{i+1}-x_0\|^p, \\ &\leq \bar{K}(v_{i+1}-v_0)^p \leq \bar{K}v_{i+1}^p < 1.\end{aligned}$$

Hence, the inverse of the linear operator $F'(x_{i+1})$ exists. Therefore, $F'(v)^{-1} \in \mathcal{L}(X_2, X_1)$ and

$$\|F'(x_{i+1})^{-1}F'(x_0)\| \leq \frac{1}{1-\bar{K}v_{i+1}^p)} \tag{11}$$

follows as a consequence of Lemma 4, where $\bar{K} = \begin{cases} K_0, & i=0 \\ L_0, & i=1,2,\ldots. \end{cases}$ The following general integral equality is implied by (NP)

$$\begin{aligned}F(x_{i+1}) &= F(x_{i+1})-F(x_i)-F'(x_i)(x_{i+1}-x_i), \\ &= \int_0^1 (F'(x_i+\xi(x_{i+1}-x_i))-F'(x_i))d\xi(x_{i+1}-x_i).\end{aligned} \tag{12}$$

Then, using the induction hypotheses, estimate (9) and condition (A3)

$$\begin{aligned}\|F'(x_0)^{-1}F(x_{i+1})\| &\leq \bar{L}\int_0^1(\xi\|x_{i+1}-x_i\|)^p d\xi \\ &\leq \frac{\bar{L}}{1+p}(v_{i+1}-v_i)^{1+p},\end{aligned} \tag{13}$$

where $\bar{L} = \begin{cases} K, & i=0 \\ L, & i=1,2,\ldots. \end{cases}$

It follows by (NP), estimates (11), (13) and the definition (3) of the sequence $\{v_n\}$

$$\begin{aligned}\|x_{i+2}-x_{i+1}\| &\leq \|F'(x_{i+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{i+1})\|, \\ &\leq \frac{\tilde{K}(v_{i+1}-v_i)^{1+p}}{2(1-\tilde{L}v_{i+1}^p)} = v_{i+2}-v_{i+1},\end{aligned}$$

where $\tilde{K} = \begin{cases} K, & i=0 \\ L, & i=1,2,\ldots \end{cases}$ and $\tilde{L} = \begin{cases} K_0, & i=0 \\ L_0, & i=1,2,\ldots. \end{cases}$ Moreover, if $v \in U[x_{i+2}, t^*-v_{i+2}]$ it follows

$$\begin{aligned}\|v-x_{i+1}\| &\leq \|v-x_{i+2}\|+\|x_{i+2}-x_{i+1}\| \\ &\leq t^*-v_{i+2}+v_{i+2}-v_{i+1} = t^*-v_{i+1}.\end{aligned}$$

So, the vector $w \in U[x_{i+1}, t^*-v_{i+1}]$ completing the induction for assertions (9) and (10). Notice that the scalar majorizing sequence $\{v_i\}$ is fundamentally convergent. Hence, the sequence $\{x_i\}$ is also convergent to some $x^* \in U[x_0, t^*]$. Furthermore, let $i \longrightarrow \infty$ in estimate (13), to conclude $F(x^*) = 0$. □

Next, the uniqueness ball for a solution is presented. Notice that not all conditions mentioned in (A) are used.

Proposition 1. *Let, for some $x_0 \in \Omega$ the center-Lipschitz condition (A2) be satisfied. Further assume that there exists $0 < R < \left(\frac{1+p}{2L_0}\right)^{\frac{1}{p}}$ such that there exists a solution $s \in U(x_0, R) \subset \Omega$ of equation (1) and such that linear operator $F'(s)$ is invertible. Let the parameter $R_1 \geq R$ be given by*

$$R_1 = \left(\frac{1+p}{L_0} - R^p\right)^{\frac{1}{p}}. \tag{14}$$

Then, the point s solves uniquely the equation $F(x) = 0$ in the set $B_2 = U(x_0, R_1) \cap \Omega$.

Proof. Define the linear operator $Q = \int_0^1 F'(\bar{s} + \xi(s - \bar{s}))d\xi$ for some point $\bar{s} \in B_2$ satisfying $F(\bar{s}) = 0$. By using the definition of R_1, set B_2 and condition (A2),

$$\begin{aligned} \|F'(x_0)^{-1}(F'(x_0) - Q)\| &\leq \int_0^1 L_0((1-\xi)^p\|x_0 - s\|^p + \xi^p\|x_0 - \bar{s}\|^p)d\xi, \\ &< \frac{L_0}{1+p}(R_1^p + R^p) = 1, \end{aligned}$$

concluding that $s = \bar{s}$, where the invertibility of the linear operator is also used together with the approximation $0 = F(s) - F(\bar{s}) = Q(s - \bar{s})$. $\square$

Remark 1. *(i) Under the conditions in (A), the existence of x^* is assured. In this case set $q = x^*$ and $R = t^*$.*
(ii) Condition (A3) can be replaced by

$$\|F'(x_0)^{-1}(F'(w_1 + \xi(w_2 - w_1)) - F'(w_1))\| \leq d_0\|\xi(w_1 - w_2)\|^p \tag{15}$$

$\forall\, w_1 \in B_1$ and $w_2 = w_1 - F'(w_1)^{-1}F(w_1) \in B_1$. This even smaller parameter d_0 can replace L in the aforementioned results. The existence of the iterate w_2 is assured by (A2) and Lemma 4. Notice that the proof of Theorem 1 goes through if condition (15) replaces stronger (A3).
(iii) Concerning the more general iteration $\{\bar{v}_n\}$ studied in [6] defined by

$$\begin{aligned} \bar{v}_0 &= 0,\ \bar{v}_1 = \eta, \\ \bar{v}_2 &= \bar{v}_1 + \int_0^1 \frac{\bar{\psi}_\theta(\theta(\bar{v}_1 - \bar{v}_0))d\theta(\bar{v}_1 - \bar{v}_0)}{1 - \bar{\psi}_1(\bar{K})}, \\ \bar{v}_{n+2} &= \bar{v}_{n+1} + \frac{\int_0^1 \psi_\theta(\theta(\bar{v}_{n+1} - \bar{v}_n))d\theta(\bar{v}_{n+1} - \bar{v}_n)}{1 - \psi_1(\bar{v}_{n+1})} \quad \forall n = 1, 2, 3, \ldots. \end{aligned} \tag{16}$$

Suppose function

$$f_\theta(t, u) = \frac{1}{t^p}\int_0^1 \frac{\psi_\theta(\theta t)d\theta}{1 - \psi_1(u)} \tag{17}$$

is nondecreasing and bounded from above by some $\bar{b} > 0$. Then, the same proof as Lemma 3 recovers the $1 + p$ order of convergence for this general iteration provided that $\bar{c} = \bar{b}^{-\frac{1}{p}}$, $\eta \leq \frac{1}{\bar{c}}$, and the conditions of Lemma 1 or Lemma 2 in [6] hold. This is due to the calculation

$$\begin{aligned} \bar{v}_{n+2} - \bar{v}_{n+1} &= \frac{\int_0^1 \psi_\theta(\theta(\bar{v}_{n+1} - \bar{v}_n))d\theta(\bar{v}_{n+1} - \bar{v}_n)^{1+p}}{(1 - \psi_1(\bar{v}_{n+1}))(\bar{v}_{n+1} - \bar{v}_n)^p} \\ &\leq \bar{b}(\bar{v}_{n+1} - \bar{v}_n)^{1+p} \ \text{ for } \forall \bar{v}_{n+1} \neq \bar{v}_n. \end{aligned}$$

Then, under the conditions of Theorem 1 and Proposition 1 in [6] the conclusions, hold for a sequence $\{x_n\}$ in this more general setting, where it is also shown that the convergence order is

$1+p$. In the case when $\bar{\psi}_\theta$, ψ_θ are constant functions, then, set $\bar{b} = \frac{L(1+\epsilon)}{1+p}$. Hence condition (17) can be realized. Notice that sequence $\bar{v}_n\}$ specializes to $\{v_n\}$ if $\bar{\psi}_1(t) = K_0 t^p$, $\bar{\psi}_\theta(t) = K(\theta t)^p$, $\psi_1(t) = L_0 t^p$ and $\psi_\theta(t) = L(\theta t)^p$. Under, these choices of functions Lemma 1 and Theorem 1 coincide with the corresponding ones in [6]. Moreover, the rest of the Lemmas in [6] show only linear convergence of majorizing sequences and, consequently of the sequence $\{x_n\}$. However, in Lemma 3, the convergence order $1+p$ is obtained.

Finally, in Lemma 2, in [6], the upper bound on η is not given explicitly in all cases, nor is the convergence order $1+p$. However, the objective of this article is to do so. That explains the approach in this article.

4. Example

The solution of a BVP is presented as an application of theory.

Example 1. *Let us consider the two-point BVP(TPBVP)*

$$\begin{aligned} u'' + u^{\frac{3}{2}} &= 0 \\ u(0) = u(1) &= 0. \end{aligned}$$

The interval $[0,1]$ is divided into j subintervals. Set $m = \frac{1}{j}$. Denote by $w_0 = 0 < w_1 < \ldots < w_j = 1$ the points of subdivision with corresponding values of the function $u_0 = u(w_0), \ldots, u_j = u(w_j)$. Then, the discretization of u'' is given by

$$u_k'' \approx \frac{u_{k-1} - 2u_k + u_{k+1}}{m^2} \text{ for } \forall k = 2, 3, \ldots j-1.$$

Further, notice that $u_0 = u_j = 0$. It follows that the following system of equations is obtained

$$\begin{aligned} m^2 u_1^{\frac{3}{2}} - 2u_1 + u_2 &= 0, \\ u_{k-1} + m^2 u_k^{\frac{3}{2}} - 2u_k + u_{k+1} &= 0 \text{ for } \forall k = 2, 3, \ldots, j-1 \\ u_{j-2} + m^2 u_{j-1}^{\frac{3}{2}} - 2u_{j-1} &= 0. \end{aligned}$$

This system can be converted into an operator equation as follows: Define operator $H : \mathbb{R}^{j-1} \longrightarrow \mathbb{R}^{j-1}$ whose derivative is given as

$$H'(u) = \begin{bmatrix} \frac{3}{2} m^2 u_1^{\frac{1}{2}} - 2 & 1 & 0 & \ldots & 0 \\ 1 & \frac{3}{2} m^2 u_2^{\frac{1}{2}} - 2 & 1 & 0 & \ldots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & \frac{3}{2} m^2 u_{j-1}^{\frac{1}{2}} - 2 \end{bmatrix}.$$

Pick $z \in \mathbb{R}^{j-1}$ be arbitrary. The norm used is $\|z\| = \max_{1 \le k \le j-1} \|z_k\|$, where as the norm for $H \in \mathbb{R}^{j-1} \times \mathbb{R}^{j-1}$ is given as

$$\|H\| = \max_{1 \le k \le j-1} \sum_{i=1}^{j-1} \|h_{k,i}\|.$$

Then, pick $u, z \in \mathbb{R}^{j-1}$ *for* $|u_k| > 0, |z_k| > 0$, for$\forall k = 1, 2, \ldots, j-1$ *to obtain in turn*

$$\begin{aligned} \|H'(u) - H'(z)\| &= \|diag\{\frac{3}{2}(u_k^{\frac{1}{2}} - z_k^{\frac{1}{2}})\}\| \\ &= \frac{3}{2}m^2\left[\max_{1\le k\le j-1}|u_k - z_k|\right]^{\frac{1}{2}} \\ &= \frac{3}{2}m^2\|u - z\|^{\frac{1}{2}}. \end{aligned}$$

Choose as an initial guess vector $130 \sin \pi x$ *to obtain after four iterations* $u_0 = [3.35740 \times 10^1, 6.5202 \times 10^1, 9.15664 \times 10^1, 1.09168 \times 10^2, 1.15363 \times 10^2, 1.09168 \times 10^2, 9.15664 \times 10^1, 6.52027 \times 10^1, 3.35740 \times 10^1]^{tr}$. *Then, the parameters are* $\|Q'(u_0)^{-1}\| \le 2.5582 \times 10^1, \eta = 9.15311 \times 10^{-5}$, $p = 0.5, K_0 = L_0 = K = L = \frac{3}{200} = 0.015$. *Then,* $K_0\eta^p = 1.4351 \times 10^{-4}$. *The following Table 1 shows that the conditions of Lemma 1 are satisfied, since* $v_n = v_{n+m}$ *for* $\forall n = 0, 1, 2, \ldots, m = 0, 1, 2, \ldots$. *Hence, the conditions of Theorem 1 hold.*

Table 1. Sequence (3).

n	1	2	3	4	5	6
v_{n+1}	0.1435×10^{-3}	0.1435×10^{-3}	0.1435×10^{-3}	0.1435×10^{-3}	0.1435×10^{-3}	0.1435×10^{-3}

By using the initial vector on (NP) the generated vector is not good enough to apply Theorem 1. However, after four iterations, the vector u_0 *is good enough. Then, the Hölder constants are obtained simply using conditions (A1)–(A3) and taking the max-norm of the resulting vector or matrix. In this paper, the conditions of Lemma 1 are verified first, which are weaker.*

Concerning the convergence order, one should verify conditions (6) of Lemma 3. Choose $\epsilon = 0.8$. *Then, using the preceding values* $\eta_5\eta < 0.47 < 0.5$. *Therefore, the convergence order is* $1 + p = 1 + 0.5 = 1.5$. *Hence, the conclusions of Theorem 1 hold. The corresponding criteria in ([Remark 2, for the Hölder case], [6]) are*

$$h_{1,p}(\gamma_p) \le 0 \text{ and, } 0 \le \delta_0 \le \gamma_p,$$

where $\delta_0 = \frac{K(v_2-v_1)^p}{(1+p)(1-K_0v_2^p)}$, $\gamma_p = \left(\frac{L}{L+(1+p)L_0}\right)^{\frac{p}{1+p}}$ *and* $h_{1,p}(t) = \frac{L}{1+p}t^p(v_2 - v_1)^p + tL_0(v_1 + (1+t))^p(v_2 - v_1)^p - t$. *However, these conditions give an implicit estimate on the smallness* η, *they are not satisfied for this example for* $p = 0.5$. *However, even if they were the convergence of the sequence* $\{x_n\}$ *is only linear. The same is true if another criterion given in [6] by* $0 \le \eta \le \min\left\{\frac{2\gamma_1}{(1+2\gamma_1)L_0}, \frac{1}{K_0+L_0}\right\}$. *That is even if it is verified the convergence order is only linear.*

5. Conclusions

The two new features are explicit upper bounds on the smallness of η. Convergence order $1 + p$ is also recovered by choosing a larger upper bound on η. New Lipschitz or Hölder parameters are smaller and specializations of previous parameters. The new theory can always replace previous ones due to a weaker a priori hypothesis. The strategy can be applied to other processes, such as Secant, Kurchatov, Stirling's, Newton-like, and multistep [2,3,5,9–11]. This will be done in future work.

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