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A Study of the Growth Results for the Hadamard Product of Several Dirichlet Series with Different Growth Indices

Hongyan Xu [1,2,3,*,†], Guangsheng Chen [4], Hari Mohan Srivastava [5,6,7,8,†], Hong Li [3], Zuxing Xuan [9] and Yongqin Cui [10]

1 College of Arts and Sciences, Suqian University, Suqian 223800, China
2 School of Mathematics and Computer Science, Shangrao Normal University, Shangrao 334001, China
3 School of Mathematics and Computer Science, Gannan Normal University, Ganzhou 341000, China; lihong@gnnu.edu.cn
4 College of Mathematics and Computer Science, Guangxi Science & Technology Normal University, Laibin 546199, China; chenguangsheng@gxstnu.edu.cn
5 Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada; harimsri@math.uvic.ca
6 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
7 Department of Mathematics and Informatics, Azerbaijan University, AZ197 Baku, Azerbaijan
8 Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy
9 Department of General Education, Beijing Union University, Beijing 100101, China; zuxing@buu.edu.cn
10 Department of Informatics and Engineering, Jingdezhen Ceramic University, Jingdezhen 333403, China; 004505@jcu.edu.cn
* Correspondence: xhy@gnnu.edu.cn
† These authors contributed equally to this work.

Abstract: In this paper, our first purpose is to describe a class of phenomena involving the growth in the Hadamard–Kong product of several Dirichlet series with different growth indices. We prove that (i) the order of the Hadamard–Kong product series is determined by the growth in the Dirichlet series with smaller indices if these Dirichlet series have different growth indices; (ii) the q_1-type of the Hadamard–Kong product series is equal to zero if p Dirichlet series are of q_j-regular growth, and $q_1 < q_2 < \cdots < q_p$, $q_j \in N_+$, $j = 1, 2, \ldots, p$. The second purpose is to reveal the properties of the growth in the Hadamard–Kong product series of two Dirichlet series—when one Dirichlet series is of finite order, the other is of logarithmic order, and two Dirichlet series are of finite logarithmic order—and obtain the growth relationships between the Hadamard–Kong product series and two Dirchlet series concerning the order, the logarithmic order, logarithmic type, etc. Finally, some examples are given to show that our results are best possible.

Keywords: Dirichlet series; Hadamard–Kong product; growth

MSC: 30B50; 30D15

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1. Introduction

The following series

$$\sum_{n=0}^{\infty} a_n e^{\lambda_n s} = a_0 e^{\lambda_0 s} + a_1 e^{\lambda_1 s} + \cdots + a_n e^{\lambda_n s} + \cdots, \qquad s = \sigma + it, \tag{1}$$

is usually called the Dirichlet series, where sequence $\{a_n\}$ is a complex number, $0 < \lambda_1 < \cdots < \lambda_n \to +\infty$ and σ and t are real variables.

If we take $e^{\lambda_n} = n$, $s = -z$ and $a_n = 1$, then series (1) becomes the famous Riemann ζ function $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$, which is useful in analytic number theory to study the properties of prime numbers. If we take $e^s = z$ and $\lambda_n = n$, then this series (1) becomes a basic power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If we take $\sigma = 0$ and $\lambda_n = n$, then this series (1) could be a complex Fourier series. It is widely known that the Dirichlet series can be used in many fields of mathematics, such as analytic number theory, functional equations, and certain areas of theoretical and applied probability (see [1–5]).

In the past 80 years, many mathematicians have paid considerable attention to the growth and the value distribution of entire functions representend by Dirichlet series that are convergent in the whole complex plane (see [6–9]). For example, Doi and Naganuma [10] studied the properties of Dirichlet series, satisfying a certain functional equation, and analytical support of the problem was given by G. Shimura [7]; X. Q. Ding, D. C. Sun, J. R. Yu explored the singular points and deficient functions of random Dirichlet series, and reveal the relationships between these Singularities and the growth of the Dirichlet series (see [11–13]); S. M. Daoud, Z. S. Gao, Y. Y. Huo, M. L. Liang discussed the growth in multiple Dirichlet series, and provided some results of the linear order, the lower order of multiple Dirichlet series (see [14–17]); M. M. Sheremeta, A. R. Reddy, C. F. Yi, J. H. Ning, H. Y. Xu explored the approximation of the Dirichlet series, and established some results regarding the relationship between error and growth (see [18–21]); H. M. Srivastava, D. Sato, S. M. Shah, S. Owa, A. R. Reddy, O.P. Juneja, D. C. Sun, Z. S. Gao investigated the Hadamard product of analytic functions and the growth in the Dirichlet series, and obtained some theorems involving the concepts of zero-order, finite p-order, and (p,q)-order, etc. (see [22–38]).

2. Some Definitions and Basic Results

Let Dirichlet series (1) satisfy

$$\limsup_{n\to+\infty} \frac{\log n}{\lambda_n} = E < +\infty, \qquad \limsup_{n\to+\infty} \frac{\log |a_n|}{\lambda_n} = -\infty, \tag{2}$$

then, in view of Refs. [6,9], we can conclude that the series (1) converges on the whole plane. Thus, the sum function $f(s)$ of (1) is an entire function. For convenience, allow D to denote the set of all functions $f(s)$ with the form (1), which is analytic in the region $\Re s < +\infty$ and the sequence $\{\lambda_n\}$ satisfy (2).

Usually, we utilize the order and type to estimate the growth in $f(s)$, which are defined as follows.

Definition 1 (see [22]). *Let $f(s) \in D$. The q-order $\rho^{[q]}$ and lower q-order $\chi^{[q]}$ of $f(s)$ are defined by*

$$\rho^{[q]} = \limsup_{\sigma\to+\infty} \frac{\log^{[q]} M(\sigma, f)}{\sigma}, \quad \chi^{[q]} = \liminf_{\sigma\to+\infty} \frac{\log^{[q]} M(\sigma, f)}{\sigma},$$

where $M(\sigma, f) = \sup\limits_{-\infty<t<+\infty} |f(\sigma + it)|$. Here and below, unless otherwise specified, q is a positive integer and $q = 2, 3, \ldots$

Furthermore, if $\rho \in (0, +\infty)$, the q-type $T^{[q]}$ and the lower q-type $\tau^{[q]}$ of $f(s)$ are defined by

$$T^{[q]} = \limsup_{\sigma\to+\infty} \frac{\log^{[q-1]} M(\sigma, f)}{e^{\rho\sigma}}, \quad \tau^{[q]} = \liminf_{\sigma\to+\infty} \frac{\log^{[q-1]} M(\sigma, f)}{e^{\rho\sigma}},$$

where $\log^{[-1]} x = e^x, \log^{[0]} x = x, \log^{[1]} x = \log^+ x, \log^{[k]} x = \log^+ \log^{[k-1]} x$, $k \in Z_+$.

Remark 1. *It is said that $f(s)$ has the growth index q, if $\rho^{[q-1]}(f) = \infty$ and $\rho^{[q]} \in [0, +\infty)$.*

Remark 2. *Generally, 2-order is always called an order, that is, $\rho^{[2]} = \rho$. Similarly, for the lower 2-order, 2-type and lower 2-type, we have $\chi^{[2]} = \chi$, $T^{[2]} = T$ and $\tau^{[2]} = \tau$.*

Remark 3. *We say that $f(s)$ is of q-regular growth if $\rho^{[q]} = \chi^{[q]}$, $f(s)$ is of q-irregular growth if $\rho^{[q]} \neq \chi^{[q]}$. Further, $f(s)$ is of perfectly q-regular growth if $\rho^{[q]} = \chi^{[q]}$ and $T^{[q]} = \tau^{[q]}$.*

To describe the growth in $f(s)$ when $\rho^{[2]} = 0$, the definitions, including the logarithmic order, logarithmic type, lower logarithmic order and lower logarithmic type, can be introduced as follows

Definition 2 (see [30,39])**.** *If $f(s) \in D$, and is of zero-order $\rho^{[2]} = 0$, then we define the logarithmic order and lower logarithmic order of $f(s)$ as follows*

$$\rho^l = \limsup_{\sigma\to+\infty} \frac{\log^+ \log^+ M(\sigma, f)}{\log \sigma}, \quad \chi^l = \liminf_{\sigma\to+\infty} \frac{\log^+ \log^+ M(\sigma, f)}{\log \sigma}.$$

Furthermore, if $1 \le \rho^l < +\infty$, the logarithmic type T^l and lower logarithmic type τ^l of $f(s)$ are defined by

$$T^l = \limsup_{\sigma\to+\infty} \frac{\log^+ M(\sigma, f)}{\sigma^{\rho_l}}, \quad \tau^l = \liminf_{\sigma\to+\infty} \frac{\log^+ M(\sigma, f)}{\sigma^{\rho_l}}.$$

Remark 4. *We say that $f(s)$ is of logarithmic regular growth if $\rho^l = \chi^l$, while $f(s)$ is of logarithmic irregular growth if $\rho^l \neq \chi^l$. Further, $f(s)$ is of perfectly logarithmic regular growth if $\rho^l = \chi^l$ and $T^l = \tau^l$.*

We will then list some results of the q-order, q-type, lower q-order, lower q-type, ... of Dirichlet series, which are used in this paper.

Theorem 1 (see [22])**.** *If $f(s) \in D$, and is of q-order $\rho^{[q]}$ and q-type $T^{[q]}$, then*

$$\rho^{[q]} = \limsup_{n\to+\infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{-\log |a_n|},$$

$$T^{[q]} = \limsup_{n\to+\infty} |a_n|^{\frac{\rho^{[q]}}{\lambda_n}} \log^{[q-2]} \left(\frac{\lambda_n}{\rho^{[q]} e}\right).$$

Theorem 2 (see [22])**.** *Let $f(s) \in D$, and be of lower q-order $\chi^{[q]}$, then*

$$\chi^{[q]} \le \liminf_{n\to\infty} \frac{\lambda_n \log^{[q-1]} \lambda_{n-1}}{\log |a_n|^{-1}},$$

the equal sign in the above inequality holds if, and only if,

$$\psi(n) = \frac{\log |a_n| - \log |a_{n+1}|}{\lambda_{n+1} - \lambda_n}$$

is a non-decreasing function of n.

Theorem 3 (see [8,22])**.** *Let $f(s) \in D$; then*

$$\tau^{[q]} \le \liminf_{n\to+\infty} |a_n|^{\frac{\rho}{\lambda_n}} \log^{[q-2]} (\frac{\lambda_{n-1}}{e\rho}),$$

the equal sign in the above inequality holds if, and only if, $\psi(n)$ is a non-decreasing function of n and $\log^{[q-2]} \lambda_{n-1} \sim \log^{[q-2]} \lambda_n (n \to \infty)$.

Theorem 4 (see [30])**.** *If $f(s) \in D$, and is of zero-order and finite-logarithmic order ρ^l, then*

$$\rho^l = 1 + \limsup_{n\to+\infty} \frac{\log \lambda_n}{\log\left(\frac{1}{\lambda_n} \log \frac{1}{|a_n|}\right)}.$$

Theorem 5 (see [30]). *Let $f(s) \in D$, and is of zero-order and finite-lower-logarithmic order χ^l, then*

$$\chi^l \le 1 + \liminf_{n\to+\infty} \frac{\log \lambda_{n-1}}{\log\left(\frac{1}{\lambda_n}\log\frac{1}{|a_n|}\right)},$$

the equal sign in the above inequality holds, if and only if, $\psi(n)$ is a non-decreasing function of n.

Theorem 6 (see [39]). *Let $f(s) \in D$, be of zero-order, logarithmic order $\rho^l(1 < \rho_l < +\infty)$, and logarithmic type T^l; then*

$$T^l = \limsup_{n\to+\infty} \frac{\lambda_n}{\rho^l\left[-\frac{\rho^l}{\rho^l-1}\frac{1}{\lambda_n}\log|a_n|\right]^{\rho^l-1}},$$

and for $\lambda_n \sim \lambda_{n-1}$,

$$\tau^l \ge \liminf_{n\to+\infty} \frac{\lambda_{n-1}}{\rho^l\left[-\frac{\rho^l}{\rho^l-1}\frac{1}{\lambda_n}\log|a_n|\right]^{\rho^l-1}}.$$

Further, if $\psi(n)$ is a non-decreasing function of n, then

$$\tau^l = \liminf_{n\to+\infty} \frac{\lambda_n}{\rho^l\left[-\frac{\rho^l}{\rho^l-1}\frac{1}{\lambda_n}\log|a_n|\right]^{\rho^l-1}}.$$

Based on the conclusions of Theorems 1–3, Kong [40], in 2009, studied the growth in rhw Dirichlet–Hadarmard product function defined by two Dirichlet series, and provided an estimation of the (lower) q-order and the (lower) q-type of this product function. To provide related results, we first define the Dirichlet–Hadamard product function as follows.

Definition 3 (see [40], Definition 2). *Let $f_1(s) = \sum_{n=1}^{\infty} a_n e^{\gamma_n s}$, $f_2(s) = \sum_{n=1}^{\infty} b_n e^{\xi_n s}$ and $f_1(s), f_2(s) \in D$, then the Dirichlet–Hadamard product function of $f_1(s)$ and $f_2(s)$ are defined by*

$$G(s) = (f_1 \Delta f_2)(u,v;s) = \sum_{n=1}^{\infty} c_n e^{\lambda_n s}, \;\; c_n = a_n^u b_n^v, \;\; \lambda_n = \frac{\gamma_n + \xi_n}{2},$$

where u, v are positive numbers and $\{a_n\}, \{b_n\} \subset \mathbb{C}$.

In view of this definition, Kong [40] obtained

Theorem 7 (see [40], Theorems 1–4). *Let $f_j(s) \in D$ be of q-order ρ_j and lower q-order χ_j, $j = 1, 2$, and if*

$$\gamma_n \sim \xi_n, \quad (n \to +\infty),$$

and $\psi_1(n) = \frac{\log|a_n| - \log|a_{n+1}|}{\gamma_{n+1}-\gamma_n}$, $\psi_2(n) = \frac{\log|b_n| - \log|b_{n+1}|}{\xi_{n+1}-\xi_n}$ are two non-decreasing functions of n.

(i) Then the (lower) q-order $\rho(\chi)$ of Dirichlet-Hadarmard product $G(s)$ satisfy

$$\frac{\chi_1\chi_2}{v\chi_1 + u\chi_2} \le \chi \le \rho \le \frac{\rho_1\rho_2}{v\rho_1 + u\rho_2},$$

(ii) if $f_j(s)(j = 1, 2)$ are of q-regular growth, then the Dirichlet–Hadarmard product function $G(s)$ is of q-regular growth, and the q-order ρ of $G(s)$ satisfies

$$\rho = \frac{\rho_1\rho_2}{v\rho_1 + u\rho_2}, \;\; \rho_1, \rho_2 \in [0, +\infty).$$

(iii) if $\rho_1, \rho_2 \in (0, +\infty)$, then the q-type of $G(s)$ satisfies

$$T \leq \begin{cases} T_1^{\frac{u\rho}{\rho_1}} T_2^{\frac{v\rho}{\rho_2}}, & q = 3, 4, 5, \dots \\ \frac{1}{\rho}[\rho_1 T_1]^{\frac{u\rho}{\rho_1}} [\rho_2 T_2]^{\frac{v\rho}{\rho_2}}, & q = 2. \end{cases}$$

Remark 5. *We can see that the conclusions of Theorem 7 were obtained under the conditions $E = 0$ in (2).*

From Theorem 7, we can see that the author only discussed the growth in the Dirichlet–Hadamard product function, which is constructed by two Dirichlet series and $f_1(s)$ and $f_2(s)$ have the same growth index q, that is, $f_1(s), f_2(s)$ of finite q-order. Following this, there have been few references focusing on the properties of the Dirichlet–Hadamard product functions. However, the following interesting questions are naturally raised:

Question 1. *Could the condition "$E = 0$" in (2) be relaxed to "$E < +\infty$" in Theorem 7?*

Question 2. *What can be said about the growth in the Dirichlet–Hadamard product function of p($\geq$2) Dirichlet series with the different growth indices?*

Question 3. *What can be said about the growth in the Dirichlet–Hadamard product function of several Dirichlet series with the logarithmic growth, and the case for some of them being of logarithmic growth and the others being of finite growth?*

Motivated by Questions 1–3, we will further explore the properties of the Dirichlet–Hadamard product function of several Dirichlet series that are convergent in the whole plane concerning the logarithmic growth and q-th growth. The paper is organized as follows. In Section 2, we will provide a definition of the Dirichlet–Hadamard product of p Dirichlet series, and describe our main results regarding the growth in Dirichlet–Hadamard product functions. After that, some examples are given in Section 3 to show that our results are correct to some extent. The details of the proofs of Theorems 8–15 are given in Sections 5–7. Finally, we provide our conclusions and some open questions in the last section.

3. Our Main Results

We first introduce the following definition of the Dirichlet–Hadamard–Kong product of a finite Dirichlet series, which is more general than the Dirichlet–Hadamard shown in Definition 3.

Definition 4. *Let $f_j(s) = \sum_{n=1}^{\infty} a_{n,j} e^{\lambda_{n,j} s} \in D$, $j = 1, 2, \dots, p, (p \geq 2, p \in \mathbb{N}_+)$, then the Hadamard–Kong product function of $f_j(s)$ is defined by*

$$G(s) = (f_1 \Delta f_2 \Delta \cdots \Delta f_p)(u_1, \dots, u_p; v_1, \dots, v_p; s) = \sum_{n=1}^{\infty} b_n e^{\lambda_n s},$$

where

$$b_n = a_{n,1}^{u_1} a_{n,2}^{u_2} \cdots a_{n,p}^{u_p}, \quad \lambda_n = v_1 \lambda_{n,1} + v_2 \lambda_{n,2} + \cdots + v_p \lambda_{n,p},$$

and $u_1, u_2, \dots, u_p, v_1, v_2, \dots, v_p$ are positive numbers.

Based on Definition 4, we obtain

Theorem 8. *Let $f_j(s) = \sum_{n=1}^{\infty} a_{n,j} e^{\lambda_{n,j} s} (\in D)$ have the growth index q_j, and be of q_j-order $\rho_j (\in [0, \infty))$, $j = 1, 2, \dots, p$, and*

$$\lambda_{n,i} \sim \lambda_{n,j}, \quad (n \to +\infty, i \neq j, i, j = 1, 2, \dots, p). \tag{3}$$

If there is a positive integer $m(1 \le m \le p)$ such that

$$q_1 = q_2 = \cdots = q_m < q_{m+1} \le \cdots \le q_p, \tag{4}$$

then Hadamard–Kong product $G(s)$ has the growth index q_1 and the q_1-order ρ of $G(s)$, satisfying

$$\rho \le V\left(\frac{u_1}{\rho_1} + \frac{u_2}{\rho_2} + \cdots + \frac{u_m}{\rho_m}\right)^{-1}, \tag{5}$$

where $V = v_1 + v_2 + \cdots + v_p$.

Remark 6. $\rho \le \frac{\rho_1}{u_1}\sum_{j=1}^{p} v_j$ *if* $m = 1$; $\rho \le (v_1 + v_2)\left(\frac{u_1}{\rho_1} + \frac{u_2}{\rho_2}\right)^{-1}$ *if* $m = p = 2$.

Remark 7. *Here and below, unless otherwise specified, we always assume $q_j, j = 1, 2, \ldots, p$ are positive integers and $q_j \ge 2$.*

Theorem 9. *Let $f_j(s) = \sum_{n=1}^{\infty} a_{n,j} e^{\lambda_{n,j} s} (\in D)$ have the growth index q_j, and be of a lower q_j-order χ_j, $j = 1, 2, \ldots, p$. If $f_j(s)$ satisfy (3), (4),*

$$\lambda_{n+1,j} - \lambda_{n,j} = \eta_j(\lambda_{n+1,1} - \lambda_{n,1}), \quad \eta_j > 0, \quad j \ne 1, \tag{6}$$

and

$$\psi_j(n) = \frac{\log|a_{n,j}| - \log|a_{n+1,j}|}{\lambda_{n+1,j} - \lambda_{n,j}}, \quad j = 1, 2, \ldots, p, \tag{7}$$

are non-decreasing functions of n. Then Hadamard-Kong product $G(s)$ has the growth index q_1, and the lower q_1-order χ of Hadamard–Kong product $G(s)$ satisfy

$$\chi \ge V\left(\frac{u_1}{\chi_1} + \frac{u_2}{\chi_2} + \cdots + \frac{u_m}{\chi_m}\right)^{-1}. \tag{8}$$

Remark 8. $\chi \ge \frac{\chi_1}{u_1}\sum_{j=1}^{p} v_j$ *if* $m = 1$; *and* $\chi \ge (v_1 + v_2)\left[\frac{u_1}{\chi_1} + \frac{u_2}{\chi_2}\right]^{-1}$ *if* $m = p = 2$.

Theorem 10. *Let $f_j(s) = \sum_{n=1}^{\infty} a_{n,j} e^{\lambda_{n,j} s} (\in D)$ be of q_j-regular growth, q_j-order ρ_j and q_j-type T_j, $j = 1, 2, \ldots, p$. If $f_j(s)$ satisfy (3), (4), (6) and (7) are non-decreasing functions of n.*

(i) Then, Hadamard–Kong product $G(s)$ is of q_1-regular growth, and the q_1-order ρ of $G(s)$ satisfy

$$\rho = V\left(\frac{u_1}{\rho_1} + \frac{u_2}{\rho_2} + \cdots + \frac{u_m}{\rho_m}\right)^{-1};$$

(ii) If there is a positive integer $m(1 \le m < p)$ satisfying (4) and $0 < \rho_j < +\infty$, $j = 1, 2, \ldots, p$, then the q_1-type $T^{[q_1]}(G)$ of $G(s)$ is equal to zero, that is, $T^{[q_1]}(G) = 0$;

(iii) If $m = p$, that is, $q_1 = q_2 = \cdots = q_p = q$, then the q-type T of $G(s)$ satisfy

$$T \le \begin{cases} T_1^{\frac{u_1\rho}{V\rho_1}} T_2^{\frac{u_2\rho}{V\rho_2}} \cdots T_p^{\frac{u_p\rho}{V\rho_p}}, & q = 3, 4, 5, \ldots \\ \frac{V}{\rho}(\rho_1 T_1)^{\frac{u_1\rho}{V\rho_1}} (\rho_2 T_2)^{\frac{u_2\rho}{V\rho_2}} \cdots (\rho_p T_p)^{\frac{u_p\rho}{V\rho_p}}, & q = 2. \end{cases}$$

Furthermore, if $f_j(s), j = 1, \ldots, p$ are of perfectly q-regular growth and

$$\log^{[q-2]} \lambda_{n-1,j} \sim \log^{[q-2]} \lambda_{n,j}, \quad n \to \infty, \quad j = 1, 2, \ldots, p, \tag{9}$$

and

$$V = v_1 + v_2 + \cdots + v_p = 1, \tag{10}$$

then $G(s)$ is of perfectly q-regular growth, and the q-type T of $G(s)$ satisfy

$$T = \begin{cases} T_1^{\frac{u_1\rho}{\rho_1}} T_2^{\frac{u_2\rho}{\rho_2}} \cdots T_p^{\frac{u_p\rho}{\rho_p}}, & q = 3,4,5,\ldots \\ \frac{1}{\rho}(\rho_1 T_1)^{\frac{u_1\rho}{\rho_1}} (\rho_2 T_2)^{\frac{u_2\rho}{\rho_2}} \cdots (\rho_p T_p)^{\frac{u_p\rho}{\rho_p}}, & q = 2. \end{cases}$$

Remark 9. *Obviously, our results are some improvements to Theorem 7, since the results in [40] are the special case of Theorems 8–10 for $p = 2$, $q_1 = q_2 = q$ and $V = 1$.*

Remark 10. *By observing Theorems 8–10, for simplicity, we allow*

$$\left(\frac{u_1}{\rho_1} + \frac{u_2}{\rho_2} + \cdots + \frac{u_m}{\rho_m}\right)^{-1} = 0,$$

if $\rho_j = 0$ for $j \in J \subseteq \{1,2,\ldots,m\}$. In fact, it should be noted that

$$\frac{u_1}{\rho_1} + \frac{u_2}{\rho_2} + \cdots + \frac{u_m}{\rho_m} \to +\infty,$$

if $\rho_j = 0$ for $j \in J \subseteq \{1,2,\ldots,m\}$. Thus, it follows that

$$\left(\frac{u_1}{\rho_1} + \frac{u_2}{\rho_2} + \cdots + \frac{u_m}{\rho_m}\right)^{-1} \to 0.$$

Similarly, let

$$\left(\frac{u_1}{\chi_1} + \frac{u_2}{\chi_2} + \cdots + \frac{u_m}{\chi_m}\right)^{-1} = 0,$$

if $\chi_j = 0$ for $j \in J \subseteq \{1,2,\ldots,m\}$.

Now, we will state the results of the growth in Dirichlet–Hadamard–Kong product function $G(s)$ of several Dirichlet series with the logarithmic growth.

Theorem 11. *Let $f_j(s) = \sum_{n=1}^{\infty} a_{n,j} e^{\lambda_{n,j} s} (\in D)$ be of zero-order and the logarithmic order ρ_j^l, $j = 1,2,\ldots,p$. If $f_j(s), j = 1,2,\ldots,p$ satisfy (3), then $G(s)$ is of zero-order and the logarithmic order ρ^l, such that*

$$\rho^l \le \rho^l_{min} =: \min\{\rho_1^l, \rho_2^l, \ldots, \rho_p^l\}.$$

Theorem 12. *Let $f_j(s) = \sum_{n=1}^{\infty} a_{n,j} e^{\lambda_{n,j} s} (\in D)$ be of zero-order and a lower logarithmic order χ_j^l, $j = 1,2,\ldots,p$. If $f_j(s)$ satisfy (3), (4), (6) and (7) are non-decreasing functions of n. Then, the lower logarithmic order χ^l of $G(s)$ satisfies*

$$\chi^l \ge \chi^l_{min} =: \min\{\chi_1^l, \chi_2^l, \ldots, \chi_p^l\}.$$

Remark 11. *In view of Theorems 11 and 12, the logarithmic growth in Dirichlet–Hadamard–Kong product $G(s)$ is determined by the Dirichlet series with the minimum logarithmic growth.*

Theorem 13. *Let $f_j(s) = \sum_{n=1}^{\infty} a_{n,j} e^{\lambda_{n,j} s} (\in D)$ be of logarithmic regular ρ_j^l and logarithmic type T_j^l, $j = 1,2,\ldots,p$. If $f_j(s)$ satisfy (3), (4), (6) and (7) are non-decreasing functions of n.*

(i) Then Dirichlet–Hadamard–Kong product $G(s)$ is of logarithmic regular growth, and the logarithmic order ρ^l of $G(s)$ satisfies

$$\rho^l = \min\{\rho_1^l, \rho_2^l, \ldots, \rho_p^l\} := \rho^l_{min};$$

(ii) If k is a positive integer and $\rho_j^l(0 < \rho_j^l < +\infty), j = 1, 2, \ldots, p$ *satisfies*

$$\rho_1^l = \rho_2^l = \cdots = \rho_k^l = \rho_{min}^l,$$

then the logarithmic type T^l *of* $G(s)$ *satisfies*

$$\begin{aligned} T^l &\leq V^{\rho^l}\left[\sum_{j=1}^{k} \frac{u_j}{(T_j^l)^{(\rho^l-1)^{-1}}}\right]^{1-\rho^l} \\ &= V^{\rho^l}\left[\frac{u_1}{(T_1^l)^{(\rho^l-1)^{-1}}} + \frac{u_2}{(T_2^l)^{(\rho^l-1)^{-1}}} + \cdots + \frac{u_k}{(T_k^l)^{(\rho^l-1)^{-1}}}\right]^{1-\rho^l}. \end{aligned}$$

Furthermore, if $f_j(s), j = 1, \ldots, p$ *are of perfectly logarithmic regular growth and satisfy (10) and* $\lambda_{n,j} \sim \lambda_{n-1,j}$, $j = 1, \ldots, p$, *then* $G(s)$ *is of perfectly logarithmic regular growth, and the logarithmic type* T^l *of* $G(s)$ *satisfies*

$$\begin{aligned} T^l &= \left[\sum_{j=1}^{k} \frac{u_j}{(T_j^l)^{(\rho^l-1)^{-1}}}\right]^{1-\rho^l} \\ &= \left[\frac{u_1}{(T_1^l)^{(\rho^l-1)^{-1}}} + \frac{u_2}{(T_2^l)^{(\rho^l-1)^{-1}}} + \cdots + \frac{u_k}{(T_k^l)^{(\rho^l-1)^{-1}}}\right]^{1-\rho^l}. \end{aligned}$$

Finally, we will pay attention to the growth in Dirichlet–Hadamard–Kong product function $G(s)$ of two Dirichlet series when one Dirichlet series is of logarithmic growth, and the other is of finite growth.

Theorem 14. *Let* $f_j(s) = \sum_{n=1}^{\infty} a_{n,j} e^{\lambda_{n,j} s}, j = 1, 2 \in D$ *satisfy* $\lambda_{n,1} \sim \lambda_{n,2}$ *as* $n \to \infty$. *If* $f_1(s)$ *is of zero-order and the logarithmic order* $\rho_1^l(1 \leq \rho_1^l < \infty)$, *and* $f_2(s)$ *is of the order* $\rho_2(0 \leq \rho_2 < \infty)$. *Then* $G(s)$ *is of zero-order and the logarithmic order* ρ^l, *such that*

$$\rho^l \leq \rho_1^l.$$

Furthermore, if $f_1(s)$ *is of a lower logarithmic order* χ_1^l, *and* $f_2(s)$ *is of lower order* χ_2. *If* $f_1(s), f_2(s)$ *satisfies (6), and (7) are non-decreasing functions of n. Then, the lower logarithmic order* χ^l *of* $G(s)$ *satisfies*

$$\chi^l \geq \chi_1^l.$$

Remark 12. *In view of the processing of the proof of Theorem 14, we can see that the conclusions still hold if the condition regarding* $f_2(s)$ *being of order* ρ_2 *is replaced by the condition of* $f_2(s)$ *having the growth index* $q(\geq 3)$.

Theorem 15. *Let* $f_j(s) = \sum_{n=1}^{\infty} a_{n,j} e^{\lambda_{n,j} s}, j = 1, 2 \in D$ *satisfy (6), and (7) be non-decreasing functions of n, and* $\lambda_{n,1} \sim \lambda_{n,2}$ *as* $n \to \infty$. *If* $f_1(s)$ *is of logarithmic regular growth, logarithmic order* ρ_1^l *and logarithmic type* T_1^l, *and* $f_2(s)$ *is of regular growth, order* ρ_2 *and type* T_2.

(i) Then Dirichlet–Hadamard–Kong product $G(s)$ *is of zero-order and logarithmic regular growth, and the logarithmic order* ρ^l *of* $G(s)$ *satisfies*

$$\rho^l = \rho_1^l;$$

(ii) If ρ_1^l, ρ_2 *satisfy* $1 < \rho_1^l < +\infty, 0 < \rho_2 < +\infty$, *then the logarithmic type* T^l *of* $G(s)$ *satisfies*

$$T^l \leq \frac{(v_1 + v_2)^{\rho_1^l}}{(u_1)^{\rho_1^l - 1}} T_1^l.$$

Furthermore, if $f_1(s)$ is of perfectly logarithmic regular growth, $f_2(s)$ is of perfectly regular growth and satisfies $v_1 + v_2 = 1$ and $\lambda_{n-1,j} \sim \lambda_{n,j}$, $j = 1,2$, then $G(s)$ is of perfectly logarithmic regular growth, and the logarithmic type T^l of $G(s)$ satisfies

$$T^l = \frac{T_1^l}{(u_1)^{\rho_1^l - 1}}.$$

Remark 13. *In view of the processing of the proof of Theorem 15, we can obtain that the conclusions still hold if the condition of $f_2(s)$ being of regular growth is replaced by the condition of $f_2(s)$ having the growth index $q(\geq 3)$.*

Remark 14. *Similar to Theorems 8–13, one can easily obtain the corresponding results if the Dirichlet–Hadamard–Kong product $G(s)$ is structured by m_1 Dirichlet series of logarithmic growth, and m_2 Dirichlet series being growth indexes q, where m_1, m_2, q are positive integers.*

4. Examples

In this section, some examples are given to show that our results are correct and precise to some extent.

Example 1. *Let $q_1 = q_2 = 4$, $q_3 = 5$, $m_j, u_j, v_j, j = 1,2,3$ be positive numbers, and let*

$$G_1(s) = (f_1 \Delta f_2 \Delta f_3)(u_1, u_2, u_3; v_1, v_2, v_3; s) = \sum_{i=1}^{\infty} b_n e^{\lambda_n s},$$

where $f_j(s) = \sum_{n=1}^{\infty} a_{n,j} e^{\lambda_{n,j} s}$, $j = 1,2,3$, and

$$b_n = a_{n,1}^{u_1} a_{n,2}^{u_2} a_{n,3}^{u_3}, \ \ \lambda_n = \sum_{j=1}^{3} v_j \lambda_{n,j}, \ \ \lambda_{n,1} = \lambda_{n,2} = \lambda_{n,3} = \frac{n}{2},$$

and

$$a_{n,1} = \left(\frac{m_1}{\log^{[2]} n} \right)^{\frac{n}{u_1}}, \ \ a_{n,2} = \left(\frac{m_2}{\log^{[2]} n} \right)^{\frac{n}{u_2}}, \ \ a_{n,3} = \left(\frac{m_3}{\log^{[3]} n} \right)^{\frac{n}{u_3}}.$$

Thus, it follows that

$$\rho_1^{[4]} = \limsup_{n\to\infty} \frac{\lambda_{n,1} \log^{[3]} \lambda_{n,1}}{-\log |a_{n,1}|} = \limsup_{n\to\infty} \frac{\frac{n}{2} \log^{[3]} \frac{n}{2}}{-\frac{n}{u_1} \log \frac{m_1}{\log^{[2]} n}} = \frac{u_1}{2}, \tag{11}$$

and

$$T_1^{[4]} = \limsup_{n\to\infty} |a_{n,1}|^{\frac{\rho_1^{[4]}}{\lambda_{n,1}}} \log^{[2]} \left(\frac{\lambda_{n,1}}{e\rho_1^{[4]}}\right) = \limsup_{n\to\infty} \left| \frac{m_1}{\log^{[2]} n} \right| \log^{[2]} \left(\frac{n/2}{eu_1/2} \right) = m_1. \tag{12}$$

Similarly, we have

$$\chi_1^{[4]} = \frac{u_1}{2}, \ \ \rho_2^{[4]} = \chi_2^{[4]} = \frac{u_2}{2}, \ \rho_3^{[5]} = \chi_3^{[5]} = \frac{u_3}{2}, \tag{13}$$

and

$$\tau_1^{[4]} = m_1, \ \ T_2^{[4]} = \tau_2^{[4]} = m_2, \ T_3^{[5]} = \tau_3^{[5]} = m_3. \tag{14}$$

Now, in view of the definition of $G_1(s)$, we can deduce that

$$\begin{aligned} \rho^{[4]}(G_1) &= \limsup_{n\to\infty} \frac{\lambda_n \log^{[3]} \lambda_n}{-\log|b_n|} \\ &= \limsup_{n\to\infty} \frac{(v_1+v_2+v_3)\frac{n}{2} \log^{[3]}[(v_1+v_2+v_3)\frac{n}{2}]}{-u_1\log|a_{n,1}|-u_2\log|a_{n,2}|-u_3\log|a_{n,3}|} \\ &= \frac{v_1+v_2+v_3}{4}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} T^{[4]}(G_1) &= \limsup_{n\to\infty} |b_n|^{\frac{\rho^{[4]}(G_1)}{\lambda_n}} \log^{[2]}\left(\frac{\lambda_n}{e\rho^{[4]}(G_1)}\right) \\ &= \limsup_{n\to\infty} \left| \frac{(m_1m_2)^{\frac{1}{2}}}{\log^{[2]} n} \left(\frac{m_3}{\log^{[3]} n}\right)^{1/2} \right| \log^{[2]}\left(\frac{(v_1+v_2+v_3)n/2}{e(v_1+v_2+v_3)/4}\right) \\ &= 0. \end{aligned} \tag{16}$$

In view of (11)–(16), we have

$$\rho^{[4]}(G_1) = \frac{v_1+v_2+v_3}{4} = (v_1+v_2+v_3)\left[\frac{u_1}{\rho_1^{[4]}} + \frac{u_2}{\rho_2^{[4]}}\right]^{-1}.$$

Therefore, this example shows that the conclusions (i) and (ii) of Theorem 10 are precise.

Example 2. *Let*

$$G_2(s) = (f_1\Delta f_2)(u_1,u_2;v_1,v_2;s) = \sum_{n=1}^{\infty} c_n e^{\lambda_n s},$$

where f_1, f_2 are stated as in Example 1. By using the same argument as in Example 1, we have

$$\begin{aligned} \rho^{[4]}(G_2) &= \limsup_{n\to\infty} \frac{\lambda_n \log^{[3]} \lambda_n}{-\log|c_n|} \\ &= \limsup_{n\to\infty} \frac{(v_1+v_2)\frac{n}{2}\log^{[3]}[(v_1+v_2)\frac{n}{2}]}{-u_1\log|a_{n,1}| - u_2\log|a_{n,2}|} \\ &= \frac{v_1+v_2}{4} = (v_1+v_2)\left(\frac{u_1}{\rho_1^{[4]}} + \frac{u_2}{\rho_2^{[4]}}\right)^{-1}, \end{aligned}$$

and

$$\begin{aligned} T^{[4]}(G_2) &= \limsup_{n\to\infty} |c_n|^{\frac{\rho^{[4]}(G_2)}{\lambda_n}} \log^{[2]}\left(\frac{\lambda_n}{e\rho^{[4]}(G_2)}\right) \\ &= \limsup_{n\to\infty} \left| \left(\frac{m_1m_2}{\log^{[2]} n}\right) \right| \log^{[2]}\left(\frac{(v_1+v_2)n/2}{e(v_1+v_2)/4}\right) \\ &= (m_1m_2)^{1/2} = (T_1^{[4]})^{\frac{u_1\rho}{V\rho_1^{[4]}}} (T_2^{[4]})^{\frac{u_2\rho}{V\rho_2^{[4]}}}. \end{aligned}$$

Therefore, this example shows that the equal sign can occur in the conclusion (iii) of Theorem 10.

Example 3. *Let $u_j, v_j, j = 1,2,3$ be positive numbers, and let*

$$G_3(s) = (f_1\Delta f_2\Delta f_3)(u_1,u_2,u_3;v_1,v_2,v_3;s) = \sum_{i=1}^{\infty} b_n e^{\lambda_n s},$$

where $f_j(s) = \sum_{n=1}^{\infty} a_{n,j} e^{\lambda_{n,j} s}, j = 1,2,3$, and

$$b_n = a_{n,1}^{u_1} a_{n,2}^{u_2} a_{n,3}^{u_3}, \quad \lambda_n = v_1\lambda_{n,1} + v_2\lambda_{n,2} + v_3\lambda_{n,3},$$

and

$$\lambda_{n,1} = \lambda_{n,2} = \lambda_{n,3} = n, \ \ a_{n,1} = e^{-n^2}, \ \ a_{n,2} = e^{-2n^2}, \ \ a_{n,3} = e^{-n^{\frac{3}{2}}}.$$

Thus, in view of Theorems 11–13, it follows that

$$\begin{aligned}\rho_1^l = \rho_1^l(f_1) &= 1 + \limsup_{n\to\infty} \frac{\log \lambda_{n,1}}{\log\left(\frac{1}{\lambda_{n,1}} \log \frac{1}{|a_{n,1}|}\right)} \\ &= 1 + \limsup_{n\to\infty} \frac{\log n}{\log(\frac{1}{n}\cdot n^2)} = 2,\end{aligned}$$

and

$$\begin{aligned}T_1^l = T_1^l(f_1) &= \limsup_{n\to+\infty} \frac{\lambda_{n,1}}{\rho_1^l\left[-\frac{\rho_1^l}{\rho_1^l-1}\frac{1}{\lambda_{n,1}}\log|a_{n,1}|\right]^{\rho_1^l-1}} \\ &= \limsup_{n\to\infty} \frac{n}{2[-2\frac{1}{n}(-n^2)]} = \frac{1}{4}.\end{aligned}$$

Similarly, we have

$$\chi_1^l = 2, \ \ \rho_2^l = \chi_2^l = 2, \ \ \rho_3^l = \chi_3^l = 3, \ \ \tau_1^l = \frac{1}{4}, \ \ T_2^l = \tau_2^l = \frac{1}{8}, \ \ T_3^l = \tau_3^l = \frac{4}{27}.$$

In view of the definition of $G_3(s)$, we have

$$\begin{aligned}\rho^l = \rho^l(G_3) \quad &= 1 + \limsup_{n\to\infty} \frac{\log \lambda_n}{\log\left(\frac{1}{\lambda_n}\log\frac{1}{|b_n|}\right)} \\ &= 1 + \limsup_{n\to\infty} \frac{\log(v_1+v_2+v_3)n}{\log(\frac{1}{(v_1+v_2+v_3)n}(u_1n^2+2u_2n^2+u_3n^{3/2}))} \\ &= 2,\end{aligned} \tag{17}$$

and

$$\begin{aligned}T^l = T^l(G_3) \quad &= \limsup_{n\to+\infty} \frac{\lambda_n}{\rho^l\left[-\frac{\rho^l}{\rho^l-1}\frac{1}{\lambda_n}\log|b_n|\right]^{\rho^l-1}} \\ &= \limsup_{n\to\infty} \frac{(v_1+v_2+v_3)n}{2[\frac{2}{(v_1+v_2+v_3)n}(u_1n^2+2u_2n^2+u_3n^{3/2})]} \\ &= \frac{(v_1+v_2+v_3)^2}{4u_1+8u_2}.\end{aligned} \tag{18}$$

Equations (17) and (18) reveal the fact that

$$\rho^l = \min\{\rho_1^l, \rho_2^l, \rho_3^l\}, \quad T^l = (v_1+v_2+v_3)^{\rho^l}\left[\frac{u_1}{(T_1^l)^{(\rho^l-1)^{-1}}} + \frac{u_2}{(T_2^l)^{(\rho^l-1)^{-1}}}\right]^{1-\rho^l}.$$

This shows that the conclusions of Theorems 11–13 are precise to some extent.

Example 4. *Let $u_j, v_j, j = 1,2$ be positive numbers, and let*

$$G_4(s) = (f_1\Delta f_2)(u_1,u_2;v_1,v_2;s) = \sum_{i=1}^{\infty} b_n e^{\lambda_n s},$$

where $f_j(s) = \sum_{n=1}^{\infty} a_{n,j}e^{\lambda_{n,j}s}, j = 1,2,$ and

$$b_n = a_{n,1}^{u_1} a_{n,2}^{u_2}, \ \ \lambda_n = v_1\lambda_{n,1} + v_2\lambda_{n,2},$$

and

$$\lambda_{n,1} = \lambda_{n,2} = n, \ \ a_{n,1} = e^{-n^{3/2}}, \ \ a_{n,2} = n^{-\frac{1}{2}n}.$$

By simple calculation, f_1 is of zero-order and (lower) logarithmic order $\rho_1^l = 3(\chi_1^l = 3)$, (lower) logarithmic type $T_1^l = \frac{4}{27}(\chi_1^l = \frac{4}{27})$, and f_2 is of (lower) order $\rho_2 = 2(\chi_2 = 2)$, (lower) type $T_2 = \frac{1}{2e}(\tau_2 = \frac{1}{2e})$. On the other hand, we can deduce that

$$\rho = \rho(G_4) = \limsup_{n\to\infty} \frac{\lambda_n \log \lambda_n}{-\log|b_n|} = \limsup_{n\to\infty} \frac{(v_1+v_2)n\log[(v_1+v_2)n]}{u_1 n^{3/2} + \frac{u_2}{2} n \log n} = 0,$$

$$\begin{aligned} \rho^l = \rho^l(G_4) &= 1 + \limsup_{n\to\infty} \frac{\log \lambda_n}{\log(\frac{1}{\lambda_n}\log\frac{1}{|b_n|})} \\ &= 1 + \limsup_{n\to\infty} \frac{\log[(v_1+v_2)n]}{\log \frac{1}{(v_1+v_2)n}[u_1 n^{3/2} + \frac{u_2}{2} n\log n]} = 3, \end{aligned}$$

and

$$\begin{aligned} T^l = T^l(G_4) &= \limsup_{n\to\infty} \frac{(v_1+v_2)n}{3[\frac{3}{2(v_1+v_2)n}(u_1 n^{3/2} + \frac{u_2}{2} n \log n)]^2} \\ &= \frac{(v_1+v_2)^3}{u_1^2}\frac{4}{27}. \end{aligned}$$

This implies that

$$\rho(G_4) = 0, \rho^l(G_4) = \rho_1^l, T^l(G_4) = \frac{(v_1+v_2)^{\rho_1^l}}{(u_1)^{\rho_1^l - 1}} T_1^l.$$

Therefore, Example 4 shows that the conclusions of Theorems 14 and 15 are the best possible to some extent.

5. Some Lemmas

To prove Theorems 8–15, we require the following lemmas.

Lemma 1. *Let $f_j(s) \in D$, $j = 1, 2, \ldots, p$, and satisfy (3). Then $G(s) \in D$, where $G(s)$ is stated as in Definition 4.*

Proof. Assume that $f_j(s)$ satisfies

$$\limsup_{n\to+\infty} \frac{\log n}{\lambda_{n,j}} = E_j, \quad j = 1, 2, \ldots, p,$$

where $E_j < +\infty$, $j = 1, 2, \ldots, p$. Thus, we have

$$\begin{aligned} \limsup_{n\to\infty} \frac{\log n}{\lambda_n} &= \limsup_{n\to\infty} \frac{\log n}{v_1\lambda_{n,1} + v_2\lambda_{n,2} + \cdots + v_p\lambda_{n,p}} \\ &\le \limsup_{n\to\infty} \frac{\log n}{\lambda_{n,1}} \frac{\lambda_{n,1}}{v_1\lambda_{n,1} + v_2\lambda_{n,2} + \cdots + v_p\lambda_{n,p}} \\ &= \frac{E_1}{V} < +\infty, \end{aligned}$$

and

$$\begin{aligned} \limsup_{n\to\infty} \frac{\log|b_n|}{\lambda_n} &= \limsup_{n\to\infty} \frac{u_1\log|a_{n,1}| + u_2\log|a_{n,2}| + \cdots + u_p\log|a_{n,p}|}{v_1\lambda_{n,1} + v_2\lambda_{n,2} + \cdots + v_p\lambda_{n,p}} \\ &\le \limsup_{n\to\infty} \left[\frac{1}{\sum_{j=1}^p v_j} \sum_{j=1}^p \frac{u_j \log|a_{n,j}|}{\lambda_{n,j}}\right] \le -\infty. \end{aligned}$$

Thus, it follows that $G(s) \in D$. Therefore, this completes the proof of Lemma 1. $\square$

Lemma 2. *Let $f_j(s) \in D$, $j = 1,2,\ldots,p$, satisfy (6), and $\psi_j(n), j = 1,2,\ldots,p$ be non-decreasing functions of n, where $\psi_j(n)$ is stated as in (7). Then*

$$\psi(n) = \frac{\log|b_n| - \log|b_{n+1}|}{\lambda_{n+1} - \lambda_n}$$

is also a non-decreasing function of n, where b_n, λ_n are stated as in Definition 4.

Proof. From the definition of $\psi(n)$, we have

$$\begin{aligned}
\psi(n) &= \frac{\log|b_n| - \log|b_{n+1}|}{\lambda_{n+1} - \lambda_n} \\
&= \left[\sum_{j=1}^{p} v_j(\lambda_{n+1,j} - \lambda_{n,j})\right]^{-1} \left(\sum_{j=1}^{p} u_j \log\left|\frac{a_{n,j}}{a_{n+1,j}}\right|\right) \\
&= \sum_{j=1}^{p} \frac{u_j(\log|a_{n,j}| - \log|a_{n+1,j}|)}{(\lambda_{n+1,j} - \lambda_{n,j})\left[v_j + \sum_{i\neq j, i=1}^{p} v_i \frac{\eta_i}{\eta_j}\right]},
\end{aligned}$$

where $\eta_1 = 1$. By combining this with $\psi_j(n)$ being non-decreasing functions of n, and $u_j, v_j, \eta_j > 0$, $j = 1,2,\ldots,p$, we obtain the idea that $\psi(n)$ is a non-decreasing function of n.

Therefore, we complete the proof of Lemma 2. $\square$

6. Proofs of Theorems about the Finite Growth Indices

In this section, we will provide the proofs of Theorems 8–10, regarding the growth in Dirichlet–Hadamard–Kong product function when Dirichlet series have the finite growth indexes.

Proof of Theorem 8. From Theorem 8, and by Lemma 1, we have $G(s) \in D$. Due to Theorem 1, we can see that

$$\rho_j = \limsup_{n\to\infty} \frac{\lambda_{n,j} \log^{[q_j-1]} \lambda_{n,j}}{-\log|a_{n,j}|}, \quad j = 1,2,\ldots,p. \tag{19}$$

Here, we only prove the case $0 < \rho_j < \infty$, $j = 1,2,\ldots,m$. For $\rho_j = 0$, one can easily prove the conclusion of Theorem 8. By virtue of (19), for any small number $\varepsilon > 0$, there are p positive integers $N_j \in \mathbb{N}_+$, $j = 1,2,\ldots,p$ such that $n > N = \max\{N_1, N_2, \ldots, N_p\}$, (it should be noted that the positive integer N, here and below, may not be the same every time)

$$\frac{\lambda_{n,j} \log^{[q_j-1]} \lambda_{n,j}}{-\log|a_{n,j}|} < \rho_j + \varepsilon, \ \ j = 1,2,\ldots,p,$$

that is,

$$\frac{\lambda_{n,j} \log^{[q_j-1]} \lambda_{n,j}}{\rho_j + \varepsilon} < -\log|a_{n,j}|, \quad j = 1,2,\ldots,p. \tag{20}$$

From the definition of $G(s)$, for $n > N$, we have that

$$-\log|b_n| = -\sum_{j=1}^{p} u_j \log|a_{n,j}| > \sum_{j=1}^{p} \frac{u_j \lambda_{n,j} \log^{[q_j-1]} \lambda_{n,j}}{\rho_j + \varepsilon}. \tag{21}$$

Thus, it follows from (21) that

$$\frac{\lambda_n \log^{[q_1-1]} \lambda_n}{-\log|b_n|} < \lambda_n \log^{[q_1-1]} \lambda_n \left(\sum_{j=1}^{p} \frac{u_j \lambda_{n,j} \log^{[q_j-1]} \lambda_{n,j}}{\rho_j+\varepsilon}\right)^{-1}. \tag{22}$$

In view of (3), (4) and $q_j \geq 2, j=1,\ldots,p$, we have

$$\lim_{n\to\infty} \frac{\lambda_{n,j} \log^{[q_j-1]} \lambda_{n,j}}{\lambda_{n,1} \log^{[q_1-1]} \lambda_{n,1}} = 1, \quad \lim_{n\to\infty} \frac{\lambda_{n,j} \log^{[q_j-1]} \lambda_{n,j}}{\lambda_{n,1} \log^{[q_1]} \lambda_{n,1}} = \infty, \quad j=1,2,\ldots,m, \tag{23}$$

and

$$\lim_{n\to\infty} \frac{\lambda_{n,j} \log^{[q_j-1]} \lambda_{n,j}}{\lambda_{n,1} \log^{[q_1-1]} \lambda_{n,1}} = 0, \quad j=m+1,m+2,\ldots,p. \tag{24}$$

Since $\lambda_n = v_1\lambda_{n,1} + v_2\lambda_{n,2} + \cdots + v_p\lambda_{n,p}$, it thus follows from (22)–(24) that

$$\rho = \rho^{[q_1]} \leq \limsup_{n\to\infty} \frac{\lambda_n \log^{[q_1-1]} \lambda_n}{\lambda_{n,1} \log^{[q_1-1]} \lambda_{n,1}} \left[\sum_{j=1}^{p} \frac{u_j \lambda_{n,j} \log^{[q_j-1]} \lambda_{n,j}}{(\rho_j+\varepsilon)\lambda_{n,1} \log^{[q_1-1]} \lambda_{n,1}}\right]^{-1},$$

By combining with the arbitrariness of ε, we have

$$\rho = \rho^{[q_1]} \leq V\left(\frac{u_1}{\rho_1} + \frac{u_2}{\rho_2} + \cdots + \frac{u_m}{\rho_m}\right)^{-1},$$

which shows that the q_1-order ρ of $G(s)$ satisfies (5).

On the other hand, since $f_j(s)$ has the growth index q_j; that is, $\rho_j^{[q_j-1]}(f_j) = \infty$, $j=1,2,\ldots,p$. Thus, for any large number $M>0$, there is a positive integer N', such that

$$\frac{\lambda_{n,j} \log^{[q_j-2]} \lambda_{n,j}}{-\log|a_{n,j}|} > M, \quad j=1,2,\ldots,p, \tag{25}$$

and

$$-\log|b_n| = -\sum_{j=1}^{p} u_j \log|a_{n,j}| < \sum_{j=1}^{p} \frac{u_j \lambda_{n,j} \log^{[q_j-2]} \lambda_{n,j}}{M}. \tag{26}$$

Thus, we can deduce from (25) and (26) that

$$\begin{aligned}\rho^{[q_1-1]}(G) &\geq \limsup_{n\to\infty} \frac{\lambda_n \log^{[q_1-2]} \lambda_n}{\lambda_{n,1} \log^{[q_1-2]} \lambda_{n,1}} \left(\sum_{j=1}^{p} \frac{u_j \lambda_{n,j} \log^{[q_j-2]} \lambda_{n,j}}{M\lambda_{n,1} \log^{[q_1-2]} \lambda_{n,1}}\right)^{-1} \\ &= \frac{MV}{u_1+u_2+\cdots+u_m},\end{aligned}$$

which implies that $\rho^{[q_1-1]}(G) = \infty$. This means that $G(s)$ has the growth index q_1.

Therefore, we complete the proof of Theorem 8. □

Proof of Theorem 9. From Theorem 8, and by Lemma 1, this yields $G(s) \in D$. By virtue of Theorem 2, we can obtain that

$$\chi_j \leq \liminf_{n\to\infty} \frac{\lambda_{n,j} \log^{[q_j-1]} \lambda_{n-1,j}}{-\log|a_{n,j}|}, \quad j=1,2,\ldots,p. \tag{27}$$

If there exists one $j \in \{1,2,\ldots,m\}$ such that $\chi_j = 0$, the conclusion of Theorem 9 holds. Hence, we only prove the case $0 < \chi_j < \infty$, $j=1,2,\ldots,p$. Due to (27), for any small number $\varepsilon > 0$ satisfying $0 < \varepsilon < \min\{\chi_1, \ldots, \chi_p\}$, there is a positive integer N such that $n > N$,

$$\frac{\lambda_{n,j} \log^{[q_j-1]} \lambda_{n-1,j}}{-\log|a_{n,j}|} > \chi_j - \varepsilon, \quad j = 1, 2, \ldots, p,$$

that is,

$$\frac{\lambda_{n,j} \log^{[q_j-1]} \lambda_{n-1,j}}{\chi_j - \varepsilon} > -\log|a_{n,j}|, \quad j = 1, 2, \ldots, p. \tag{28}$$

Thus, noting with the definition of $G(s)$, for $n > N$, we can obtain that

$$-\log|b_n| = -\sum_{j=1}^{p} u_j \log|a_{n,j}| < \sum_{j=1}^{p} \frac{u_j \lambda_{n,j} \log^{[q_j-1]} \lambda_{n-1,j}}{\chi_j - \varepsilon}. \tag{29}$$

Similar to the argument in the proof of Theorem 8, by combining this with the assumptions of Theorem 9, it follows from (23), (24) and (28) that

$$\chi = \chi^{[q_1]} \geq \liminf_{n\to\infty} \frac{\lambda_n \log^{[q_1-1]} \lambda_{n-1}}{\lambda_{n,1} \log^{[q_1-1]} \lambda_{n-1,1}} \left[\sum_{j=1}^{p} \frac{u_j \lambda_{n,j} \log^{[q_j-1]} \lambda_{n-1,j}}{(\chi_j - \varepsilon)\lambda_{n,1} \log^{[q_1-1]} \lambda_{n,1}}\right]^{-1}.$$

By combining this with the arbitrariness of ε, we have

$$\chi = \chi^{[q_1]} \geq V\left(\frac{u_1}{\chi_1} + \frac{u_2}{\chi_2} + \cdots + \frac{u_m}{\chi_m}\right)^{-1},$$

which implies that the lower q_1-order χ of $G(s)$ satisfies (8). By combining this with Theorem 8, we can obtain the conclusions of Theorem 9.

Therefore, we complete the proof of Theorem 9. □

Proof of Theorem 10. (i) In view of (4) in Theorem 8 and (8) in Theorem 9, it follows that Dirichlet–Hadamard–Kong product $G(s)$ has the growth index q_1 and the q_1-order ρ of $G(s)$ satisfies

$$\begin{aligned} V\left(\tfrac{u_1}{\chi_1} + \tfrac{u_2}{\chi_2} + \cdots + \tfrac{u_m}{\chi_m}\right)^{-1} &\leq \chi \leq \rho \\ &\leq V\left(\tfrac{u_1}{\rho_1} + \tfrac{u_2}{\rho_2} + \cdots + \tfrac{u_m}{\rho_m}\right)^{-1}. \end{aligned} \tag{30}$$

Since $f_j(s), (j = 1, \ldots, p)$ be of q_j-regular growth, that is, $\chi_j = \rho_j$, $j = 1, \ldots, p$, thus we have

$$V\left(\frac{u_1}{\chi_1} + \frac{u_2}{\chi_2} + \cdots + \frac{u_m}{\chi_m}\right)^{-1} = V\left(\frac{u_1}{\rho_1} + \frac{u_2}{\rho_2} + \cdots + \frac{u_m}{\rho_m}\right)^{-1},$$

This implies from (30) that

$$\chi = \rho = V\left(\frac{u_1}{\rho_1} + \frac{u_2}{\rho_2} + \cdots + \frac{u_m}{\rho_m}\right)^{-1}. \tag{31}$$

Therefore, this completes the proof of Theorem 10(i).

(ii) Since $f_j(s)$ is of q_j-order $\rho_j (0 < \rho_j < +\infty)$ and of q_j-type T_j, $j = 1, 2, \ldots, p$, in view of Theorem 1, then, for any small $\varepsilon > 0$, there exists a positive integer N, such that

$$|a_{n,j}|^{\frac{\rho_j}{\lambda_{n,j}}} \leq \frac{T_j + \varepsilon}{\log^{[q_j-2]}\left(\frac{\lambda_{n,j}}{e\rho_j}\right)}, \quad j = 1, 2, \ldots, p, \tag{32}$$

hold for $n > N$. Now, we will divide into two cases below.

Case (ii_1) $q_1 \geq 3$. In view of the fact that (4) holds for the positive integer $m(1 \leq m < p)$, it thus follows that $q_j \geq 3, j = 1, 2, \ldots, p$. By combining with $\lambda_{n,i} \sim \lambda_{n,j}$, $i, j = 1, 2 \ldots, p$, we have that

$$\log^{[q_j-2]}\left(\frac{\lambda_{n,i}}{K}\right) \sim \log^{[q_j-2]}\left(\frac{\lambda_{n,j}}{K}\right), \quad i, j = 1, 2 \ldots, p, \tag{33}$$

holds for any positive constant K. From Theorem 10(i), we have that $G(s)$ is of q_1-order ρ. Thus, in view of (1) and the definition of b_n, we can deduce that

$$\begin{aligned} T^{[q_1]}(G) = \;& \limsup_{n\to\infty} |b_n|^{\frac{\rho}{\lambda_n}} \log^{[q_1-2]}(\tfrac{\lambda_n}{e\rho}) \\ \leq \;& \limsup_{n\to\infty} \log^{[q_1-2]}(\tfrac{\lambda_n}{e\rho}) \left[\left(\frac{T_1+\varepsilon}{\log^{[q_1-2]}(\frac{\lambda_{n,1}}{e\rho_1})}\right)^{\frac{u_1\lambda_{n,1}}{\rho_1}} \cdots \left(\frac{T_m+\varepsilon}{\log^{[q_m-2]}(\frac{\lambda_{n,m}}{e\rho_m})}\right)^{\frac{u_m\lambda_{n,m}}{\rho_m}} \right. \\ & \left. \times \left(\frac{T_{m+1}+\varepsilon}{\log^{[q_{m+1}-2]}(\frac{\lambda_{n,m+1}}{e\rho_{m+1}})}\right)^{\frac{u_{m+1}\lambda_{n,m+1}}{\rho_{m+1}}} \cdots \left(\frac{T_p+\varepsilon}{\log^{[q_p-2]}(\frac{\lambda_{n,p}}{e\rho_p})}\right)^{\frac{u_p\lambda_{n,p}}{\rho_p}} \right]^{\frac{\rho}{\lambda_n}} \\ \leq \;& \limsup_{n\to+\infty} \log^{[q_1-2]}(\tfrac{\lambda_n}{e\rho}) \textstyle\prod_{j=1}^{p} (T_j+\varepsilon)^{\frac{\rho u_j \lambda_{n,j}}{\rho_j \lambda_n}} \left[\prod_{j=1}^{m} \left(\log^{[q_j-2]} \lambda_{n,j}\right)^{\frac{u_j\rho\lambda_{n,j}}{\lambda_n\rho_j}} \right]^{-1} \times \\ & \times \left[\textstyle\prod_{j=m+1}^{p} \left(\log^{[q_j-2]} \lambda_{n,j}\right)^{\frac{u_j\rho\lambda_{n,j}}{\lambda_n\rho_j}} \right]^{-1}. \end{aligned} \tag{34}$$

In view of (31) and $\lambda_{n,i} \sim \lambda_{n,j}$ for $i, j = 1, 2, \ldots, p$, we have that

$$\lim_{n\to\infty} \sum_{j=1}^{m} \frac{u_j\rho\lambda_{n,j}}{\lambda_n\rho_j} = 1, \qquad \prod_{j=m+1}^{p} \left(\log^{[q_j-2]} \lambda_{n,j}\right)^{\frac{u_j\rho}{V\rho_j}} \longrightarrow \infty, \quad n \to \infty. \tag{35}$$

Thus, we can deduce from (32)–(35) that

$$T^{[q_1]}(G) \leq \limsup_{n\to+\infty} \prod_{j=1}^{p} (T_j+\varepsilon)^{\frac{\rho u_j}{\rho_j V}} \left[\prod_{j=m+1}^{p} \left(\log^{[q_j-2]} \lambda_{n,j}\right)^{\frac{u_j\rho}{V\rho_j}} \right]^{-1}, \tag{36}$$

In view of the arbitrariness of ε, it follows that $T^{[q_1]}(G) \leq 0$, by combining with the fact that $T^{[q_1]}(G) \geq 0$, we have $T^{[q_1]}(G) = 0$.

Case (ii_2). $q_1 = 2$. In view of (4) and (30), for any small $\varepsilon > 0$, there is a positive integer N, such that

$$|a_{n,j}|^{\frac{\rho_j}{\lambda_{n,j}}} \leq \frac{e\rho_j(T_j+\varepsilon)}{\lambda_{n,j}}, \quad j = 1, 2, \ldots, m, \tag{37}$$

and

$$|a_{n,j}|^{\frac{\rho_j}{\lambda_{n,j}}} \leq \frac{T_j+\varepsilon}{\log^{[q_j-2]}\left(\frac{\lambda_{n,j}}{e\rho_j}\right)}, \quad j = m+1, \ldots, p, \tag{38}$$

hold for $n > N$. From Theorem 10(i), we have that $G(s)$ is of 2-order ρ. Similar to the argument in (35), we can see from (37) and (38) that

$$
\begin{aligned}
T^{[2]}(G) &= \limsup_{n\to\infty} |b_n|^{\frac{\rho}{\lambda_n}} (\frac{\lambda_n}{e\rho}) \\
&\le \limsup_{n\to\infty} (\frac{\lambda_n}{e\rho}) \left[\left(\frac{e\rho_1(T_1+\varepsilon)}{\lambda_{n,1}} \right)^{\frac{u_1\lambda_{n,1}}{\rho_1}} \cdots \left(\frac{e\rho_m(T_m+\varepsilon)}{\lambda_{n,m}} \right)^{\frac{u_m\lambda_{n,m}}{\rho_m}} \right. \\
&\quad \times \left. \left(\frac{T_{m+1}+\varepsilon}{\log^{[q_{m+1}-2]}(\frac{\lambda_{n,m+1}}{e\rho_{m+1}})} \right)^{\frac{u_{m+1}\lambda_{n,m+1}}{\rho_{m+1}}} \cdots \left(\frac{T_p+\varepsilon}{\log^{[q_p-2]}(\frac{\lambda_{n,p}}{e\rho_p})} \right)^{\frac{u_p\lambda_{n,p}}{\rho_p}} \right]^{\frac{\rho}{\lambda_n}} \\
&\le \limsup_{n\to+\infty} \frac{V}{\rho} \prod_{j=1}^{p} [\rho_j(T_j+\varepsilon)]^{\frac{\rho u_j}{V\rho_j}} \left[\prod_{j=m+1}^{p} \left(\log^{[q_j-2]} \lambda_{n,j} \right)^{\frac{\rho u_j}{V\rho_j}} \right]^{-1}.
\end{aligned}
$$

In view of the arbitrariness of ε, it follows that $T^{[2]}(G) \le 0$, by combining this with the fact that $T^{[2]}(G) \ge 0$, we can obtain $T^{[2]}(G) = 0$. In view of Case (ii_1) and Case (ii_2), we have $T^{[q_1]}(G) = 0$ for $q_1 = 2, 3, 4, \ldots$.

Therefore, this completes the conclusion of Theorem 10(ii).

(iii) Since $m = p$ and $q_1 = q_2 = \cdots = q_p = q$, it follows that $G(s)$ is of q-order ρ

$$
\chi = \rho = V\left(\frac{u_1}{\rho_1} + \frac{u_2}{\rho_2} + \cdots + \frac{u_p}{\rho_p} \right)^{-1}. \tag{39}
$$

If $q \ge 3, q \in N_+$, similar to the argument in Case (ii_1), we can deduce that

$$
\begin{aligned}
T = T^{[q]}(G) &= \limsup_{n\to\infty} |b_n|^{\frac{\rho}{\lambda_n}} \log^{[q-2]}(\frac{\lambda_n}{e\rho}) \\
&\le \limsup_{n\to\infty} \log^{[q-2]} \lambda_n \left[(\frac{T_1+\varepsilon}{\log^{[q-2]} \lambda_{n,1}})^{\frac{u_1\lambda_{n,1}}{\rho_1}} \cdots (\frac{T_p+\varepsilon}{\log^{[q-2]} \lambda_{n,p}})^{\frac{u_p\lambda_{n,p}}{\rho_p}} \right]^{\frac{\rho}{\lambda_n}} \\
&\le \limsup_{n\to+\infty} \log^{[q-2]} \lambda_n \prod_{j=1}^{p} (T_j+\varepsilon)^{\frac{\rho u_j}{V\rho_j}} \left[\prod_{j=1}^{p} \left(\log^{[q-2]} \lambda_{n,j} \right)^{\frac{\rho u_j}{V\rho_j}} \right]^{-1}.
\end{aligned}
$$

In view of the arbitrariness of ε, and by combining this with the equality (39), we have

$$
T \le T_1^{\frac{u_1\rho}{V\rho_1}} T_2^{\frac{u_2\rho}{V\rho_2}} \cdots T_p^{\frac{u_p\rho}{V\rho_p}}. \tag{40}
$$

If $q = 2$, similar to the argument in Case (ii_2), we can deduce that

$$
\begin{aligned}
T = &T^{[2]}(G) = \limsup_{n\to\infty} (\frac{\lambda_n}{e\rho}) |b_n|^{\frac{\rho}{\lambda_n}} \\
&\le \limsup_{n\to\infty} (\frac{\lambda_n}{e\rho}) \left[\left(\frac{e\rho_1(T_1+\varepsilon)}{\lambda_{n,1}} \right)^{\frac{u_1\lambda_{n,1}}{\rho_1}} \cdots \left(\frac{e\rho_p(T_p+\varepsilon)}{\lambda_{n,p}} \right)^{\frac{u_p\lambda_{n,p}}{\rho_p}} \right]^{\frac{\rho}{\lambda_n}} \\
&\le \frac{V}{\rho} \prod_{j=1}^{p} [\rho_j(T_j+\varepsilon)]^{\frac{\rho u_j}{V\rho_j}}.
\end{aligned}
$$

In view of the arbitrariness of ε, it follows that

$$
T \le \frac{V}{\rho} (\rho_1 T_1)^{\frac{u_1\rho}{V\rho_1}} (\rho_2 T_2)^{\frac{u_2\rho}{V\rho_2}} \cdots (\rho_p T_p)^{\frac{u_p\rho}{V\rho_p}}. \tag{41}
$$

Since $f_j(s), j = 1, \ldots, p$ are of perfectly q_j regular growth, by combining this with Theorems 3 and 10, we have $T_j = \tau_j$, $j = 1, 2, \ldots, p$. Thus, for any small $\varepsilon > 0$, there is a positive integer N, such that

$$|a_{n,j}|^{\frac{\rho_j}{\lambda_{n,j}}} \geq \frac{\tau_j - \varepsilon}{\log^{[q-2]}\left(\frac{\lambda_{n-1,j}}{e\rho_j}\right)}, \quad j = 1, 2, \ldots, p,$$

hold for $n > N$.

In view of the conclusion of Theorem 10(i), it follows that $G(s)$ is of q-regular growth. In view of (9) and (10), it follows that $\lambda_n \sim \lambda_{n-1}$ as $n \to \infty$. Assuming that $G(s)$ is of lower q-type τ, similar to the argument in the above, then we can deduce from Theorem 3 that

$$\begin{aligned}
\tau =& \tau^{[q]}(G) = \liminf_{n\to\infty} |b_n|^{\frac{\rho}{\lambda_n}} \log^{[q-2]}\left(\frac{\lambda_{n-1}}{e\rho}\right) \\
&\geq \liminf_{n\to\infty} \log^{[q-2]} \lambda_{n-1} \left[\left(\frac{\tau_1 - \varepsilon}{\log^{[q-2]} \lambda_{n-1,1}}\right)^{\frac{u_1\lambda_{n,1}}{\rho_1}} \cdots \left(\frac{\tau_p - \varepsilon}{\log^{[q-2]} \lambda_{n-1,p}}\right)^{\frac{u_p\lambda_{n,p}}{\rho_p}} \right]^{\frac{\rho}{\lambda_n}} \\
&\geq \liminf_{n\to+\infty} \log^{[q-2]} \lambda_{n-1} \prod_{j=1}^{p} (\tau_j - \varepsilon)^{\frac{\rho u_j}{\rho_j}} \left[\prod_{j=1}^{p} \left(\log^{[q-2]} \lambda_{n-1,j}\right)^{\frac{\rho u_j}{\rho_j}} \right]^{-1}
\end{aligned}$$

holds for $q \geq 3, q \in \mathbb{N}_+$. In view of (9) and (39), and by combining this with the arbitrariness of ε, we can see that

$$\tau \geq \tau_1^{\frac{u_1\rho}{\rho_1}} \tau_2^{\frac{u_2\rho}{\rho_2}} \cdots \tau_p^{\frac{u_p\rho}{\rho_p}} \tag{42}$$

holds for $q \geq 3, q \in \mathbb{N}_+$. Similarly, for $q = 2$, in view of (9) and (39), it follows that

$$\begin{aligned}
\tau =& \tau^{[2]}(G) = \liminf_{n\to\infty} \left(\frac{\lambda_{n-1}}{e\rho}\right) |b_n|^{\frac{\rho}{\lambda_n}} \\
&\geq \liminf_{n\to\infty} \left(\frac{\lambda_{n-1}}{e\rho}\right) \left[\left(\frac{e\rho_1(\tau_1 - \varepsilon)}{\lambda_{n-1,1}}\right)^{\frac{u_1\lambda_{n,1}}{\rho_1}} \cdots \left(\frac{e\rho_p(\tau_p - \varepsilon)}{\lambda_{n-1,p}}\right)^{\frac{u_p\lambda_{n,p}}{\rho_p}} \right]^{\frac{\rho}{\lambda_n}} \\
&\geq \frac{1}{\rho} \prod_{j=1}^{p} [\rho_j(\tau_j - \varepsilon)]^{\frac{\rho u_j}{\rho_j}}.
\end{aligned}$$

Based on the arbitrariness of ε, we can see that

$$\tau \geq \frac{1}{\rho} (\rho_1\tau_1)^{\frac{u_1\rho}{\rho_1}} (\rho_2\tau_2)^{\frac{u_2\rho}{\rho_2}} \cdots (\rho_p\tau_p)^{\frac{u_p\rho}{\rho_p}}, \quad q = 2. \tag{43}$$

By combining this with the fact that (10), $T \geq \tau$ and $T_j = \tau_j, j = 1, \ldots, p$, we can easily obtain the conclusions of Theorem 10(iii) from (40)–(43).

Therefore, we can complete the proof of Theorem 10. □

7. Proofs of Theorems about the Logarithmic Growth

In this section, we will provide details of the proof of Theorems 11–13, which are related to the growth in Dirichlet–Hadamard–Kong product function when Dirichlet series are of logarithmic growth.

Proof of Theorem 11. Since $f_j(s) \in D$, $j = 1, 2, \ldots, p$, we have $G(s) \in D$ by Lemma 1. Since $f_j(s)$ is of zero-order and logarithmic order ρ_j^l, $j = 1, \ldots, p$, we find that $G(s)$ is of zero-order from the conclusions of Theorem 8. Moreover, in view of Theorem 4, it follows that

$$\rho_j^l = 1 + \limsup_{n\to\infty} \frac{\log \lambda_{n,j}}{\log\left(-\frac{1}{\lambda_{n,j}} \log|a_{n,j}|\right)}, \quad j = 1, 2, \ldots, p. \tag{44}$$

Due to (44) and $\rho_j^l \geq 1$, $j = 1, \ldots, p$, for any small number $\varepsilon > 0$, there are p positive integers $N_j \in \mathbb{N}_+$, $j = 1, 2, \ldots, p$ such that $n > N = \max\{N_1, N_2, \ldots, N_p\}$,

$$\log|a_{n,j}| < -(\lambda_{n,j})^{\frac{\rho_j^l+\varepsilon}{\rho_j^l-1+\varepsilon}}, \quad j = 1, \ldots, p. \tag{45}$$

Without losing generality, we can assume that there exists a positive integer m_1, such that $1 \leq m_1 \leq p$ and

$$\rho_1^l = \rho_2^l = \cdots = \rho_{m_1}^l < \rho_{m_1+1}^l \leq \cdots \leq \rho_p^l. \tag{46}$$

Thus, it follows that

$$\frac{\rho_j^l + \varepsilon}{\rho_j^l - 1 + \varepsilon} < \frac{\rho_1^l + \varepsilon}{\rho_1^l - 1 + \varepsilon}, \quad j = m_1 + 1, \ldots, p. \tag{47}$$

In view of (45) and the fact that $b_n = \prod_{j=1}^p (a_{n,j})^{u_j}$, it follows that

$$\begin{aligned}
\frac{\log\lambda_n}{\log(-\frac{1}{\lambda_n}\log|b_n|)} &= \frac{\log\lambda_n}{\log(-\frac{1}{\lambda_n}\sum_{j=1}^p u_j \log|a_{n,j}|)} \\
&\leq \frac{\log\lambda_n}{\log(\frac{1}{\lambda_n}\sum_{j=1}^p u_j (\lambda_{n,j})^{(\rho_j^l+\varepsilon)/(\rho_j^l-1+\varepsilon)})} \\
&< \log\lambda_n \left\{ \log\left(\frac{(\lambda_{n,1})^{\frac{\rho_1^l+\varepsilon}{\rho_1^l-1+\varepsilon}}}{\lambda_n} \left[\sum_{j=1}^p \frac{u_j(\lambda_{n,j})^{(\rho_j^l+\varepsilon)/(\rho_j^l-1+\varepsilon)}}{(\lambda_{n,1})^{(\rho_1^l+\varepsilon)/(\rho_1^l-1+\varepsilon)}} \right] \right) \right\}^{-1}.
\end{aligned} \tag{48}$$

Based on the condition $\lambda_{n,i} \sim \lambda_{n,j}$ as $n \to \infty$, and combining with (47), we have $\log\lambda_{n,i} \sim \log\lambda_{n,j} \sim \log\lambda_n$ as $n \to \infty$, $i, j = 1, \ldots, p$ and

$$\lim_{n\to\infty} \frac{(\lambda_{n,j})^{(\rho_j^l+\varepsilon)/(\rho_j^l-1+\varepsilon)}}{(\lambda_{n,1})^{(\rho_1^l+\varepsilon)/(\rho_1^l-1+\varepsilon)}} = 0, \quad j = m_1 + 1, \ldots, p. \tag{49}$$

Thus, by applying Theorem 4, we can deduce from (48) and (49) that

$$\begin{aligned}
\rho^l &= \rho^l(G) = 1 + \limsup_{n\to\infty} \frac{\log\lambda_n}{\log(-\frac{1}{\lambda_n}\log|b_n|)} \\
&\leq 1 + \limsup_{n\to\infty} \frac{\log\lambda_n}{\frac{1}{\rho_1^l-1+\varepsilon}\log\lambda_n + \log\sum_{j=1}^{m_1} u_j} \\
&= \rho_1^l + \varepsilon,
\end{aligned} \tag{50}$$

In view of the arbitrariness of ε, we can obtain the conclusion of Theorem 11 from (50). Therefore, we complete the proof of Theorem 11. $\square$

Proof of Theorem 12. Similar to the argument in the proof of Theorem 9, and combining this with the conclusion of Theorem 5, one can easily prove Theorem 12. $\square$

Proof of Theorem 13. (i) From the assumptions of Theorem 13, by combining with the conclusions of Theorems 11 and 12, we find that $G(s)$ is of zero-order and the (lower) logarithmic order $\rho^l(\chi^l)$ satisfy

$$\chi^l_{min} =: \min\{\chi^l_1, \chi^l_2, \ldots, \chi^l_p\} \le \chi^l \le \rho^l \le \rho^l_{min} =: \min\{\rho^l_1, \rho^l_2, \ldots, \rho^l_p\}. \tag{51}$$

Since $f_j(s), j = 1, \ldots, p$ are of logarithmic regular growth, that is, $\rho^l_j = \chi^l_j$, $j = 1, \ldots, p$, it follows from (51) that

$$\chi^l = \rho^l = \rho^l_{min} =: \min\{\rho^l_1, \rho^l_2, \ldots, \rho^l_p\}.$$

Therefore, this completes the proof of Theorem 13(i).

(ii) Since $f_j(s)$ is of zero-order and logarithmic order $\rho^l_j(1 \le \rho_j < +\infty)$ and of logarithmic type T^l_j, $j = 1, 2, \ldots, p$, in view of Theorem 6, for any small $\varepsilon > 0$, there is a positive integer N, such that

$$\begin{aligned} \log|a_{n,j}| &\le -\frac{\rho^l_j - 1}{\rho^l_j} \lambda_{n,j} \left(\frac{\lambda_{n,j}}{\rho^l_j (T^l_j + \varepsilon)}\right)^{\frac{1}{\rho^l_j - 1}} \\ &\le -(\rho^l_j - 1)\left[(\rho^l_j)^{\rho^l_j}(T^l_j + \varepsilon)\right]^{-\frac{1}{\rho^l_j - 1}} (\lambda_{n,j})^{\frac{\rho^l_j}{\rho^l_j - 1}} \end{aligned} \tag{52}$$

hold for $n > N$ and $j = 1, 2, \ldots, p$. From the conclusion of Theorem 13(i), it follows that $G(s)$ is of logarithmic order ρ^l. By applying Theorem 6, we have

$$\begin{aligned} \frac{\lambda_n}{\rho^l \left[-\frac{\rho^l}{\rho^l - 1} \frac{1}{\lambda_n} \log|b_n|\right]^{\rho^l - 1}} &= \frac{\lambda_n}{\rho^l}\left[-\frac{\rho^l}{\lambda_n(\rho^l - 1)} \sum_{j=1}^p u_1 \log|a_{n,j}|\right]^{-(\rho^l - 1)} \\ &\le \frac{\lambda_n}{\rho^l}\left[\frac{\rho^l}{\lambda_n(\rho^l - 1)} \sum_{j=1}^p \frac{u_j(\rho^l_j - 1)}{[(\rho^l_j)^{\rho^l_j}(T^l_j + \varepsilon)]^{(\rho^l_j - 1)^{-1}}} (\lambda_{n,j})^{\frac{\rho^l_j}{\rho^l_j - 1}}\right]^{-(\rho^l - 1)} \\ &\le \frac{\lambda_n}{\rho^l}\left[\frac{\rho^l (\lambda_{n,j})^{\frac{\rho^l_1}{\rho^l_1 - 1}}}{\lambda_n(\rho^l - 1)} \Psi(n)\right]^{-(\rho^l - 1)}, \end{aligned} \tag{53}$$

where

$$\Psi(n) = \sum_{j=1}^p \frac{u_j(\rho^l_j - 1)}{[(\rho^l_j)^{\rho^l_j}(T^l_j + \varepsilon)]^{(\rho^l_j - 1)^{-1}}} (\lambda_{n,j})^{\frac{\rho^l_j}{\rho^l_j - 1} - \frac{\rho^l_1}{\rho^l_1 - 1}}.$$

In view of the conclusion of Theorem 13(i), it follows that $\rho^l = \rho^l_1$ and

$$\frac{\rho^l_j}{\rho^l_j - 1} - \frac{\rho^l_1}{\rho^l_1 - 1} \le 0 \tag{54}$$

hold for $j = k + 1, \ldots, p$. Thus, we obtain

$$\lim_{n\to\infty} \Psi(n) = (\rho^l)^{-\frac{\rho^l}{\rho^l - 1}} (\rho^l - 1) \sum_{j=1}^k \frac{u_j}{(T^l_j + \varepsilon)^{(\rho^l - 1)^{-1}}}. \tag{55}$$

Due to (53), (55) and Theorem 6, and by combining this with the arbitrariness of ε, we can deduce that

$$\begin{aligned} T^l = T^l(G) &\leq \limsup_{n\to\infty} \frac{\lambda_n}{\rho^l\left[-\frac{\rho^l}{\rho^l-1}\frac{1}{\lambda_n}\log|b_n|\right]^{\rho^l-1}} \\ &\leq V\rho^l\left[\sum_{j=1}^{k}\frac{u_j}{(T_j^l)^{(\rho^l-1)^{-1}}}\right]^{1-\rho^l} \\ &= V\rho^l\left[\frac{u_1}{(T_1^l)^{(\rho^l-1)^{-1}}}+\frac{u_2}{(T_2^l)^{(\rho^l-1)^{-1}}}+\cdots+\frac{u_k}{(T_k^l)^{(\rho^l-1)^{-1}}}\right]^{1-\rho^l}. \end{aligned} \tag{56}$$

Since $f_j(s), j = 1,\ldots,p$ are of perfectly logarithmic regular growth, by combining this with the conclusion of Theorem 6 and the assumptions of Theorem 13, we have $T_j^l = \tau_j^l$, $j = 1,2,\ldots,p$. Thus, for any small $\varepsilon > 0$, there is a positive integer N, such that

$$\log|a_{n,j}| \geq -(\rho_j^l-1)\left[(\rho_j^l)^{\rho_j^l}(\tau_j^l-\varepsilon)\right]^{-\frac{1}{\rho_j^l-1}}(\lambda_{n,j})^{\frac{\rho_j^l}{\rho_j^l-1}} \tag{57}$$

hold for $n > N$.

In view of the conclusion of Theorem 13(i), it follows that $G(s)$ is of logarithmic regular growth. In view of (9) and (10), it follows that $\lambda_n \sim \lambda_{n-1}$ as $n \to \infty$. Assume that $G(s)$ is of lower logarithmic type τ^l, similarly to the argument in the above, then we can deduce that

$$\begin{aligned} \frac{\lambda_{n-1}}{\rho^l\left[-\frac{\rho^l}{\rho^l-1}\frac{1}{\lambda_n}\log|b_n|\right]^{\rho^l-1}} &= \frac{\lambda_{n-1}}{\rho^l}\left[-\frac{\rho^l}{\lambda_n(\rho^l-1)}\sum_{j=1}^{p}u_1\log|a_{n,j}|\right]^{-(\rho^l-1)} \\ &\geq \frac{\lambda_{n-1}}{\rho^l}\left[\frac{\rho^l(\lambda_{n,j})^{\rho_1^l/(\rho_1^l-1)}}{\lambda_n(\rho^l-1)}\Phi(n)\right]^{-(\rho^l-1)}, \end{aligned} \tag{58}$$

where

$$\begin{aligned} \lim_{n\to\infty}\Phi(n) &= \lim_{n\to\infty}\sum_{j=1}^{p}\frac{u_j(\rho_j^l-1)}{[(\rho_j^l)^{\rho_j^l}(\tau_j^l-\varepsilon)]^{(\rho_j^l-1)^{-1}}}(\lambda_{n,j})^{\frac{\rho_j^l}{\rho_j^l-1}-\frac{\rho_1^l}{\rho_1^l-1}} \\ &= \sum_{j=1}^{k}\frac{u_j(\rho_j^l-1)}{[(\rho_j^l)^{\rho_j^l}(\tau_j^l-\varepsilon)]^{(\rho_j^l-1)^{-1}}}. \end{aligned} \tag{59}$$

Thus, in view of Theorem 13 and (58) and (59), and by combining with the arbitrariness of ε, we have

$$\begin{aligned} \tau^l = \tau^l(G) &= \liminf_{n\to\infty}\frac{\lambda_{n-1}}{\rho^l\left[-\frac{\rho^l}{\rho^l-1}\frac{1}{\lambda_n}\log|b_n|\right]^{\rho^l-1}} \\ &\geq \left[\sum_{j=1}^{k}\frac{u_j}{(\tau_j^l)^{(\rho^l-1)^{-1}}}\right]^{1-\rho^l} \\ &= \left[\frac{u_1}{(\tau_1^l)^{(\rho^l-1)^{-1}}}+\frac{u_2}{(\tau_2^l)^{(\rho^l-1)^{-1}}}+\cdots+\frac{u_k}{(\tau_k^l)^{(\rho^l-1)^{-1}}}\right]^{1-\rho^l}. \end{aligned} \tag{60}$$

And since $T_j^l = \tau_j^l$, $j = 1,2,\ldots,p$ and $T^l \geq \tau^l$, from (10), (56) and (60), we obtain that $G(s)$ is of perfectly logarithmic regular growth and

$$\tau^l = T^l = \left[\frac{u_1}{(T_1^l)^{(\rho^l-1)^{-1}}}+\frac{u_2}{(T_2^l)^{(\rho^l-1)^{-1}}}+\cdots+\frac{u_k}{(T_k^l)^{(\rho^l-1)^{-1}}}\right]^{1-\rho^l}.$$

This completes the proof of Theorem 13. □

8. Proofs of Theorems about the Mixed Case

In this section, we will provide details of proof of Theorems 14 and 15 regarding the growth in Dirichlet–Hadamard–Kong product function under the mixed case that one Dirichlet series is of logarithmic growth and the other is of a finite order.

Proof of Theorem 14. Firstly, we only prove the case $\rho_1^l > 1$ and $\rho_2 > 0$. For the case $\rho_1^l = 1$ or $\rho_2 = 0$, using the same argument, one can easily obtain the conclusions. Since $f_j(s) = \sum_{n=1}^{\infty} a_{n,j} e^{\lambda_{n,j} s}$, $j = 1, 2 \in D$ satisfy $\lambda_{n,1} \sim \lambda_{n,2}$ as $n \to \infty$, it follows that $G(s) \in D$. Since $f_1(s)$ is of zero-order and the logarithmic order ρ_1^l, and $f_2(s)$ is of order ρ_2, for any small $\varepsilon > 0$, there is a positive integer N, such that for $n > N$,

$$\log|a_{n,1}| < -(\lambda_{n,1})^{\frac{\rho_1^l+\varepsilon}{\rho_1^l-1+\varepsilon}}, \quad \log|a_{n,2}| < -\frac{\lambda_{n,2}\log^+\lambda_{n,2}}{\rho_2+\varepsilon}. \tag{61}$$

Thus, we can deduce from (61) that

$$\begin{aligned} \frac{\lambda_n \log\lambda_n}{-\log|b_n|} &= \frac{\lambda_n\log\lambda_n}{-u_1\log|a_{n,1}| - u_2\log|a_{n,2}|} \\ &< \frac{\lambda_n\log\lambda_n}{u_1(\lambda_{n,1})^{(\rho_1^l+\varepsilon)/(\rho_1^l-1+\varepsilon)} + \frac{u_2\lambda_{n,2}\log\lambda_{n,2}}{\rho_2+\varepsilon}} \end{aligned} \tag{62}$$

and

$$\begin{aligned} \frac{\log\lambda_n}{\log(-\frac{1}{\lambda_n}\log|b_n|)} &= \frac{\log\lambda_n}{\log[-\frac{1}{\lambda_n}(u_1\log|a_{n,1}| + u_2\log|a_{n,2}|)]} \\ &< \frac{\log\lambda_n}{\log[\frac{1}{\lambda_n}(u_1(\lambda_{n,1})^{(\rho_1^l+\varepsilon)/(\rho_1^l-1+\varepsilon)} + u_2\frac{\lambda_{n,2}\log^+\lambda_{n,2}}{\rho_2+\varepsilon})]}. \end{aligned} \tag{63}$$

By combining with $\lambda_{n,1} \sim \lambda_{n,2}$ as $n \to \infty$, we have $\log\lambda_{n,1} \sim \log\lambda_{n,2} \sim \log\lambda_n$ as $n \to \infty$. Applying this for (62) and (63), we obtain

$$\begin{aligned} \rho(G) &= \limsup_{n\to\infty} \frac{\lambda_n\log\lambda_n}{-\log|b_n|} \\ &\le \limsup_{n\to\infty} \frac{(\lambda_n\log\lambda_n)/(\lambda_{n,1}\log\lambda_{n,1})}{u_1(\lambda_{n,1})^{(\rho_1^l-1+\varepsilon)^{-1}}/\log\lambda_{n,1} + \frac{u_2\lambda_{n,2}\log\lambda_{n,2}}{(\rho_2+\varepsilon)\lambda_{n,1}\log\lambda_{n,1}}} \to 0, \end{aligned} \tag{64}$$

and

$$\begin{aligned} \rho^l = \rho^l(G) &= 1 + \limsup_{n\to\infty} \frac{\log\lambda_n}{\log(-\frac{1}{\lambda_n}\log|b_n|)} \\ &\le 1 + \limsup_{n\to\infty} \frac{\log\lambda_n}{\log[(\lambda_{n,1})^{(\rho_1^l+\varepsilon)/(\rho_1^l-1+\varepsilon)}(\lambda_n)^{-1}K]} \\ &\le \rho_1^l + \varepsilon, \end{aligned} \tag{65}$$

since

$$\lim_{n\to\infty}(\lambda_{n,1})^{(\rho_1^l-1+\varepsilon)^{-1}}/\log\lambda_{n,1} = \infty,$$

and

$$\lim_{n\to\infty} K = \lim_{n\to\infty}\left(u_1 + \frac{u_2}{\rho_2+\varepsilon} \times \frac{\lambda_{n,2}\log\lambda_{n,2}}{(\lambda_{n,1})^{(\rho_1^l+\varepsilon)/(\rho_1^l-1+\varepsilon)}}\right) = u_1.$$

In view of (64) and (65), and by combining this with the arbitrariness of ε, we find that $\rho(G) = 0$ and $\rho^l(G) \le \rho_1^l$.

Furthermore, if $f_1(s)$ is of lower logarithmic order χ_1^l, and $f_2(s)$ is of lower order χ_2, we can only prove the conclusions for the case $\chi_1^l > 1$ and $\chi_2 > 0$. By using the same argument, one can easily prove the same conclusion. In view of the assumptions of Theorem 14 and the conclusions of Theorems 2 and 5, for any small $\varepsilon(0 < \varepsilon < \min\{\chi_1^l - 1, \chi_2\}$, there is a positive integer N, such that, for $n > N$,

$$\log|a_{n,1}| > -\lambda_{n,1}(\lambda_{n-1,1})^{\frac{1}{\chi_1^l - 1 - \varepsilon}}, \quad \log|a_{n,2}| > -\frac{\lambda_{n,2}\log^+ \lambda_{n-1,2}}{\chi_2 - \varepsilon}. \tag{66}$$

In view of (66), by using the same argument as in the above, we have

$$\begin{aligned}
\chi^l = \chi^l(G) &= 1 + \liminf_{n\to\infty} \frac{\log \lambda_{n-1}}{\log(-\frac{1}{\lambda_n}\log|b_n|)} \\
&\geq 1 + \liminf_{n\to\infty} \frac{\log \lambda_{n-1}}{\log\Big((\lambda_n)^{-1}[u_1\lambda_{n,1}(\lambda_{n-1,1})^{1/(\chi_1^l - 1 - \varepsilon)} + \frac{u_2\lambda_{n,2}\log\lambda_{n-1,2}}{\chi_2 - \varepsilon}]\Big)} \\
&\geq \chi_1^l - \varepsilon,
\end{aligned}$$

by combining this with the arbitrariness of ε, we have that $\chi^l \geq \chi_1^l$.

Therefore, this completes the proof of Theorem 14. □

Proof of Theorem 15. (i) From the conclusions of Theorem 14, we find that $G(s)$ is of zero order and the (lower) logarithmic order $\rho^l(\chi^l)$ satisfies

$$\chi_1^l \leq \chi^l \leq \rho^l \leq \rho_1^l. \tag{67}$$

Thus, by combining this with the condition that f_1 is of logarithmic regular growth, it follows from (67) that

$$\chi^l = \rho^l = \rho_1^l,$$

which means that $G(s)$ is also of logarithmic regular growth.

This completes the proof of Theorem 15(i).

(ii) Since ρ_1^l, ρ_2 satisfy $1 < \rho_1^l < +\infty, 0 < \rho_2 < +\infty$, in view of Theorems 1 and 6, for any small $\varepsilon(0 < \varepsilon < \min\{\chi_1^l - 1, \chi_2\})$, there is a positive integer N, such that, for $n > N$,

$$\log|a_{n,1}| \leq -(\lambda_{n,1})^{\frac{\rho_1^l}{\rho_1^l - 1}} \frac{\rho_1^l - 1}{\rho_1^l}[\rho_1^l(T_1^l + \varepsilon)]^{-\frac{1}{\rho_1^l - 1}}, \ \log|a_{n,2}| \leq -\frac{\lambda_{n,2}}{\rho_2}\log\frac{\lambda_{n,2}}{e\rho_2(T_2 + \varepsilon)}. \tag{68}$$

Since $G(s)$ is of logarithmic regular growth, we have

$$\begin{aligned}
\frac{\lambda_n}{\rho^l\Big[-\frac{\rho^l}{\rho^l - 1}\frac{1}{\lambda_n}\log|b_n|\Big]^{\rho^l - 1}} &= \frac{\lambda_n}{\rho^l\Big[-\frac{\rho^l}{\rho^l - 1}\frac{1}{\lambda_n}(u_1\log|a_{n,1}| + u_2\log|a_{n,2}|)\Big]^{\rho^l - 1}} \\
&\leq \frac{\lambda_n}{\rho^l\Big[\Upsilon_1(n)\frac{\rho^l}{\rho^l - 1}\frac{1}{\lambda_n}(\lambda_{n,1})^{\rho_1^l/(\rho_1^l - 1)}\Big]^{\rho^l - 1}}
\end{aligned} \tag{69}$$

where

$$\Upsilon_1(n) = u_1\frac{\rho_1^l - 1}{\rho_1^l}[\rho_1^l(T_1^l + \varepsilon)]^{-\frac{1}{\rho_1^l - 1}} + u_2\frac{\lambda_{n,2}}{\rho_2}\log\frac{\lambda_{n,2}}{e\rho_2(T_2 + \varepsilon)}(\lambda_{n,1})^{-\rho_1^l/(\rho_1^l - 1)}.$$

In view of $\rho_1^l > 1$ and $\lambda_{n,1} \sim \lambda_{n,2}$ as $n \to \infty$, it follows

$$\lim_{n\to\infty}\Upsilon_1(n) = u_1\frac{\rho_1^l - 1}{\rho_1^l}[\rho_1^l(T_1^l + \varepsilon)]^{-\frac{1}{\rho_1^l - 1}}. \tag{70}$$

In view of (69) and (70), and combining this with $\rho^l = \rho_1^l$, $\lambda_n = v_1\lambda_{n,1} + v_2\lambda_{n,2}$ and the arbitrariness of ε, we can deduce that

$$T^l = \limsup_{n\to\infty} \frac{\lambda_n}{\rho^l\left[-\frac{\rho^l}{\rho^l-1}\frac{1}{\lambda_n}\log|b_n|\right]^{\rho^l-1}} \le \frac{(v_1+v_2)^{\rho^l}}{(u_1)^{\rho^l-1}}T_1^l. \tag{71}$$

On the other hand, from the assumptions of Theorem 15, we know that f_1 satisfies the conditions of Theorem 6 and f_2 satisfies the conditions of Theorem 3. Thus, for any small $\varepsilon(0<\varepsilon<\min\{\tau_1^l,\tau_2\})$, there is a positive integer N, such that, for $n>N$,

$$\log|a_{n,1}| \ge -\lambda_{n,1}\frac{\rho_1^l-1}{\rho_1^l}\left(\frac{\lambda_{n-1,1}}{\rho_1^l(\tau_1^l-\varepsilon)}\right)^{(\rho_1^l-1)^{-1}}, \quad \log|a_{n,2}| \ge -\frac{\lambda_{n,2}}{\rho_2}\log\frac{\lambda_{n-1,2}}{e\rho_2(\tau_2-\varepsilon)}.$$

Since $G(s)$ is of logarithmic regular growth, we have

$$\frac{\lambda_{n-1}}{\rho^l\left[-\frac{\rho^l}{\rho^l-1}\frac{1}{\lambda_n}\log|b_n|\right]^{\rho^l-1}} > \frac{\lambda_{n-1}}{\rho^l\left[\Upsilon_2(n)\frac{\rho^l}{\rho^l-1}\frac{\lambda_{n,1}}{\lambda_n}(\lambda_{n-1,1})^{1/(\rho_1^l-1)}\right]^{\rho^l-1}} \tag{72}$$

where

$$\Upsilon_2(n) = u_1\frac{\rho_1^l-1}{\rho_1^l}[\rho_1^l(\tau_1^l-\varepsilon)]^{-\frac{1}{\rho_1^l-1}} + u_2(\lambda_{n-1,1})^{-1/(\rho_1^l-1)}\frac{\lambda_{n,2}}{\rho_2\lambda_{n,1}}\log\frac{\lambda_{n-1,2}}{e\rho_2(\tau_2-\varepsilon)}.$$

In view of $\rho_1^l>1$ and $\lambda_{n,1}\sim\lambda_{n,2}$ as $n\to\infty$, it follows

$$\lim_{n\to\infty}\Upsilon_2(n) = u_1\frac{\rho_1^l-1}{\rho_1^l}[\rho_1^l(\tau_1^l-\varepsilon)]^{-\frac{1}{\rho_1^l-1}}. \tag{73}$$

Due to $v_1+v_2=1$ and $\lambda_{n-1,j}\sim\lambda_{n,j}$, $j=1,2$, as $n\to\infty$, and combining with $\rho^l=\rho_1^l$, $\lambda_n = v_1\lambda_{n,1}+v_2\lambda_{n,2}$ and the arbitrariness of ε, it follows from (72) and (73) that

$$\tau^l = \tau^l(G) = \liminf_{n\to\infty}\frac{\lambda_{n-1}}{\rho^l\left[-\frac{\rho^l}{\rho^l-1}\frac{1}{\lambda_n}\log|b_n|\right]^{\rho^l-1}} \ge \frac{\tau_1^l}{(u_1)^{\rho^l-1}}.$$

In view of $T_1^l = \tau_1^l$, and combining this with the fact $T^l(G)\ge\tau^l(G)$, we have $\tau^l = T^l = \frac{T_1^l}{(u_1)^{\rho^l-1}}$.

Therefore, we complete the proof of Theorem 15. □

9. Conclusions

In this paper, our main aims are to supplement and improve the article by Kong [40] on entire functions represented by the Hadamard product of Dirichlet series in three ways. Firstly, the condition that $\limsup_{n\to\infty}\frac{\log n}{\lambda_n} = E < +\infty$ is more relaxed than $\limsup_{n\to\infty}\frac{\log n}{\lambda_n} = 0$ given by Kong [40]. Secondly, the form of the Dirichlet–Hadamard product in Definition 4 is more general than the form in Definition 3, since the form in Definition 3 is a special case of $p=2$ and $v_1=v_2=\frac{1}{2}$ in Definition 4. Thirdly, the results of this article are more abundant, including the Dirichlet–Hadamard–Kong product of some Dirichlet series, which have different growth indexes (see Theorems 8–10), logarithmic growths (see Theorems 11–13), or the mixed case of logarithmic growth and finite growth (see Theorems 14 and 15).

In view of Theorems 8–15 and Examples 1–4, some demonstrate that the growth in the Dirichlet–Hadamard–Kong product series may be determined by the Dirichlet series with smaller growth (see Theorems 8, 9, 11, 12 and 14), and the others show that the growth in

Dirichlet–Hadamard–Kong product series could be algebraic expressions of the growth indexes of some Dirichlet series (see Theorems 10, 13 and 15).

By observing the results in this paper, we can see that these conclusions hold under the condition that $\lambda_{n,i} \sim \lambda_{n,j}$ and Dirichlet series $f_j(s)$ converge on the whole plane; that is, $f_j(s) \in D$ for $i, j = 1, 2, \ldots, p$. In fact, many Dirichlet series convergent at the half complex plane, such as $f(s) = \sum_{n=1}^{\infty} (\log n) e^{ns}$, $f(s) = \sum_{n=1}^{\infty} n e^{ns}, \ldots$ Thus, the following questions can be raised:

Question 4. *What would happen to the growth in the Hadamard–Kong product series of the Dirichlet series when some of them converge in the whole plane and the others converge at the half-complex plane, or all series converge at the half-complex plane?*

Question 5. *What can be said regatding the properties of the Hadamard–Kong product series of the Dirichlet series if the exponents $\lambda_{n,j}$ have other relationships, such as: (i) $\lambda_{n,1} = \xi_j \lambda_{n,j}$, (ii) $\log \lambda_{n,1} = \xi_j \log \lambda_{n,j}, \ldots$, where $\xi_j > 0$, $j = 2, 3, \ldots, p$?*

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