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Impulsive Memristive Cohen–Grossberg Neural Networks Modeled by Short Term Generalized Proportional Caputo Fractional Derivative and Synchronization Analysis

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Abstract: The synchronization problem for impulsive fractional-order Cohen–Grossberg neural networks with generalized proportional Caputo fractional derivatives with changeable lower limit at any point of impulse is studied. We consider the cases when the control input is acting continuously as well as when it is acting instantaneously at the impulsive times. We defined the global Mittag–Leffler synchronization as a generalization of exponential synchronization. We obtained some sufficient conditions for Mittag–Leffler synchronization. Our results are illustrated with examples.

Keywords: generalized proportional Caputo fractional derivatives; impulses; Cohen–Grossberg neural networks; Mittag–Leffler synchronization

MSC: 34A08; 34K37

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1. Introduction

Recently, differential equations with various types of fractional derivatives have been widely studied because of their applications in various areas of science and engineering (see, for example, ref. [1] (star clusters), ref. [2](viscoelasticity), ref. [3] (optics), ref. [4] (dynamics of a free particle)). In the literature there are various types of fractional derivatives with different properties. The main common property of fractional derivatives is connected with the memory which differs from integer-order derivatives (see, for example, ref. [5] and the cited therein references). Recently [6,7] generalized proportional integrals and derivatives were introduced and applied to differential equations (see, for example, refs. [8,9]). These integrals and derivatives generalize the classical Riemann–Liouville and Caputo fractional integrals and derivatives. At the same time, to describe more adequate dynamics of processes with sudden, discontinuous jumps, impulses are involved in fractional differential equations (see, for example, refs. [10–12]).

In the past decades, complex networks have been intensively studied. Synchronization has always been a hot research topic in complex systems. The exponential synchronization for various types of neural networks with ordinary derivatives are studied, for example, in [13] (for inertial Cohen–Grossberg delayed neural networks), in [14] (for chaotic delayed neural networks with impulsive effects), in [15] (for Cohen–Grossberg neural networks with mixed time-delays), in [16] (for Cohen–Grossberg neural networks with impulse controller).

Recently, there were some kinds of synchronization have been investigated for neural networks with various type of fractional derivatives (see, for example, refs. [17–19]). The Cohen–Grossberg neural network is one of the most typical and popular neural network models because it contains some well-known neural networks such as recurrent neural networks, cellular neural networks, and Hopfield neural networks as a special case.

The Cohen–Grossberg neural network models have been widely applied within various engineering and scientific fields such as optimization problems, system control, signal processing, associative memory, pattern recognition, and new class of artificial neural networks. In order to synchronize nonlinear dynamical systems, impulsive control strategy, as an important control means, has been widely concerned. However, to the best of the authors' knowledge, there are only a few corresponding results on the impulsive generalizations of fractional Cohen–Grossberg neural networks reported in the existing literature. Very recently, in [20] an example of an impulsive control for the exponential synchronization of Cohen–Grossberg neural networks with Caputo fractional derivative was presented. In the paper [21] a class of impulsive control memristive Cohen - Grossberg neural networks with state feedback and Caputo fractional derivative is introduced, and a synchronization analysis is studied. In both papers the Caputo fractional order derivative with a fixed lower limit at the initial time is applied.

Note that in the application of impulses to the fractional differential equation it is very important to take care of the connection between the lower limit of the fractional derivative and the impulses. There are two basic types of impulsive fractional differential Equations [22]:

- Fixed lower limit of the fractional derivative at the initial time point;
- Changeable lower limit of the fractional derivative at any impulsive time.

Note both types of impulsive fractional differential equations have different properties and different methods for investigation and we are not able to mix these two types of fractional impulsive equations. For example, when Caputo fractional derivative with fixed lower limit at the initial time is applied and we consider the following scalar impulsive fractional differential equation

$$ {}_{t_0}^{C}D^{\alpha}u(t) = \lambda u(t), \ \ t \in (t_k, t_{k+1}], \ k \in \mathbb{Z}_0, \quad u(t_k+0) = b_k u(t_k), \ \ k \in \mathbb{Z}_+, \tag{1} $$

where λ, b_k are constants and the points $t_k: \ t_{k-1} < t_k < t_{k+1}, \ k \in Z_+$ are initially given.

Then on $(t_0, t_1]$ the solution $u(t) = u(t_0)E_\alpha(\lambda(t-t_0)^\alpha)$. However, on $(t_1, t_2]$ the solution is not given by $u(t) = u(t_1+)E_\alpha(\lambda(t-t_1)^\alpha$ (similar is the situation with other types of fractional derivatives, such as see, for example, the proof of Theorem 2 [20], Equation (28) [21]).

At the same time, if we consider Caputo fractional derivative with changeable lower limit and we consider the following scalar impulsive fractional differential equation

$$ {}_{t_k}^{C}D^{\alpha}u(t) = \lambda u(t), \ \ t \in (t_k, t_{k+1}], \ k \in \mathbb{Z}_0, \quad u(t_k+0) = b_k u(t_k), \ \ k \in \mathbb{Z}_+, \tag{2} $$

then on $(t_1, t_2]$ the solution is given by $u(t) = u(t_1+)E_\alpha(\lambda(t-t_1)^\alpha)$.

In connection with the above-given discussions, we will study Cohen–Grossberg neural networks modeled by differential equations with generalized proportional Caputo fractional derivatives and impulses at initially given time points. We define Mittag-Lefller stabilization. We will apply the quadratic Lyapunov functions to obtain conditions for Mittag–Leffler stabilization which is a fractional generalization of exponential stabilization (see Remark 5).

In studying the stability or stabilization of differential equations one of the most useful methods is the Lyapunov method. The most applied functions are absolute values functions and quadratic functions. When the absolute value Lyapunov function is applied, for example, with Caputo fractional derivative, then the inequality ${}_{t_0}^{C}D_t^{\alpha}|u(t)| \le \mathrm{sign}(u(t))\, {}_{t_0}^{C}D_t^{\alpha}u(t)$ is not true for any continuous and differentiable function.

In this paper, a generalized proportional Caputo fractional differential model of Cohen–Grossberg neural networks with impulses is studied. We study the case when the lower limit of the fractional derivative is changing after each impulsive time. To the best of our knowledge this is the first model of neural networks with impulses and generalized proportional Caputo fractional derivative studied in the literature. Both cases of continuously acting control and impulsive control are studied. Mittag–Leffler synchronization is defined and studied. It is a generalization of the exponential synchronization. Our sufficient conditions naturally depend significantly on the fractional order of the model.

2. Preliminary Notes on Generalized Proportional Fractional Derivatives

We recall that the generalized proportional fractional integral and the generalized Caputo proportional fractional derivative of a function $u : [a,b] \to \mathbb{R}$, $(b \leq \infty)$, are defined, respectively, by (as long as all integrals are well defined, see [6,7])

$$({}_a\mathcal{I}^{\alpha,\rho}u)(t) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}u(s)\,ds, \quad t \in (a,b], \ \alpha \geq 0, \ \rho \in (0,1],$$

and

$$\begin{aligned}({}_a^C\mathcal{D}^{\alpha,\rho}u)(t) &= ({}_a\mathcal{I}^{1-\alpha,\rho}(\mathcal{D}^{1,\rho}u))(t) \\ &= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-\alpha}(\mathcal{D}^{1,\rho}u)(s)\,ds, \\ &\text{for } \ t \in (a,b], \ \ \alpha \in (0,1), \ \rho \in (0,1],\end{aligned}$$

where $(\mathcal{D}^{1,\rho}u)(t) = (\mathcal{D}^{\rho}u)(t) = (1-\rho)u(t) + \rho u'(t)$ and $\Gamma(z) = \int_0^{\infty} s^{z-1}e^{-s}ds$.

Remark 1. *If $\rho = 1$, then the generalized Caputo proportional fractional derivative is reduced to the classical Caputo fractional derivative.*

Denote by $C^{\alpha,\rho}[a,b] = \{u : [a,b] \to \mathbb{R} : \ ({}_a\mathcal{D}^{\alpha,\rho}u)(t) \ \text{exists on } (a,b]\}$.

Remark 2. *The generalized proportional Caputo fractional derivative could be generalized for $u : [a,b] \to \mathbb{R}^n$ component-wise.*

Lemma 1 (Theorem 5.3 [6])**.** *Let $u \in C^{\alpha,\rho}[a,b]$, $\rho \in (0,1]$ and $\alpha \in (0,1)$. Then we have*

$$({}_a\mathcal{I}^{\alpha,\rho}({}_a^C\mathcal{D}^{\alpha,\rho}u))(t) = u(t) - u(a)e^{\frac{\rho-1}{\rho}(t-a)}, \ \ t \in (a,b].$$

Corollary 1 ([6])**.** *Let $u \in C^{\alpha,\rho}[a,b]$, $\rho \in (0,1]$ and $\rho \in (0,1]$, $\alpha \in (0,1)$. Then*

$$({}_a^C\mathcal{D}^{\alpha,\rho}({}_a\mathcal{I}^{\alpha,\rho}u))(t) = u(t), \quad t \in (a,b].$$

Lemma 2 (Theorem 5.2 [6])**.** *For $\rho \in (0,1]$ and $\alpha \in (0,1)$, $\rho \in (0,1]$, we have*

$$({}_a\mathcal{I}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1})(\tau) = \frac{\Gamma(\beta)}{\rho^{\alpha}\Gamma(\beta+\alpha)})e^{\frac{\rho-1}{\rho}\tau}(\tau-a)^{\beta-1+\alpha}, \quad \beta > 0.$$

Remark 3. *The generalized proportional Caputo fractional derivative of a constants is not zero for $\rho \in (0,1)$ (compare with the Caputo fractional derivative of a constant).*

Corollary 2 (Remark 3.2 [6])**.** *For $\rho \in (0,1]$ and $\alpha \in (0,1)$ the equality*

$$({}_a^C\mathcal{D}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}(.)})(t) = 0 \ \ \text{for} \ \ t > a$$

holds.

Lemma 3 (Lemma 3.2 [9])**.** *If $u \in C^1([a,\infty),\mathbb{R})$, $\rho \in (0,1]$ and $\alpha \in (0,1)$ then*

$$({}_a^C\mathcal{D}^{\alpha,\rho}u^2)(t) \leq 2u(t)({}_a^C\mathcal{D}^{\alpha,\rho}u)(t) \quad \text{for } t > a.$$

From Lemma 1, we have the following result for the initial value problem for the generalized proportional Caputo fractional differential equation

$$\begin{aligned} &({}_a^C\mathcal{D}^{\alpha,\rho}u)(t) = f(t,u(t)), \quad t > a, \\ &u(a) = u_0, \ \ \alpha \in (0,1), \ \rho \in (0,1]. \end{aligned} \tag{3}$$

Lemma 4. *For $\rho \in (0,1]$ and $\alpha \in (0,1)$ the solution $u(t)$ of (3) satisfies the integral equation*

$$u(t) = u_0 e^{\frac{\rho-1}{\rho}(t-a)} + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1} f(s,u(s))ds, \ \ t \in (a,b].$$

We will use the explicit form of the solution of the initial value problem for the scalar linear generalized proportional Caputo fractional differential equation which is given in Example 5.7 [6] and which is (with necessary slight corrections):

Lemma 5. *The solution of the scalar linear generalized proportional Caputo fractional initial value problem*

$$({}_a^C\mathcal{D}^{\alpha,\rho}u)(t) = \lambda u(t), \ \ u(a) = u_0, \ \ \alpha \in (0,1), \ \rho \in (0,1] \tag{4}$$

has a solution

$$u(t) = u_0 e^{\frac{\rho-1}{\rho}(t-a)} E_\alpha(\lambda(\frac{t-a}{\rho})^\alpha),$$

where $E_\alpha(t)$ is the Mittag–Leffler function of one parameter.

We will use the following result

Lemma 6 ([9])**.** *Let $u \in C^1([a,\xi],\mathbb{R})$, $u(\xi) = 0$ and $\rho \in (0,1]$ and $\alpha \in (0,1)$, then*

$$({}_a^C\mathcal{D}^{\alpha,\rho}u)(t)|_{t=\xi} = \frac{\rho^\alpha}{\Gamma(-\alpha)} \int_a^\xi e^{\frac{\rho-1}{\rho}(\xi-s)} \frac{u(s)}{(\xi-s)^{\alpha+1}} ds - \frac{\rho^\alpha}{\Gamma(1-\alpha)} e^{\frac{\rho-1}{\rho}(t-a)} \frac{u(a)}{(\xi-a)^\alpha}. \tag{5}$$

From Lemma 6 and the inequality $\Gamma(-\alpha) < 0$, $\alpha \in (0,1)$ we obtain the following result:

Corollary 3. *Let $u \in C^1([a,\xi],\mathbb{R})$, $u(t) < 0$ for $t \in [a,\xi)$, $u(\xi) = 0$ and $\rho \in (0,1]$, $\alpha \in (0,1)$, then $({}_a^C\mathcal{D}^{\alpha,\rho}u)(t)|_{t=\xi} > 0$.*

Furthermore, we will use the following comparison result:

Lemma 7. *The solution $v(.)$ of the scalar linear generalized proportional Caputo fractional differential inequality*

$$({}_a^C\mathcal{D}^{\alpha,\rho}v)(t) \le \lambda vs.(t), \ \ v(a) \le u_0, \ \ \alpha \in (0,1), \ \rho \in (0,1] \tag{6}$$

satisfies the inequality

$$v(t) \le u(t) = u_0 e^{\frac{\rho-1}{\rho}(t-a)} E_\alpha(\lambda(\frac{t-a}{\rho})^\alpha), \tag{7}$$

where $u(t)$ is the solution of initial value problem (4).

Proof. Let $\varepsilon > 0$ be an arbitrary number. Define the function $m(t) = v(t) - u(t) - \varepsilon e^{\frac{\rho-1}{\rho}(t-a)}$, $t \in (a,b]$. For $t = a$ the inequality $m(a) \le u_0 - u_0 - \varepsilon < 0$ Assume there exist a point $t^* \in (a,b]$ such that $m(t) < 0$ for $t \in (a,t^*)$ and $m(t^*) = 0$. Then according to Corollary 3 the inequality $0 < ({}_a^C\mathcal{D}^{\alpha,\rho}m)(t)|_{t=t^*} = ({}_a^C\mathcal{D}^{\alpha,\rho}v)(t)|_{t=t^*} - ({}_a^C\mathcal{D}^{\alpha,\rho}u)(t)|_{t=t^*} - \varepsilon({}_a^C\mathcal{D}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}(t-a)}|_{t=t^*} = ({}_a^C\mathcal{D}^{\alpha,\rho}v)(t)|_{t=t^*} - \lambda u(t^*) = ({}_a^C\mathcal{D}^{\alpha,\rho}v)(t)|_{t=t^*} - \lambda v(t^*)$ holds. It contradicts (6). Therefore, $v(t) < u(t) + \varepsilon e^{\frac{\rho-1}{\rho}(t-a)}$, $t \in (a,b]$. Since ε is an arbitrary number we obtain (7). $\square$

3. Statement of the Problem

Let a sequence $\{t_k\}_{k=1}^{\infty}$: $0 \leq t_{k-1} < t_k \leq t_{k+1}$, $\lim_{k\to\infty} t_k = \infty$ be given. Let $t_0 \neq t_k, k = 1,2,\dots$ be the given initial time. Without loss of generality we can assume $t_0 \in [0, t_1)$.

Denote by $\mathbb{Z}_0$ the set of all non-negative integers, by $\mathbb{Z}+$ the set of all natural numbers, by $\mathbb{Z}[a,b]$ the set of all integers $k: \ a \leq k \leq b$ where $a, b \in \mathbb{Z}_0$.

We restrict $\alpha \in (0,1)$, $\rho \in (0,1]$ everywhere in the paper due to many applications in science and engineering.

In this paper, we will study the dynamics of the Cohen–Grossberg neural networks modeled by generalized proportional Caputo fractional derivative with impulses at initially given points. We will consider the case when the lower limit of the fractional derivative is changed after each impulsive time.

We will study the following model:

$$\begin{aligned}
&({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}x_i)(t) = -d_i(x_i(t))\left(c_i(x_i(t)) - \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t)) - I_i\right), \\
&\qquad t \in (t_k, t_{k+1}],\ k \in \mathbb{Z}_0 \\
&x_i(t_k + 0) = \psi_k(x_i(t_k)),\ \ k \in \mathbb{Z}_+,\ \ i = 1,2,\dots,N,
\end{aligned} \tag{8}$$

where N represents the number of neurons in the network, $x(t) = (x_1(t), x_2(t), \dots, x_N(t))^T$ denotes the variable neuron's state; $d_i(x_i(t))$ is the amplification function of the i-th neuron; $c_i(x_i(t))$ is well behaved function; $f_j(x_j(t))$ are the activation function of the j-th neuron; I_i is the external input; $\psi_k(x_i(t_k))$ are the impulsive functions at impulsive time t_k, $k \in \mathbb{Z}_+$; $a_{ij}(t)$ are neural connection memristive weights of the j-th neuron on the i-th neuron at time t.

We consider the system (8) as a driven system and the responce system is as follows

$$\begin{aligned}
&({}_{a}^{C}\mathcal{D}^{\alpha,\rho}y_i)(t) = -d_i(y_i(t))\left(c_i(y_i(t)) - \sum_{j=1}^{n} a_{ij}(t) f_j(y_j(t)) - I_i\right) + u_i(t), \\
&\qquad t \in (t_k, t_{k+1}],\ k \in \mathbb{Z}_0, \\
&y_i(t_k + 0) = \psi_k(y_i(t_k)) + w_{ik},\ k \in \mathbb{Z}_+,\ i = 1,2,\dots,N,
\end{aligned} \tag{9}$$

where $u_i(t)$ is the input continuous control and w_{ik} is the input impulsive control.

Remark 4. *Throughout this paper, we assume the solutions of the systems (8) and (9) are left continuous, i.e.,* $x(t_k) = \lim_{t\to t_k, t<t_k} x(t)$.

We will introduce the following assumptions

Assumption 1. *There exist positive numbers* $M_{i,j}$, $i,j = 1,2,\dots,N$ *such that* $|a_{i,j}(t)| \leq M_{i,j}$ *for* $t > t_0$.

Assumption 2. *There exist positive constants* $\tilde{d}_i$, $\hat{d}_i$, D_i *such that the aplification function satisfy*

$$0 < \hat{d}_i \leq d_i(x) \leq \tilde{d}_i < \infty, \quad |d_i(y) - d_i(x)| \leq D_i|y - x|, \quad x, y \in R$$

Assumption 3. *For well behaved function $c_i(x)$ and amplification function $d_i(x)$ there exists a positive constant A_i such that*

$$\frac{d_i(y)c_i(y) - d_i(x)c_i(x)}{y - x} \geq A_i, \ \ i \in \mathbb{Z}[1, m], \ \ x, y \in \mathbb{R}, \ x \neq y.$$

Assumption 4. *The activation functions are bounded, i.e., there exist constants $C_j > 0$ such that $|f_j(x)| \leq C_j$, $x \in \mathbb{R}$, $j \in \mathbb{Z}[1, N]$ and it satisfies the Lipschitz condition with a constant $F_j > 0$, i.e.,*

$$|f_j(y) - f_j(x)| \leq F_j|y - x|, \ \ y, x \in \mathbb{R}, \ \ j \in \mathbb{Z}[1, n].$$

Assumption 5. *For the impulsive functions $\psi_k(x)$ there exists a positive constant L_k such that*

$$|\psi_k(y) - \psi_k(x)| \leq L_k|y - x|, \ \ k \in \mathbb{Z}[1, N], \ \ x, y \in \mathbb{R}, \ x \neq y.$$

4. Main Results

We will obtain some sufficient conditions for achievements of finite time synchronization and exponential synchronization of (8) and (9) with different controllers.

Definition 1. *The driven system of impulsive generalized proportional Caputo fractional differential Equation (8) and the responce system of impulsive generalized proportional Caputo fractional differential Equation (9) are globally Mittag–Leffler synchronized if for any initial values $u_i^0, v_i p \in \mathbb{R}$ there exist constants $C, K, \beta > 0$ such that*

$$||x(t; t_0, x^0) - y(t; t_0, y^0)|| \leq Km(x^0 - y^0)\Big(E_\alpha(-C(t - t_k)^\alpha)\prod_{j=0}^{k-1} E_\alpha(-C(t_{j+1} - t_j)^\alpha)\Big)^\beta,$$
$$t \in (t_k, t_{k+1}], \ k \in \mathbb{Z}_0$$

where $m \in C(\mathbb{R}^n_+, \mathbb{R}_+)$ (with $m(0) = 0$) is Lipschitz, $||x|| = \sqrt{\sum_{i=1}^N x_i^2}$, $x \in \mathbb{R}^N$, $x = (x_1, x_2, \dots, x_N)$.

Remark 5. *Mittag–Leffler stabilization is a fractional generalization of the exponential one. Indeed, from inequality $E_\alpha(-\lambda(t - t_0)^\alpha) \leq e^{\frac{-\lambda}{\alpha}(t-t_0)}$, $t > t_0$ (see Equation (16) [21]) it follows that if inequality (10) holds, then $||x(t; t_0, x^0) - y(t; t_0, y^0)|| \leq Km(x^0 - y^0)e^{\frac{-C}{\alpha}(t-t_k)}\prod_{j=0}^{k-1} e^{\frac{-C}{\alpha}(t_{j+1}-t_j)} = Km(x^0 - y^0)e^{\frac{-C}{\alpha}(t-t_0)}$.*

4.1. Mittag–Leffler Synchronization under State Feedback Control

The continuous state feedback controller $u_i(t)$ will be designed to enable the controlled model (8) and (9) to synchronize.

In this section, we will assume that the input impulsive control is zero, i.e., $w_{ik} = 0$, $k \in \mathbb{Z}_+$, $k = 1, 2, \dots, N$.

Define the synchronization error $e_i(t) = y_i(t) - x_i(t)$ and the control gains

$$u_i(t) = k_i e_i(t), \ \ i \in \mathbb{Z}[1, N]. \tag{10}$$

Theorem 1. *Let the following conditions be fulfilled:*

1. *Assumptions 1–5 are satisfied.*
2. *There exists a constant $\gamma > 0$ such that*

$$2\sum_{i=1}^N \Big[A_i - k_i - D_i I_i - D_i \sum_{j=1}^N M_{ij}C_j - 0.5\tilde{d}_i \sum_{j=1}^N M_{ij}F_j - 0.5F_i \sum_{j=1}^N \tilde{d}_j M_{ji}\Big] \geq \gamma. \tag{11}$$

3. *There exists a constant $\mathcal{L} > 0$ such that the sequence $\{\prod_{i=1}^{k} L_i\}_{k=1}^{\infty}$ is nondecreasing and bounded, i.e., $\prod_{i=1}^{\infty} L_i \leq \mathcal{L}$.*

Then the driven system (8) and the responce system (9) with $w_{ik} = 0$ are globally Mittag–Leffler synchronized under the control (10).

Proof. Define the quadratic Lyapunov function $V(x) = x\,x^T = \sum_{i=1}^{N} x_i^2$.

Let $t \in (t_k, t_{k+1}],\ \ k \in \mathbb{Z}_0$. Then according to condition A2, Lemma 3 and inequality $2ab \leq a^2 + b^2$ we can write

$$
\begin{aligned}
&({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}V(e))(t) \leq 2\sum_{i=1}^{N} e_i(t)({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}e_i)(t) = 2\sum_{i=1}^{N} e_i(t)\Big(({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}y_i)(t) - ({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}x_i)(t)\Big) \\
&\leq 2\sum_{i=1}^{N}\Big[-A_i e_i^2(t) + D_i e_i^2(t) I_i \\
&\quad + e_i(t)\sum_{j=1}^{N}\Big(d_i(y_i(t))a_{ij}(t)f_j(y_j(t)) - d_i(x_i(t))a_{ij}(t)f_j(x_j(t))\Big) + k_i e_i^2(t)\Big] \\
&\leq 2\sum_{i=1}^{N}\Big[-A_i + D_i I_i + k_i\Big]e_i^2(t) \\
&\quad + 2|e_i(t)|\sum_{j=1}^{N}\Big[|d_i(y_i(t))|\,|a_{ij}(t)|\Big|f_j(y_j(t)) - f_j(x_j(t))\Big| \\
&\quad + \Big|d_i(y_i(t)) - d_i(x_i(t))\Big|\,|a_{ij}(t)|\,|f_j(x_j(t))|\Big] \\
&\leq 2\sum_{i=1}^{N}\Big[-A_i + D_i I_i + k_i\Big]e_i^2(t) + 2|e_i(t)|\sum_{j=1}^{N}\Big[\tilde{d}_i M_{ij} F_j |e_j(t)| \\
&\quad + D_i e_i^2(t) M_{ij} C_j\Big] \\
&\leq -2\sum_{i=1}^{N}\Big[A_i - k_i - D_i I_i - D_i\sum_{j=1}^{N} M_{ij}C_j\Big]e_i^2(t) \\
&\quad + e_i^2(t)\tilde{d}_i\sum_{j=1}^{N} M_{ij}F_j + \tilde{d}_i\sum_{j=1}^{N} M_{ij}F_j e_j^2(t) \\
&\leq -2\sum_{i=1}^{N}\Big[A_i - k_i - D_i I_i - D_i\sum_{j=1}^{N} M_{ij}C_j - 0.5\tilde{d}_i\sum_{j=1}^{N} M_{ij}F_j - 0.5F_i\sum_{j=1}^{N}\tilde{d}_j M_{ji}\Big]e_i^2(t) \\
&\leq -\gamma V(e(t)),\ \ t \in (t_k, t_{k+1}],\ k \in \mathbb{Z}_0.
\end{aligned} \tag{12}
$$

For any $k \in \mathbb{Z}_+$ we have

$$
\begin{aligned}
V(e(t_k + 0)) &= \sum_{i=1}^{N} e_i^2(t_k + 0) = \sum_{i=1}^{N}(y_i(t_k + 0) - x(t_k + 0))^2 \\
&= \sum_{i=1}^{N}(\psi_k(y_i(t_k)) - \psi_k(x_i(t_k)))^2 \leq L_k^2\sum_{i=1}^{N} e_i^2(t_k) = L_k^2 V(e(t_k))
\end{aligned} \tag{13}
$$

Apply Lemma 7 with $a = t_k$, $u_0 = V(e(t_k + 0))$, $\lambda = -\gamma$ to inequality (12), use inequality (13) and obtain

$$
\begin{aligned}
V(e(t)) &< V(e(t_k + 0))e^{\frac{\rho-1}{\rho}(t-t_k)}E_\alpha(-\gamma(\frac{t-t_k}{\rho})^\alpha) \\
&\leq L_k^2 V(e(t_k))e^{\frac{\rho-1}{\rho}(t-t_k)}E_\alpha(-\gamma(\frac{t-t_k}{\rho})^\alpha),\ \ t \in (t_k, t_{k+1}],\ k \in \mathbb{Z}_0.
\end{aligned} \tag{14}
$$

From inequality (12) for $t \in (t_{k-1}, t_k]$, Lemma 7 with $a = t_{k-1}$, $u_0 = V(e(t_{k-1}+0))$, $\lambda = -\gamma$ and inequality (13) for $k-1$ and obtain

$$\begin{aligned} V(e(t)) &< V(e(t_{k-1}+0))e^{\frac{\rho-1}{\rho}(t-t_{k-1})}E_\alpha(-\gamma(\frac{t-t_{k-1}}{\rho})^\alpha) \\ &\leq L_{k-1}^2 V(e(t_{k-1}))e^{\frac{\rho-1}{\rho}(t-t_{k-1})}E_\alpha(-\gamma(\frac{t-t_{k-1}}{\rho})^\alpha), \ \ t \in (t_{k-1}, t_k], \ k \in \mathbb{Z}_0. \end{aligned} \tag{15}$$

Therefore, from inequality (15) for $t = t_k$ we have

$$V(e(t_k)) \leq L_{k-1}^2 V(e(t_{k-1}))e^{\frac{\rho-1}{\rho}(t_k-t_{k-1})}E_\alpha(-\gamma(\frac{t_k-t_{k-1}}{\rho})^\alpha). \tag{16}$$

From inequalities (14) and (16) we get

$$\begin{aligned} V(e(t)) \leq L_k^2 L_{k-1}^2 V(e(t_{k-1}))e^{\frac{\rho-1}{\rho}(t_k-t_{k-1})}E_\alpha(-\gamma(\frac{t_k-t_{k-1}}{\rho})^\alpha)e^{\frac{\rho-1}{\rho}(t-t_k)}E_\alpha(-\gamma(\frac{t-t_k}{\rho})^\alpha) \\ \text{for } \ t \in (t_k, t_{k+1}], \ k \in \mathbb{Z}_0. \end{aligned} \tag{17}$$

Continue this process and applying Lemma 7 inductive with $a = t_{k-1}, t_{k-2}, \ldots, t_0$ we obtain

$$\begin{aligned} V(e(t)) &\leq V(e(t_{k-1}))L_k^2 L_{k-1}^2 e^{\frac{\rho-1}{\rho}(t-t_{k-1})}E_\alpha(-\gamma(\frac{t_k-t_{k-1}}{\rho})^\alpha)E_\alpha(-\lambda(\frac{t-t_k}{\rho})^\alpha) \\ &\leq \ldots\ldots \\ &\leq V(e(t_0))\Big(\prod_{i=0}^{k} L_i^2\Big)\Big(\prod_{i=0}^{k-1} E_\alpha(-\gamma(\frac{t_{i+1}-t_i}{\rho})^\alpha)\Big)e^{\frac{\rho-1}{\rho}(t-t_0)}E_\alpha(-\gamma(\frac{t-t_k}{\rho})^\alpha) \\ &= \Big(\prod_{i=0}^{k} L_i^2\Big)\sum_{i=1}^{n}(x_i^0-y_i^0)^2\Big(\prod_{i=0}^{k-1} E_\alpha(-\gamma(\frac{t_{i+1}-t_i}{\rho})^\alpha)\Big)e^{\frac{\rho-1}{\rho}(t-t_0)}E_\alpha(-\gamma(\frac{t-t_k}{\rho})^\alpha) \end{aligned}$$

Thus,

$$\begin{aligned} &\|x(t)-y(t)\| \\ &\leq \|x_0-y_0\|\Big(\prod_{i=0}^{k} L_i\Big)e^{\frac{\rho-1}{2\rho}(t-t_0)}\sqrt{\Big(\prod_{i=0}^{k-1} E_\alpha(-\gamma(\frac{t_{i+1}-t_i}{\rho})^\alpha)\Big)E_\alpha(-\gamma(\frac{t-t_k}{\rho})^\alpha)}, \\ &\qquad t \in (t_k, t_{k+1}]. \end{aligned} \tag{18}$$

□

Corollary 4. *Let the conditions of Theorem 1 are satisfied. Then the driven system (8) and the responce system (9) with $w_{ik} = 0$ are globally exponentially synchronized under the control (10).*

The proof follows from Remark 5.

4.2. Synchronization under Impulsive Control

In some applications, it is necessary for small control gains and control to be activated only in some isolated points. This type of control is well described by impulses and the control activated at impulsive times, which can not only realize the synchronization target but also save the control costs. In this section, an impulsive controller w_{ik} will be designed such that the given synchronization criteria reduce this conservativeness, i.e., in this section we will assume that $u_i(t) \equiv 0$.

Define the synchronization error $e_i(t) = y_i(t) - x_i(t)$ and the impulsive control gains

$$w_{i,k} = K_{ik}e_i(t_k), \ \ i \in \mathbb{Z}[1, N], \ k \in \mathbb{Z}_+, \tag{19}$$

where $K_{ik},\ i \in \mathbb{Z}[1,N],\ k \in \mathbb{Z}_+$ are constants.

Theorem 2. *Let the following conditions be fulfilled:*

1. *Assumptions 1–5 are satisfied.*
2. *There exist constants $\gamma,\ \mathcal{L} > 0$ such that*

$$2\sum_{i=1}^{N}\Big[A_i - D_i I_i - D_i\sum_{j=1}^{N} a_{ij}C_j - 0.5\tilde{d}_i\sum_{j=1}^{N} M_{ij}F_j - 0.5F_i\sum_{j=1}^{N}\tilde{d}_j M_{ji}\Big] \geq \gamma;$$

the sequence $\{\prod_{i=1}^{k} B_i\}_{k=1}^{\infty}$ is increasing bounded, i.e., $\prod_{i=1}^{\infty} B_i \leq \mathcal{B}$, where $B_k = 2(L_k^2 + \max_{i\in\mathbb{Z}_{[1,N]}} K_{ik}^2)$.

Then the driven system (8) and the responce system (9) with $u_i(t) \equiv 0$ are globally Mittag–Leffler synchronized under the impulsive control (19).

Proof. Define the quadratic Lyapunov function $V(x) = x\,x^T = \sum_{i=1}^{N} x_i^2$.

Let $t \in (t_k, t_{k+1}],\ \ k \in \mathbb{Z}_0$. Then similar to the proof of Theorem 1 and inequality (12) we prove

$$\begin{aligned}
&({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}V(e))(t)\\
&\leq -2\sum_{i=1}^{N}\Big[A_i - D_i I_i - D_i\sum_{j=1}^{N} a_{ij}M_j - 0.5\tilde{d}_i\sum_{j=1}^{N} a_{ij}(t)F_j - 0.5F_i\sum_{j=1}^{N}\tilde{d}_j a_{ji}(t)\Big]e_i^2(t)\\
&\leq -\gamma V(e(t)),\ \ t \in (t_k, t_{k+1}],\ k \in \mathbb{Z}_0.
\end{aligned} \tag{20}$$

For any $k \in \mathbb{Z}_+$ we have

$$\begin{aligned}
&V(e(t_k+0)) = \sum_{i=1}^{N} e_i^2(t_k+0) = \sum_{i=1}^{N}(y_i(t_k+0) + K_{ik}e_i(t_k) - x(t_k+0))^2\\
&= \sum_{i=1}^{N}(y_i(t_k+0) - x(t_k+0))^2 + \sum_{i=1}^{N} k_i^2 e_i^2(t_k) + 2\sum_{i=1}^{N}(y_i(t_k+0) - x(t_k+0))K_{ik}e_i(t_k)\\
&\leq 2\sum_{i=1}^{N}(y_i(t_k+0) - x(t_k+0))^2 + 2\sum_{i=1}^{N} K_{ik}^2 e_i^2(t_k)\\
&= 2\sum_{i=1}^{N}(\psi_k(y_i(t_k)) - \psi_k(x_i(t_k)))^2 + 2\sum_{i=1}^{N} K_{ik}^2 e_i^2(t_k)\\
&\leq 2(L_k^2 + \max_{i\in\mathbb{Z}_{[1,N]}} K_{ik}^2)\sum_{i=1}^{N} e_i^2(t_k) = B_k V(e(t_k))
\end{aligned} \tag{21}$$

According to Lemma 7 with $a = t_k$, $u_0 = V(e(t_k+0))$ we have

$$\begin{aligned}
&\sum_{i=1}^{N}(x_i(t;t_0,x_0) - y_i(t;t_0,y_0))^2 = V(e(t)) < V(e(t_k+0))e^{\frac{\rho-1}{\rho}(t-t_k)}E_\alpha(-\gamma(\frac{t-t_k}{\rho})^\alpha)\\
&\leq B_k V(e(t_k))e^{\frac{\rho-1}{\rho}(t-t_k)}E_\alpha(-\gamma(\frac{t-t_k}{\rho})^\alpha)
\end{aligned} \tag{22}$$

Continue this process and applying Lemma 7 inductively with $a = t_{k-1}, t_{k-2}, \ldots, t_0$ we obtain

$$V(e(t)) \leq V(e(t_{k-1}))B_k B_{k-1} e^{\frac{\rho-1}{\rho}(t_k - t_{k-1})} E_\alpha(-\gamma(\frac{t_k - t_{k-1}}{\rho})^\alpha) e^{\frac{\rho-1}{\rho}(t-t_k)} E_\alpha(-\gamma(\frac{t-t_k}{\rho})^\alpha)$$
$$\leq \ldots\ldots$$
$$\leq V(e(t_0))\Big(\prod_{i=0}^{k} B_i\Big)\Big(\prod_{i=0}^{k-1} e^{\frac{\rho-1}{\rho}(t_{i+1}-t_i)} E_\alpha(-\gamma(\frac{t_{i+1}-t_i}{\rho})^\alpha)\Big) e^{\frac{\rho-1}{\rho}(t-t_k)} E_\alpha(-\gamma(\frac{t-t_k}{\rho})^\alpha)$$
$$= \mathcal{B}\sum_{i=1}^{n}(x_i^0 - y_i^0)^2 \Big(\prod_{i=0}^{k-1} E_\alpha(-\gamma(\frac{t_{i+1}-t_i}{\rho})^\alpha)\Big) e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha(-\gamma(\frac{t-t_k}{\rho})^\alpha)$$

and thus

$$\|x(t) - y(t)\|$$
$$\leq \|x_0 - y_0\| e^{\frac{\rho-1}{2\rho}(t-t_0)} \sqrt{\mathcal{B}\Big(\prod_{i=0}^{k-1} E_\alpha(-\gamma(\frac{t_{i+1}-t_i}{\rho})^\alpha)\Big) E_\alpha(-\gamma(\frac{t-t_k}{\rho})^\alpha)|}, \; t \in (t_k, t_{k+1}].$$

□

5. Example

We will provide a partial case of impulsive memristive Cohen–Grossberg neural networks (8) to illustrate the application of the obtained sufficient conditions. Let $\alpha = 0.3$, $\rho = 0.8,\ t_k = k,\ k \in \mathbb{Z}_0$ and consider the driven system (8) and the responce system (9) with $w_{ik} = 0$, $N = 3$, $c_i(t) \equiv c_i$, with $I_i = 0$, the activation functions $f_j(s) = 0.5\ tanh(s)$ with $F_j = 0.5\ C_j = 0.5$, and $|a_{ij}(t)| \leq M_{ij},\ i,j = 1,2,3,\ t \geq 0$ where $M = \{M_{ij}\}$is given by

$$M = \begin{pmatrix} 0.1 & 0.5 & 0.3 \\ 0.2 & 0.3 & 0.2 \\ 0.4 & 0.2 & 0.1 \end{pmatrix}.$$

Let the impulsive functions be $\psi_k(x) = \sin(x)$ with $L_i = 1$ and $\mathcal{L} = 1$, $c_i(u) = 1.5u + \sin(u)$, $i = 1,2,3$, $d_i(u) = 1 + \frac{1}{1+u^2}, i = 1,2,3$ with $\hat{d}_i = 1, \tilde{d}_i = 2$ and $D_i = 0.67$. Then $\frac{d_i(y)c_i(y) - d_i(x)c_i(x)}{y-x} = \frac{(1+\frac{1}{1+y^2})(1.5y+\sin(y)) - (1+\frac{1}{1+x^2}(1.5x+\sin(x))}{y-x} \geq A_i = 5$.

Let the control gain be $u_i(t) = k_i e_i(t),\ \ i = 1,2,3$ with $k_1 = k_2 = k_3 = 4$.

Thus, the inequality (10) is reduced to

$$2\sum_{i=1}^{3}\Big[5 - 4 - (0.67) * 0 - 0.835\sum_{j=1}^{3} M_{ij} - 0.5\sum_{j=1}^{3} M_{ji}\Big] \geq \gamma$$

$$2\sum_{i=1}^{3}\Big[1 - 0.835(0.7) - 0.5(0.6)\Big] \geq \gamma = 0.693$$

Therefore, the conditions of Theorem 1 are satisfied and then the driven system (8) and its corresponding responce system (9) in this partial case are globally Mittag–Leffler synchronized under the defined above control, i.e., the inequality (18) is reduced to

$$\begin{aligned}
\|x(t)-y(t)\| &= \sqrt{\sum_{i=1}^{3}(x(t)-y(t))^2} \\
&\leq \sqrt{\sum_{i=1}^{3}(x_i^0-y_i^0)^2 e^{-0.125t}}\sqrt{\Big(\prod_{i=0}^{k-1}E_{0.3}(-0.693(\frac{1}{0.8})^{0.3})\Big)E_{0.3}(-0.693(\frac{t}{0.8})^{0.3})} \\
&\leq \sqrt{\sum_{i=1}^{3}(x_i^0-y_i^0)^2(0.730885)^k e^{-0.125t}}\sqrt{E_{0.3}(-0.534193t^{0.3})}.
\end{aligned} \tag{23}$$

6. Conclusions

In this paper, a memristive Cohen–Grossberg neural network with impulsive effects at initially given impulsive times and generalized proportional Caputo fractional derivatives with lower limits at the impulsive time is studied. Some sufficient conditions for the global Mittag–Leffler synchronization are obtained. We consider two types of controllers, continuous controller and discrete controller acting at the impulsive time. The obtained results are significant for various applications in engineering and technology.

Note, the results and the studied type of neural network could extend to the case of non-Lipschitz discontinuous activation functions. Furthermore, both approaches for the interpretation of solutions of fractional equations with impulses could be applied. It will give wider possibilities for adequate modeling of the connections between neurons in the networks. This topic goes beyond the scope of this paper and will be a challenging issue for future research.

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