Article

Mathematical Modelling of Qualitative System Development

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Abstract: Many scientific fields need to know how human systems develop. From an economic point of view, the main factors of system output change are changes in the quantity of inputs (extensive factors) and changes in efficiency (input quality and productivity, intensive factors). The growth accounting (GA) method is used for the calculation of the impact of both factors on GDP change. However, its interpretation is sometimes difficult, and GA does not cover all of the possible situations of system (country economy) development. This article uses mathematical tools to derive new indicators (dynamic intensity indicator and dynamic extensity indicator) that clearly count and express how the changes in intensive or extensive factors contribute to the output change in any system. The indicators come from the complex system development typology analyzed in the text. The typology covers all of the relationships among the inputs, their efficiency, and their output. The article shows the use of these indicators in macroeconomics when examining the intensity of GDP development in the World's major economies during the period of 1961–2021 and in microeconomics while investigating the intensity of the development Apple in the period of 1999–2021. We further discuss how indicators reduce managerial risk and uncertainty and their pros and cons.

Keywords: nonlinear model; decision analysis; typology of system development; input; output; efficiency; dynamic intensity and extensity parameters

MSC: 93A30

1. Introduction

The aim of any system should be its preservation and development [1]. Systems mostly compete with each other, and if a system does not pay attention to its current situation and future development, there is a high probability of its failure in favor of other systems. Successful human systems that want to stand out among the competition therefore need to prepare for potential threats. The perpetual analysis of possible failures must be an integral part of risk management [2]. Studies in the literature [3–5] mention many factors contributing to the failure of a human system, such as [6] its costs (inputs) exceeding its output (performance, revenue); bad management [7], including bad managerial decisions due to stress [8]; insufficient output quality [9] or insufficient logistics [10]; the system being too small or too big to be able to compete with others [11–13]; large redistribution within the system, reducing the willingness of the system members to achieve the optimal system output to enable its further existence [14], etc.

The stagnation (no development) of a human system does not guarantee its success in competition with other systems, and development seems to be necessary, but not the only condition [7]. If the system develops (usually extends), it can use two pure ways to determine how to succeed [15]. The former is purely extensive—it only changes a system's inputs, and thus, there are no qualitative changes in its outputs. The latter is purely intensive—the number of inputs does not change, but due to changes in their quality,

the system is able to generate higher outputs. The extensive form is connected to the law of diminishing marginal yields when, in one moment, an additional unit of input results in a smaller output in comparison to previous units [16,17]. This results in a situation where there is no additional output or where the value of a generated output is lower than the value of the inputs necessary for its production. The number of inputs is always further limited, and it cannot be extended to infinity. Even if a further extension of the system is possible, a system achieves a point where it is "too big to be successfully managed" [12]. The pure extensive method of expansion is therefore only possible for a certain period of time. The intensive form is, on the other hand, usually connected with the invention and discovery of new inputs and thus contains both intensive and extensive factors.

Mathematics can help to express how the intensive and extensive factors that a system uses contribute to its development. The growth accounting approach [18] is used on the level of the national economy. However, this method has numerous limitations, for instance, it does not cover all possible situations of system's (country economy) development, and it is not used for companies or other organizations (e.g., firms)—for details, see [19–22]. The aim of this article is to present a new method that, unlike growth accounting, intuitively and clearly counts how two different factors (e.g., extensive and intensive) contribute to achieving a certain value of the system output and to changing this value in time using minimal data. These methods can be used not only in the case of measuring the impact of extensive and intensive factors on system behavior, but for many other situations—e.g., determining how a change in (total) revenue depends on the change in output and price. Economists usually use a math expression called elasticity [17]. Its interpretation can, however, sometimes be difficult. The approach introduced in our article provides a comparison with elasticity as well as easily interpretable results, and it can thus contribute to a better decision-making process. Behavioral economics emphasizes that if the addressees of the results must use a lot of cognitive effort to understand them, then they will use simpler procedures (heuristics) that may lead to misinterpretation and erroneous conclusions [23,24]. Risk management is no exception. Risk managers typically must process a lot of information, make decisions under stress, and have little time to make decisions [2]. The tendency to simplify decisions has logical reasons, but it leads to mistakes. It is thus necessary to look for ways to eliminate them. The method introduced in our article offers a better approach to the risks and uncertainties connected with managerial decisions concerning the intensity and extensity of system development or the elasticity of price changes and the evaluation of these risks.

2. Definition of the Issue

This article deals with the behavior of a general dynamical system. In our case, we will not solve the issue of the system structure. The behavior of the system is manifested by changes in the input quantities provided by the sequences int_t and output quantities provided by the sequence out_t.

Definition 1. *Input and output sequence: Both the input sequence in_t and the output sequence out_t have n + 1 members: t = 0; 1; 2; . . . ; n − 1; n. The index t is called the moment of the sequence. The input sequence represents the time series n + 1 of real positive numbers R+, and the output sequence is represented by the time series n + 1 for non-negative real numbers R+. For the first member of the sequence, t = 0. For the last number of the sequence, t = n. The number of members in the sequence n is a natural number. The time interval between the individual members of both the input and output sequences is the same for both sequences and is constant, and it is called a step. The step can be a minute, an hour, a year, etc.*

$$in_t > 0, \text{ for } t = 0; 1; \ldots ; n \tag{1}$$

$$out_t \geq 0, \text{ for } t = 0; 1; \ldots ; n \tag{2}$$

Definition 2. *Efficiency: The share of corresponding members of the output and input sequence at the moment t is defined as the efficiency:*

$$Ef_t = out_t/in_t, \text{ for } t = 0; 1; \ldots ; n \tag{3}$$

We do not predict zero inputs, but the value of the output can be zero; thus,

$$Ef_t \geq 0, \text{ for } t = 0; 1; \ldots ; n \tag{4}$$

Efficiency Ef_t expresses the number of units per unit of inputs at moment t. According to the specific interpretation of the inputs and outputs, efficiency takes the form of productivity, effectiveness, or speed if the input is time and the output is the path traveled, or the price if the input is the number of units sold and the output is revenue, etc. The time series of the efficiency values defined by Equation (3) represents the sequence of efficiency.

Definition 3. *Profit: Profit is defined as the difference R_t of members of the output and input sequences at moment t.*

$$R_t = out_t - in_t, \text{ for } t = 0; 1; \ldots ; n \tag{5}$$

The domain of a function of profit is all real numbers

$$-\infty < R_t < +\infty \text{ for } t = 0; 1; \ldots ; n \tag{6}$$

The time series of profits defined by Equation (5) represents the sequence of profits. However, in many specific cases, profit does not have a real interpretation.

Definition 4. *Operator: An operator expresses changes in the time development of members of an input sequence or output sequence or efficiency or profit. We define three different operators. (1) absolute change Δ; (2) coefficient α; and (3) relative change β. The definition of an operator is made using an example of the input sequence in_t (similar relations apply for output out_t, efficiency Ef_t, and the profit R_t):*

$$\text{Absolute change: } \Delta in_t = in_t - in_{t-1}, \text{ for } t = 1; \ldots ; n \tag{7}$$

$$\text{Coefficient: } \alpha in_t = in_t/in_{t-1} = \beta in_t + 1, \text{ for } t = 1; \ldots ; n \tag{8}$$

$$\text{Relative change: } \beta in_t = \Delta in_t/in_{t-1} = \alpha in_t - 1, \text{ for } t = 1; \ldots ; n \tag{9}$$

This applies to each sequence: the number of operators corresponding to one step is n, while there are $n + 1$ members of the sequence. The expressions of the operators also determine the relations between these operators. It is clear from Equation (9) that the relative change at moment t is equal to the absolute change at moment t divided by a member of the sequence at moment t_{-1}. It is further clear from Equation (8) that the coefficient at moment t is equal to the relative change at moment t plus the value 1. The relative change at moment t is equal to the coefficient at the same moment as *t* minus the value 1.

For the coefficients, it can be derived from Equations (1) and (2):

$$\alpha in_t > 0; \alpha out_t \geq 0 \tag{10}$$

and for relative change:

$$\beta in_t > -1; \beta out_t \geq -1 \tag{11}$$

Theorem 1. *The relation between the operators of relative change in the input sequence, output sequence, and efficiency sequence: The relative change in the members of the sequence of outputs at the moment t can be expressed from Equations (3) and (9) as a function of the relative changes*

in the members of the sequence of inputs and the relative changes in the members of the sequence of efficiency:

$$\beta out_t = \beta in_t + \beta Ef_t + \beta in_t \cdot \beta Ef_t \qquad (12)$$

Theorem 2. *The relations between the operators of the coefficients of the input sequence, output sequence, and efficiency sequence: From Equations (3) and (8), the following relation can be derived between the coefficients of the input, output, and efficiency sequences at moment t:*

$$\alpha out_t = \alpha in_t \cdot \alpha Ef_t \qquad (13)$$

The operators in Definition 4 are defined for one step since t only changes by one sequence member there. The same relations apply to the multi-step operators, where t grows by more than 1:

$$\text{Absolute change: } \Delta in_{t,0} = in_t - in_0, \text{ for } t = 2; \ldots ; n \qquad (14)$$

$$\text{Coefficient: } \alpha in_{t,0} = in_t/in_0 = \beta in_t + 1, \text{ for } t = 2; \ldots ; n \qquad (15)$$

$$\text{Relative change: } \beta in_{t,0} = \Delta in_t/in_0 = \alpha in_t - 1, \text{ for } t = 2; \ldots ; n \qquad (16)$$

In this case, it is also possible to define the average operator. The definition of the average operator is expressed in the example of the input sequence in_t.

Definition 5. *Average operator.*

$$\text{Average absolute change}: \overline{\Delta} in_{t,0} = \frac{in_t - in_0}{t} \text{ for } t = 2; \ldots ; n \qquad (17)$$

$$\text{Average coefficient}: \overline{\alpha} in_{t,0} = \sqrt[t]{\frac{in_t}{in_0}} \text{ for } t = 2; \ldots ; n \qquad (18)$$

$$\text{Mean of relative change}: \overline{\beta} in_{t,0} = \overline{\alpha} in_{t,0} - 1 \text{ for } t = 2; \ldots ; n \qquad (19)$$

Equations (17)–(19) can be used both for the members of the input sequence in_t and the output sequence out_t as well as for the members of the efficiency sequence Ef_t. If the average operator over the entire period n is multiplied by the first member of the sequence, then the value of the last member of that sequence, i.e., in period n, must be obtained.

Definition 6. *Purely extensive development: Purely extensive development during a step or a period occurs if there is no change in the value of efficiency at that step/period. In this case, the values of the input sequence change at the same rate as the output sequence values do, i.e., they have the same coefficients*

$$\alpha in_t = \alpha out_t \qquad (20a)$$

and the same relative changes

$$\beta in_t = \beta out_t \qquad (20b)$$

Thus, for purely extensive development

$$\Delta Ef_t = 0; \ \alpha Ef_t = 1 \ a \ \beta Ef_t = 0 \qquad (20c)$$

Definition 7. *Purely intensive development: Purely intensive development during a step or a period occurs if the values of the input sequence do not change at all, but there is a change in the values of the output sequence. Then, changes in the output only happen due to efficiency. Then, apply*

$$\alpha Ef_t = \alpha out_t \qquad (21a)$$

$$\beta Ef_t = \beta out_t \qquad (21b)$$

Thus, for purely extensive development,

$$\Delta int_t = 0; \; \alpha int_t = 1 \text{ a } \beta int_t = 0 \tag{21c}$$

Purely extensive or purely intensive development can be either concurrent growth or a concurrent decrease in the values of the sequence of outputs and the values of another sequence (in the case of purely extensive development, it is the sequence of inputs; in the case of purely intensive development, it is the sequence of efficiency). Pure development means that the values of the third sequence (in the case of a purely extensive development, it is the sequence of efficiency; in the case of purely intensive development, it is the sequence of inputs) do not change. It must be emphasized that neither purely extensive nor purely intensive development are as frequent as developments where both extensive and intensive factors act simultaneously on and somehow contribute to the changes in the values of the sequence of outputs. The problem solved by this article is finding a way to calculate the share (percentage) expressing how much a change in outputs is caused by extensive factors and how much is caused by intensive factors.

3. The Typology of the System Development

An appropriate way of expressing how both a change in system inputs (extensive factors) and how a change in their quality or efficiency (intensive factors) contribute to a change in the system outputs represents dynamic indicators of extensity and intensity. Before deriving them, we created a typology of development. It shows all of the possible combinations of values for the growth, decrease, and stagnation in the sequence of inputs, sequence of outputs, and sequence of efficiency and assigns them a specific name. In real life, the values of a sequence of inputs or outputs can grow, decrease, or stagnate. The specific development of both sequences results in the relevant development of the efficiency sequence.

Definition 8. *Typology of developments: The typology of developments is the complete expression of all possible relations between the development of the members of the input sequence in_t of the system, the output sequence out_t of the system, and the sequence of efficiency Ef_t, which is determined by Equation (3), in which it is always $t = 1; \ldots ; n$.*

Definition 9. *Default relation for deriving the typology of developments: The default relation for deriving the development typology is the expression of the members of the output sequence out_t as the product of the members of the input sequence in_t and the members of the efficiency sequence*

$$out_t = in_t \cdot Ef_t \tag{22}$$

which can be derived from Equation (3). Equation (22) shows the growth, decrease, or stagnation in the members of the output sequence out_t, which are denoted by the change in the members of just one of the sequences on the right side, with no size changes being observed in the members of the remaining sequences. Both sequences may also change. If members of both sequences change, both changes may be in the same direction as the change in output (e.g., both the members of the input sequence and the members of the efficiency sequence grow). However, the changes in the members of the input sequence and the members of the efficiency sequence can occur in opposite directions. This can result in output stagnation, in which the members of one sequence grow while the members of the other sequence decrease such that there is no change in the members of the output sequence.

Definition 10. *Dynamization of the default relation for deriving the typology of developments: To quantify the effect of a change in the members of the input sequence in_t (extensive factors) or a change in the members of the efficiency sequence Ef_t (intensive factors) on the change in the members*

of the output sequence out_t, it is necessary to dynamize Equation (22), that is, to find an analogy to Equation (24) for the operators, which is called the coefficient.

$$\alpha out_t = \alpha Ef_t \cdot \alpha in_t \tag{23}$$

Definition 11. *Space for displaying the typology of developments: The relations between the changes in the extensive and intensive factors and the changes in the members of the output sequence αout_t can be effectively expressed using a figure of the coordinates for αin_t on the x-axis and the coordinates for αEf on the y-axis—see Figure 1. The figure also contains the isoquants of αout_t, i.e., the isoquants representing all of values of αEf_t and αin_t, which lead to αout_t having the same specific value (here, 0.5; 1; 2; and 3). These isoquants can be expressed by the relation*
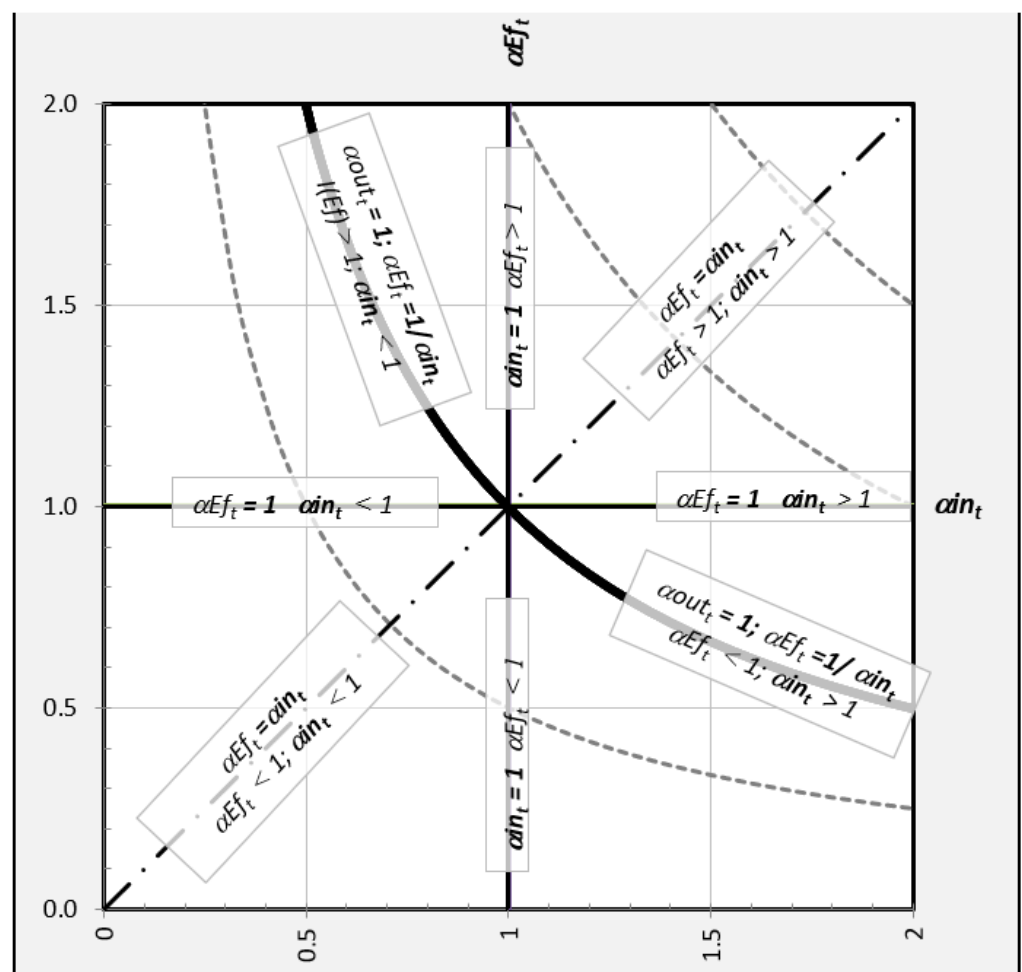
$$\alpha Ef_t = \alpha out_t / \alpha in_t \tag{24}$$

Figure 1. Space to display and describe the basic types of development observed in αout_t, αin_t, and αEf_t. Note: The range of the coefficients of both factors (αEf_t and αin_t) in Figure 1 is selected to be in the interval from 0 to 2, i.e., from a decrease in output to zero through stagnation ($\alpha Ef_t = 1$ or $\alpha in_t = 1$) to double growth. If using logarithmic coordinates, these isoquants would be linear.

It is clear from Equation (24) and Figure 1 that the isoquants of the steady (same) development of output αout_t are equal–axial hyperbolas. There is constant elasticity on these hyperbolas. Of particular importance is the hyperbola of output stagnation, which passes through the origin of the coordinates (1; 1). All of the isoquants above represent output

growth, and all of the isoquants below represent a decrease. For example, the isoquant with value of 2 in Figure 1 shows all of the combinations of αin_t a αEf_t, which results in the doubling of the outputs. Figure 1 illustrates all of the basic types of relations between the development of extensive and intensive factors on the one hand and the development of output on the other. These basic types of relations (basic developments) include:

Definition 12. *Pure developments: These are located on the coordinate axes of Figure 1. Growth or decreases in output occur solely due to one of the factors considered, either a purely extensive factor or purely intensive factor. The second factor does not change, i.e., $\alpha in_t = 1$ (for a purely intensive change) or $\alpha Ef_t = 1$ (for a purely extensive change).*

Definition 13. *Balanced developments: There are two factors that are considered to act the same, i.e., $\alpha in_t = \alpha Ef_t$. These developments are located in quadrants I and III on a line at a 45-degree angle that intersects the origin of the coordinate axes, i.e., point (1; 1).*

Definition 14. *Compensation developments: Here both considered factors completely compensate for output stagnation, i.e., $\alpha out_t = 1$; therefore, $\alpha Ef_t = 1/\alpha in_t$. These developments are found on the hyperbolic isoquant of the stagnation outputs (see above).*

Definition 15. *Complete stagnation: Complete stagnation (zero development) is characterized by $\alpha out_t = \alpha in_t = \alpha Ef_t = 1$, i.e., none of the considered quantities changed during the given period. This situation corresponds to the origin of the coordinates at point (1; 1).*

If we characterize each of these developments in greater detail, the following applies:

Definition 16. *Pure extensive growth and decline: Pure developments can be differentiated into pure growth and pure decline. For a pure extensive development, where $\alpha Ef_t = 1$, pure extensive growth ($\alpha in_t > 1$) is found on the positive ray of the x-axis, and pure extensive decline ($\alpha in_t < 1$) is represented by the negative ray of the x-axis. For pure intensive development, $\alpha in_t = 1$; then, the following applies analogously: pure intensive growth ($\alpha Ef_t > 1$) is shown by the positive ray of the y-axis, and pure intensive decline ($\alpha Ef_t < 1$) is shown by the negative ray of the y-axis.*

Definition 17. *Balanced intensive–extensive growth and decline: For balanced developments ($\alpha Ef_t = \alpha in_t$), intensive–extensive growth ($\alpha out_t > 1$) is represented by the positive part of the line below the 45° angle intersecting the origin of the coordinate axes (i.e., the part in quadrant I), and intensive–extensive decline ($\alpha out_t < 1$) is represented by the negative part of the line at a 45° angle intersecting the origin of the coordinate axes (i.e., the part in quadrant III).*

Definition 18. *Intensive–extensive and extensive–intensive compensation. For compensatory development ($\alpha out_t = 1$, so $\alpha Ef_t = 1/\alpha in_t$), it can be distinguished by intensive–extensive compensation—observed in the upper half of the stagnation hyperbola, where $\alpha Ef_t > 1$ and $\alpha in_t < 1$ apply, or by extensive–intensive compensation—observed in the lower half of the stagnation hyperbola, where $\alpha Ef_t < 1$ and $\alpha in_t > 1$ apply.*

The basic types of developments are crucial for deriving the general typology of the developments, but they are rare in reality. It is not very likely that the output of a system would grow or decline purely intensively or purely extensively or that both factors (αin_t and αEf_t) would act on the growth or decline of the output exactly at the same rate, nor is it frequent that the output does not change at all ($\alpha out_t = 1$) because of the fully compensatory action of both factors. Therefore, it is necessary to focus on mixed types of developments. These are all of the other situations that can arise apart from the basic ones that have just been defined. Graphically, it applies for those with representations in Figure 1 that lie outside the coordinate axes, outside the line at a 45-degree angle in quadrants I and III intersecting the origin of the coordinate axes, and outside the hyperbolic isoquant for stagnation. These are eight separate spaces that can always be characterized by a triad of

inequalities that concurrently determine whether the product grows or not, i.e., $\alpha out_t > 1$ or $\alpha out_t < 1$. The first one of the three inequalities determines the relation between αEf_t and αin_t or (for compensatory development) between one quantity and the inverted value of the other. The second inequality determines whether there is growth or a decrease in the inputs in_t, i.e., $\alpha in_t > 1$ or $\alpha in_t < 1$. The third inequality determines whether there is growth or a decline in the efficiency Ef_t, i.e., $\alpha Ef_t > 1$ or $\alpha Ef_t < 1$, for example, a space in which $\alpha out_t > 1$ and concurrently $\alpha Ef_t > 1/\alpha in_t$ where $\alpha Ef_t > 1$ and $\alpha in_t < 1$ is applied represents mixed development, as shown on the area between the positive direction of the *y*-axis and the top of the stagnation hyperbola. It expresses the situation of the output growth even though inputs are decreasing. This means that the growth of Ef_t not only compensates for the decline in_t, but is also sufficient to be the cause of output growth, i.e., $\alpha out_t > 1$.

The relations αEf_t and αin_t for all of the basic and mixed development types are shown in Figure 2 and in Table 1.
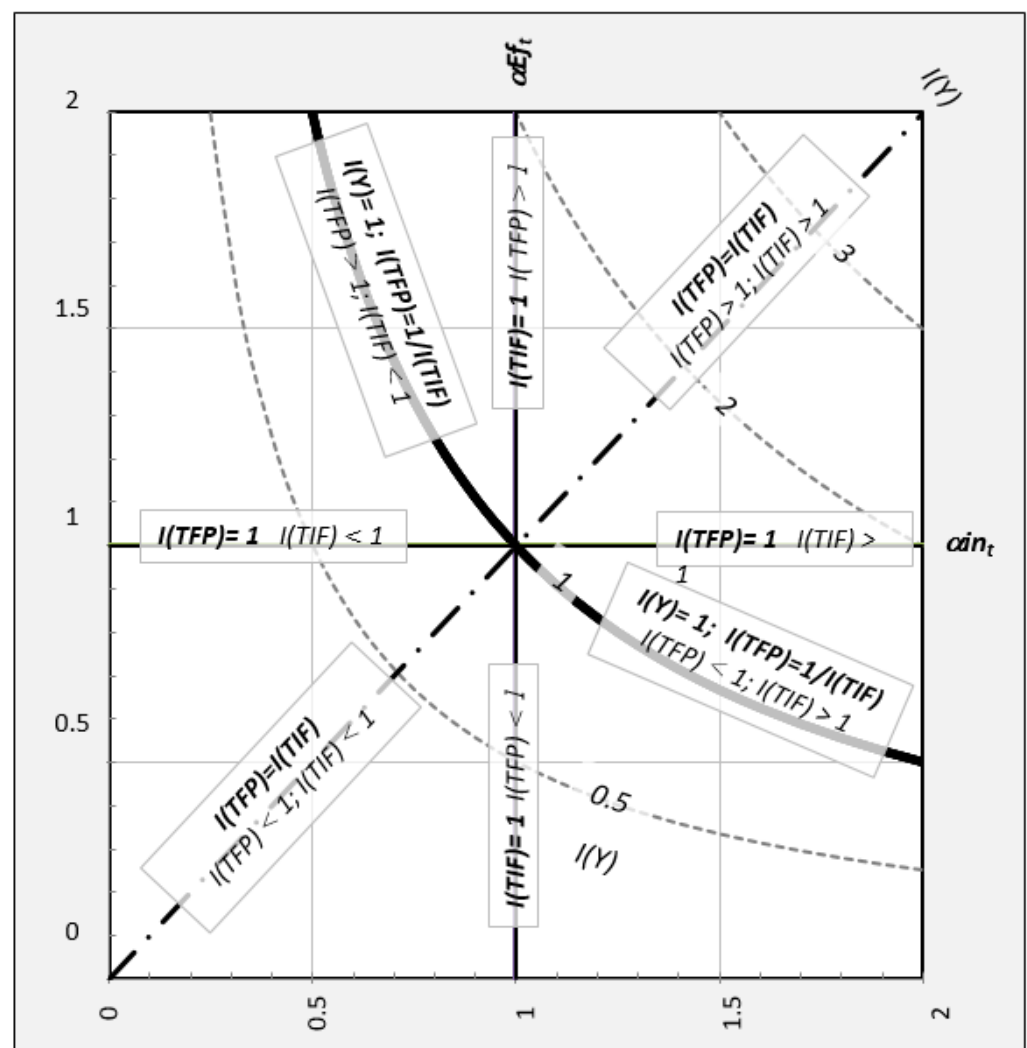

Figure 2. Detailed description of pure and combined types of all possible developments of a system.

Table 1. All system development types and corresponding values of dynamic intensity and extensity indicators.

	Change in Extensive Factors αin_t	Change in Intensive Factors αEf_t	Change in Output (αout_t)	Value of Intensity (int_t) and Extensity (ext_t)	Type of Development
1.	growth, $\alpha in_t > 1$	unchanged, $\alpha Ef_t = 1$	growth, $\alpha out_t > 1$	$ext_t = 1$; $int_t = 0$	pure extensive growth
2.	unchanged, $\alpha in_t = 1$	growth, $\alpha Ef_t > 1$	growth, $\alpha out_t > 1$	$ext_t = 0$; $int_t = 1$	pure intensive growth
3.	grows at the same rate as the intensive, $\alpha in_t > 1$, $\alpha in_t = \alpha Ef$	grows at the same rate as the extensive, $\alpha Ef_t > 1$, $\alpha Ef_t = \alpha in_t$	growth, $\alpha out_t > 1$	$ext_t = 0{,}5$; $int_t = 0.5$	pure intensive extensive growth
4.	grow faster than intensive, $\alpha in_t > 1$, $\alpha in_t > \alpha Ef_t$	grow slower than extensive, $\alpha Ef_t > 1$, $\alpha Ef_t < \alpha in_t$	growth, $\alpha out_t > 1$	$ext_t > 0$; $int_t > 0$; $ept_t > int_t$	mainly extensive growth
5.	grow slower than intensive, $\alpha in_t > 1$, $\alpha in_t < \alpha Ef_t$	grow faster than extensive, $\alpha Ef_t > 1$, $\alpha Ef_t > \alpha in_t$	growth, $\alpha out_t > 1$	$ext_t > 0$; $int_t > 0$; $int_t > ext_t$	mainly intensive growth
6.	is greater than the inverted value of the intensive, $\alpha in_t > 1$, $\alpha in_t > 1/\alpha Ef_t$	is greater than the inverted value of the extensive, $\alpha Ef_t < 1$, $\alpha Ef_t > 1/\alpha in_t$	growth, $\alpha out_t > 1$	$ext_t > 0$; $int_t < 0$; $ext_t > /int_t/$	extensive–intensive compensatory growth
7.	is greater than the inverted value of the intensive, $\alpha in_t < 1$, $\alpha in_t > 1 I \alpha Ef_t$	is greater than the inverted value of the extensive, $\alpha Ef_t > 1$, $\alpha Ef_t > 1/\alpha in_t$	growth, $\alpha out_t > 1$	$ext_t < 0$; $int_t > 0$; $int_t > /ext_t/$	intensive–extensive compensatory growth
8.	equals the inverted value of the intensive, $\alpha in_t > 1$, $\alpha in_t = 1/\alpha Ef_t$	equals to the inverted value of the extensive, $\alpha Ef_t < 1$, $\alpha Ef_t = 1/\alpha in_t$	no change, stagnation $\alpha out_t = 1$	$ext_t = 0.5$; $int_t = -0.5$	pure extensive–intensive compensation
9.	equals the inverted value of the intensive, $\alpha in_t < 1$, $\alpha in_t = 1/\alpha Ef_t$	equals to the inverted value of the extensive, $\alpha Ef_t > 1$, $\alpha Ef_t = 1/\alpha in_t$	no change, stagnation $\alpha out_t = 1$	$ext_t = -0.5$; $int_t = 0.5$	pure intensive extensive compensation
10.	is less than the inverted value of the intensive, $\alpha in_t < 1$, $\alpha in_t < 1/\alpha Ef_t$	is less than the inverted value of the extensive, $\alpha Ef_t > 1$, $\alpha Ef_t < 1/\alpha in_t$	decline, $\alpha out_t < 1$	$ext_t < 0$; $int_t > 0$; $int_t < /ext_t/$	intensive– extensive compensatory decline
11.	is less than the inverted value of intensive, $\alpha in_t > 1$, $\alpha in_t < 1/\alpha Ef_t$	is less than the inverted value of the extensive, $\alpha Ef_t < 1$, $\alpha Ef_t < 1/\alpha in_t$	decline, $\alpha out_t < 1$	$ext_t > 0$; $int_t < 0$; $ext_t < /int_t/$	extensive–intensive compensatory decline
12.	decreases faster than the intensive decrease, $\alpha in_t < 1$, $\alpha in_t < \alpha Ef_t$	decreases more slowly than extensive, $\alpha Ef_t < 1$, $\alpha Ef_t > \alpha in_t$	decline, $\alpha out_t < 1$	$ext_t < 0$; $int_t < 0$; $|ext_t /> /int_t/$	mainly extensive decline
13.	decreases slower than intensive, $\alpha in_t < 1$, $\alpha in_t > \alpha Ef_t$	decreases faster than extensive, $\alpha Ef_t < 1$, $\alpha Ef_t < \alpha in_t$	decline, $\alpha out_t < 1$	$ext_t < 0$; $int_t < 0$; $/int_t /> /ext_t/$	maily intensive decline
14.	decreases at the same rate as the intensive, $\alpha in_t < 1$, $\alpha in_t = \alpha Ef_t$	decreases at the same rate as the extensive, $\alpha out_t < 1$	decline, $\alpha out_t < 1$	$exp_t = -0.5$; $int_t = -0.5$	mainly intensive extensive decline

Table 1. *Cont.*

	Change in Extensive Factors αin_t	Change in Intensive Factors αEf_t	Change in Output (αout_t)	Value of Intensity (int_t) and Extensity (ext_t)	Type of Development
15.	decrease, $\alpha in_t < 1$	unchanged, $\alpha Ef_t = 1$	decline, $\alpha out_t < 1$	$ext_t = -1$; $int_t = 0$	pure extensive decline
16.	unchanged, $\alpha in_t = 1$	decrease, $\alpha Ef_t < 1$	decline, $\alpha out_t < 1$	$ext_t = 0$; $int_t = -1$	pure intensive decline

The individual spaces shown in Figure 2 are named so that the names convey reality as accurately as possible. The nomenclature of all of the basic and mixed developments types is based on the following principles:

- The nomenclature must cover all development types.
- If output grows, the term growth is used; if it falls, the term decline is used; if it does not change, the term pure compensation is used.
- All basic developments are referred to as pure.
- If both factors (both intensive and extensive) act on growth or if both act on a decline in output, but not equally, the word mainly or predominantly is used, whereas the name of the predominant factor is also used, i.e., mainly intensive growth shows a situation where both factors (αin_t and αEf_t) act on growth, but the influence of intensive factors is greater than the influence of extensive factors. Similarly, the term mainly extensive decline describes a situation where both factors (αin_t and αEf_t) decline, yet the influence of extensive factors is greater than the influence of intensive factors.
- In the designation of opposite developments, where one factor acts on the growth and the other acts on the decline in output, the words compensation or compensatory are used.
- If the words intensive and extensive are used in the case of mixed compensatory or pure compensatory developments, the first one used is the one that acts on the growth, and the second one acts on the decline. As an example, the term intensive–extensive compensatory growth thus refers to the situation where intensive factors grow so rapidly that they even compensate for the decline in extensive factors, and they contribute to the output growth, such as in the above-mentioned situation where $\alpha out_t > 1$ at the same time as $\alpha Ef_t > 1/\alpha in_t$ while $\alpha in_t < 1$ and $\alpha Ef_t > 1$. Similarly, an intensive–extensive compensatory decline reflects a situation where intensive factors increase while extensive factors decrease at a higher rate, which results in a decline in the output. Pure intensive–extensive compensation using this logic then reflects a state where the intensive factors act on the growth and the extensive factors act on a decline at the same time, such that the result is the stagnation of the output (pure extensive–intensive compensation defines the situation when extensive factors act on growth and intensive factors act on decline such that they once again result in the stagnation of the output).

The names of all of the basic and mixed developments are provided in Figure 3, which also contains the value of the dynamic intensity indicator and the dynamic extensity indicator (which are derived and explained in Section 4) related to the development. To make the figure clear, we use the following symbols: *i* represents the dynamic intensity indicator, and *e* represents the dynamic extensity indicator.

Depending on the location of the point in the appropriate combination of zones, the development of the analyzed system can be clearly characterized. Figure 4 expresses the main different zones qualitatively in the form of eight spaces displaying the quality of the development. The analyzed zone is always displayed in gray and marked with a number. The pictures in the columns (the first one is above the remaining picture, and the second one is below the remaining picture) show zones that complement each other.
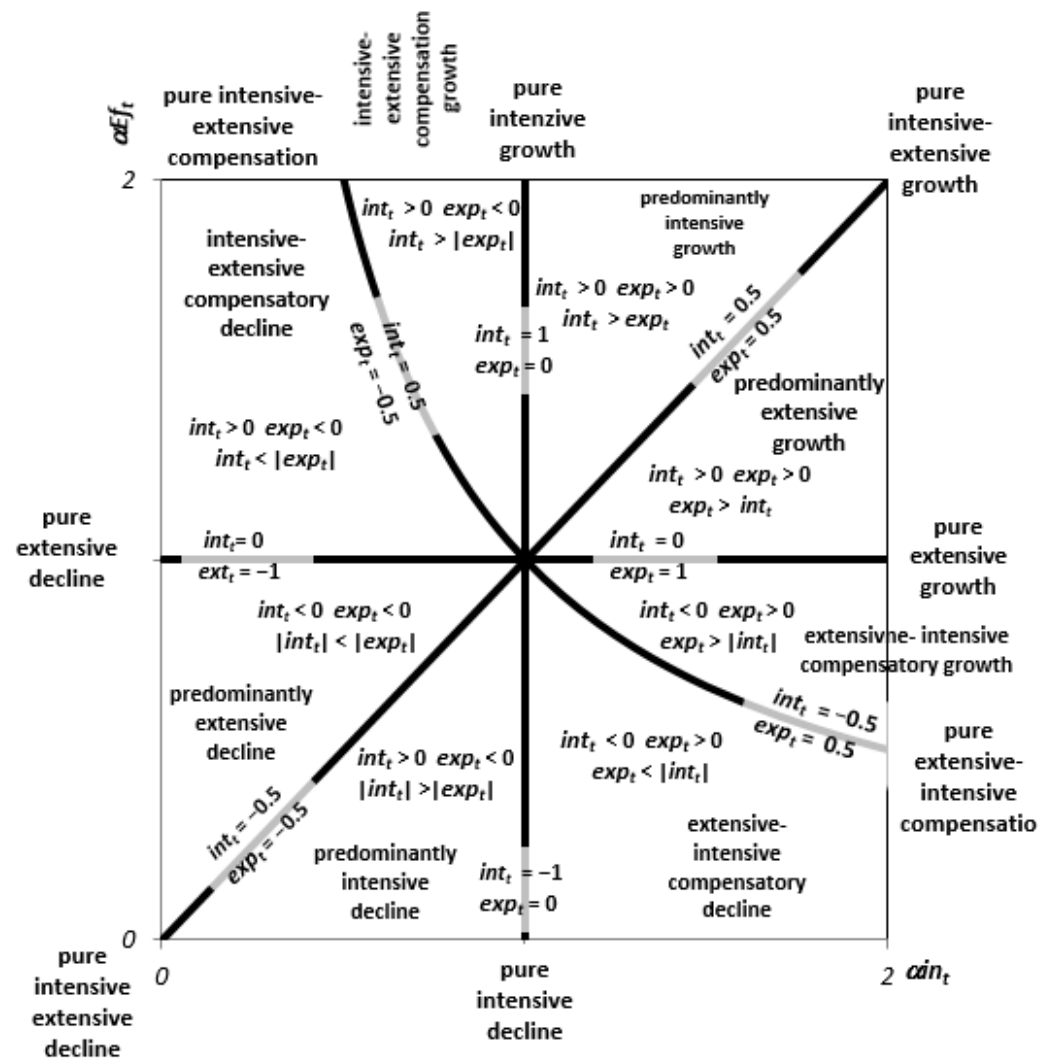
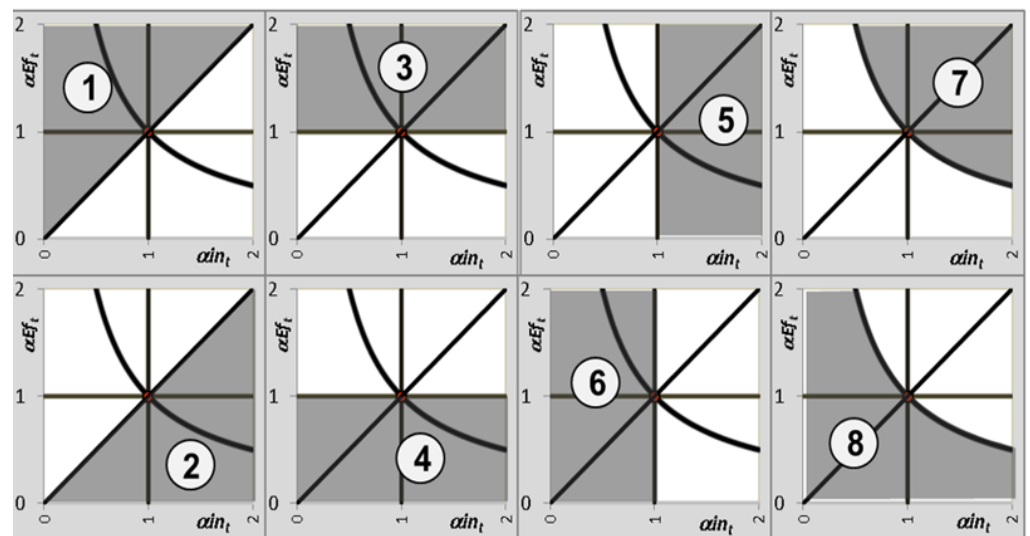**Figure 3.** Complex typology of all possible system developments.

Figure 4. Zones with possible system development.

Definition 19. *Characteristics and definition of the zones of a system development.*

Zone 1: $int_t \geq exp_t$; intensive factors exceed extensive ones or are equal at the diagonal boundary of the zones;
Zone 2: $int_t \leq exp_t$; extensive factors exceed intensive ones or are equal at the diagonal boundary of the zones;
Zone 3: $int_t \geq 0$; intensive factors contribute to the output growth or are zero at the horizontal boundary of the zones;
Zone 4: $int_t \leq 0$; intensive factors contribute to the output decline or are zero at the horizontal boundary of the zones;
Zone 5: $exp_t \geq 0$; extensive factors contribute to the output growth or are zero at the vertical boundary of the zones;
Zone 6: $exp_t \leq 0$; extensive factors contribute to the output decline or are zero at the vertical boundary of the zones;
Zone 7: $\alpha out_t \geq 1$ or $\beta out_t \geq 0$; output grows or stagnates at the hyperbolic boundary of the zones;
Zone 8: $\alpha out_t \leq 1$ or $\beta out_t \leq 0$; output decreases or stagnates at the hyperbolic boundary of the zones.

Each specific point is always located in four zones. For example, if there is a point in the overlap of zones 1, 3, 6, and 7, this means that the output growth is the result of efficiency. The growth of the intensive factors (efficiency) not only compensates for the decline in extensive factors (inputs), but results in output growth.

4. Derivation of Dynamic Indicators of Extensity and Intensity Based on the Identity of the Coefficients

As mentioned in the previous section, dynamic intensity and extensity indicators express how the changes in intensive factors (the change in efficiency, *Ef*) or extensive factors (the change in inputs, *int*) contribute to a change in output (*out*). The key relation for deriving the indicator is Equation (13), which can be logarithmically converted to the following additive relation (25):

$$\ln \alpha out_t = \ln \alpha in_t + \ln \alpha Ef_t \tag{25}$$

The dynamic intensity indicator expressing the share of the influence of the intensive factor on the system development can be written for situations when both the output as well as the efficiency and inputs grow as:

$$int_t = \ln \alpha Ef_t \; / \; \ln \alpha out_t \tag{26}$$

$$\alpha Ef_t = (\alpha out_t)^{int_t} \tag{27}$$

The extensity indicator can be expressed analogously

$$ext_t = \ln \alpha in_t \; / / \ln \alpha out_t \tag{28}$$

or as

$$\alpha in_t = (\alpha out_t)^{ext_t} \tag{29}$$

Since only output growth *(out_t)*, *(in_t)* i *(Ef_t)* is currently being considered, it is possible to derive the value of the intensity indicator for pure intensive and extensive growth. For pure intensive growth**,** the output coefficients *(out_t)* and the efficiency *(Ef_t)* are equal to

$$\alpha out_t = \alpha Ef_t \tag{30}$$

and the value of the intensity indicators *(int_t)* in Equation (26) should acquire a magnitude of 1 or 100% for pure intensive growth. This is clear from the fact that in Equation (26), the numerator and denominator will be identical, and therefore, $int_t = 1$. At the same time,

it applies $\alpha in_t = 1$, which can only be fulfilled using Equation (28) when extensity $ext_t = 0$ or 0%, i.e., the numerator of Equation (28) is equal to 0, whereas the denominator is not equal to zero.

For pure extensive growth, the following applies:

$$out_t = \alpha in_t \tag{31}$$

the value of the extensity indicator (ext_t) in Equation (28) for a pure extensive development must have the value of 1 or 100%. The numerator and denominator of Equation (28) are identical in situations with pure extensive development. At the same time, $\alpha Ef_t = 1$, which can only be fulfilled as part of Equation (26) in cases where $int_t = 0$ or when the intensity is 0%. This is because Equation (26) has a null numerator and a non-zero denominator.

For symmetric pure intensive–extensive growth,

$$\alpha out_t = (\alpha in_t)^2 \tag{32}$$

the value of the intensity indicator (int_t) in Equation (26) is 0.5 or 50% for this development, meaning that the extensity indicator (ext_t) in Equation (28) will have a magnitude of 0.5 or 50%.

If we also analyze declines using the same logic, we can once again start from Equations (26) and (28). Only one factor is involved in the fall of both pure declines and causes the output (out_t) to decrease. In a way, pure intensive decline is the opposite of pure intensive growth, which has been assigned values of $int_t = 1$, i.e., 100%, and $ext_t = 0$, i.e., 0%. The opposite development should be expressed with the opposite indicator value, and once again, only the intensive factor contributes 100% to its development, albeit by its decline. Thus, Equations (26) and (28) must be adjusted so as not to change the very logical results for growth, but so that the equations generate $int_t = -1$, i.e., −100%, and $ext_t = 0$ for the pure intensive decrease. In Equations (26) and (28), we can use the αout_t from Equation (25) once again, whereas we assigned both logarithms in the denominator to have an absolute value. This does not change anything with the previous results, but for the pure intensive decrease, the equations that have been adjusted in this way will generate the required values for the intensity indicator, i.e., $int_t = -1$.

Definition 20. *Dynamic intensity indicator.*

$$int_t = \frac{\ln \alpha Ef_t}{|\ln \alpha Ef_t| + |\ln \alpha in_t|} \tag{33}$$

Definition 21. *Dynamic extensity indicator.*

$$ext_t = \frac{\ln \alpha in_t}{|\ln \alpha Ef_t| + |\ln \alpha in_t|} \tag{34}$$

We will now verify whether Equations (33) and (34) will generate values of the dynamic intensity and extensity indicators in the remaining developments according to the general typology of developments. For a pure extensive decrease, which is subject to the equation

$$\alpha out_t = \alpha in_t \tag{35}$$

Equations (33) and (34) generate dynamic indicator values of $int_t = 0$ and $ext_t = -1$, i.e., −100 %. An intensive–extensive decrease caused by the same decrease in both factors is subject to the equation

$$\alpha Ef_t = \alpha in_t \tag{36}$$

and Equations (27) and (29) transition into the relations in Equations (37) and (38), which differ from Equations (27) and (29) by the sign of the exponent. Equations (37) and (38) apply to all of the points in quadrant III:

$$\alpha E f_t = (\alpha out_t)^{-\alpha int_t} \tag{37}$$

$$\alpha in_t = (\alpha out_t)^{-\alpha ext_t} \tag{38}$$

It remains to be seen which values both indicators take in cases of complete compensation (on the compensatory hyperbola, $\alpha out_t = 1 = \alpha in_t \cdot \alpha E f_t$), in which the following equation in applied:

$$\ln \alpha E f_t + \ln \alpha in_t = 0 \tag{39}$$

In pure intensive–extensive compensation, there is a positive expression, $\ln \alpha E f_t$, and a negative expression, $\ln \alpha in_t$. With pure extensive–intensive compensation, it is the opposite. If the sum of the logarithms of $\alpha E f_t$ and αin_t equal 0, then they must be equal in terms of their absolute value. Therefore, in Equations (33) and (34), the denominator is always twice the value in the numerator. The values of the dynamic indicators (int_t) and (ext_t) always take values of 0.5 with pure compensations, and the sign is determined by the numerator. For pure intensive– extensive compensation, $int_t = 0.5$, i.e., 50%, and $ext_t = -0.5$, i.e., −50%. For pure extensive–intensive compensation, $int_t = -0.5$, i.e., −50%, and $ext_t = 0.5$, i.e., 50%.

Figure 5 shows the values of the intensity and extensity indicators for all of the basic and mixed developments in relation to each other. The indicator (ext_t) is drawn on the *x*-axis, and the indicator (int_t) is drawn on the *y*-axis. The figure is based on the relation between the indicators resulting from Equations (33) and (34), which takes the following form:

$$int_t \cdot \text{sgn}(\beta E f_t) + ext_t \cdot \text{sgn}(\beta in_t) = 1 \text{ or } l \ int_t \ I + I \ ext_t \ l = 1 \tag{40}$$
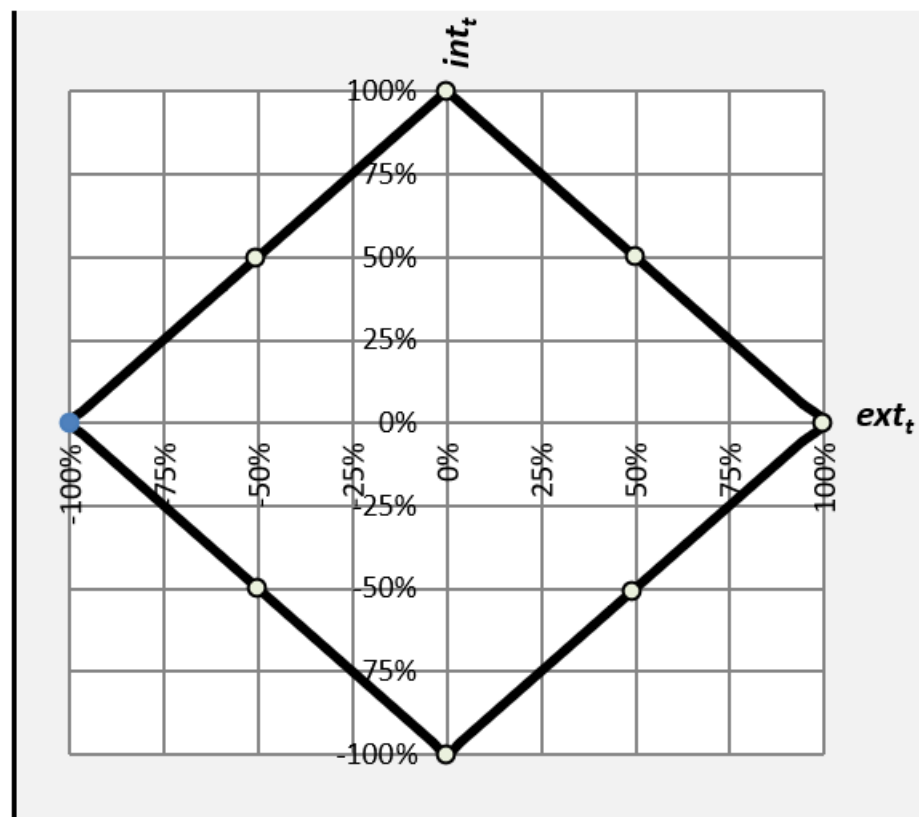

Figure 5. Combination of all possible values of dynamic intensity or extensity indicators according to Equations (34) and (35).

The derivation of the indicators comes from the nonlinear model of reality. It is clear from Figure 5 that the magnitude of the dynamic indicators of intensity int_t and extensity ext_t is normed at an interval of $(-1; 1)$. For all of the basic developments, it is possible for the values of both dynamic indicators to be any of the following numbers: -1, -0.5, 0, 0.5, or 1, and in all cases, the values correspond to the derived typology of developments. The values of the indicators are symmetrically distributed around the axis of quadrants I and III. The indicators can be used wherever there are the changes (different values) in the output and input variables and where there are the changes in efficiency that are measurable by changes in the shares of the outputs and inputs. The only fundamental prerequisite is that the values of the members of the time sequence of the inputs must be positive rational numbers, i.e., $0 < in_t$, and that the values of the members of the time sequence of the outputs must also be non-negative, i.e., $0 \leq out_t$.

These indicators can be used as a suitable base for decision-making models. Contemporary models (see, e.g., [25] for details) do not pay sufficient attention to issues of intensity and extensity during system development. The intensity of a system can be seen as proof of appropriate system development, and it should be added to inventories of other signs of successful development, such as system reliability. It is not sufficient to rely only on efficiency as defined by Equation (3). The value of efficiency can, for instance, still be the same, but the number of inputs and outputs decrease proportionally. In that case, the system shrinks and may be too small to be able to compete with other systems. These indicators generally describe what happens in a system. If the correct number of inputs and outputs is used, then the indicators are able to reveal the positive/negative development of the system as well as major events within the system, including periods of system development to be investigated by the theory of catastrophes (see, [26–28] for details). Long periods of time in which the dynamic intensity indicator has a value of zero or a negative value indicates that the system was hit or has experienced problems, including serious catastrophes such as sudden sharp changes in the price of the inputs or outputs, [29], traffic accidents [30], etc.

5. Examples of the Use of Dynamic Indicators of Intensity and Extensity

Dynamic indicators of intensity and extensity are mainly used in economics (see [15]) to examine the intensity and extent of the development of an economic system (e.g., country or firm development). The examples presented here are drawn from macroeconomics (Section 5.1) and business analysis (Section 5.2).

5.1. Intensity of the Development of the Major World Economies

Dynamic indicators of intensity and extensity can easily describe, analyze, and compare the quality of the development of the world's major economies, e.g., the USA, China, Russia, and the EU15 (Austria, Belgium, Denmark, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, Netherlands, Portugal, Spain, Sweden, and the United Kingdom). We chose the sixty-year period of 1961–2021 and a period covering the last ten years: 2011–2021. Each economy is an economic system that uses inputs and produces output. The input for each year is the Total Input Factor (*TIF*), i.e., the weighted average of labor and capital (capital stock)—the value 0.5 was chosen as the weight (for details, see [31]). The output for each year is the gross domestic product (*GDP*). In general:

$$HDP_t \approx out_t;\ TIF_t \approx in_t;\ TFP_t \approx Ef_t; \tag{41}$$

$$TIF = L^{\alpha} \cdot K^{(1-\alpha)},\ \alpha = 0.5 \tag{42}$$

The absolute values of GDP, labor, and capital stock for each economy and each year were obtained from the databases of the International Monetary Fund, the European Union, the International Labor Organization, and other sources. The relative changes in βHDP_t,

βL_t, and βK_t for each year-on-year change were counted according to these absolute values. For the relative change in *TIF* (e.g., βTIF_t), the following equation was used:

$$\beta TIF_t = \{[\beta L_t + 1]^{\alpha} \cdot [\beta K_t + 1]^{(1-\alpha)}\} - 1 \tag{43}$$

The relative change in the total factor productivity βTFP_t was calculated using the equation

$$\beta TFP_t = \{[\beta HDP_t + 1]/[\beta TIF_t + 1]\} - 1 \tag{44}$$

The counted values of the relative changes in βTIF_t and βTFP_t allow for the calculation of both dynamic indicators, int_t and ext_t, using expressions (33) and (35). Equation (45)

$$\beta (K/L)_t = \alpha K_t / \alpha L_t - 1, \tag{45}$$

was used for the calculation of the relative change in the capital labor share $\beta (K/L)_t$. Table 2 and Figure 6 show the average growth rate (relative change, β) of the GDP, labor, capital, capital labor share, total input factor (*TIF*), total factor productivity (*TFP*), and average values of the dynamic indicator of intensity and the dynamic indicator of extensity for each country for the period of 1961–2011. Table 3 and Figure 7 contain the average growth rates and the average values of the dynamic indicator of intensity and the dynamic indicator for the period of 2011–2021.

Table 2. Relative changes and the indicators int_t and int_t for the period from 1961 to 2021.

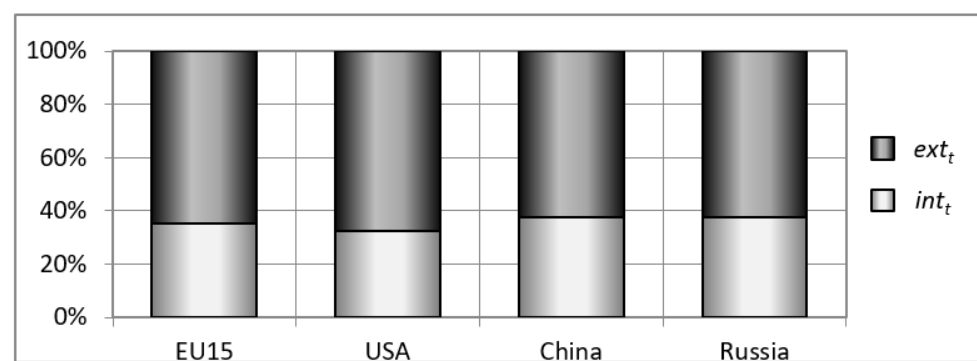|	$\overline{\beta HDP}_{61,21}$	$\overline{\beta L}_{61,21}$	$\overline{\beta K}_{61,21}$	$\overline{\beta (\frac{K}{L})}_{61,21}$	$\overline{\beta TIF}_{61,21}$	$\overline{\beta TFP}_{61,21}$	$int_{61,21}$	$ext_{61,21}$
China	7.47%	1.80%	7.46%	5.56%	4.60%	2.74%	38%	62%
USA	2.96%	1.42%	2.57%	1.14%	2.00%	0.95%	32%	68%
EU	2.36%	0.37%	2.68%	2.30%	1.52%	0.83%	35%	65%
USSR a Russia	1.79%	0.32%	1.92%	1.59%	1.12%	0.67%	38%	62%

Figure 6. Averrage values of dynamic intensity and extensity indicators for the period 1961–2021.

Table 3. Relative changes and the indicators int_t and int_t for the period from 2011 to 2021.

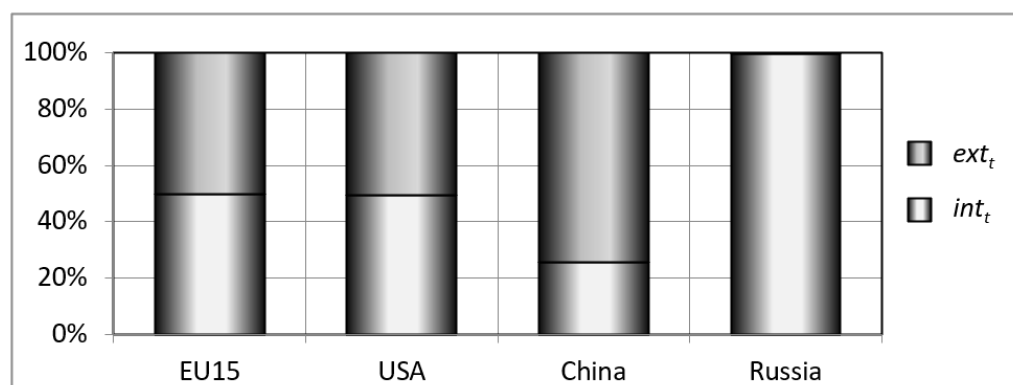|	$\overline{\beta HDP}_{11,21}$	$\overline{\beta L}_{11,21}$	$\overline{\beta K}_{11,21}$	$\overline{\beta (\frac{K}{L})}_{11,21}$	$\overline{\beta TIF}_{11,21}$	$\overline{\beta TFP}_{11,21}$	$int_{11,21}$	$ext_{11,21}$
China	6.39%	−0.17%	9,83%	10.02%	4.71%	1.61%	26%	74%
USA	2.34%	1.01%	1.36%	0.35%	1.18%	1.15%	49%	51%
EU	1.23%	0.20%	1.04%	0.83%	0.62%	0.61%	50%	50%
USSR a Russia	1.29%	−0.50%	0.52%	1.02%	0.01%	1.29%	100%	0%

Figure 7. Averrage values of dynamic intensity and extensity indicators for the period 2011–2021.

Despite the fact that the analysis is based on a relatively small amount of input data, it provides a rich indication of the quality of the development in several dimensions. The change in the output of the analyzed countries is characterized by the average annual relative change in the GDP at constant prices. China reported the highest average year-on-year relative change over 60 years (the period of 1961–2021) and in the past 10 years (the period of 2011–2021), followed by the US. The EU15 ranked third over 60 years, but was overtaken by Russia in the period of 2011–2021. The differences in the intensities are very small over the last 60 years, in the range of 6 percentage points. This is a consequence of technical progress being a worldwide phenomenon. Sooner or later, innovation will be realized globally. Significant differences in intensity are evident for the period of 2011–2021. In Europe and the US, the influence of intensive factors continues to grow, approaching 50%. China, on the other hand, showed a decrease in intensity against the 60-year average. China experienced major innovation in previous decades, especially after 1976, when China became one of the leaders in technology and innovation, something that slowed down in the period of 2011–2021. The intensive growth seen in Russia during the selected period is due to the collapse of the USSR and the need to face economic sanctions. Russia also became more independent, especially in the manufacturing industry. Figure 8 provides a comprehensive picture of the quality of the development of the analyzed countries over the past 60 years. All countries lie in the intensity range from 30 to 40%. The substantial differences are in the relative changes in the gross domestic product (GDP), especially for China, which has a higher change value (= growth rate of GDP). However, China started its GDP development from a substantially lower level than all of the other countries.

Figure 9 illustrates the quality of GDP development in the period of 2011–2021. The EU15 and US moved to an intensity of 50%, but with a lower average relative change in GDP. Russia only showed a slight reduction in the average relative change in the GDP, but is showing purely intensive growth, whereas its total factor productivity *TFP* is growing at roughly the same rate as the US and the sum of all of the other countries. China has maintained almost the same average relative change in the total input factor *TIF*, but with a substantially lower average relative change in total factor productivity *TFP*, with a change from 2.74% to 1.61%. For this reason, China saw a decrease in intensity from 28% to 26%. Our analysis confirms that the most important factor in terms of contributions to economic growth are the initial innovation period that took place in China in the 1980s and 1990s, as also seen in [18]. Gradually, the effect of subsequent innovations diminishes, which is manifested in a decrease in the value of *TFP*.
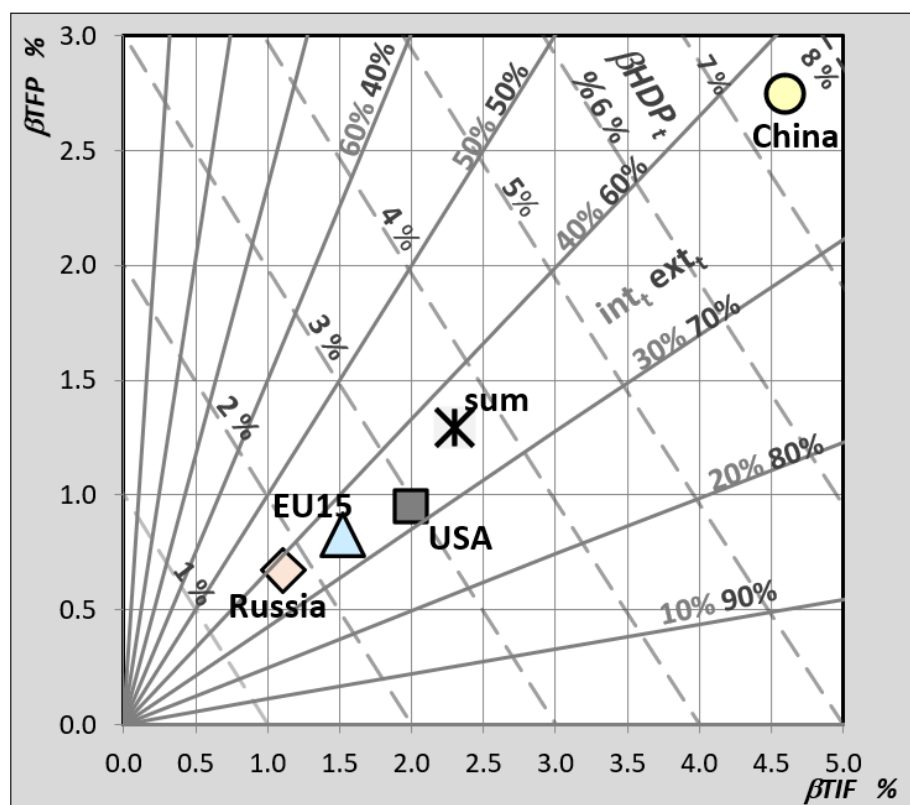
Figure 8. Qaulity of GDP development of anylyzed countries for the period of 1961–2021.
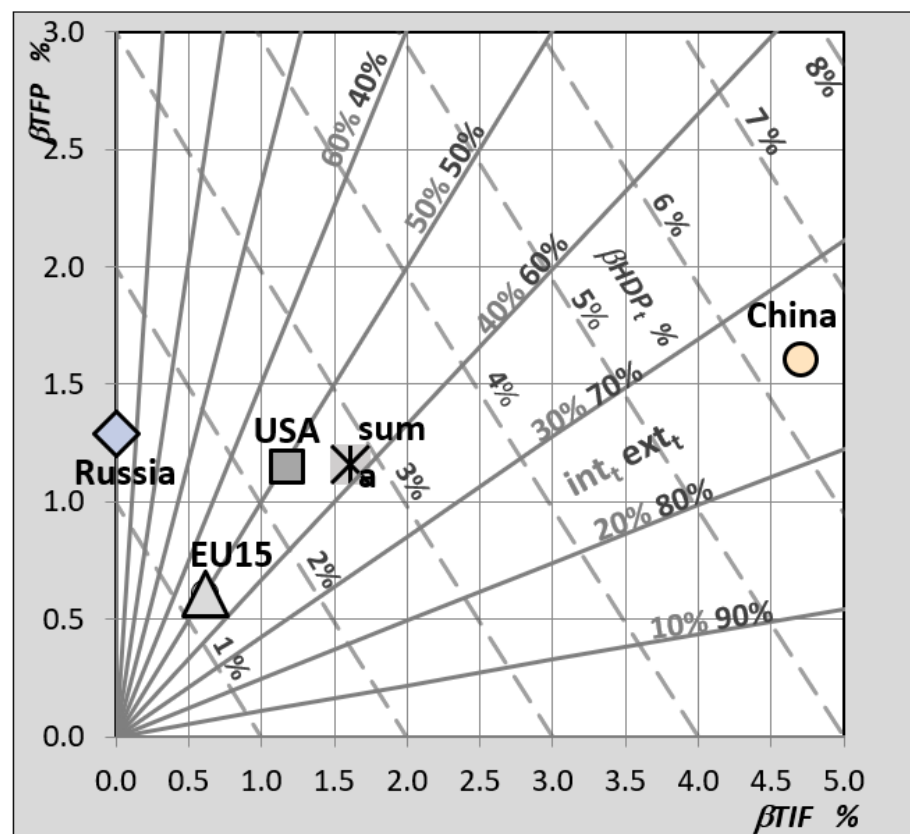

Figure 9. Quality of the development of important economic entities for the period of 1960–2021.

5.2. Intensity of Apple Company Development

Another example using dynamic intensity and extensity indicators is the analysis of the quality of the development experienced by Apple in the period from 1999 to 2021, mainly in terms of its achieved intensity. From the macroeconomic perspective, this period includes the so-called dot-com bubble, recovery from it, the financial crisis after 2007 (Great Recession), a period of moderate GDP growth after the year 2010, and the COVID-19 crisis (2020, 2021). The source for the time series of revenue TR_t and costs TN_t for the investigated period were taken from the annual reports of the company. From these data, we calculated the time series of efficiency Ef_t and the corresponding operators of relative change (βTR_t, βTC_t a βEf_t.). All data, including dynamic indicators of intensity int_t and extensity exp_t, are listed in Table 4. Due to the length of the time series, the table is divided into Part A and Part B. In the last column of the table in Part B, the average year-on-year data for the entire analyzed period is presented. The average year-on-year intensity and extensity can be also found in Figure 10.

Table 4. Intensity of Apple's development for the period of 1999–2021.

| years	00/99	01/00	02/01	03/02	04/03	05/04	06/05	07/06	08/07	09/08	10/09	11/10
t	1	2	3	4	5	6	7	8	9	10	11	12
βTR_τ	30%	−33%	7%	8%	33%	68%	39%	24%	35%	32%	52%	66%
βTC_t	29%	−24%	0%	8%	29%	57%	37%	18%	35%	13%	50%	59%
βEf_t	0.7%	−12.2%	6.7%	−0.3%	3.3%	7.0%	0.9%	5.0%	0.4%	17.2%	1.1%	4.4%
int_t	3%	−33%	95%	−3%	11%	13%	3%	22%	1%	57%	3%	9%
ext_t	97%	−67%	5%	97%	89%	87%	97%	78%	99%	43%	97%	91%
years	12/11	13/12	14/13	15/14	16/15	17/16	18/17	19/18	20/19	21/20	21/99	
t	13	14	15	16	17	18	19	20	21	22	average	
βTR_τ	45%	9%	7%	28%	−8%	6%	16%	−2%	6%	33%	20%	
βTC_t	36%	20%	7%	25%	−4%	8%	16%	1%	6%	23%	19%	
βEf_t	6.3%	−9.3%	0.1%	2.5%	−3.7%	−1.5%	−0.1%	−2.8%	−0.6%	8.0%	1.3%	
int_t	17%	−34%	1%	10%	−46%	−16%	−1%	−78%	−9%	27%	7%	
ext_t	83%	66%	99%	90%	−54%	84%	99%	22%	91%	73%	93%	
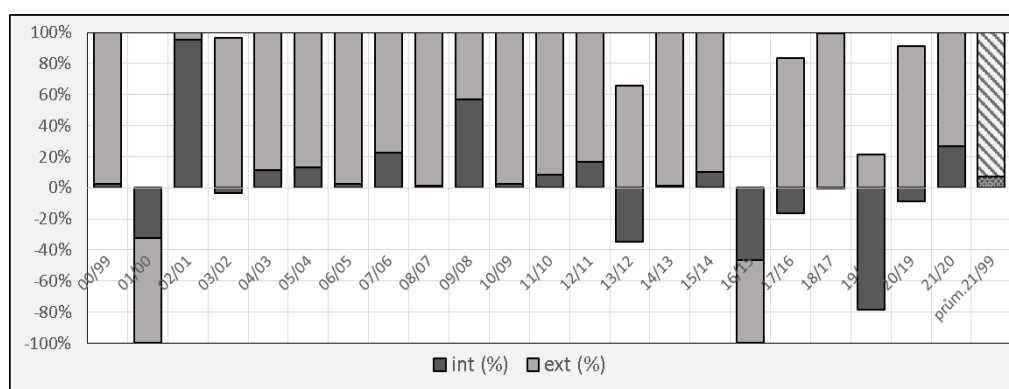
Figure 10. Apple's intensity and extensity development 1999–2021.

The relative changes in revenue βTR_t, cost βTC_t, and efficiency βEf_t are quite volatile in the analyzed period. Revenue grew the fastest in 2005/2004 and in 2011/2010. Efficiency grew at its fastest rate in 2009/2008. The time series, however, shows decreases as well. This fluctuation is also reflected in the development of intensity and extensity indicators. Of the 22 year-on-year periods, the intensity is negative in eight cases. In three cases, i.e., 2001/2000, 2016/2015, and 2019/2018, negative intensity due to a decrease in revenue is observed. Negative intensity is always a crisis trend caused by internal or external factors. For example, in 2000, the company was hit by the dot-com crisis, which was the main reason for the higher relative change in costs than in revenues. The company responded to

this by reducing its costs in 2001, but the consequences of the crisis also manifested in a significant decline in revenue.

A peak intensity of 95% is reported for the period of 2002/2001, which immediately follows the crisis year 2001/2000. However, this intensity is not that significant, as it was achieved with relatively low revenue growth of only 7%. The most successful period in Apple's development lasted from 2003 to 2012. High revenue growth rates (relative changes) that increased from 24% to 66% were accompanied by intensities ranging from 1% to 57%. In these years, sales growth was at least partially caused by innovative activities that increased the competitiveness of the company. Our analysis generally reveals that despite the fact that Apple achieved quite a long intensive period during the analyzed years and even though its revenues grew by 20% on average between the years 1999 and 2021, the average year-on-year intensity is 7%, which mostly consists of extensive development. In terms of process and organizational innovation, it can be concluded that the firm has made only reasonable progress. This does not mean it has not been successful in terms of product and marketing innovation. However, Apple only partially fulfills one of the characteristics of innovation—cutting costs and increasing output. We are aware of the fact that the firm produces its products outside the US as well. The cost trends of the firm may be influenced by the exchange rate trend of USD against the currencies of the countries from which the firm imports. Despite this fact, however, our analysis implies that the company should focus more on process and organizational innovation.

6. Discussion

How accurate are the dynamic intensity and extensity indicators? Our article shows that indicators can easily and understandably describe how a system develops and if the change in its output mainly depends on the change in the number of inputs (the quantitative or extensive change) or the quality of inputs (the qualitative or intensive changes). To be able to clearly state that a system is mainly developing extensively or intensively, it is necessary to analyze not only the quantities (amount) of outputs and inputs, but also their quality. It is possible that neither the outputs nor inputs for a specific period are comparable to the outputs and inputs for another period. This can lead to situations where the indicators do not provide accurate results. In cases where the inputs grow faster than outputs, the dynamic intensity indicator is negative. However, when the quality of the outputs improves significantly, and even when there is a smaller amount of output, greater satisfaction can be achieved for system members or other subjects. Overall, this kind of development can be considered as intensive, but the intensity indicator shows the opposite. The solution in these situations lies in the appropriate choice of output. These should not be just physical units, but units embodying a quality. However, especially in cases of economic systems with inputs and outputs that are usually expressed in monetary units, their value can be biased by inflation or other factors. To obtain accurate values of the dynamic intensity and extensity indicators, stable input and output prices should be used, or other adjustments reflecting the processes in the given economic system and leading to the correct values of input and output should be realized. Other suitable data can include research and development expenditure, patent applications, high-technology exports, the number of researchers in R&D, etc. We did not use these data in our investigation in Section 5 due to the difficulty of obtaining them. However, it is necessary to emphasize that the indicators that come from data can only correctly describe whether a system has developed intensively or extensively if the data accurately and truthfully describe reality.

Questionable results can be also achieved if the changes in the inputs and outputs of a system are quite small. Just imagine a situation where the output changes from 2 to 2.2 (by 10%) and where the input grows from 1 to 1.02 (by 2%). Indicators mainly indicate intensity development, with the dynamic indicator achieving intensity levels of 88.39%. However, such small absolute change cannot confirm that the change is due to effects of intensive factors. If there is almost no system development, then there cannot be reliable intensity or extensity values.

Another problem is if the analysis focuses on changes in the short run period. In the case of the analysis, if the output of a company or the country's GDP changed intensively or extensively, then misleading values mainly come from the changes between the next periods (e.g., year-on-year). The inputs of economic systems are mostly fixed, and they cannot be changed in a short period of time. The amount of output depends on the demand for system products. If demand falls for a reason beyond the system's control and the number of inputs does not change, the intensity indicator is negative. However, this may be a short-term fluctuation that does not require special attention. Indicators truthfully describe that something has happened, but they depend on other assessment to determine whether the system should respond to the event and how.

The above-mentioned situations show that values of the indicators collected over a long period of time (in the case of economic systems, 3 or more years) are much more accurate. Generally, it must be emphasized that the value of dynamic intensive indicators should be positive in the long run. A negative value in long run clearly indicates that the system is not developing optimally and will face, sooner or later, serious problems.

Our classification demonstrates that the development of a system's performance may be positive and its output increases, even when the value of dynamic intensity indicator is negative. This situation is shown in row 6 of Table 1 (extensive–intensive compensatory growth)—the decline in intensive factors is offset by an increase in extensive factors. Similarly, the situation shown in row 8 of Table 1 (pure intensive–extensive compensation) is also dangerous, as intensive factors are declining, but extensive factors are increasing at the same rate, thereby offsetting the decline in intensive factors. In this case, the system's output does not change. The management of the system can remain complacent in both situations, resulting in the belief that everything is in order. Neither extensive–intensive compensatory growth nor pure intensive–extensive compensation is sustainable in the long run. As we already mentioned, the amount of the inputs will become depleted at some point, and the system will not have other resources for its development. Other situations described in Table 1 may also be alarming, such as the situation in row 4, especially if the value of the dynamic extensity parameter is a much higher rate than the value of the dynamic intensity parameter in the long term. This situation indicates possible stagnation and the probability of the failure in the competition with other systems.

Row 11 of Table 1 shows a situation where the growth of extensive factors cannot offset the decline in intensive factors; rows 12 and 14 show a decline in both intensive and extensive factors; while row 15 describes a decline in extensive factors and no change in intensive factors. All of these situations mean a decrease in the system's output. In that case, a system should consider steps to increase the value of the dynamic intensity parameter. If the system is a firm, it must be emphasized that standard business evaluation methods such as financial analysis (e.g., [32]) need not reveal that the firm has not developed intensively (for details see [33]). Therefore, the indicators should be used as an additional source for the firm's analysis.

Indicators can be also used for other issues. If a firm wants to know how changes in output and price contribute to the change in total revenues (*TR*), one parameter (e.g., dynamic intensity indicator) can express the impact of the price changes, and the second one can express the impact of quantity changes. The results are, again, compared to the elasticity that is used for this task, easily understandable, and cover all possible situations, such as if price decreased but quantity increased, resulting in the *TR* increasing, or if price increased but quantity also increased thus, increasing *TR*. Another example of using the parameters is investigating how changes in speed (speed can be seen as an intensity parameter) contributes to changes in distance.

7. Conclusions

This article introduces a complex methodology of system development in which the main factors affecting system change are considered as input changes (extensive factors) and efficiency changes (intensive factors). The typology reveals that changes in the output

system can be positive (i.e., output increases) but that the changes in efficiency are negative or zero. This development indicates future problems. System management therefore needs some clear indicators revealing how the system develops, such as whether its changes are mainly based on intensive or extensive factors. Math represents a useful tool for solving this issue. The dynamic intensity and extensity parameters are derived from the coefficient of a change or from the relative change (growth rate). The values of the indicators lie in the interval of $(-1; 1)$, and the sum of their absolute values always equals 1. The indicator can thus be interpreted as the expression percentage of how the factors related to the indicator affect the growth or decline of the system output and in which direction. A positive value of the indicator means that changes in the factor lead to output growth and the opposite development of the negative value.

What can happen is that the changes in one factor are offset by the changes in the remaining factor, resulting in system stagnation (no change, i.e., no development). Additionally, the changes in one factor contribute to output growth at a higher rate than the changes in the remaining factors contributing to output decline. The overall output grows (declines) as a result. The indicators clearly and understandably describe the situations mentioned here and other situations. They can be easily counted using basic data expressing system development (a volume of input and output is a sufficient set of data). Their calculation and analysis help to reduce of the risk of negative system development—i.e., a system that mainly develops extensively. The parameters also reduce the uncertainty of system management during decision making and other managerial processes and can be seen as a valuable way to improve managerial tasks.

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