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Stochastic Analysis of a Hantavirus Infection Model

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Abstract: In this paper, a stochastic Hantavirus infection model is constructed. The existence, uniqueness, and boundedness of the positive solution of the stochastic Hantavirus infection model are derived. The conditions for the extinction of the Hantavirus infection from the stochastic system are obtained. Furthermore, the criteria for the presence of a unique ergodic stationary distribution for the Hantavirus infection model are established using a suitable Lyapunov function. Finally, the importance of environmental noise in the Hantavirus infection model is illustrated using the Milstein method.

Keywords: Hantavirus; Milstein method; stochastic; ergodic stationary distribution; biodiversity

MSC: 34D20; 37N25; 92D25; 37A50

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1. Introduction

Hantaviruses may be transmitted to humans through the saliva of rodents, such as mice and rats, their urine or feces, or through contact and inhalation of air contaminated with droplets of rodent saliva or dust contaminated with their dry droppings. This may result in fatal diseases in humans, such as pulmonary infection syndrome and hemorrhagic fever. Hantavirus pulmonary infection syndrome is rare, but fatal [1]. The Southwest USA experienced a Hantavirus outbreak in 1993, which led to a high mortality rate. Mathematical modeling of the spread of the Hantavirus infection is one of the important tools for understanding and interpreting different interactions between susceptible and infected mice. A simple mathematical model was developed by Abramson [1] to simulate the propagation of the virus, and it was shown to be capable of simulating some features of infection. In real life, rodents and so-called 'alien' species share the resources available in the environment. Therefore, rodents do not only share resources among themselves. Biodiversity and the competition between "alien" species and rodents should be taken into account. According to Peixoto [2] and Solomon [3], biodiversity plays an important role in controlling the spread of Hantavirus. The rodents and nonhost species can exert pressure on one another through the level of their respective interspecific competition. In order to account for the biodiversity effect, Peixoto [2] extended the basic Abramson model by including a nonhost alien species. Yusof et al. [4] extended the Peixoto model to include the effects of harvesting. Some studies of the modeling of Hantavirus infection include [5–17]. According to [18,19], the intrinsic growth rate, mortality rate, carrying capacity, competition coefficients, and other system parameters would be impacted by environmental changes. Following [20], one can estimate the birth and death rates by an average value plus errors. In general, by the well-known central limit theorem, the error term follows a normal distribution; thus, for a short correlation time, one can assume that the birth and death rates are subjected to the Gaussian white noise. The stochastic effect, which can be significant because the environmental conditions for its transmission are

subject to ecological randomness, is not taken into account by the deterministic Peixoto Hantavirus infection model. The primary purpose of this paper is to formulate a stochastic dynamic model to predict Hantavirus infection and identify the key factors that significantly affect the disease spread and control of Hantavirus infection. Hence, our goal in this paper is to provide a comprehensive analysis of the stochastic Hantavirus infection model, especially the existence and uniqueness of a positive global solution and the conditions for the extinction of the Hantavirus infection. This approach has recently been used in many papers for the analysis of stochastic predator–prey systems [21–25], stochastic epidemic models [26–33], and stochastic analysis methods [34–36]. This paper is arranged as follows: In Section 3, the existence and uniqueness of a positive global solution of the stochastic Hantavirus infection model are investigated, and sufficient conditions for the extinction of the infection from the stochastic system are obtained. In Section 4, some numerical simulations are presented to verify the obtained theoretical results. Finally, Section 5 contains the conclusion.

2. Hantavirus Model

In this section, we first present Abramson's model that investigates the spread of Hantavirus infection. In this model, the total population of rodents is divided into susceptible mice $x_1(t)$ and infected mice $x_2(t)$. The Abramson model's equations are as follows [1]:

$$\begin{aligned} \frac{dx_1}{dt} &= b(x_1 + x_2) - cx_1 - \frac{x_1}{k}(x_1 + x_2) - ax_1x_2, \\ \frac{dx_2}{dt} &= ax_1x_2 - \frac{x_2}{k}(x_1 + x_2) - cx_2, \end{aligned} \tag{1}$$

where b is the birth rate, c is the death rate, k is related to the carrying capacity of the environment, and a is the constant infection rate. The Peixoto model of competition Hantavirus dynamics including the nonhost alien species $x_3(t)$ takes the form [2]:

$$\begin{aligned} \frac{dx_1}{dt} &= b(x_1 + x_2) - cx_1 - \frac{x_1}{k}(x_1 + x_2 + \rho x_3) - ax_1x_2, \\ \frac{dx_2}{dt} &= ax_1x_2 - \frac{x_2}{k}(x_1 + x_2 + \rho x_3) - cx_2, \\ \frac{dx_3}{dt} &= (\beta - \gamma)x_3 - \frac{x_3}{k}(x_3 + \delta(x_1 + x_2)), \end{aligned} \tag{2}$$

where β and γ are the alien population's birth and death rates, respectively. ρ is the interspecific competition strength exerted by the alien population onto the mouse population, and δ is the interspecific competition strength exerted by the mouse population onto the alien population. In the present paper, the Hantavirus infection model (2) will extend to include the stochastic effects as follows:

$$\begin{aligned} dx_1 &= \left[b(x_1 + x_2) - cx_1 - \frac{x_1}{k}(x_1 + x_2 + \rho x_3) - ax_1x_2\right]dt + \sigma_1 x_1\, dB_1, \\ dx_2 &= \left[ax_1x_2 - \frac{x_2}{k}(x_1 + x_2 + \rho x_3) - cx_2\right]dt + \sigma_2 x_2\, dB_2, \\ dx_3 &= \left[(\beta - \gamma)x_3 - \frac{x_3}{k}(x_3 + \delta(x_1 + x_2))\right]dt + \sigma_3 x_3\, dB_3, \end{aligned} \tag{3}$$

where $B = \{B_1, B_2, B_3, t \geq 0\}$ represents the three-dimensional standard Brownian motions. The stochastic extension of the deterministic Abramson model (1) is recovered by setting $\rho = 0$ and ignoring the third equation of system (3).

3. Dynamics of the Stochastic Model

Firstly, we shall demonstrate the existence and uniqueness of a positive global solution of the Hantavirus infection model (3) in the following theorem.

Theorem 1. *There exists a unique solution of the Hantavirus infection model (3) for positive initial values, and the positive global solution remains in $\mathbb{R}^3_+$ with probability one.*

Proof. Assume $(x_1(t), x_2(t), x_3(t))$ is the solution to the Hantavirus infection model (3) for $t \in [0, \tau_e)$, where τ_e is the explosion time. Using the following variables

$$X_1(t) = \ln x_1(t),\ X_2(t) = \ln x_2(t),\ X_3(t) = \ln x_3(t),$$

one obtains

$$\begin{aligned} d\,X_1(t) &= \left[b(1+\frac{e^{X_2}}{e^{X_1}}) - c - \frac{1}{k}\left[e^{X_1} + e^{X_2} + \rho e^{X_3}\right] - ae^{X_2} - \frac{\sigma_1^2}{2}\right]dt + \sigma_1\, dB_1, \\ d\,X_2(t) &= \left[ae^{X_1} - \frac{1}{k}\left[e^{X_1} + e^{X_2} + \rho e^{X_3}\right] - c - \frac{\sigma_2^2}{2}\right]dt + \sigma_2\, dB_2, \\ d\,X_3(t) &= \left[(\beta-\gamma) - \frac{1}{k}\left[e^{X_3} + \delta(e^{X_1} + e^{X_2}\right] - \frac{\sigma_3^2}{2}\right]dt + \sigma_3\, dB_3. \end{aligned} \tag{4}$$

The transformed system (4) has a a unique local solution on $[0, \tau_e)$, as the coefficients satisfy the local Lipschitz conditions. Next, we prove that $\tau_e = \infty$ almost surely. Let $s_0 > 0$ be sufficiently large for every coordinate in the interval $[\frac{1}{s_0}, s_0]$. For each integer $s > s_0$, we can define

$$\tau_s = \inf\left\{t \in [0,\tau_e) : \min\{x_1(t), x_2(t), x_3(t)\} \notin (\frac{1}{s}, s) \text{ or } \max\{x_1(t), x_2(t), x_3(t)\} \notin (\frac{1}{s}, s)\right\}. \tag{5}$$

Using the following positive definite C^2 function $V_1(x_1, x_2, x_3)$ as

$$V_1(x_1, x_2, x_3) = (x_1 + 1 - \ln x_1) + (x_2 + 1 - \ln x_2) + (x_3 + 1 - \ln x_3), \tag{6}$$

one obtains

$$\begin{aligned} dV_1 =& \left[(x_1-1)\left(b + \frac{bx_2}{x_1} - c - \frac{1}{k}[x_1 + x_2 + \rho x_3] - ax_2\right) + (x_2-1)\left(ax_1 - \frac{1}{k}[x_1 + x_2 + \rho x_3] - c\right)\right. \\ &\left. + (x_3-1)\left(\beta - \gamma - \frac{1}{k}(x_3 + \delta(x_1 + x_2))\right) + \frac{1}{2}\sum_{i=1}^{3}\sigma_i^2\right]dt + \sigma_1(x_1-1)dB_1 + \sigma_2(x_2-1)dB_2 + \sigma_3(x_3-1)dB_3 \\ \le& \left[(b + \frac{2+\delta}{k})x_1 + (b + a + \frac{2+\delta}{k})x_2 + (\beta - \gamma + \frac{2\rho+1}{k})x_3 + 2c + \frac{1}{2}\sum_{i=1}^{3}\sigma_i^2\right]dt + \sigma_1(x_1-1)dB_1 \\ &+ \sigma_2(x_2-1)dB_2 + \sigma_3(x_3-1)dB_3. \end{aligned}$$

Using the inequality $A \le 2(A + 1 - \ln A)$, for any $A > 0$, one obtains

$$\begin{aligned} dV_1 \le& \left[2(b + \frac{2+\delta}{k})(x_1 + 1 - \ln x_1) + 2(b + a + \frac{2+\delta}{k})(x_2 + 1 - \ln x_2) + 2(\beta - \gamma + \frac{2\rho+1}{k})(x_3 + 1 - \ln x_3)\right. \\ &\left. + \left(2c + \frac{1}{2}\sum_{i=1}^{3}\sigma_i^2\right)\right]dt + \sigma_1(x_1-1)dB_1 + \sigma_2(x_2-1)dB_2 + \sigma_3(x_3-1)dB_3, \end{aligned}$$

which means that

$$dV_1 \le K(1 + V_1)dt + \sigma_1(x_1-1)dB_1 + \sigma_2(x_2-1)dB_2 + \sigma_3(x_3-1)dB_3, \tag{7}$$

where

$$K = \max\left\{2(b + \frac{2+\delta}{k}), 2(b + a + \frac{2+\delta}{k}), 2(\beta - \gamma + \frac{2\rho+1}{k}), 2c + \frac{1}{2}\sum_{i=1}^{3}\sigma_i^2\right\}. \tag{8}$$

Integrating form 0 to $t_1 \wedge \tau_s$ and taking the expectation, one obtains

$$EV_1(x_1(t_1 \wedge \tau_s), x_2(t_1 \wedge \tau_s), x_3(t_1 \wedge \tau_s)) \leq V_1(x_1(0), x_2(0), x_3(0)) + KT + K \int_0^{t_1 \wedge \tau_s} EV_1 \, dt.$$

Following [18,37], using Grownwall's inequality, one obtains

$$EV_1(x_1(t_1 \wedge \tau_s), x_2(t_1 \wedge \tau_s), x_3(t_1 \wedge \tau_s)) \leq [V_1(x_1(0), x_2(0), x_3(0)) + KT]e^{KT} = K_2.$$

The remaining part of the proof is similar to [37,38]; therefore, it can be omitted. □

Theorem 1 shows that the stochastic Hantavirus infection model (3) has a positive global solution remaining in $\mathbb{R}^3_+$ with probability one. Next, we establish the boundedness property of the Hantavirus infection model (3).

Theorem 2. *Let $H(t) = x_1(t) + x_2(t) + x_3(t)$; then, the following inequality holds:*

$$\lim_{t \to \infty} sup \;\; E[H(t)] \leq \frac{\beta_1^2 k}{2\beta_2}$$

almost surely, where $\beta_1 = max\{b, \beta\}$, and $\beta_2 = min\{c, \gamma\}$.

Proof. According to the stochastic Hantavirus infection model, (3)

$$\begin{aligned}
dH(t) \leq & \left[b(x_1 + x_2) - c(x_1 + x_2) - \frac{1}{k}(x_1 + x_2)^2 - \frac{\rho x_3}{k}(x_1 + x_2) + (\beta - \gamma)x_3 - \frac{x_3}{k}(x_3 + \delta(x_1 + x_2)) \right] dt \\
& + \sigma_1 x_1 \, dB_1 + \sigma_2 x_2 \, dB_2 + \sigma_3 x_3 \, dB_3 \\
\leq & \frac{-1}{k}\left[(x_1 + x_2)^2 - bk(x_1 + x_2) + \frac{b^2 k^2}{4} \right] + \frac{b^2 k}{4} - \frac{1}{k}\left[x_3^2 - \beta k x_3 + \frac{\beta^2 k^2}{4} \right] + \frac{\beta^2 k}{4} - c(x_1 + x_2) - \gamma x_3 \\
& + \sigma_1 x_1 \, dB_1 + \sigma_2 x_2 \, dB_2 + \sigma_3 x_3 \, dB_3 \\
\leq & \frac{\beta_1^2 k}{2} - \beta_2(x_1 + x_2 + x_3) + \sigma_1 x_1 \, dB_1 + \sigma_2 x_2 \, dB_2 + \sigma_3 x_3 \, dB_3.
\end{aligned}$$

Consequently,

$$H(t) \leq H(0) + \frac{\beta_1^2 k}{2} t - \beta_2 \int_0^t H(s) ds + \int_0^t [\sigma_1 x_1 dB_1 + \sigma_2 x_2 dB_2 + \sigma_3 x_3 dB_3] ds.$$

Using the strong law of large numbers, one obtains

$$E[H(t)] \leq H(0) + \frac{\beta_1^2 k}{2} t - \beta_2 \int_0^t E(H(s)) ds.$$

Consequently,

$$\frac{dE[H(t)]}{dt} + \beta_2 E[H(t)] \leq \frac{\beta_1^2 k}{2}.$$

Thus, one obtains

$$\lim_{t \to \infty} \sup \;\; E[H(t)] \leq \frac{\beta_1^2 k}{2\beta_2}.$$

□

According to Theorem 2, the solution of the Hantavirus infection model (3) is uniformly bounded in mean, and as a result, the deterministic Hantavirus infection model (2) is uniformly bounded.

Theorem 3. *If* $\sigma_1^2 + 2b + 1 < 2c$, $\sigma_2^2 + 1 < 2c$, $\sigma_3^2 + 2\beta + 1 < 2\gamma$, *then the solutions of* (3) *are stochastically ultimate bounded.*

Proof. For $(x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^3_+$, we define the following function

$$V_2(x(t), y(t), z(t)) = x(t)^2 + y(t)^2 + z(t)^2. \tag{9}$$

By the Itô formula, one has

$$dV_2 = LV_2 dt + 2\sigma_1 x_1^2 dB_1 + 2\sigma_2 x_2^2 dB_2 + 2\sigma_3 x_3^2 dB_3, \tag{10}$$

where

$$\begin{aligned} LV_2(x_1, x_2, x_3) =& 2\left[bx_1^2 + bx_1x_2 - cx_1^2 - \frac{x_1^2}{2}(x_1 + x_2 + \rho x_3) - ax_1^2x_2 + ax_1x_2^2 - cx_2^2 - \frac{x_2^2}{2}(x_1 + x_2 + \rho x_3)\right] \\ &+ 2x_3^2\left((\beta - \gamma) - \frac{1}{k}(x_3 + \delta(x_1 + x_2))\right) + \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 x_3^2 \\ \leq& (\sigma_1^2 + 2b - 2c + 1)x_1^2 + \left(\sigma_2^2 - 2c + 1\right)x_2^2 + \left(\sigma_3^2 + 2(\beta - \gamma) + 1\right)x_3^2 + 2ax_1x_2^2 - (x_1^2 + x_2^2 + x_3^2). \end{aligned}$$

Assume $f_1(x_1, x_2, x_3) = (\sigma_1^2 + 2b - 2c + 1)x_1^2 + (\sigma_2^2 - 2c + 1)x_2^2 + (\sigma_3^2 + 2(\beta - \gamma) + 1)x_3^2 + 2ax_1x_2^2$. According to Theorem 2, one can find that the function $f_1(x_1, x_2, x_3)$ has an upper bound. Let $M = \sup f_1(x_1, x_2, x_3) + 1$. As a result,

$$dV_2 = (M - V_2)dt + 2\sigma_1 x_1^2 dB_1 + 2\sigma_2 x_2^2 dB_2 + 2\sigma_3 x_3^2 dB_3. \tag{11}$$

By the Itô formula, one obtains

$$d(e^t V_2) \leq e^t N_1 dt + e^t\left[2\sigma_1 x_1^2 dB_1 + 2\sigma_2 x_2^2 dB_2 + 2\sigma_3 x_3^2 dB_3\right].$$

Consequently,

$$e^t V_2(x_1(t), x_2(t), x_3(t)) \leq V_2(x_1(0), x_2(0), x_3(0)) + Me^t - M;$$

hence,

$$\limsup_{t\to\infty} \mathbb{E}[|X(t)|^2] \leq M.$$

According to Chebyshev's inequality, one obtains

$$\mathbb{P}[|X(t)| \geq \eta] \leq \frac{\mathbb{E}[|X(t)|^2]}{\eta^2},$$

where $\eta = \frac{\sqrt{M}}{\sqrt{\nu}}$, $\nu > 0$. Then,

$$\limsup_{t\to\infty} \mathbb{P}[|X(t)| \geq \eta] \leq \frac{M}{\eta^2} = \nu.$$

This completes the proof. □

Next, we establish the conditions for the extinction of the Hantavirus infection model (3).

Theorem 4. *For any positive initial conditions, if* $b < c$ *and* $\beta < \frac{\sigma_3^2}{2} + \gamma$, *then the populations of the Hantavirus infection model* (3) *will be extinct with probability one.*

Proof. Using the Itô formula, one obtains

$$\begin{aligned} d(\ln(x_1+x_2)) =& \left[b-\frac{(x_1+x_2)}{k}-\frac{\rho x_3}{k}-c-\frac{1}{2(x_1+x_2)^2}\left(\sigma_1^2x_1^2+\sigma_2^2x_2^2\right)\right]dt \\ &+\frac{\sigma_1x_1}{(x_1+x_2)}dB_1+\frac{\sigma_2x_2}{(x_1+x_2)}dB_2 \\ \leq& (b-c)dt+\frac{\sigma_1x_1}{(x_1+x_2)}dB_1+\frac{\sigma_2x_2}{(x_1+x_2)}dB_2. \end{aligned}$$

As a result,

$$\ln[x_1(t)+x_2(t)] \leq \ln[x_1(0)+x_2(0)]+(b-c)\,t,$$

which implies that

$$\limsup_{t\to\infty}\frac{\ln[x_1(t)+x_2(t)]}{t} \leq b-c<0 \quad almostsurely.$$

Hence,

$$\lim_{t\to\infty}[x_1(t)+x_2(t)]=0.$$

According to the third equation of the Hantavirus infection system (3), one obtains

$$d(\ln x_3(t)) = \left[(\beta-\gamma)-\frac{1}{k}(x_3+\delta(x_1+x_2))-\frac{\sigma_3^2}{2}\right]dt+\sigma_3\,dB_3. \tag{12}$$

Consequently,

$$\ln x_3(t) = \ln x_3(0)-\frac{1}{k}\int_0^t[x_3(s)+\delta(x_1(s)+x_2(s))]ds+((\beta-\gamma)-\frac{\sigma_3^2}{2})\,t+\sigma_3\,B_3, \tag{13}$$

and it follows that

$$\limsup_{t\to\infty}\frac{\ln x_3(t)}{t} \leq (\beta-\gamma)-\frac{\sigma_3^2}{2}<0 \quad almostsurely.$$

Thus, $\lim_{t\to\infty}x_3(t)=0$. As a result, if $b<c$, and $\beta<\frac{\sigma_3^2}{2}+\gamma$, then the populations of the Hantavirus infection model (3) will be extinct with probability one. □

The asymptotic stability of the Hantavirus infection system (3) is established in the following theorem.

Theorem 5. *For any positive initial conditions, the trivial solution of the Hantavirus infection model (3) is stochastically asymptotically stable in probability if* $\frac{\sigma_1^2}{2}+b<c$, $\frac{\sigma_3^2}{2}+\beta<\gamma$, *and* $(\frac{\sigma_1^2}{2}+b-c)(\frac{\sigma_2^2}{2}-c)>\frac{b^2}{4}$.

Proof. The first step is to consider the following linearized Hantavirus infection model about the origin

$$\begin{aligned} dx_1 &= [(b-c)x_1+bx_2]dt+\sigma_1x_1\,dB_1, \\ dx_2 &= -cx_2dt+\sigma_2x_2\,dB_2, \\ dx_3 &= (\beta-\gamma)x_3dt+\sigma_3x_3\,dB_3. \end{aligned} \tag{14}$$

Consider the following Lyapunov function

$$V_3=\frac{1}{2}\left[x_1^2(t)+x_2^2(t)+x_3^2(t)\right]. \tag{15}$$

One can compute

$$LV_3 = \left[\frac{\sigma_1^2}{2} + b - c\right]x_1^2 + \left[\frac{\sigma_2^2}{2} - c\right]x_2^2 + \left[\frac{\sigma_3^2}{2} + \beta - \gamma\right]x_3^2 + bx_1x_2. \tag{16}$$

One can rewrite LV_3 to be $LV_3 = \frac{1}{2}X^TQX$, where $X = (x_1, x_2, x_3)$ and

$$Q = \begin{pmatrix} 2(\frac{\sigma_1^2}{2} + b - c) & b & 0 \\ b & 2(\frac{\sigma_2^2}{2} - c) & 0 \\ 0 & 0 & 2(\frac{\sigma_3^2}{2} + \beta - \gamma) \end{pmatrix}.$$

The matrix Q will be negative definite if $\frac{\sigma_1^2}{2} + b < c$, $\frac{\sigma_3^2}{2} + \beta < \gamma$, and $(\frac{\sigma_1^2}{2} + b - c)(\frac{\sigma_2^2}{2} - c) > \frac{b^2}{4}$. As indicated by [39], the linearized stochastic Hantavirus infection model (14) is stochastically stable in the large if LV_3 is a negative-definite function. According to Arnold [40], the trivial solution of the nonlinear stochastic Hantavirus infection model (3) is stochastically asymptotically stable if the linear stochastic Hantavirus model (14) is stochastically asymptotically stable. □

The equilibrium point $E = (x_0, 0, 0)$, where $x_0 = k(b - c)$ is the Hantavirus-free equilibrium of the deterministic Hantavirus infection model (2), but it may be not an equilibrium of the stochastic Hantavirus infection model (3). Next, we investigate the asymptotic property around E for the stochastic system.

Theorem 6. *The stochastic Hantavirus infection model* (3) *has the following property*

$$\lim_{t\to\infty} sup\frac{1}{t}E\int_0^t \left[(x_1(u) - x_0)^2 + x_2(u)^2 + x_3(u)^2\right]du \le \left[(b + \frac{x_0}{k}) + (\beta - \gamma + \frac{x_0\rho}{k})\right]\frac{\beta_1^2k^2}{2\beta_2} + \left(\frac{\sigma_1^2}{2} - (b - c)\right)x_0k.$$

Proof. In order to prove Theorem 6, one can define the following function

$$V_4(x_1, x_2, x_3) = \left(x_1 - x_0 + \ln(\frac{x_1}{x_0})\right) + x_2 + x_3. \tag{17}$$

Applying the Itô formula leads to

$$\begin{aligned} dV_4 =& \left[(x_1 - x_0)\left(b + \frac{x_2}{x_1} - c - \frac{1}{k}(x_1 + x_2 + \rho x_3) - ax_2\right) + \frac{x_0\sigma_1^2}{2} + \left(-cx_2 - \frac{x_2}{k}(x_1 + x_2 + \rho x_3) + ax_1x_2\right)\right] \\ &+\left((\beta - \gamma)x_3 - \frac{x_3}{k}(x_3 + \delta(x_1 + x_2))\right) + \sigma_1(x_1 - x_0)dB_1 + \sigma_2x_2dB_2 + \sigma_3x_3\,dB_3 \\ &\le \left[\frac{-1}{k}(x_1 - x_0)^2 - \frac{1}{k}x_2^2 - \frac{1}{k}x_3^2 + (b + \frac{x_0}{k})x_2 + (\beta - \gamma + \frac{x_0q}{k})x_3 + \left(\frac{x_0\sigma_1^2}{2} - (b - c)x_0\right)\right]dt + \sigma_1(x_1 - x_0)dB_1 \\ &+\sigma_2x_2dB_2 + \sigma_3x_3dB_3 \\ &\le \left[\frac{-1}{k}(x_1 - x_0)^2 - \frac{1}{k}x_2^2 - \frac{1}{k}x_3^2 + (b + \frac{x_0}{k})x_2 + (\beta - \gamma + \frac{x_0\rho}{k})x_3 + \left(\frac{x_0\sigma_1^2}{2} - (b - c)x_0\right)\right]dt + \sigma_1(x_1 - x_0)dB_1 \\ &+\sigma_2x_2dB_2 + \sigma_3x_3dB_3. \end{aligned}$$

Consequently,

$$\begin{aligned} 0 \le& E[V_4(x_1(t), x_2(t), x_3(t))] \le E[V_4(x_1(0), x_2(0), x_3(0))] \\ &+E\int_0^t \left[-\frac{1}{k}(x_1(u) - x_0)^2 - \frac{1}{k}x_2(u)^2 - \frac{1}{k}x_3(u)^2 + (b + \frac{x_0}{k})x_2(u) + (\beta - \gamma + \frac{x_0\rho}{k})x_3(s) + \left(\frac{x_0\sigma_1^2}{2} - (b - c)x_0\right)\right]du, \end{aligned}$$

using Theorem 2, one obtains

$$E\int_0^t\left[\frac{1}{k}(x_1(u)-x_0)^2+\frac{1}{k}x_2(u)^2+\frac{1}{k}x_3(u)^2\right]du \le E(V_4(x_1(0),x_2(0),x_3(0)))+(b+\frac{x_0}{k})\frac{\beta_1^2k}{2\beta_2}t$$
$$+(\beta-\gamma+\frac{x_0\rho}{k})\frac{\beta_1^2k}{2\beta_2}t+\left(\frac{x_0\sigma_1^2}{2}-(b-c)x_0\right)t.$$

Therefore,

$$\limsup_{t\to\infty}\ \frac{1}{t}E\int_0^t\left[(x_1(u)-x_0)^2+x_2(u)^2+x_3(u)^2\right]du \le \left[(b+\frac{x_0}{k})+(\beta-\gamma+\frac{x_0\rho}{k})\right]\frac{\beta_1^2k^2}{2\beta_2}+\left(\frac{\sigma_1^2}{2}-(b-c)\right)x_0k.$$

□

From Theorem 6, one can see that the Hantavirus infection will tend to die out when the intensity of the stochastic perturbations σ_1 is small enough. In the following theorem, we establish the criteria for the existence of an ergodic stationary distribution in the stochastic Hantavirus infection model (3) using the method of Khasminskii [41]. According to [42,43], one can investigate the stationary distribution for the Hantavirus infection model (3) instead of asymptotically stable equilibria. Before giving the main theorem, we first state the following Lemma

Lemma 1 ([41]). *The Markov process $X(t)$ has a unique ergodic stationary distribution $\pi(.)$ if there exists a bounded closed domain $U_1 \subset \mathbb{R}^d$ with regular boundary Γ, having the following properties:*

H_1: *there is a positive number M_0 such that $\sum_{i,j=1}^d a_{ij}(x)\eta_i\eta_j \ge M_0|\eta^2|, x \in U_1, \eta \in \mathbb{R}^d$,*

H_2: *there exists a nonnegative C^2 function V such that LV is negative on $\mathbb{R}^d \backslash U_1$.*

Remark 1. *The positive equilibrium point $E = (x_1^*, x_2^*, x_3^*)$ of the deterministic Peixoto system (2) satisfies*

$$(b-c)=\frac{1}{k}(x_1^*+x_2^*+\rho x_3^*)+ax_2^*-b\frac{x_2^*}{x_1^*},\ \ c=ax_2^*-\frac{1}{k}(x_1^*+x_2^*+\rho x_3^*),\ \ (\beta-\gamma)=\frac{1}{k}(x_3^*+\delta(x_1^*+x_2^*)),$$

where

$$x_1^*=\frac{b}{a},\ \ x_2^*=\frac{ak(b-c)-b(1-\rho\delta)-ak\rho(\beta-\gamma)}{a(1-\rho\delta)},\ \ x_3^*=\frac{k[(\beta-\gamma)-\delta(b-c)]}{1-\rho\delta},$$

$0<\rho<1$, $0<\delta<1$, $(\beta-\gamma)>\delta(b-c)$ *and* $\beta>\gamma$, $(\beta-\gamma)>\delta(b-c)$, $ak(b-c)>b(1-\rho\delta)+ak\rho(\beta-\gamma)$.

Theorem 7. *Assume $(ak-1)(1-\rho)>\delta$, $\rho+\delta+2ak\rho<1$, $b>c$, $\beta>\gamma$, $0<\rho\delta<1$, $(\beta-\gamma)>\delta(b-c)$, $ak(b-c)>b(1-\rho\delta)+ak\rho(\beta-\gamma)$, and*

$$m<min\left\{\frac{((ak-1)(1-\rho)-\delta)}{k}(x_1^*)^2,\frac{(ak+ak\rho+1)}{k}(x_2^*)^2,\frac{(1-\rho-\delta-2ak\rho)}{k}(x_3^*)^2\right\},$$

where $m=b(\frac{b^2k}{4c}+x_1^*)+\frac{(ak-1)x_1^*\sigma_1^2}{2}+\frac{(ak-1)x_2^*\sigma_2^2}{2}+\frac{x_3^*\sigma_3^2}{2}$; then, the stochastic Hantavirus infection model (3) has an ergodic stationary distribution for any given positive initial values.

Proof. In order to prove Theorem 7, one needs only to validate conditions H_1 and H_2 of Lemma 1. The first step is to validate condition H_1 of Lemma 1. Following [44], one can define the following nonnegative C^2 function

$$V_5(x_1,x_2,x_3)=\alpha_1\left(x_1-x_1^*-x_1^*\ln\frac{x_1}{x_1^*}\right)+\alpha_2\left(x_2-x_2^*-x_2^*\ln\frac{x_2}{x_2^*}\right)+\alpha_3\left(x_3-x_3^*-x_3^*\ln\frac{x_3}{x_3^*}\right). \tag{18}$$

Applying the Itô formula leads to

$$\begin{aligned} LV_5 =& \alpha_1(x_1 - x_1^*)\left(b - c + \frac{bx_2}{x_1} - \frac{1}{k}(x_1 + x_2 + \rho x_3) - ax_2\right) + \alpha_2(x_2 - x_2^*)\left(ax_1 - \frac{1}{k}(x_1 + x_2 + \rho x_3) - c\right) \\ &+ \alpha_3(x_3 - x_3^*)\left(\beta - \gamma - \frac{1}{k}(x_3 + \delta(x_1 + x_2))\right) + \frac{\alpha_1 x_1^* \sigma_1^2}{2} + \frac{\alpha_2 x_2^* \sigma_2^2}{2} + \frac{\alpha_3 x_3^* \sigma_3^2}{2} \\ \leq& -\frac{\alpha_1}{k}(x_1 - x_1^*)^2 - \frac{\alpha_2}{k}(x_2 - x_2^*)^2 - \frac{\alpha_3}{k}(x_3 - x_3^*)^2 - \left[\alpha_1(\frac{ak+1}{k}) - \alpha_2(\frac{ak-1}{k})\right](x_1 - x_1^*)(x_2 - x_2^*) \\ &- \left(\frac{\alpha_1\rho}{k} + \frac{\alpha_3\delta}{k}\right)(x_1 - x_1^*)(x_3 - x_3^*) - \frac{\rho}{k}(\alpha_2 + \alpha_3)(x_2 - x_2^*)(x_3 - x_3^*) + \alpha_1 b(x_1 - x_1^*)\left(\frac{x_2}{x_1} - \frac{x_2^*}{x_1^*}\right) \\ &+ \frac{\alpha_1 x_1^* \sigma_1^2}{2} + \frac{\alpha_2 x_2^* \sigma_2^2}{2} + \frac{\alpha_3 x_3^* \sigma_3^2}{2}. \end{aligned} \tag{19}$$

Taking $\alpha_1 = ak - 1$, $\alpha_2 = ak + 1$, and $\alpha_3 = 1$, therefore,

$$\begin{aligned} LV_5 \leq& -\frac{((ak-1)(1-\rho) - \delta)}{k}(x_1 - x_1^*)^2 - \frac{(ak + ak\rho + 1)}{k}(x_2 - x_2^*)^2 - \frac{(1 - \rho - \delta - 2ak\rho)}{k}(x_3 - x_3^*)^2 \\ &+ b(\frac{b^2 k}{4c} + x_1^*) + \frac{(ak-1)x_1^* \sigma_1^2}{2} + \frac{(ak+1)x_2^* \sigma_2^2}{2} + \frac{x_3^* \sigma_3^2}{2}. \end{aligned} \tag{20}$$

Following [18,44–46], when

$$m < \min\left\{\frac{((ak-1)(1-\rho) - \delta)}{k}(x_1^*)^2, \frac{(ak + ak\rho + 1)}{k}(x_2^*)^2, \frac{(1 - \rho - \delta - 2ak\rho)}{k}(x_3^*)^2\right\},$$

then the ellipsoid

$$-\frac{((ak-1)(1-\rho) - \delta)}{k}(x_1 - x_1^*)^2 - \frac{(ak + ak\rho + 1)}{k}(x_2 - x_2^*)^2 - \frac{(1 - \rho - \delta - 2ak\rho)}{k}(x_3 - x_3^*)^2 + m = 0,$$

lies entirely in $\mathbb{R}^3_+$. One can take U_1 to be a neighborhood of the ellipsoid, which satisfies $\bar{U_1} \subseteq \mathbb{R}^3_+$; hence, $LV_5 < 0$ for $(x_1, x_2, x_3) \in \mathbb{R}^3_+ \setminus \bar{U_1}$. This implies that the first condition H_1 of the method of Khasminskii [41] is satisfied. The second step is to validate condition H_2 of Lemma 1. The diffusion matrix A_1 of the stochastic Hantavirus infection model (3) is as follows

$$A_1 = \begin{pmatrix} \sigma_1^2 x_1^2 & 0 & 0 \\ 0 & \sigma_2^2 x_2^2 & 0 \\ 0 & 0 & \sigma_3^2 x_3^2 \end{pmatrix}.$$

Following [37,46,47], we choose $M_0 = \min\{\sigma_1^2 x_1^2, \sigma_2^2 x_2^2, \sigma_3^2 x_3^2\}$; then, one can find a positive number M_0 such that

$$\sum_{i,j=1}^{3} a_{ij}(x_1, x_2, x_3)\xi_i \xi_j = \sigma_1^2 x_1^2 \xi_1^2 + \sigma_2^2 x_2^2 \xi_2^2 + \sigma_3^2 x_3^2 \xi_3^2 \geq M_0 |\xi^2|,$$

for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $(x_1, x_2, x_3) \in U_1$. This implies condition H_2 in Lemma 1 is satisfied. As a result, the stochastic Hantavirus infection model (3) has an ergodic stationary distribution for any given positive initial values. □

Remark 2: If we assume $y(t) = x_1(t) + x_2(t)$ and add the first and second equations of system (3), one obtains

$$\begin{aligned} dy(t) =& \left[(b-c)y - \frac{1}{k}y^2 - \frac{\rho\, y\, x_3}{k}\right]dt + \sigma_1 x_1\, dB_1 + \sigma_2 x_2\, dB_2 \\ \leq& (b-c)\, y(1 - \frac{y}{k(b-c)})dt + \sigma_1 x_1\, dB_1 + \sigma_2 x_2\, dB_2. \end{aligned} \tag{21}$$

According to Liu and Wang [48], if the intensities of the white noises are sufficiently small and $b - c > \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}$, then there is a stationary distribution to the following equation for positive initial values

$$dy(t) = (b-c)\, y(1 - \frac{y}{k(b-c)})dt + \sigma_1\, y\, dB_1 + \sigma_2\, y\, dB_2, \tag{22}$$

and it has an ergodic property. From the third equation of the stochastic Hantavirus infection system (3), one obtains

$$dx_3(t) \leq (\beta - \gamma)x_3\left(1 - \frac{x_3}{k(\beta-\gamma)}\right)dt + \sigma_3 x_3\, dB_3, \tag{23}$$

According to [49,50], $x_3(t)$ neither reaches zero nor infinity in finite time, and provided $(\beta - \gamma) > \frac{\sigma_3^2}{2}$, the process has been shown to have a stationary distribution. Moreover, it has been shown that

$$0 < \liminf_{t\to\infty}\ x_3(t) \leq \limsup_{t\to\infty}\ x_3(t) < \infty \quad almostsurely.$$

4. Numerical Simulations

In order to demonstrate the above theoretical results for the stochastic Hantavirus system, we use the following parameters [2,51]:

$$a = 0.1;\ b = 1;\ c = 0.6;\ \beta = 1;\ \gamma = 0.5;\ \rho = 0.2;\ \delta = 0.1;\ k = 50.$$

To give some numerical finding to the stochastic Hantavirus system (3), we use the Milstein method mentioned in [52,53]. The stochastic Hantavirus infection system (3) reduces to the following discrete system

$$\begin{aligned}
x_{1(j+1)} &= x_{1j} + h\left(b(x_{1j}+y_{1j}) - cx_{1j} - \frac{x_{1j}}{k}(x_{1j}+x_{2j}+\rho x_{3j}) - ax_{1j}x_{2j}\right) + \sigma_1 x_{1j}\sqrt{h}\epsilon_{1j} + \frac{\sigma_1^2}{2}x_{1j}\left[\epsilon_{1j}^2 - 1\right]h \\
x_{2(j+1)} &= x_{2j} + h\left(ax_{1j}x_{2j} - \frac{x_{2j}}{k}(x_{1j}+x_{2j}+\rho x_{3j}) - cx_{2j}\right) + \sigma_2 x_{2j}\sqrt{h}\epsilon_{2j} + \frac{\sigma_2^2}{2}x_{2j}\left[\epsilon_{2j}^2 - 1\right]h \\
x_{3(j+1)} &= x_{3j} + h\left((\beta-\gamma)x_{3j} - \frac{x_{3j}}{k}(x_{3j} + \delta(x_{1j}+x_{2j}))\right) + \sigma_3 x_{3j}\sqrt{h}\epsilon_{3j} + \frac{\sigma_3^2}{2}x_{3j}\left[\epsilon_{3j}^2 - 1\right]h,
\end{aligned} \tag{24}$$

where $\epsilon_{ij}, (i,j = 1,2,3)$ are independent random Gaussian variables $N(0,1)$, and h is a positive time increment. In the stochastic Hantavirus infection model (3), if one gradually increases the values of σ_i and keeps the remaining parameters unchanged, the fluctuations become larger around the positive equilibrium point for the values of $\sigma_i = 0.2$, as shown in Figure 1. The infected mice $y(t)$ are represented by the black line when $(\sigma_i = 0)$, as seen in Figure 1. The conditions of Theorem 4 for the given parameters are verified, and the populations will be extinct with probability one, if $b < c$ and $\beta < \frac{\sigma_3^3}{2} + \gamma$ as indicated in Figure 2, when $b = 0.55$ and $\beta = 0.5$. Moreover, the trivial solution of the Hantavirus infection model (3) is stochastically asymptotically stable in probability if the conditions of Theorem 5 are verified, i.e., $\frac{\sigma_1^2}{2} + b < c$, $\frac{\sigma_3^2}{2} + \beta < \gamma$, and $(\frac{\sigma_1^2}{2} + b - c)(\frac{\sigma_2^2}{2} - c) > \frac{b^2}{4}$ as indicated in Figure 3. The stochastic form of the Abramson model (2) is recovered by setting $\rho = 0$ and ignoring the third equation of system (3). Following [2,51], there is a critical value k_c for the carrying capacity k. The population of the infected $y(t)$ will die away when $k < k_c$. The Hantavirus disease will spread and increase in rodents if $k > k_c$. Figure 4 represents the dynamical behavior of the Abramson model (2) and verifies the statement of [1]. For $k = 20$, the free Hantavirus equilibrium point $E = (8,0,0)$ is locally asymptotically stable in the deterministic model. while the solutions of the stochastic

Hantavirus infection model (3) oscillate around the equilibrium point, which coincides with Theorem 6. The oscillation amplitude will be

$$\limsup_{t\to\infty} \frac{1}{t} E \int_0^t \left[(x_1(u) - x_0)^2 + x_2(u)^2 + x_3(u)^2 \right] du \leq 80\,\sigma_1^2 + 388\,\alpha_2\alpha_1^2 + 96.$$

The histograms of the density function for the Hantavirus infection model (3) are shown in Figure 5, and the system (3) has a unique stationary distribution and has an ergodic property according to the conditions of Theorem 7.

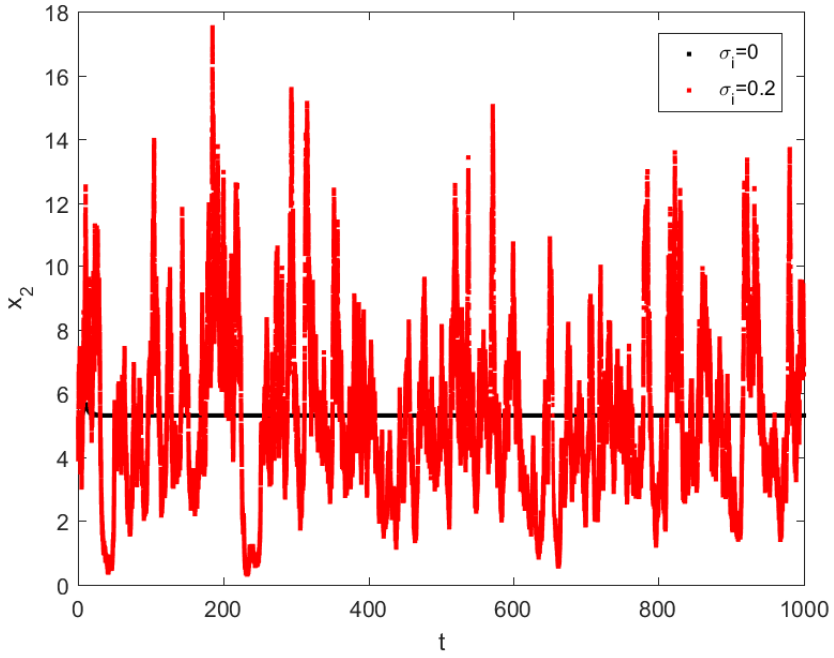

Figure 1. The stochastic Hantavirus system (3) with respect to $\sigma_i = 0$ and $\sigma_i = 0.2$.

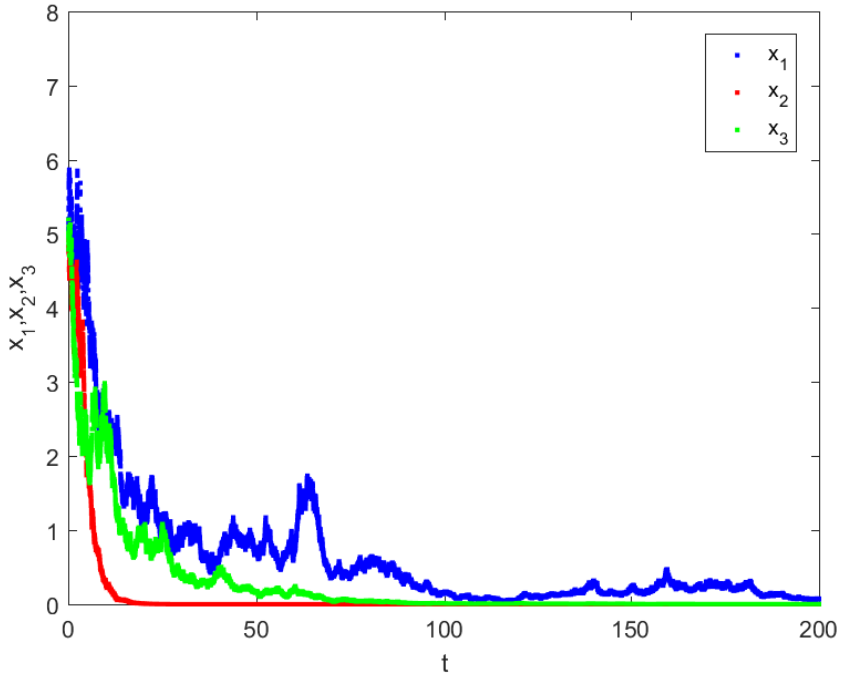

Figure 2. The extinct behavior of the solutions to the Hantavirus infection system (3).

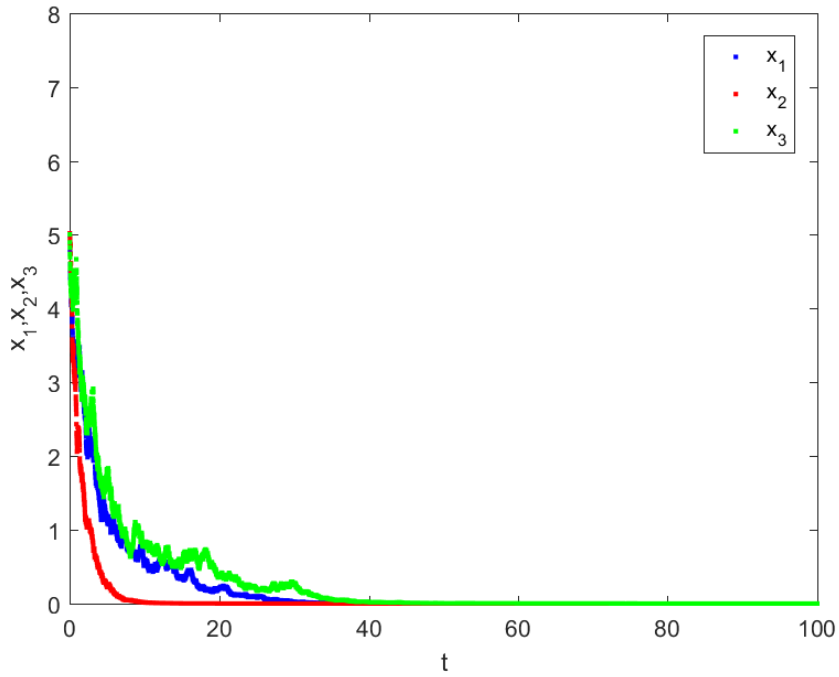

Figure 3. The stochastic trajectories for the trivial solutions of the stochastic Hantavirus infection system (3).

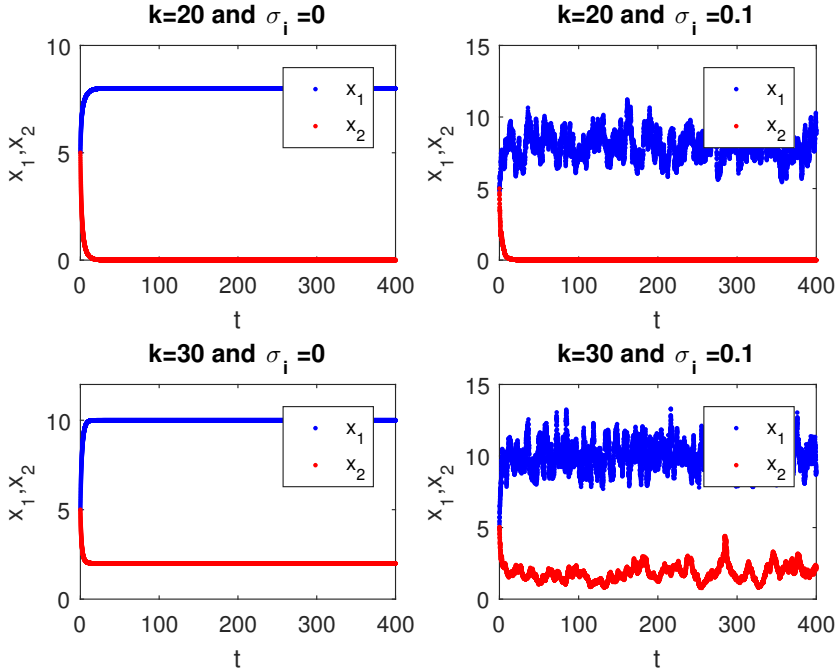

Figure 4. The stochastic behavior of the Abramson system for $k = 20, 30$, and $\sigma_i = 0, 0.1$.

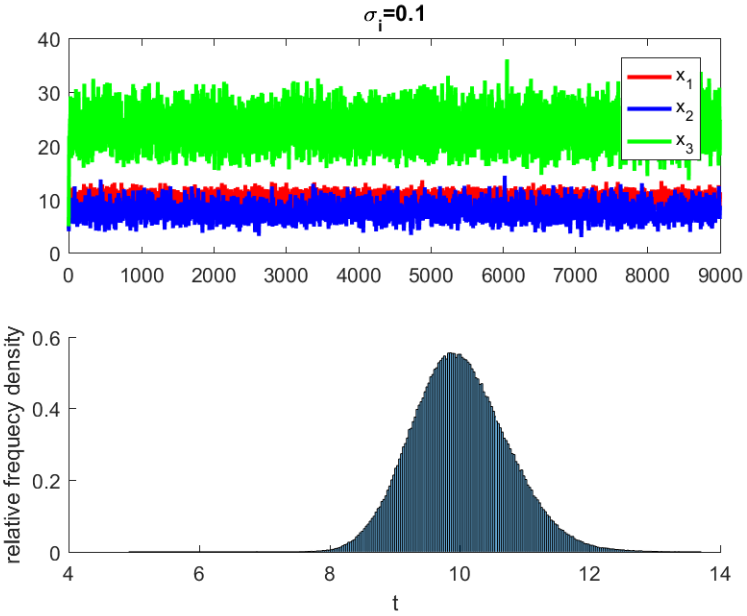

Figure 5. The density function of susceptible mice of the Hantavirus infection system (3).

5. Conclusions

This paper mainly analyzed a stochastic Hantavirus infection model. The existence and boundedness of the positive solution of the stochastic Hantavirus infection model were derived. The conditions for the extinction of the Hantavirus infection from the stochastic system were obtained using stochastic analysis tools. Furthermore, the criteria for the presence of a unique ergodic stationary distribution for the Hantavirus infection model were established using a suitable Lyapunov function. The numerical Milstein method was used to simulate the significance of environmental noise in the Hantavirus infection model. When intensities of fluctuation $\sigma_i = 0$, one can obtain the results for the deterministic model introduced by Peixoto [2]. One can note that the movement of rodents cannot be neglected; consequently, it is interesting to investigate the spatial effects for the stochastic Hantavirus infection model, which will be future work. Moreover, the authors wish to consider the fractionalization of the stochastic Hantavirus infection model given that there is much current research and example publications in this area, with regard to the SIR model, for example [54]. In future work, the authors wish to conduct a detailed analysis of the stochastic Hantavirus infection model using boundary methods as introduced in the following papers [44,55–61].

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