mathematics

MDPI

Article

Some Generalized Properties of Poly-Daehee Numbers and Polynomials Based on Apostol–Genocchi Polynomials

Talha Usman [1], Nabiullah Khan [2], Mohd Aman [2], Shrideh Al-Omari [3], Kamsing Nonlaopon [4,*] and Junesang Choi [5]

[1] Department of General Requirements, University of Technology and Applied Sciences-Sur, Sur 411, Oman; talha.sur@cas.edu.om

[2] Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh 202002, India; nkhan.am@amu.ac.in (N.K.); aman.rs@amu.ac.in (M.A.)

[3] Department of Scientific Basic Sciences, Faculty of Engineering Technology, Al-Balqa Applied University, Amman 11134, Jordan; shridehalomari@bau.edu.jo

[4] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

[5] Department of Mathematics, Dongguk University, Gyeongju 38066, Korea; junesang@dongguk.ac.kr

* Correspondence: nkamsi@kku.ac.th

Abstract: Numerous polynomial variations and their extensions have been explored extensively and found applications in a variety of research fields. The purpose of this research is to establish a unified class of Apostol–Genocchi polynomials based on poly-Daehee polynomials and to explore some of their features and identities. We investigate these polynomials via generating functions and deduce various identities, summation formulae, differential and integral formulas, implicit summation formulae, and several characterized generating functions for new numbers and polynomials. Finally, by using an operational version of Apostol–Genocchi polynomials, we derive some results in terms of new special polynomials. Due to the generic nature of the findings described here, they are used to reduce and generate certain known or novel formulae and identities for relatively simple polynomials and numbers.

Keywords: Bernoulli polynomials; Daehee polynomials; poly-Daehee polynomials; Apostol polynomials; differential operator

MSC: 05A15; 11B68; 26B10; 33E20

check for updates

Citation: Usman, T.; Khan, N.; Aman, M.; Al-Omari, S.; Nonlaopon, K.; Choi, J. Some Generalized Properties of Poly-Daehee Numbers and Polynomials Based on Apostol–Genocchi Polynomials. *Mathematics* **2022**, *10*, 2502. https://doi.org/10.3390/math10142502

Academic Editor: Valery Karachik

Received: 12 June 2022
Accepted: 13 July 2022
Published: 18 July 2022

1. Introduction

The study of special functions is a notable subject of mathematics which has attracted various mathematicians in the recent past. Some known special functions, including Bernoulli numbers, polynomials, hypergeometric functions of Euler and Gauss, Euler's gamma and beta functions, Abel's, Weierstrass' and Jacobi's elliptic functions, Bessel functions, Legendre polynomials, Jacobi, Laguerre, and Hermite, are thoroughly discussed in the literature. Some of these functions were introduced to solve specific problems and some others were used to solve general problems. In recent years, generalized and multivariable forms of special functions of mathematical physics have also undergone significant evolutions (see [1–12] for more details). The theory of orthogonal polynomials and special functions is of intrinsic interest to many parts of mathematics. Moreover, it can be used to explain many physical and chemical phenomena. For example, the vibrations of a drum head can be explained in terms of special functions known as Bessel functions. Furthermore, the solutions of the Schrodinger equation for a harmonic oscillator can be described using orthogonal polynomials known as Hermite polynomials. Furthermore, the eigenfunctions for the Schrodinger operator associated with the hydrogen atom are described in terms orthogonal polynomials known as Laguerre polynomials.

The subject of special polynomials of two variables, in particular, enabled the development of novel methods for solving vast classes of partial differential equations that are often encountered in physical issues. The majority of special functions of mathematical physics and their generalization have been inspired by physical problems. There is an abundance of remarkable characteristics and correlations with special generalized polynomials in the literature (see, for details, [13–24]).

2. Background and Preliminaries

The following polynomials and numbers are required for the current investigation: The κ-th polylogarithm function $\mathrm{Li}_\kappa(w)$ is defined by (see, e.g., [25], ([26], Section 2.4); see also [21,23])

$$\begin{aligned}\mathrm{Li}_\kappa(w) &:= \sum_{s=1}^{\infty} \frac{w^s}{s^\kappa} \quad (w \in \mathbb{C},\ |w| \leqslant 1;\ \kappa \in \mathbb{N} \setminus \{1\}) \\ &= \int_0^w \frac{\mathrm{Li}_{\kappa-1}(t)}{t}\, dt \quad (\kappa \in \mathbb{N} \setminus \{1\}),\end{aligned} \tag{1}$$

where

$$\mathrm{Li}_1(t) := -\log(1-t). \tag{2}$$

Here and in the following, $\mathbb{N}$ and $\mathbb{C}$ denote the sets of positive integers and complex numbers, respectively. Furthermore, put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Kim and Kim [27] explored the Daehee polynomials $\mathcal{D}_r(u)$ which are generated by (see also [19,20,27–29])

$$\sum_{r=0}^{\infty} \mathcal{D}_r(u) \frac{\varsigma^r}{r!} = (1+\varsigma)^u \frac{\log(1+\varsigma)}{\varsigma}. \tag{3}$$

Here $\mathcal{D}_r := \mathcal{D}_r(0)$ are called the Daehee numbers. We find that

$$\mathcal{D}_r = (-1)^r \frac{r!}{r+1} \quad (r \in \mathbb{N}_0). \tag{4}$$

The first few are

$$\mathcal{D}_0 = 1, \quad \mathcal{D}_1 = -\frac{1}{2}, \quad \mathcal{D}_2 = \frac{2}{3}, \quad \mathcal{D}_3 = -\frac{3}{2}, \quad \mathcal{D}_4 = \frac{24}{5}, \ldots.$$

Lim and Kwon [28] introduced and investigated the poly-Daehee polynomials $\mathcal{D}_r^{(\kappa)}(u)$ which are given by the following generating function:

$$\frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}(1+\varsigma)^u = \sum_{r=0}^{\infty} \mathcal{D}_r^{(\kappa)}(u) \frac{\varsigma^r}{r!} \quad (\kappa \in \mathbb{N}). \tag{5}$$

Then $\mathcal{D}_r^{(\kappa)} := \mathcal{D}_r^{(\kappa)}(0)$ are called the poly-Daehee numbers. In view of (2), it is easy to find that

$$\mathcal{D}_r^{(1)}(u) = \mathcal{D}_r(u) \quad (r \in \mathbb{N}_0). \tag{6}$$

The Bernoulli polynomials $\mathcal{B}_r(u)$ (see, e.g., [15], ([26], Section 1.7)) and their second kind $b_n(x)$ (see, e.g., [24]) are defined by the following generating functions:

$$\frac{\varsigma}{e^\varsigma - 1} e^{u\varsigma} = \sum_{r=0}^{\infty} \mathcal{B}_r(u) \frac{\varsigma^r}{r!} \quad (|\varsigma| < 2\pi), \tag{7}$$

and

$$\sum_{r=0}^{\infty} b_r(u) \frac{\varsigma^r}{r!} = \frac{\varsigma}{\log(1+\varsigma)}(1+\varsigma)^u. \tag{8}$$

By combining (3) and (8), we get

$$\sum_{s=0}^{r} \binom{r}{s} b_{r-s}\, \mathcal{D}_s(u) = r! \binom{2u}{r} \quad (r \in \mathbb{N}_0). \tag{9}$$

Kaneko [21] committed their research on the poly-Bernoulli numbers $\mathcal{B}_r^{(\kappa)}$ which are generated by the following function:

$$\frac{\mathrm{Li}_\kappa(1-e^{-\varsigma})}{1-e^{-\varsigma}} = \sum_{r=0}^{\infty} \mathcal{B}_r^{(\kappa)} \frac{\varsigma^r}{r!}. \tag{10}$$

When $\kappa = 1$, $\mathcal{B}_r^{(1)}$ are generated by

$$\sum_{r=0}^{\infty} \mathcal{B}_r^{(1)} \frac{\varsigma^r}{r!} = \frac{\varsigma\, e^{\varsigma}}{e^{\varsigma}-1} = \sum_{r=0}^{\infty} \mathcal{B}_r(1) \frac{\varsigma^r}{r!}. \tag{11}$$

From (11), the following relationship between the poly-Bernoulli numbers $\mathcal{B}_r^{(1)}$ and the Bernoulli polynomials $\mathcal{B}_r(1)$ holds:

$$\mathcal{B}_r^{(1)} = \mathcal{B}_r(1) \quad (r \in \mathbb{N}_0). \tag{12}$$

The poly-Bernoulli numbers $\mathcal{B}_r^{(\kappa)}$ are given explicitly by the following identity (see ([21], Theorem 1)):

$$\mathcal{B}_r^{(\kappa)} = (-1)^r \sum_{s=0}^{r} \frac{(-1)^s\, s!\, S(r,s)}{(s+1)^\kappa} (r \in \mathbb{N}_0,\ \kappa \in \mathbb{Z}), \tag{13}$$

where (elsewhere) $\mathbb{Z}$ denote the set of integers, and $S(r,s)$ are the Stirling numbers of the second kind which are explicitly given by (see, e.g., ([26], Section 1.6))

$$S(r,s) = \frac{(-1)^s}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} j^r. \tag{14}$$

The first few of $\mathcal{B}_r^{(\kappa)}$ are

$$\mathcal{B}_0^{(\kappa)} = 1, \quad \mathcal{B}_1^{(\kappa)} = \frac{1}{2^\kappa}, \quad \mathcal{B}_2^{(\kappa)} = \frac{2}{3^\kappa} - \frac{1}{2^\kappa}, \quad \mathcal{B}_3^{(\kappa)} = \frac{1}{2^\kappa} + \frac{6}{2^{2\kappa}} - \frac{6}{3^\kappa}.$$

In the usual way, poly-Bernoulli polynomials $\mathcal{B}_r^{(\kappa)}(u)$ can be defined by the following function:

$$\frac{\mathrm{Li}_\kappa(1-e^{-\varsigma})}{1-e^{-\varsigma}} e^{u\varsigma} = \sum_{r=0}^{\infty} \mathcal{B}_r^{(\kappa)}(u) \frac{\varsigma^r}{r!}. \tag{15}$$

Then, obviously, $\mathcal{B}_r^{(\kappa)} = \mathcal{B}_r^{(\kappa)}(0)$ $(r \in \mathbb{N}_0)$.

The classical Genocchi polynomials are defined by (see, e.g., [30–32], ([26], Section 1.7))

$$\frac{2\varsigma}{e^{\varsigma}+1} e^{u\varsigma} = \sum_{r=0}^{\infty} \mathcal{G}_r(u) \frac{\varsigma^r}{r!} \quad (|\varsigma| < \pi). \tag{16}$$

As usual, $\mathcal{G}_r := \mathcal{G}_r(0)$ are referred to as Genocchi numbers generated by

$$\frac{2\varsigma}{e^{\varsigma}+1} = \sum_{r=0}^{\infty} \mathcal{G}_r \frac{\varsigma^r}{r!} \quad (|\varsigma| < \pi), \tag{17}$$

which have a significant role in number theory.

Luo and Srivastava [33] introduced the generalized Apostol–Bernoulli polynomials $\mathcal{B}_r^{(m)}(u;\lambda)$ of order $m \in \mathbb{C}$ which are generated by (see also ([26], Section 1.8))

$$\left(\frac{\varsigma}{\lambda e^{\varsigma}-1}\right)^m e^{u\varsigma} = \sum_{r=0}^{\infty} \mathcal{B}_r^{(m)}(u;\lambda)\frac{\varsigma^r}{r!} \tag{18}$$

$$(|\varsigma| < 2\pi,\ \text{when } \lambda = 1;\ |\varsigma| < |\log\lambda|,\ \text{when } \lambda \neq 1;\ 1^m := 1).$$

Furthermore, Luo [34,35] investigated the generalized Apostol–Euler polynomials $\mathcal{E}_r^{(m)}(u;\lambda)$ of order $m \in \mathbb{C}$ and the generalized Apostol–Genocchi polynomials $\mathcal{G}_r^{(m)}(u;\lambda)$ of order $m \in \mathbb{C}$ which are defined by

$$\left(\frac{2}{\lambda e^{\varsigma}+1}\right)^m e^{u\varsigma} = \sum_{r=0}^{\infty} \mathcal{E}_r^{(m)}(u;\lambda)\frac{\varsigma^r}{r!} \tag{19}$$

$$(|\varsigma| < \pi,\ \text{when } \lambda = 1;\ |\varsigma| < |\log(-\lambda)|, \text{when } \lambda \neq 1;\ 1^m := 1).$$

and

$$\left(\frac{2\varsigma}{\lambda e^{\varsigma}+1}\right)^m e^{u\varsigma} = \sum_{r=0}^{\infty} \mathcal{G}_r^{(m)}(u;\lambda)\frac{\varsigma^r}{r!} \tag{20}$$

$$(|\varsigma| < \pi,\ \text{when } \lambda = 1;\ |\varsigma| < |\log(-\lambda)|, \text{when } \lambda \neq 1;\ 1^m := 1).$$

Setting $u = 0$ in (18)–(20) results in the generalized Apostol–Bernoulli, generalized Apostol–Euler, and generalized Apostol–Genocchi numbers, respectively, which are defined as follows:

$$\begin{aligned}\mathcal{B}_r^{(m)}(\lambda) &:= \mathcal{B}_r^{(m)}(0;\lambda);\\ \mathcal{E}_r^{(m)}(\lambda) &:= \mathcal{E}_r^{(m)}(0;\lambda);\\ \mathcal{G}_r^{(m)}(\lambda) &:= \mathcal{G}_r^{(m)}(0;\lambda).\end{aligned} \tag{21}$$

Obviously , the following relations hold:

$$\begin{aligned}\mathcal{B}_r^{(m)}(u;1) &= \mathcal{B}_r^{(m)}(u);\\ \mathcal{E}_r^{(m)}(u;1) &= \mathcal{E}_r^{(m)}(u);\\ \mathcal{G}_r^{(m)}(u;1) &= \mathcal{G}_r^{(m)}(u).\end{aligned} \tag{22}$$

3. Generalized Apostol–Genocchi-Based Poly-Daehee Polynomials

This section introduces and investigates a unified class of polynomials called the Apostol–Genocchi-based poly-Daehee polynomials. Certain identities and explicit formulae for these polynomials are derived.

Definition 1. *The Apostol–Genocchi-based poly-Daehee polynomials ${}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,v;\lambda)$ (abbreviated by AGPD) are defined by the following generating function:*

$$\frac{\log(1+\varsigma)}{\mathrm{Li}_{\kappa}(1-e^{-\varsigma})}(1+\varsigma)^u\left(\frac{2\varsigma}{\lambda e^{\varsigma}+1}\right)^m e^{v\varsigma} = \sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,v;\lambda)\frac{\varsigma^r}{r!} \tag{23}$$

$$\begin{gathered}(m,\ u,\ v \in \mathbb{C};\ \kappa \in \mathbb{N};\\ |\varsigma| < 1,\ \textit{when } \lambda = 1;\ |\varsigma| < \min\{|\log(-\lambda)|, 1\},\ \textit{when } \lambda \neq 1;\ 1^m := 1).\end{gathered}$$

Furthermore, ${}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(\lambda) := {}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(0,0,\lambda)$ *are called Apostol–Genocchi-based poly-Daehee numbers.*

Remark 1. *Let the generating function on the left-member of* (23) *be denoted by*

$$g(m,\, u,\, v,\, \kappa,\, \lambda;\, \varsigma) = g_1(\kappa;\, \varsigma)\, g_2(m,\, u,\, v,\, \lambda;\, \varsigma), \tag{24}$$

where

$$g_1(\kappa;\, \varsigma) := \frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}$$

and

$$g_2(m,\, u,\, v,\, \lambda;\, \varsigma) := (1+\varsigma)^u \left(\frac{2\varsigma}{\lambda e^{\varsigma}+1}\right)^m e^{v\varsigma}.$$

The right-member of (23) *is the Maclaurin series centered at* $\varsigma = 0$*. So the generating function on the left-member of* (23) *should be analytic at* $\varsigma = 0$*. In view of* (1), $\mathrm{Li}_\kappa(1-e^{-\varsigma})|_{\varsigma=0} = 0$ *and* $\varsigma = 0$ *may be a singular point of the generating function. Here we find*

$$\frac{d}{d\varsigma}\mathrm{Li}_\kappa(1-e^{-\varsigma})|_{\varsigma=0} = \sum_{s=1}^{\infty} \frac{(1-e^{-\varsigma})^{s-1}}{s^{\kappa-1}}\, e^{-\varsigma}\Big|_{\varsigma=0} = 1.$$

Furthermore, by using L'Hospital's rule,

$$\lim_{\varsigma\to 0} g_1(\kappa;\, \varsigma) = \lim_{\varsigma\to 0} \frac{1}{1+\varsigma} \Big/ \frac{d}{d\varsigma}\mathrm{Li}_\kappa(1-e^{-\varsigma}) = 1.$$

Note that $g_2(m,\, u,\, v,\, \lambda;\, \varsigma)$ *is analytic at* $\varsigma = 0$*. We thus find that* $\varsigma = 0$ *is a removable singular point of the generating function. Therefore,* $\varsigma = 0$ *can be an analytic point of the generating function.*

As noted in (25), the poly-Daehee numbers $\mathcal{D}_r^{(\kappa)}$ are given by the following generating function:

$$\frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})} = \sum_{r=0}^{\infty} \mathcal{D}_r^{(\kappa)} \frac{\varsigma^r}{r!} \quad (\kappa \in \mathbb{N}). \tag{25}$$

In order to use later in this work, we introduce the other sequence of numbers, which are similar to the poly-Daehee numbers, in the following definition.

Definition 2. *The sequence of numbers* $\Omega_r^{(\kappa)}$ $(r \in \mathbb{N}_0)$ *is defined by the following generating function*

$$g(\kappa;\varsigma) := \frac{\varsigma}{\mathrm{Li}_\kappa(1-e^{-\varsigma})} := \sum_{r=0}^{\infty} \Omega_r^{(\kappa)}\, \varsigma^r \quad (\kappa \in \mathbb{N}). \tag{26}$$

Remark 2. *We observe the following properties for the numbers* $\Omega_r^{(\kappa)}$*:*

(i) *We find*

$$\lim_{\varsigma\to 0} \frac{\varsigma}{\mathrm{Li}_\kappa(1-e^{-\varsigma})} = 1 = \Omega_0^{(\kappa)}.$$

This means that $g(\kappa;\varsigma)$ *is analytic at* $\varsigma = 0$ *and so can be expanded as the Maclaurin series in a neighborhood (possibly small) of* 0 *as in the right member of* (26)*.*

(ii) *By the help of Mathematica, we compute*

$$\begin{aligned}
\Omega_1^{(\kappa)} &= \frac{1}{2} - 2^{-\kappa}, \\
\Omega_2^{(\kappa)} &= \frac{1}{6} + 2^{1-2\kappa} - 2 \times 3^{-\kappa}, \\
&\cdots
\end{aligned}$$

(iii) *Here let*

$$\mathrm{Li}_\kappa(1-e^{-\varsigma}) := \sum_{j=0}^{\infty} \Lambda_j^{(\kappa)} \varsigma^j. \tag{27}$$

By the aid of Mathematica, we find

$$\begin{aligned}
\Lambda_0^{(\kappa)} &= 0,\\
\Lambda_1^{(\kappa)} &= 1,\\
\Lambda_2^{(\kappa)} &= -\frac{1}{2} + 2^{-\kappa},\\
\Lambda_3^{(\kappa)} &= \frac{1}{6} - 2^{-\kappa} + 3^{-\kappa},\\
&\cdots
\end{aligned}$$

(iv) *From* (26) *and* (27), *we have*

$$\varsigma = \sum_{r=0}^{\infty} \Lambda_r^{(\kappa)} \varsigma^r \sum_{s=0}^{\infty} \Omega_s^{(\kappa)} \varsigma^s \quad = \sum_{r=0}^{\infty} \sum_{s=0}^{r} \Lambda_{r-s}^{(\kappa)} \Omega_s^{(\kappa)} \varsigma^r,$$

from which we obtain

$$\sum_{s=0}^{r} \Lambda_{r-s}^{(\kappa)} \Omega_s^{(\kappa)} = 0 \quad (r \in \mathbb{N} \setminus \{1\}). \tag{28}$$

Due to the AGPD's generic nature, they may reduce to a number of new and known polynomials, some of which are included in Table 1.

Table 1. Some known polynomials occurring as special cases of AGPD.

Case	$m, \kappa, u, v, \varsigma$	Generating Function	Name of the Polynomials
I.	$m = 0 = v$	$\frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}(1+\varsigma)^u = \mathcal{D}_r^{(\kappa)}(u)\frac{\varsigma^r}{r!}$	poly-Daehee polynomials [28]
II.	$m = 0 = v;\ \kappa = 1$	$\frac{\log(1+\varsigma)}{\varsigma}(1+\varsigma)^u = \mathcal{D}_r(u)\frac{\varsigma^r}{r!}$	Daehee polynomials [27,28]
III.	$m = 0 = v;\ \kappa = 1;\ \lambda \in \mathbb{N}$	$\left(\frac{\log(1+\varsigma)}{\varsigma}\right)^\lambda (1+\varsigma)^u = \mathcal{D}_r^\lambda(u)\frac{\varsigma^r}{r!}$	Higher order Daehee polynomials [19]
IV.	$m = 0 = v;\ \kappa = 1; \varsigma = \xi\varsigma$	$\left(\frac{\log(1+\xi\varsigma)}{\xi\varsigma}\right)(1+\xi\varsigma)^u = \mathcal{D}_{r,\xi}(u)\frac{\varsigma^r}{r!}$	r-th twisted Daehee polynomials [29]
V.	$m = 0 = v;\ \kappa = 1; u \to 1-u$	$(1+\varsigma)\left(\frac{\log(1+\varsigma)}{\varsigma}\right)\frac{1}{(1+\varsigma)^u} = \mathcal{D}_{r,\xi}^\lambda(u)\frac{\varsigma^r}{r!}$	Daehee polynomials of second kind [27]

Theorem 1. *The Apostol–Genocchi-based poly-Daehee polynomials are explicitly given by*

$${}_{G}\mathcal{D}_{r,m}^{(\kappa)}(u,v;\lambda) = \sum_{s=0}^{r} \binom{r}{s} \mathcal{D}_{r-s}^{(\kappa)}(u)\, \mathcal{G}_s^{(m)}(v;\lambda) \quad (r \in \mathbb{N}_0) \tag{29}$$

Here, the constraints of parameters and variable would be modified relative to those in (23).

Proof. We first recall the following well-known double series manipulation: Let $f, g : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{C}$ be functions and $p \in \mathbb{N}$. Then

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} f(k, n-pk), \tag{30}$$

where the involved double series is assumed to be absolutely convergent.

It is straightforward from (23) and (20) that

$$\sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,v;\lambda)\frac{\varsigma^r}{r!} = \frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}(1+\varsigma)^u\left(\frac{2\varsigma}{\lambda e^\varsigma+1}\right)^m e^{v\varsigma}$$

$$= \left(\sum_{r=0}^{\infty} \mathcal{D}_r^{(\kappa)}\frac{\varsigma^r}{r!}\right)\left(\sum_{s=0}^{\infty} \mathcal{G}_s^{(m)}(v;\lambda)\frac{\varsigma^s}{s!}\right).$$

Using the series rearrangement for the case $p = 1$ in (30), we now obtain

$$\sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,v;\lambda)\frac{\varsigma^r}{r!} = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \mathcal{D}_{r-s}^{(\kappa)}(u)\frac{\varsigma^{r-s}}{(r-s)!}\,\mathcal{G}_s^{(m)}(v;\lambda)\frac{\varsigma^s}{s!},$$

which, upon equating the coefficients of like powers of ς, immediately yields the desired assertion of Theorem 1. □

Theorem 2. *The following identity for* AGPD *holds true:*

$$ {}_{\mathcal{G}}\mathcal{D}_{r,m}^{\kappa}(u,v;\lambda) = \frac{{}_{\mathcal{G}}\mathcal{D}_{r+1,m}^{(\kappa)}(u+1;v) - {}_{\mathcal{G}}\mathcal{D}_{r+1,m}^{(\kappa)}(u,v)}{r+1} \quad (r \in \mathbb{N}_0). \tag{31}$$

Here, the constraints of parameters and variable would be adjusted with respect to those in (23).

Proof. Using (23), we write

$$\begin{aligned}
&\sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u+1,v;\lambda)\frac{\varsigma^r}{r!} - \sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,v;\lambda)\frac{\varsigma^r}{r!}\\
&= \left\{\frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}(1+\varsigma)^{u+1}\left(\frac{2\varsigma}{\lambda e^\varsigma+1}\right)^m e^{v\varsigma}\right\}\\
&\quad - \left\{\frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}(1+\varsigma)^{u}\left(\frac{2\varsigma}{\lambda e^\varsigma+1}\right)^m e^{v\varsigma}\right\}\\
&= \sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,v;\lambda)\frac{\varsigma^{r+1}}{r!},
\end{aligned}$$

from which, we have

$$\sum_{r=1}^{\infty}\left[{}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u+1,v;\lambda) - {}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,v;\lambda)\right]\frac{\varsigma^r}{r!} = \sum_{r=1}^{\infty} {}_{\mathcal{G}}\mathcal{D}_{r-1,m}^{(\kappa)}(u,v;\lambda)\frac{\varsigma^r}{(r-1)!}. \tag{32}$$

Now, equating the coefficients of ς^r in both sides of (32) yields the desired identity (31). □

Theorem 3. AGPD *satisfy the following addition property:*

$$ {}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u+\alpha,v;\lambda) = \sum_{s=0}^{r}\binom{r}{s}\langle\alpha\rangle_s\, {}_{\mathcal{G}}\mathcal{D}_{r-s,m}^{(\kappa)}(u,v) \quad (r \in \mathbb{N}_0), \tag{33}$$

where $\langle\alpha\rangle_s$ *is the well known falling factorial defined as*

$$\langle\alpha\rangle_s := \alpha(\alpha-1)\cdots(\alpha-s+1). \tag{34}$$

Here $\alpha \in \mathbb{C}$, *and the restrictions of the other parameters and variable would be modified in light of those in* (23).

Proof. Substituting $u+\alpha$ for u in (23) gives

$$\begin{aligned}\sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(r+\alpha,v;\lambda)\frac{\varsigma^r}{r!} &= \frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}(1+\varsigma)^u\left(\frac{2\varsigma}{\lambda e^{\varsigma}+1}\right)^m e^{v\varsigma}\,(1+\varsigma)^\alpha \\ &= \left(\sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,v;\lambda)\frac{\varsigma^r}{r!}\right)\left(\sum_{s=0}^{\infty}\langle\alpha\rangle_s\frac{\varsigma^s}{s!}\right) \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{r} {}_{\mathcal{G}}\mathcal{D}_{r-s,m}^{(\kappa)}(u,v;\lambda)\langle\alpha\rangle_s\frac{\varsigma^r}{(r-s)!\,s!},\end{aligned}$$

for the last equality of which the case $p=1$ in (30) is used. Finally, comparing the coefficient of ς^r on both sides offers the desired identity. □

Theorem 4. *For $r\in\mathbb{N}_0$, the following correlation holds true:*

$$\sum_{s=0}^{r}\binom{r}{s}\mathcal{B}_s\,{}_{\mathcal{G}}\mathcal{D}_{r-s,m}(u,v;\lambda)=\sum_{s=0}^{r}\binom{r}{s}\mathcal{B}_s^{(\kappa)}\,{}_{\mathcal{G}}\mathcal{D}_{r-s,m}^{(\kappa)}(u,v;\lambda). \tag{35}$$

Here, the restrictions of the parameters and variable would be modified in light of those in (23).

Proof. Using (10) and (23) reveals

$$\begin{aligned}&\frac{\log(1+\varsigma)}{e^{\varsigma}-1}(1+\varsigma)^u\left(\frac{2\varsigma}{\lambda e^{\varsigma}+1}\right)^m e^{v\varsigma} \\ &=\left\{\frac{\mathrm{Li}_\kappa(1-e^{-\varsigma})}{e^{\varsigma}-1}\right\}\left\{\frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}(1+\varsigma)^u\left(\frac{2\varsigma}{\lambda e^{\varsigma}+1}\right)^m e^{v\varsigma}\right\} \\ &=\frac{\mathrm{Li}_\kappa(1-e^{-\varsigma})}{e^{\varsigma}-1}\,\frac{\log(1+\varsigma)}{L_{i\kappa}(1-e^{-\varsigma})}(1+\varsigma)^u\left(\frac{2u}{\lambda e^{\varsigma}+1}\right)^m e^{v\varsigma} \\ &=\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{r}\binom{r}{s}\mathcal{B}_s^{(\kappa)}\,{}_{\mathcal{G}}\mathcal{D}_{r-s,m}^{(\kappa)}(u,v;\lambda)\right\}\frac{\varsigma^r}{r!}.\end{aligned} \tag{36}$$

Using (7) and rewriting the left hand side of (36) leads to the Apostol–Genocchi-based poly-Daehee polynomials

$$\begin{aligned}&\frac{\log(1+\varsigma)}{e^{\varsigma}-1}(1+\varsigma)^u\left(\frac{2\varsigma}{\lambda e^{\varsigma}+1}\right)^m e^{v\varsigma} \\ &=\frac{\varsigma}{e^{\varsigma}-1}\,\frac{\log(1+\varsigma)}{\varsigma}(1+\varsigma)^u\left(\frac{2u}{\lambda e^{u}+1}\right)^m e^{v\varsigma} \\ &=\left(\sum_{s=0}^{\infty}\mathcal{B}_s\frac{\varsigma^s}{s!}\right)\left(\sum_{r=0}^{\infty}{}_{\mathcal{G}}\mathcal{D}_{r,m}(u,v;\lambda)\frac{\varsigma^r}{r!}\right) \\ &=\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{r}\binom{r}{s}\mathcal{B}_s\,{}_{\mathcal{G}}\mathcal{D}_{r-m,m}(u,v;\lambda)\right\}\frac{\varsigma^r}{r!}.\end{aligned} \tag{37}$$

Therefore, in view of (36) and (37), we can easily arrive at the desired result. □

Theorem 5. *For $r\in\mathbb{N}_0$, the following relation holds true:*

$$\mathcal{G}_{r,m}(v;\lambda)=\sum_{s=0}^{r}\binom{r}{s}b_s^{(\kappa)}(-u)\,{}_{\mathcal{G}}\mathcal{D}_{r-s,m}^{(\kappa)}(u,v,\lambda). \tag{38}$$

Proof. From (20) and (23), we can write

$$\sum_{r=0}^{\infty} \mathcal{G}_{r,m}(v;\lambda)\frac{\varsigma^r}{r!} = \left(\frac{2\varsigma}{\lambda e^{\varsigma}+1}\right)^m e^{v\varsigma}$$
$$= \frac{\mathrm{Li}_\kappa(1-e^{-\varsigma})}{\log(1+\varsigma)}(1+\varsigma)^{-u}\sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}^{\kappa}_{r,m}(u,v,\lambda)\frac{\varsigma^r}{r!}.$$

Now using (8), we have

$$\begin{aligned}\sum_{r=0}^{\infty} \mathcal{G}_{r,m}(v;\lambda)\frac{\varsigma^r}{r!} &= \left(\sum_{s=0}^{\infty} b_s^{(\kappa)}(-u)\frac{\varsigma^s}{s!}\right)\left(\sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m}(u,v,\lambda)\frac{\varsigma^r}{r!}\right)\\ &= \sum_{n=0}^{\infty}\left\{\sum_{s=0}^{r}\binom{r}{s} b_s^{(\kappa)}(-u)\, {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r-z,m}(u,v,\lambda)\right\}\frac{\varsigma^r}{r!}.\end{aligned} \tag{39}$$

Using the series rearrangement technique in (30) and equating the coefficients of like powers of ς in (39), yields (38). □

Theorem 6. *The following formula for the Apostol–Genocchi-based poly-Daehee polynomials holds*

$${}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m+\beta}(u,v+\mu;\lambda) = \sum_{s=0}^{r}\binom{r}{s}\, {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r-s,m}(u,v+\mu;\lambda)\, G_s^{(\beta)}(\mu;\lambda). \tag{40}$$

Proof. By replacing v by $v+\mu$ and m by $m+\beta$ in (23) and using (20), we get

$$\begin{aligned}\sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m+\beta}(u,v+\mu;\lambda)\frac{\varsigma^r}{r!} &= \frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}(1+\varsigma)^u\left(\frac{2\varsigma}{\lambda e^{\varsigma}+1}\right)^{m+\beta} e^{(v+\mu)\varsigma}\\ &= \left(\sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m}(u,v;\lambda)\frac{\varsigma^r}{r!}\right)\left(\sum_{s=0}^{\infty} \mathcal{G}_s^{(\beta)}(\mu;\lambda)\frac{varsigma^s}{S1}\right)\\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{r} {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r-s,m}(u,v;\lambda)\mathcal{G}_s^{(\beta)}(\mu;\lambda)\,\frac{\varsigma^r}{(r-s)!s!}\end{aligned}$$

which yields the required result (40). □

Theorem 7. *The following correlation holds*

$${}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m}(u,v;\lambda) = \sum_{s=0}^{r}\binom{r}{s}\, \mathcal{D}^{(\kappa)}_{r-s,m}(u,v-\alpha)\, \mathcal{G}_s^{(m)}(\alpha;\lambda). \tag{41}$$

Proof. By (23), we can write

$$\begin{aligned}&\frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}(1+\varsigma)^u\left(\frac{2\varsigma}{\lambda e^{\varsigma}+1}\right)^{m} e^{(v-\alpha)\varsigma+\alpha\varsigma}\\ &= \left(\sum_{r=0}^{\infty} \mathcal{D}^{(\kappa)}_{r}(u,v-\alpha)\frac{\varsigma^r}{r!}\right)\left(\sum_{s=0}^{\infty} \mathcal{G}_s^{(m)}(\alpha;\lambda)\frac{\varsigma^s}{s!}\right).\end{aligned}$$

By using the series manipulation for the case $p=1$ in (30), we get

$$\sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m}(u,v;\lambda)\frac{\varsigma^r}{r!} = \sum_{r=0}^{\infty}\left(\sum_{s=0}^{r}\binom{r}{s}\, \mathcal{D}^{(\kappa)}_{r-s,m}(u,v-\alpha)\, \mathcal{G}_s^{(m)}(\alpha;\lambda)\right)\frac{\varsigma^r}{r!},$$

which, upon equating the coefficients of the similar powers of ς, leads to the desired identity. □

Theorem 8. *The following summation formula holds true:*

$$ {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m}(u,v+1;\lambda) = \sum_{s=0}^{r} \binom{r}{s} {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r-s,m}(u,v;\lambda). \tag{42} $$

Proof. Replace the parameter v with $v+1$ in (23). By using similar process as those in previous theorems, we can get the identity (42). Therefore, the details are omitted. □

4. Differential Formulas

This section establishes two differential formulas for AGPD with respect to the parameters u and v.

Theorem 9. *The following differential formula holds true.*

$$ \frac{\partial}{\partial u} {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m}(u,v;\lambda) = \sum_{s=0}^{r-1} \frac{(-1)^{r-s-1} r!}{(r-s)\, s!} {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{s,m}(u,v;\lambda) \quad (r \in \mathbb{N}). \tag{43} $$

Proof. Differentiating both sides of (23) with respect to u and using the notation in (24), with the aid of the case $p=1$ in (30), we have

$$ \begin{aligned} \sum_{r=0}^{\infty} \frac{\partial}{\partial u} {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m}(u,v;\lambda) \frac{\varsigma^r}{r!} &= \log(1+u) \cdot g(m,\, u,\, v,\, \kappa,\, \lambda;\, \varsigma) \\ &= \left(\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} \varsigma^r \right) \left(\sum_{s=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{s,m}(u,v;\lambda) \frac{\varsigma^s}{s!} \right) \\ &= \sum_{r=0}^{\infty} \sum_{s=0}^{r-1} \frac{(-1)^{r-s-1}}{r-s} {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{s,m}(u,v;\lambda) \frac{\varsigma}{s!}, \end{aligned} $$

which, upon equating the coefficients of ς^r, yields the desired identity. □

Theorem 10. *The following differential formula holds true.*

$$ \frac{\partial}{\partial v} {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m}(u,v;\lambda) = r\, {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r-1,m}(u,v;\lambda) \quad (r \in \mathbb{N}). \tag{44} $$

Proof. Differentiating both sides of (23) with respect to v and using the similar process as in the proof of Theorem 10, we may obtain (44). So the specifics are omitted. □

5. Integral Formulas

This section establishes two integral formulas for AGPD.

Theorem 11. *The following integral formula holds true.*

$$ \int_0^1 {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m}(u,v;\lambda)\, dv = \frac{1}{r+1} \left\{ {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r+1,m}(u,1;\lambda) - {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r+1,m}(u,0;\lambda) \right\} \quad (r \in \mathbb{N}_0). \tag{45} $$

Proof. Integrating both sides of (23) with respect to the parameter v from 0 to 1, we by using (44) get

$$ \frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})} (1+\varsigma)^u \left(\frac{2\varsigma}{\lambda e^\varsigma + 1} \right)^m \cdot \frac{1}{\varsigma}(e^\varsigma - 1) = \sum_{r=0}^{\infty} \int_0^1 {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m}(u,v;\lambda) dv \, \frac{\varsigma^r}{r!}. \tag{46} $$

Multiplying both sides of (46) by ς and using (23), we obtain

$$\sum_{r=0}^{\infty}\left\{{}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,1;\lambda)-{}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,0;\lambda)\right\}\frac{\varsigma^r}{r!}=\sum_{r=1}^{\infty}\int_0^1 {}_{\mathcal{G}}\mathcal{D}_{r-1,m}^{(\kappa)}(u,v;\lambda)dv\,\frac{\varsigma^r}{(r-1)!}. \tag{47}$$

Equating the coefficients of ς^r on both sides of (47), we derive

$$\int_0^1 {}_{\mathcal{G}}\mathcal{D}_{r-1,m}^{(\kappa)}(u,v;\lambda)dv=\frac{1}{r}\left\{{}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,1;\lambda)-{}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,0;\lambda)\right\},$$

which, upon setting $r=r'+1$ and dropping the prime on r, yields the desired formula. □

Theorem 12. *The following integral formula holds true.*

$$\int_0^1 {}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,v;\lambda)\,du=\sum_{s=0}^{r}\frac{r!}{s!}\,\Omega_{r-s}^{(\kappa)}\,\mathcal{G}_s^{(m)}(v;\lambda)\quad(r\in\mathbb{N}_0), \tag{48}$$

where $\mathcal{G}_r^{(m)}(u;\lambda)$ *are the polynomials in* (20) *and* $\Omega_r^{(\kappa)}$ *are the numbers in* (26).

Proof. We find

$$\int_0^1 (1+\varsigma)^u\,du=\frac{\varsigma}{\log(1+\varsigma)}. \tag{49}$$

Integrating both sides of (23) with respect to the parameter u from 0 to 1 and using (49), we obtain

$$\frac{\varsigma}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}\cdot\left(\frac{2\varsigma}{\lambda e^{\varsigma}+1}\right)^m e^{v\varsigma}=\sum_{r=0}^{\infty}\int_0^1 {}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,v;\lambda)du\,\frac{\varsigma^r}{r!}. \tag{50}$$

Employing (26) and (20) for the first and the second factors, respectively, in the left-member of (50), with similar process of proofs of the previous formulas, we derive

$$\begin{aligned}\sum_{r=0}^{\infty}\int_0^1 {}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,v;\lambda)du\,\frac{\varsigma^r}{r!}&=\sum_{r=0}^{\infty}\Omega_r^{(\kappa)}\,\varsigma^r\sum_{s=0}^{\infty}\mathcal{G}_s^{(m)}(v;\lambda)\frac{\varsigma^s}{s!}\\&=\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{1}{s!}\,\Omega_{r-s}^{(\kappa)}\mathcal{G}_s^{(m)}(v;\lambda)\,\varsigma^r,\end{aligned} \tag{51}$$

on the first and last members of which, upon equating the coefficients of ς^r, we obtain the desired formula. □

6. An Implicit Summation Formula

This section explores an implicit summation formula for AGPD.

Theorem 13. *The Apostol–Genocchi-based poly-Daehee polynomials satisfy the following implicit summation formula:*

$$ {}_{\mathcal{G}}\mathcal{D}_{q+l,m}^{(\kappa)}(u,\alpha;\lambda)=\sum_{r=0}^{q}\sum_{p=0}^{l}\binom{q}{r}\binom{l}{p}(\alpha-v)^{r+p}\,{}_{\mathcal{G}}\mathcal{D}_{q+l-p-r,m}^{(\kappa)}(u,v;\lambda) \tag{52}$$

$$(l,\,q\in\mathbb{N}_0;\ \alpha\in\mathbb{C}).$$

Here the restrictions of the other parameters and variable would be modified in light of those in (23).

Proof. We first recall the following series manipulation formula (consult, for example, ([36], p. 52, Equation (2)) and [37–41]):

$$\sum_{R=0}^{\infty}f(R)\frac{(u+v)^R}{R!}=\sum_{r,s=0}^{\infty}f(r+s)\frac{u^r}{r!}\frac{v^s}{s!}. \tag{53}$$

Replacing ς by $\varsigma+\mu$ in (23) gives

$$\begin{aligned}&\frac{\log(1+(\varsigma+\mu))}{\mathrm{Li}_\kappa(1-e^{-(\varsigma+\mu)})}(1+(\varsigma+\mu))^u\left(\frac{2(\varsigma+\mu)}{\lambda e^{\varsigma+\mu}+1}\right)^m\\&\qquad=e^{-v(\varsigma+\mu)}\sum_{r=0}^{\infty}{}_{\mathcal{G}}\mathcal{D}_{r,m}^{(\kappa)}(u,v;\lambda)\frac{(\varsigma+\mu)^r}{r!}.\end{aligned}\tag{54}$$

Employing (53) in the series on the right-hand side of (54), we obtain

$$\begin{aligned}&\frac{\log(1+(\varsigma+\mu))}{\mathrm{Li}_\kappa(1-e^{-(\varsigma+\mu)})}(1+(\varsigma+\mu))^u\left(\frac{2(\varsigma+\mu)}{\lambda e^{\varsigma+\mu}+1}\right)^m\\&\qquad=e^{-v(\varsigma+\mu)}\sum_{q,l=0}^{\infty}{}_{\mathcal{G}}\mathcal{D}_{q+l,m}^{(\kappa)}(u,v,\lambda)\frac{\varsigma^q}{q!}\frac{\mu^l}{l!}.\end{aligned}\tag{55}$$

Note that the left-member of (55) is independent of the parameter v and so, for any $\alpha\in\mathbb{C}$,

$$e^{-v(\varsigma+\mu)}\sum_{q,l=0}^{\infty}{}_{\mathcal{G}}\mathcal{D}_{q+l,m}^{(\kappa)}(u,v,\lambda)\frac{\varsigma^q}{q!}\frac{\mu^l}{l!}=e^{-\alpha(\varsigma+\mu)}\sum_{q,l=0}^{\infty}{}_{\mathcal{G}}\mathcal{D}_{q+l,m}^{(\kappa)}(u,\alpha,\lambda)\frac{\varsigma^q}{q!}\frac{\mu^l}{l!}.$$

Or, equivalently,

$$e^{(\alpha-v)(\varsigma+\mu)}\sum_{q,l=0}^{\infty}{}_{\mathcal{G}}\mathcal{D}_{q+l,m}^{(\kappa)}(u,v;\lambda)\frac{\varsigma^q}{q!}\frac{\mu^l}{l!}=\sum_{q,l=0}^{\infty}{}_{\mathcal{G}}\mathcal{D}_{q+l,m}^{(\kappa)}(u,\alpha;\lambda)\frac{\varsigma^q}{q!}\frac{\mu^l}{l!},\tag{56}$$

for any $\alpha\in\mathbb{C}$. Using (53), we get

$$e^{(\alpha-v)(\varsigma+\mu)}=\sum_{R=0}^{\infty}\frac{[(\alpha-v)(\varsigma+\mu)]^R}{R!}=\sum_{r,p=0}^{\infty}\frac{(\alpha-v)^{r+p}\varsigma^r\mu^p}{r!p!}.\tag{57}$$

Setting (57) in (56), we have

$$\sum_{q,r=0}^{\infty}\sum_{l,p=0}^{\infty}\frac{(\alpha-v)^{r+p}\varsigma^r\mu^p}{r!p!}\,{}_{\mathcal{G}}\mathcal{D}_{q+l,m}^{(\kappa)}(u,v;\lambda)\frac{\varsigma^q}{q!}\frac{\mu^l}{l!}=\sum_{q,l=0}^{\infty}{}_{\mathcal{G}}\mathcal{D}_{q+l,m}^{(\kappa)}(u,\alpha;\lambda)\frac{\varsigma^q}{q!}\frac{\mu^l}{l!}.\tag{58}$$

Here, using the series manipulation technique for the case $p=1$ in (30) in each one of two double series in the left-member of (58), we find

$$\begin{aligned}&\sum_{q,l=0}^{\infty}\sum_{r=0}^{q}\sum_{p=0}^{l}(\alpha-v)^{r+p}\,{}_{\mathcal{G}}\mathcal{D}_{q+l-r-p,m}^{(\kappa)}(u,v;\lambda)\frac{\varsigma^q}{r!(q-r)!}\frac{\mu^l}{p!(l-p)!}\\&=\sum_{q,l=0}^{\infty}{}_{\mathcal{G}}\mathcal{D}_{q+l,m}^{(\kappa)}(u,\alpha;\lambda)\frac{\varsigma^q}{q!}\frac{\mu^l}{l!}.\end{aligned}\tag{59}$$

Finally, equating the coefficients of $\varsigma^q\mu^l$ on both sides of (59), we prove the desired identity. □

Remark 3. *It may be interesting to observe that the left-member of* (52) *is independent of the parameter v in the right-member of* (52)*. In particular,*

$$\begin{aligned}&\sum_{r=0}^{q}\sum_{p=0}^{l}\binom{q}{r}\binom{l}{p}(\alpha-v)^{r+p}\,{}_{\mathcal{G}}\mathcal{D}_{q+l-p-r,m}^{(\kappa)}(u,v;\lambda)\\&\qquad=\sum_{r=0}^{q}\sum_{p=0}^{l}\binom{q}{r}\binom{l}{p}(\alpha)^{r+p}\,{}_{\mathcal{G}}\mathcal{D}_{q+l-p-r,m}^{(\kappa)}(u,0;\lambda).\end{aligned}$$

for any $v \in \mathbb{C}$.

7. Concluding Remarks

The polynomials defined in (23) arises from the well known Apostol–Genocchi polynomials defined in (20). They exhibit a close relationship with Apostol–Euler and Apostol–Bernoulli polynomials. Therefore, we can explore other hybrid polynomials and obtain their corresponding properties as well as some new results. Table 2 below illustrates some hybrid polynomials similar to the Apostol–Genocchi-based poly-Daehee polynomials in (23).

Table 2. Members similar to the polynomials ${}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m}(u,v;\lambda)$.

S. No.	Name of Polynomial	$\mathcal{A}(\varsigma)$	Generating Function
I.	Apostol–Euler-based poly-Daehee polynomials	$\left(\frac{2}{\lambda e^{\varsigma}+1}\right)^m \frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}(1+\varsigma)^u$	$\sum_{r=0}^{\infty} {}_{\mathcal{E}}\mathcal{D}^{(\kappa)}_{r,m}(u,v;\lambda)\frac{\varsigma^r}{r!} = \left(\frac{2}{\lambda e^{\varsigma}+1}\right)^m \frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}(1+\varsigma)^u e^{v\varsigma}$
II.	Apostol–Bernoulli-based poly-Daehee polynomials	$\left(\frac{\varsigma}{\lambda e^{\varsigma}-1}\right)^m \frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}(1+\varsigma)^u$	$\sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m}(u,v;\lambda)\frac{\varsigma^r}{r!} = \left(\frac{\varsigma}{\lambda e^{\varsigma}-1}\right)^m \frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}(1+\varsigma)^u e^{v\varsigma}$
III.	Apostol–Bernoulli-based Daehee polynomials	$\left(\frac{\varsigma}{\lambda e^{\varsigma}-1}\right)^m \frac{\log(1+\varsigma)}{\varsigma}(1+\varsigma)^u$	$\sum_{r=0}^{\infty} {}_{\mathcal{B}}\mathcal{D}_{r,m}(u,v;\lambda)\frac{\varsigma^r}{r!} = \left(\frac{\varsigma}{\lambda e^{\varsigma}-1}\right)^m \frac{\log(1+\varsigma)}{\varsigma}(1+\varsigma)^u e^{v\varsigma}$
IV.	Apostol–Euler-based Daehee polynomials	$\left(\frac{2}{\lambda e^{\varsigma}+1}\right)^m \frac{\log(1+\varsigma)}{\varsigma}(1+\varsigma)^u$	$\sum_{r=0}^{\infty} {}_{\mathcal{E}}\mathcal{D}_{r,m}(u,v;\lambda)\frac{\varsigma^r}{r!} = \left(\frac{2}{\lambda e^{\varsigma}+1}\right)^m \frac{\log(1+\varsigma)}{\varsigma}(1+\varsigma)^u e^{v\varsigma}$
V.	Apostol–Genocchi-based Daehee polynomials	$\left(\frac{2\varsigma}{\lambda e^{\varsigma}+1}\right)^m \frac{\log(1+\varsigma)}{\varsigma}(1+\varsigma)^u$	$= \sum_{r=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}_{r,m}(u,v;\lambda)\frac{\varsigma^r}{r!} = \left(\frac{2\varsigma}{\lambda e^{\varsigma}+1}\right)^m \frac{\log(1+\varsigma)}{\varsigma}(1+\varsigma)^u e^{v\varsigma}$

Example 1 may show how to define some polynomials by means od operational forms.

Example 1. *Using the ordinary derivative operator $\hat{D}_u$, an operational form to define the generalized Apostol–Genocchi polynomials in (20) could be*

$$\left(\frac{2\hat{D}_u}{\lambda e^{\hat{D}_u}+1}\right)^m v^r = \mathcal{G}^{(m)}_r(v;\lambda). \tag{60}$$

Similarly, the generalized Apostol–Euler polynomials in (19) can be cast as

$$\left(\frac{2}{\lambda e^{\hat{D}_u}+1}\right)^m v^r = \mathcal{E}^{(m)}_r(v;\lambda), \tag{61}$$

which, on comparing with (60), provides the relation

$$\mathcal{G}^{(m)}_r(v;\lambda) = (\hat{D}_u)^m \mathcal{E}^{(m)}_r(v;\lambda) = \langle r\rangle_m\, \mathcal{E}^{(m)}_{r-m}(v;\lambda). \tag{62}$$

From (23) and (60), we can write

$${}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m}(u,v;\lambda) = \frac{\log(1+\varsigma)}{\mathrm{Li}_\kappa(1-e^{-\varsigma})}(1+\varsigma)^u\left(\frac{2\varsigma}{\lambda e^{\varsigma}+1}\right)^m v^r, \tag{63}$$

which, in view of (62), provides

$$ {}_{\mathcal{G}}\mathcal{D}^{(\kappa)}_{r,m}(u,v;\lambda) = \langle r \rangle_m \, \mathcal{E}^{(\kappa)}_{r-m,m}(u,v;\lambda). \tag{64} $$

Author Contributions: Conceptualization, T.U. and S.A.-O.; methodology, K.N.; software, T.U.; validation, J.C., T.U. and M.A.; formal analysis, T.U.; investigation, T.U.; resources, S.A.-O.; data curation, S.A.-O.; writing—original draft preparation, N.K.; writing—review and editing, S.A.-O.; writing—review and editing, J.C.; visualization, M.A.; supervision, S.A.-O.; project administration, S.A.-O.; funding acquisition, K.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research has received funding support from the National Science, 43 Research and Innovation Fund (NSRF), Thailand.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Al-Omari, S.; Suthar, D.; Araci, S. A fractional q-integral operator associated with certain class of q-Bessel functions and q-generating series. *Adv. Contin. Discret. Model. Theory Appl.* **2021**, *2021*, 441. [CrossRef]
2. Chandak, S.; Suthar, D.L.; Al-Omari, S.K.; Gulyaz-Ozyurt, S. Estimates of classes of generalized special functions and their application in the fractional (k,s)-calculus theory. *J. Funct. Spaces* **2021**, *2021*, 9582879. [CrossRef]
3. Al-Omari, S. Extension of generalized Fox's H-function operator to certain set of generalized integrable functions. *Adv. Differ. Equ.* **2020**, *2020*, 448. [CrossRef]
4. Khan, N.; Usman, T.; Aman, M.; Al-Omari, S.; Choi, J. Integral transforms and probality distributions involving generalized hypergeometric function. *Georgian J. Math.* **2021**, *28*, 883–894. [CrossRef]
5. Al-Omari, S.K.Q. On Some Variant of a Whittaker Integral Operator and its representative in a Class of Square Integrable Boehmians. *Bol. Soc. Parana. Mat.* **2020**, *38*, 173–183. [CrossRef]
6. Al-Omari, S.K. Estimation of a modified integral associated with a special function kernel of Fox's H-function type. *Commun. Korean Math. Soc.* **2020**, *35*, 125–136.
7. Al-Omari, S. On some Whittaker transform of a special function kernel for a class of generalized functions. *Nonlinear Stud.* **2019**, *26*, 435–443.
8. Al-Omari, S. A revised version of the generalized Krätzel-Fox integral operators. *Mathematics* **2018**, *6*, 222. [CrossRef]
9. Srivastava, H.M.; Masjed-Jamei, M.; Beyki, M.R. Some New Generalizations and Applications of the Apostol-Bernoulli, Apostol–Euler and Apostol-Genocchi Polynomials. *Rocky Mt. J. Math.* **2019**, *49*, 681–697. [CrossRef]
10. Masjed-Jamei, M.; Beyki, M.R.; Koepf, W. An extension of the Euler–Maclaurin quadrature formula using a parametric type of Bernoulli polynomials. *Bull. Sci. Math. Matiques* **2019**, *156*, 102798. [CrossRef]
11. Bayad, A.; Hajli, M. On the multidimensional zeta functions associated with theta functions, and the multidimensional Appell polynomials. *Math. Methods Appl. Sci.* **2020**, *43*, 2679–2694. [CrossRef]
12. Masjed-Jamei, M. A basic class of symmetric orthogonal polynomials using the extended Sturm–Liouville theorem for symmetric functions. *J. Math. Anal. Appl.* **2007**, *325*, 753–775. [CrossRef]
13. Araci, S.; Acikgoz, M. A note on the Frobenius–Euler numbers and polynomials associated with Bernstein polynomials. *Adv. Stud. Contemp. Math.* **2012**, *22*, 399–406.
14. Carlitz, L. A note on Bernoulli and Euler polynomials of the second kind. *Scr. Math.* **1961**, *25*, 323–330.
15. Dattoli, G.; Lorenzutta, S.; Cesarano, C. Finite sums and generalized forms of Bernoulli polynomials. *Rend. Math.* **1999**, *19*, 385–391.
16. Khan, N.U.; Usman, T.; Choi, J. A new class of generalized polynomials. *Turk. J. Math.* **2018**, *42*, 1366–1379. [CrossRef]
17. Usman, T.; Aman, M.; Khan, O.; Nisar, K.S.; Araci, S. Construction of partially degenerate Laguerre-Genocchi polynomials with their applications. *AIMS Math.* **2020**, *5*, 4399–4411. [CrossRef]
18. Kim, T. On the multiple q-Genocchi and Euler numbers. *Russ. J. Math. Phys.* **2008**, *15*, 481–486. [CrossRef]
19. Kim, D.S.; Kim, T.; Lee, S.-H.; Seo, J.-J. Higher-order Daehee numbers and polynomials. *Int. J. Math. Anal. (Ruse)* **2014**, *8*, 273–283. [CrossRef]
20. Kim, T.; Lee, S.-H.; Mansour, T.; Seo, J.-J. A note on q-Daehee polynomials and numbers. *Adv. Stud. Contemp. Math.* **2014**, *24*, 155–160. [CrossRef]
21. Kaneko, M. Poly-Bernoulli numbers. *J. Théor. Nombres Bordx.* **1997**, *9*, 221–228. [CrossRef]
22. Khan, N.U.; Usman, T.; Aman, M. Certain generating function of generalized Apotol type Legendre-based polynomials. *Note Mat.* **2017**, *37*, 21–43.
23. Khan, N.U.; Usman, T.; Aman, M. Generating functions for Legendre-Based Poly-Bernoulli numbers and polynomials. *Honam Math. J.* **2017**, *39*, 217–231.

24. Kim, T.; Kwon, H.I.; Lee, S.-H.; Seo, J.-J. A note on poly-Bernoulli numbers and polynomials of the second kind. *Adv. Differ. Equ.* **2014**, *2014*, 219. [CrossRef]
25. Lewin, L. *Polylogarithms and Associated Functions*; Elsevier (North-Holland): New York, NY, USA; London, UK; Amsterdam, The Netherlands, 1981.
26. Srivastava, H.M.; Choi, J. *Zeta and q-Zeta Functions and Associated Series Integrals*; Elsevier Science Publishers: Amsterdam, The Netherlands; London, UK; New York, NY, USA, 2012.
27. Kim, D.S.; Kim, T. Daehee numbers and polynomials. *Appl. Math. Sci.* **2013**, *7*, 5969–5976. [CrossRef]
28. Lim, D.S.; Kwon, J. A note on poly-Daehee numbers and polynomials. *Proc. Jangjeon Math. Soc.* **2016**, *19*, 219–224.
29. Park, J.-W.; Rim, S.-H.; Kwon, J. The twisted Daehee numbers and polynomials. *Adv. Differ. Equ.* **2014**, *2014*, 1. [CrossRef]
30. Comtet, L. *Advanced Combinatorics*, revised and enlarged edition; Reidel: Dordrecht, The Netherlands, 1974.
31. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions I*; McGraw-Hill Book Company, Inc.: New York, NY, USA, 1953.
32. Sándor, J.; Crstici, B. *Handbook of Number Theory II*; Kluwer Acad. Publ.: Dordrecht, The Netherlands, 2004.
33. Luo, Q.M.; Srivastava, H.M. Some generalizations of the Apostol-Bernoulli and Apostol–Euler polynomials. *J. Math. Anal. Appl.* **2005**, *308*, 290–302. [CrossRef]
34. Luo, Q.-M. Extensions for the Genocchi polynomials and their Fourier expansions and integral representations. *Osaka J. Math.* **2011**, *48*, 291–309.
35. Luo, Q.M. Apostol–Euler polynomials of higher order and the Gaussian hypergeometric function. *Taiwan. J. Math.* **2006**, *10*, 917–925. [CrossRef]
36. Srivastava, H.M.; Manocha, H.L. *A Treatise on Generating Functions*; Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons: New York, NY, USA; Chichester, UK; Brisbane, Australia; Toronto, ON, Canada, 1984.
37. Kim, D.S.; Kim, T. A study on the integral of the product of several Bernoulli polynomials. *Rocky Mt. J. Math.* **2014**, *44*, 1251–1263. [CrossRef]
38. Kim, D.S.; Kim, T. Some identities involving Genocchi polynomials and numbers. *Ars Comb.* **2015**, *121*, 403–412.
39. Kim, D.S.; Lee, N.; Na, J.; Park, K.H. Identities of symmetry for higher-order Euler polynomials in three variables (I). *Adv. Stud. Contemp. Math. (Kyungshang)* **2012**, *22*, 51–74. [CrossRef]
40. Komatsu, T.; Luca, F. Some relationships between poly-Cauchy numbers and poly-Bernoulli numbers. *Ann. Math. Inform.* **2013**, *41*, 99–105.
41. Rainville, E.D. *Special Functions*; Macmillan Company: New York, NY, USA, 1960; Reprinted by Chelsea Publishing Company: Bronx, NY, USA, 1971.