mathematics

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Some Integral Inequalities in $\mathcal{V}$-Fractional Calculus and Their Applications

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Abstract: We consider the Steffensen–Hayashi inequality and remainder identity for $\mathcal{V}$-fractional differentiable functions involving the six parameters truncated Mittag–Leffler function and the Gamma function. In view of these, we obtain some integral inequalities of Steffensen, Hermite–Hadamard, Chebyshev, Ostrowski, and Grüss type to the $\mathcal{V}$-fractional calculus.

Keywords: $\mathcal{V}$-fractional derivative; $\mathcal{V}$-fractional integral; truncated Mittag–Leffler function

Citation: Srivastava, H.M.; Mohammed, P.O.; Almutairi, O.; Kashuri, A.; Hamed, Y.S. Some Integral Inequalities in $\mathcal{V}$-Fractional Calculus and Their Applications. *Mathematics* **2022**, *10*, 344. https://doi.org/10.3390/math10030344

Academic Editors: Dimplekumar N. Chalishajar, Christopher Goodrich and Ivo Petráš

Received: 7 December 2021
Accepted: 20 January 2022
Published: 24 January 2022

1. Introduction and Preliminaries

One useful and important branch of science which involves derivatives and integrals taken to fractional orders is fractional calculus, in general [1–4].

Various fractional derivatives are given until now, some of them are Riemann-Liouville, Caputo, Hadamard, Caputo–Hadamard, Riesz, and many others can be found in [3,5].

Sousa and Oliveira [6] defined M-fractional derivative via Mittag–Leffler function of one parameter [7]. Most recently, Sousa and Oliveira [8] introduced the $\mathcal{V}$-fractional derivative involving the six parameters truncated Mittag–Leffler function and the Gamma function.

Let us recall the six truncated Mittag–Leffler function and the truncated $\mathcal{V}$-fractional derivative which will be used in the sequel.

Six parameters truncated Mittag–Leffler function is defined by:

$$ {}_{\ell}\mathbf{E}^{\varrho,\delta,q}_{\gamma,\theta,p}(\xi) = \sum_{k=0}^{\ell} \frac{(\varrho)_{qk}}{(\delta)_{pk}} \frac{\xi^k}{\Gamma(\gamma k + \theta)}, \tag{1} $$

for $\theta, \gamma, \varrho, \delta \in \mathbb{C}$ and $p, q > 0$ such that $\Re(\theta) > 0, \Re(\gamma) > 0, \Re(\varrho) > 0, \Re(\delta) > 0, \Re(\gamma) + p \geq q$, where $(\varrho)_{qk}$ and $(\delta)_{pk}$ are given by the symbol of Pochhammer:

$$ (\varrho)_{qk} = \frac{\Gamma(\varrho + qk)}{\Gamma(\varrho)}, \quad (\delta)_{qk} = \frac{\Gamma(\delta + qk)}{\Gamma(\delta)}. \tag{2} $$

Remark 1. *It is easy to see that*

(i) *when $\ell \to \infty$, then*

$$\mathbf{E}^{\varrho,\delta,q}_{\gamma,\theta,p}(\xi) = \lim_{\ell\to\infty} {}_{\ell}\mathbf{E}^{\varrho,\delta,q}_{\gamma,\theta,p}(\xi) = \sum_{k=0}^{\infty} \frac{(\varrho)_{q\,k}}{(\delta)_{p\,k}} \frac{\xi^k}{\Gamma(\gamma\,k+\theta)}.$$

(ii) *From* (1), *we can obtain directly by determining some parameters to be* 1, *some particular cases regarding the following truncated Mittag–Leffler functions:*

(a) *For $p = 1$, we get the five parameters truncated Mittag–Leffler function*

$${}_{\ell}\mathbf{E}^{\varrho,\delta,q}_{\gamma,\theta}(\xi) = \sum_{k=0}^{\ell} \frac{(\varrho)_{q\,k}}{(\delta)_{k}} \frac{\xi^k}{\Gamma(\gamma\,k+\theta)}.$$

(b) *With $p = \delta = 1$, we get the four parameters truncated Mittag–Leffler function*

$${}_{\ell}\mathbf{E}^{\varrho,q}_{\gamma,\theta}(\xi) = \sum_{k=0}^{\ell} \frac{(\varrho)_{q\,k}}{k!} \frac{\xi^k}{\Gamma(\gamma\,k+\theta)}.$$

(c) *In the case $p = \delta = q = 1$, we get the three parameters truncated Mittag–Leffler function*

$${}_{\ell}\mathbf{E}^{\varrho}_{\gamma,\theta}(\xi) = \sum_{k=0}^{\ell} \frac{(\varrho)_{k}}{k!} \frac{\xi^k}{\Gamma(\gamma\,k+\theta)}.$$

(d) *For $p = \delta = q = \varrho = 1$, we get the two parameters truncated Mittag–Leffler function*

$${}_{\ell}\mathbf{E}_{\gamma,\theta}(\xi) = \sum_{k=0}^{\ell} \frac{\xi^k}{\Gamma(\gamma\,k+\theta)}.$$

(e) *With $p = \delta = q = \varrho = \theta = 1$, we get the one parameter truncated Mittag–Leffler function*

$${}_{\ell}\mathbf{E}_{\gamma}(\xi) = \sum_{k=0}^{\ell} \frac{\xi^k}{\Gamma(\gamma\,k+1)}.$$

(f) *Particularly, for $p = \delta = q = \varrho = \theta = \gamma = 1$, we get the truncated exponential function*

$${}_{\ell}\mathbf{E}(\xi) = \sum_{k=0}^{\ell} \frac{\xi^k}{\Gamma(k+1)}.$$

For more general Mittag–Leffler type functions that have been investigated rather systematically and extensively, see, for details, [9–12]).

Definition 1 ([8,13] ($\mathcal{V}$-fractional derivative))**.** *Let $\mu \in (0,1)$ with the function $f\colon [0,\infty) \to \mathbb{R}$, and $\theta, \gamma, \varrho, \delta \in \mathbb{C}$ for $p, q > 0$, and $\Re(\theta) > 0, \Re(\gamma) > 0, \Re(\varrho) > 0, \Re(\delta) > 0, \Re(\gamma) + p \geq q$. The truncated $\mathcal{V}$-fractional derivative of f of order $\mu > 0$, is given as:*

$${}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f(t) := \lim_{\varepsilon\to 0} \frac{f\Big(t\,{}_{\ell}\mathcal{H}^{\varrho,\delta,q}_{\gamma,\theta,p}(\varepsilon\,t^{-\mu})\Big) - f(t)}{\varepsilon} \qquad \text{for all } t > 0, \tag{3}$$

where the truncated function ${}_{\ell}\mathcal{H}^{\varrho,\delta,q}_{\gamma,\theta,p}(\cdot)$ is defined as follows

$${}_{\ell}\mathcal{H}^{\varrho,\delta,q}_{\gamma,\theta,p}(\epsilon\,t^{-\mu})(\xi) := \Gamma(\theta)\,{}_{\ell}\mathbf{E}^{\varrho,\delta,q}_{\gamma,\theta,p}(\xi) = \Gamma(\theta) \sum_{k=0}^{\ell} \frac{(\varrho)_{q\,k}}{(\delta)_{p\,k}} \frac{\xi^k}{\Gamma(\gamma\,k+\theta)}. \tag{4}$$

It should be mentioned that if f is differentiable, then

$$ {}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f(\xi) = \frac{t^{1-\mu}\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p} f'(\xi). \tag{5} $$

Definition 2 ([8,13] ($\mathcal{V}$-fractional integral)). *Let $\mu \in (0,1)$ with the function $f\colon [a_1,\infty) \to \mathbb{R}$ and $a_1 \geq 0$. Also, let $\theta,\gamma,\varrho,\delta \in \mathbb{C}$, where $p,q > 0$, and $\Re(\theta) > 0, \Re(\gamma) > 0, \Re(\varrho) > 0, \Re(\delta) > 0, \Re(\gamma) + p \geq q$. The $\mathcal{V}$-fractional integral of f of order $\mu > 0$, is given by:*

$$ {}^{\varrho}_{a_1}\mathcal{I}^{\delta,p,q}_{\gamma,\theta,\mu} f(\xi) := \int_{a_1}^{\xi} f(t)\mathsf{d}_\mu t := \frac{\Gamma(\gamma+\theta)(\delta)_p}{\Gamma(\theta)(\varrho)_q} \int_{a_1}^{\xi} f(t)t^{\mu-1}dt \quad \textit{for all } \xi > 0. \tag{6} $$

Theorem 1 ([8] (Integrating by Parts)). *Let $\mu \in (0,1)$, $\theta,\gamma,\varrho,\delta \in \mathbb{C}$, where $p,q > 0$, and $\Re(\theta) > 0, \Re(\gamma) > 0, \Re(\varrho) > 0, \Re(\delta) > 0$ with $\Re(\gamma) + p \geq q$. If the functions $f,g\colon [a_1,a_2] \to \mathbb{R}$ are both differentiable with $a_2 > a_1 \geq 0$, then*

$$ \int_{a_1}^{a_2} f(\xi)\left({}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} g(\xi)\right)\mathsf{d}_\mu\xi = f(\xi)\,g(\xi)\Big|_{a_1}^{a_2} - \int_{a_1}^{a_2} g(\xi)\left({}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f(\xi)\right)\mathsf{d}_\mu\xi. $$

Remark 2. *Several results similar to the results found in the classical calculus are obtained from the truncated $\mathcal{V}$-fractional derivative using the six parameters truncated Mittag–Leffler function ${}_{\ell}\mathbf{E}^{\varrho,\delta,q}_{\gamma,\theta,p}(\xi)$ and the well-known gamma function $\Gamma(\theta)$. We can mention here the fact that the truncated $\mathcal{V}$-fractional derivative is linear and continuous. For more details, see [8,13] (Section 3).*

Motivated by above results and literatures, the main motivation of this article is to derive the fractional Steffensen–Hayashi inequality and some interesting applications to various inequalities involving $\mathcal{V}$-fractional operators in the proposed framework, such as Steffensen, Chebyshev, Ostrowski, Grüss and Hermite–Hadamard type integral inequalities.

The structure of this article is organized as follows. We derive the fractional Steffensen–Hayashi inequality and Remainder identity in Section 2. In Section 3, we give some interesting applications to various inequalities involving $\mathcal{V}$-fractional operators. Section 4 is devoted to discussion and conclusion of our article.

2. Fractional Steffensen–Hayashi Inequality and Remainder Identity

For more details about the well-known Steffensen's inequality and Hayashi's inequality, see [14–17]. Many further results have been derived from these; however, so far such kind of interesting inequalities have not been extended, improved and investigated using Mittag–Leffler kernels. Based on this motivation, in the present section, we will focus on our attention to the study of fractional Steffensen–Hayashi inequality.

Lemma 1. *Let $\mu \in (0,1)$, where $a_1,a_2 \in \mathbb{R}$ and $0 \leq a_1 < a_2$, with $\theta,\gamma,\varrho,\delta \in \mathbb{C}$, where $p,q > 0$, and $\Re(\theta) > 0, \Re(\gamma) > 0, \Re(\varrho) > 0, \Re(\delta) > 0$ and $\Re(\gamma) + p \geq q$. Also, let $g\colon [a_1,a_2] \to [0,A]$ with $A > 0$, be a $\mathcal{V}$-fractional integrable function on $[a_1,a_2]$. If $\varpi \in [0,a_2-a_1]$ holds, where ϖ is defined by*

$$ \varpi := \frac{\mu(a_2-a_1)\Gamma(\theta)(\varrho)_q}{A\left(a_2^{\mu}-a_1^{\mu}\right)\Gamma(\gamma+\theta)(\delta)_p} \int_{a_1}^{a_2} g(\xi)\mathsf{d}_\mu\xi, \tag{7} $$

then

$$ \int_{a_2-\varpi}^{a_2} A\,\mathsf{d}_\mu\xi \leq \int_{a_1}^{a_2} g(\xi)\mathsf{d}_\mu\xi \leq \int_{a_1}^{a_1+\varpi} A\,\mathsf{d}_\mu\xi. \tag{8} $$

Proof. From definition of $\mathcal{V}$-fractional integral and (7), we have

$$0 \leq \varpi = \frac{\mu(a_2 - a_1)\Gamma(\theta)(\varrho)_q}{A\left(a_2^\mu - a_1^\mu\right)\Gamma(\gamma+\theta)(\delta)_p} \int_{a_1}^{a_2} g(\xi)\mathrm{d}_\mu\xi \leq \frac{\mu(a_2 - a_1)\Gamma(\theta)(\varrho)_q}{\left(a_2^\mu - a_1^\mu\right)\Gamma(\gamma+\theta)(\delta)_p} \int_{a_1}^{a_2} \mathrm{d}_\mu\xi$$
$$= a_2 - a_1 \tag{9}$$

for $g(\xi) \in [0, A]$ and $\xi \in [a_1, a_2]$. Employing the facts that the function $\xi^{\mu-1}$ is a decreasing on $[a_1, a_2]$, or $(0, a_2]$ for $\mu \in (0,1)$ and $\mathrm{d}_\mu\xi = \xi^{\mu-1}d\xi$, implies

$$\frac{1}{\varpi}\int_{a_2-\varpi}^{a_2} \mathrm{d}_\mu\xi \leq \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \mathrm{d}_\mu\xi \leq \frac{1}{\varpi}\int_{a_1}^{a_1+\varpi} \mathrm{d}_\mu\xi.$$

Hence, we get

$$\int_{a_2-\varpi}^{a_2} A\,\mathrm{d}_\mu\xi \leq \frac{\varpi}{a_2 - a_1}\int_{a_1}^{a_2} A\,\mathrm{d}_\mu\xi \leq \int_{a_1}^{a_1+\varpi} A\,\mathrm{d}_\mu\xi.$$

Combining the above inequality with (9), we obtain the desired inequality (8). □

The main results of the section concerning fractional Steffensen–Hayashi inequality is provided as follows.

Theorem 2 (Fractional Steffensen–Hayashi Inequality)**.** *Let $\mu \in (0,1)$, where $a_1, a_2 \in \mathbb{R}$ with $0 \leq a_1 < a_2$, and $\theta, \gamma, \varrho, \delta \in \mathbb{C}$, where $p, q > 0$ with $\Re(\theta) > 0, \Re(\gamma) > 0, \Re(\varrho) > 0, \Re(\delta) > 0$, and $\Re(\gamma) + p \geq q$. Let $f\colon [a_1, a_2] \to \mathbb{R}$ and $g\colon [a_1, a_2] \to [0, A]$ with $A > 0$ are $\mathcal{V}$-fractional integrable functions on $[a_1, a_2]$. If*

(i) *f is non-negative and non-increasing, then it holds*

$$A\int_{a_2-\varpi}^{a_2} f(\xi)\mathrm{d}_\mu\xi \leq \int_{a_1}^{a_2} f(\xi)\,g(\xi)\mathrm{d}_\mu\xi \leq A\int_{a_1}^{a_1+\varpi} f(\xi)\mathrm{d}_\mu\xi. \tag{10}$$

(ii) *f is non-positive and non-decreasing, then the inequalities in* (10) *are reversed.*

Proof. (i) We will prove only the left-hand side of (10) because the proof of the right-hand side is similar. Let $f\colon [a_1, a_2] \to \mathbb{R}$ be a non-negative and non-increasing function. It follows from Lemma 1 ($a_1 \leq a_2 - \varpi \leq a_2$) that

$$\begin{aligned}
&\int_{a_1}^{a_2} f(\xi)\,g(\xi)\mathrm{d}_\mu\xi - A\int_{a_2-\varpi}^{a_2} f(\xi)\mathrm{d}_\mu\xi \\
&= \int_{a_1}^{a_2-\varpi} f(\xi)\,g(\xi)\mathrm{d}_\mu\xi + \int_{a_2-\varpi}^{a_2} f(\xi)\,g(\xi)\mathrm{d}_\mu\xi - A\int_{a_2-\varpi}^{a_2} f(\xi)\mathrm{d}_\mu\xi \\
&= \int_{a_1}^{a_2-\varpi} f(\xi)\,g(\xi)\mathrm{d}_\mu\xi - \int_{a_2-\varpi}^{a_2} (A - g(\xi))f(\xi)\mathrm{d}_\mu\xi \\
&\geq \int_{a_1}^{a_2-\varpi} f(\xi)\,g(\xi)\mathrm{d}_\mu\xi - f(a_2 - \varpi)\int_{a_2-\varpi}^{a_2} (A - g(\xi))\mathrm{d}_\mu\xi.
\end{aligned}$$

Applying Lemma 1 again, we utilize the facts, f, g are non-negative and f is non-increasing, to find

$$\begin{aligned}
\int_{a_1}^{a_2} f(\xi)\,g(\xi)\mathrm{d}_\mu\xi - A\int_{a_2-\varpi}^{a_2} f(\xi)\mathrm{d}_\mu\xi &\geq \int_{a_1}^{a_2-\varpi} f(\xi)\,g(\xi)\mathrm{d}_\mu\xi - f(a_2 - \varpi)\int_{a_1}^{a_2-\varpi} g(\xi)\mathrm{d}_\mu\xi \\
&= \int_{a_1}^{a_2-\varpi} (f(\xi) - f(a_2 - \varpi))g(\xi)\mathrm{d}_\mu\xi \geq 0.
\end{aligned}$$

This means that the first inequality of (10) is valid.

(ii) Whereas, let the function $f\colon [a_1,a_2] \to \mathbb{R}$ be a non-positive and non-decreasing. By the same manner of assertion (i), it reads

$$\begin{aligned}
&\int_{a_1}^{a_2} f(\xi)\, g(\xi) \mathsf{d}_\mu \xi - A \int_{a_1}^{a_1+\varpi} f(\xi) \mathsf{d}_\mu \xi \\
&= \int_{a_1+\varpi}^{a_2} f(\xi)\, g(\xi) \mathsf{d}_\mu \xi + \int_{a_1}^{a_1+\varpi} (g(\xi) - A) f(\xi) \mathsf{d}_\mu \xi \\
&\geq \int_{a_1+\varpi}^{a_2} f(\xi)\, g(\xi) \mathsf{d}_\mu \xi + f(a_1+\varpi) \int_{a_1}^{a_1+\varpi} (g(\xi) - A) \mathsf{d}_\mu \xi \\
&\geq \int_{a_1+\varpi}^{a_2} f(\xi)\, g(\xi) \mathsf{d}_\mu \xi - f(a_1+\varpi) \int_{a_1+\varpi}^{a_2} g(\xi) \mathsf{d}_\mu \xi \\
&= \int_{a_1+\varpi}^{a_2} (f(\xi) - f(a_1+\varpi)) g(\xi) \mathsf{d}_\mu \xi \geq 0,
\end{aligned}$$

where we have used the facts that g is non-negative. So, the right-hand side of the reversed (10) holds, which completes the proof. □

Furthermore, we shall invoke the above inequalities to establish several significant equalities, remainder identity.

Lemma 2 (Remainder Identity)**.** *Let $\mu \in (0,1]$, where $\theta, \gamma, \varrho, \delta \in \mathbb{C}$ with $p, q > 0$ and $\Re(\theta) > 0, \Re(\gamma) > 0, \Re(\varrho) > 0, \Re(\delta) > 0$, and $\Re(\gamma) + p \geq q$. If $f\colon [0,\infty) \to \mathbb{R}$ is a $\mathcal{V}$-fractional differentiable function, then*

$$\begin{aligned}
&\int_{a_1}^{a_2} {}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f(s) \left(\frac{\Gamma(\gamma+\theta)(\delta)_p (\xi^\mu - s^\mu)}{\mu\, \Gamma(\theta)(\varrho)_q} \right) \mathsf{d}_\mu s \\
&= \int_{a_1}^{a_2} f(s) \mathsf{d}_\mu s + \left(\frac{\Gamma(\gamma+\theta)(\delta)_p}{\mu\, \Gamma(\theta)(\varrho)_q} \right) \left[\left(\xi^\mu - a_2^\mu \right) f(a_2) - \left(\xi^\mu - a_1^\mu \right) f(a_1) \right], \qquad (11)
\end{aligned}$$

holds for all $\xi \in [0,\infty)$.

Proof. Integrating by parts using Theorem 1, we have

$$\begin{aligned}
&\int_{a_1}^{a_2} {}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f(s) \left(\frac{\Gamma(\gamma+\theta)(\delta)_p (\xi^\mu - s^\mu)}{\mu\, \Gamma(\theta)(\varrho)_q} \right) \mathsf{d}_\mu s \\
&= \left(\frac{\Gamma(\gamma+\theta)(\delta)_p (\xi^\mu - s^\mu)}{\mu\, \Gamma(\theta)(\varrho)_q} \right) f(s) \Big|_{a_1}^{a_2} + \int_{a_1}^{a_2} f(s) \mathsf{d}_\mu s \\
&= \int_{a_1}^{a_2} f(s) \mathsf{d}_\mu s + \left(\frac{\Gamma(\gamma+\theta)(\delta)_p}{\mu\, \Gamma(\theta)(\varrho)_q} \right) \left[\left(\xi^\mu - a_2^\mu \right) f(a_2) - \left(\xi^\mu - a_1^\mu \right) f(a_1) \right]
\end{aligned}$$

for all $\xi \in [0,\infty)$, which completes the proof. □

Corollary 1. *Taking, respectively, $\xi = a_1$ and $\xi = a_2$ in Lemma 2, we get the following*

$$\begin{aligned}
\int_{a_1}^{a_2} {}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f(s) \left(\frac{\Gamma(\gamma+\theta)(\delta)_p \left(a_1^\mu - s^\mu \right)}{\mu\, \Gamma(\theta)(\varrho)_q} \right) \mathsf{d}_\mu s &= \int_{a_1}^{a_2} f(s) \mathsf{d}_\mu s \\
&\quad + \left(\frac{\Gamma(\gamma+\theta)(\delta)_p \left(a_1^\mu - a_2^\mu \right)}{\mu\, \Gamma(\theta)(\varrho)_q} \right) f(a_2),
\end{aligned}$$

and

$$\int_{a_1}^{a_2} {}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f(s) \left(\frac{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^\mu - s^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q} \right) \mathrm{d}_\mu s = \int_{a_1}^{a_2} f(s)\mathrm{d}_\mu s - \left(\frac{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q} \right) f(a_1).$$

3. Applications to Various Inequalities Involving $\mathcal{V}$-Fractional Operators

In the section, we shall employ the previous results obtained in Section 2 to explore various inequalities involving $\mathcal{V}$-fractional operators.

3.1. Steffensen Inequality

Theorem 3. *Let $\mu \in (0,1]$ and $f\colon [a_1,a_2] \to \mathbb{R}$ be a $\mathcal{V}$-fractional differentiable function.*

(i) *If ${}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f$ is increasing function and f is decreasing on $[a_1,a_2]$, then*

$$f\left(\frac{a_1+a_2}{2}\right) \le \frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} f(s)\mathrm{d}_\mu s \le f(a_1) + f(a_2) - f\left(\frac{a_1+a_2}{2}\right). \quad (12)$$

(ii) *If ${}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f$ is decreasing function and f is increasing on $[a_1,a_2]$, then inequalities (12) are reversed.*

Proof. Here, we just prove the assertion (i), because the second conclusion could be obtained easily by the similar way. Let ${}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f$ be increasing function and f be decreasing on $[a_1,a_2]$. So, the function $F := -\,{}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f$ is decreasing on $[a_1,a_2]$. Denote

$$g(\xi) := \frac{a_2^\mu - \xi^\mu}{a_2^\mu - a_1^\mu} \in [0,1], \quad \xi \in [a_1,a_2].$$

Since F and g satisfy the assumptions of Theorem 2 (i) with $A = 1$, then

$$\varpi := \frac{\mu(a_2-a_1)\Gamma(\theta)(\varrho)_q}{\left(a_2^\mu - a_1^\mu\right)\Gamma(\gamma+\theta)(\delta)_p} \int_{a_1}^{a_2} g(\xi)\mathrm{d}_\mu \xi = \frac{a_2-a_1}{2},$$

and

$$-\int_{a_2-\varpi}^{a_2} {}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f(\xi)\mathrm{d}_\mu\xi \le -\int_{a_1}^{a_2} {}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f(\xi) \left(\frac{a_2^\mu - \xi^\mu}{a_2^\mu - a_1^\mu} \right) \mathrm{d}_\mu\xi \le -\int_{a_1}^{a_1+\varpi} {}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} \mathrm{d}_\mu\xi.$$

From Corollary 1, we get

$$f(\xi)\Big|_{a_1}^{a_1+\varpi} \le \left(\frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^\mu - a_1^\mu\right)} \right) \int_{a_1}^{a_2} f(s)\mathrm{d}_\mu s - f(a_1) \le f(\xi)\Big|_{a_2-\varpi}^{a_2}.$$

Hence, we obtain

$$f\left(\frac{a_1+a_2}{2}\right)-f(a_1)\le\left(\frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^{\mu}-a_1^{\mu}\right)}\right)\int_{a_1}^{a_2}f(s)\mathrm{d}_{\mu}s-f(a_1)$$
$$\le f(a_2)-f\left(\frac{a_1+a_2}{2}\right),$$

which completes the proof. □

3.2. Chebyshev Inequality

In the subsection, we are devoted to investigate Chebyshev's inequality with $\mathcal{V}$-fractional integrals.

Theorem 4 (Chebyshev Inequality)**.** *Let $\theta,\gamma,\varrho,\delta\in\mathbb{C}$, where $p,q>0$, and $\Re(\gamma)>0,\Re(\theta)>0,\Re(\varrho)>0,\Re(\delta)>0$ with $\Re(\gamma)+p\ge q$. If f and g are both increasing or both decreasing functions on $[a_1,a_2]$, and $\mu\in(0,1]$, then*

$$\int_{a_1}^{a_2}f(\xi)\,g(\xi)\,\mathrm{d}_{\mu}\xi\ge\frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^{\mu}-a_1^{\mu}\right)}\int_{a_1}^{a_2}f(\xi)\,\mathrm{d}_{\mu}\xi\int_{a_1}^{a_2}g(\xi)\,\mathrm{d}_{\mu}\xi.\tag{13}$$

If f and g are monotone functions with opposite monotonicity, then inequality (13) is reversed.

Proof. By using the similar arguments of the proof of classical Chebyshev's inequality (i.e., $\mu=1$), it is not difficult to show that the above fractional Chebyshev's inequality is true. So, we omit here the proof. □

The following theorem, extends the recent result [18] for q-calculus to the case of $\mathcal{V}$-fractional.

Theorem 5. *Let $\mu\in(0,1]$ and the function $f:[a_1,a_2]\to\mathbb{R}$ be a $\mathcal{V}$-fractional differentiable. If ${}_{\ell}^{\varrho}\mathcal{V}_{\gamma,\theta,\mu}^{\delta,p,q}f$ is increasing on $[a_1,a_2]$, then*

$$\frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^{\mu}-a_1^{\mu}\right)}\int_{a_1}^{a_2}f(s)\mathrm{d}_{\mu}s\le\frac{f(a_1)+f(a_2)}{2}.\tag{14}$$

Moreover, if ${}_{\ell}^{\varrho}\mathcal{V}_{\gamma,\theta,\mu}^{\delta,p,q}f$ is decreasing on $[a_1,a_2]$, then inequality (14) is reversed.

Proof. Assume that ${}_{\ell}^{\varrho}\mathcal{V}_{\gamma,\theta,\mu}^{\delta,p,q}f$ is increasing. Let $\theta,\gamma,\varrho,\delta\in\mathbb{C}$, where $p,q>0$ and $\Re(\gamma)>0$, $\Re(\theta)>0,\Re(\varrho)>0,\Re(\delta)>0$ with $\Re(\gamma)+p\ge q$. Also, denote

$$F(\xi):={}_{\ell}^{\varrho}\mathcal{V}_{\gamma,\theta,\mu}^{\delta,p,q}f(\xi),\quad G(\xi):=\frac{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^{\mu}-\xi^{\mu}\right)}{\mu\,\Gamma(\theta)(\varrho)_q}.$$

Since F is increasing and G is decreasing, it follows from inequality (13), that

$$\int_{a_1}^{a_2}F(\xi)\,G(\xi)\,\mathrm{d}_{\mu}\xi\le\frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^{\mu}-a_1^{\mu}\right)}\int_{a_1}^{a_2}F(\xi)\,\mathrm{d}_{\mu}\xi\int_{a_1}^{a_2}G(\xi)\,\mathrm{d}_{\mu}\xi.\tag{15}$$

From the facts,

$$\int_{a_1}^{a_2} F(\xi)\,\mathrm{d}_\mu\xi = f(a_2) - f(a_1),$$

$$\int_{a_1}^{a_2} G(\xi)\,\mathrm{d}_\mu\xi = \frac{1}{2}\left(\frac{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}\right)^2,$$

and Corollary 1, it finds

$$\int_{a_1}^{a_2} F(\xi)\,G(\xi)\,\mathrm{d}_\mu\xi = \int_{a_1}^{a_2} f(s)\mathrm{d}_\mu s - \left(\frac{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}\right) f(a_1).$$

The latter combined with (15) implies

$$\begin{aligned}
&\int_{a_1}^{a_2} f(s)\mathrm{d}_\mu s - \left(\frac{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}\right) f(a_1)\\
&\le \frac{\mu\,\Gamma(\theta)(\varrho)_q(f(a_2)-f(a_1))}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)} \cdot \frac{1}{2}\left(\frac{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}\right)^2\\
&= \frac{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}{2\mu\,\Gamma(\theta)(\varrho)_q}(f(a_2)-f(a_1)).
\end{aligned}$$

Hence, we get

$$\frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} f(s)\mathrm{d}_\mu s \le \frac{f(a_1)+f(a_2)}{2},$$

which completes the proof. □

Furthermore, by the use of Theorems 3 and 5, we have the following result.

Corollary 2 (Hermite–Hadamard inequality). *Let $\mu \in (0,1]$ and the function $f\colon [a_1,a_2] \to \mathbb{R}$ be a $\mathcal{V}$-fractional differentiable. If ${}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f$ is increasing and f is decreasing on $[a_1,a_2]$, then*

$$f\left(\frac{a_1+a_2}{2}\right) \le \frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} f(s)\mathrm{d}_\mu s \le \frac{f(a_1)+f(a_2)}{2}. \tag{16}$$

3.3. Ostrowski Inequality

In the subsection, we will utilize a Montgomery identity obtain establish the Ostrowski's type inequality involving $\mathcal{V}$-fractional integral. For more detail on Ostrowski's inequalities, the reader is welcome to consult [19].

Lemma 3 (Montgomery Identity). *Let $\theta, \gamma, \varrho, \delta \in \mathbb{C}$, where $p, q > 0$ and $\Re(\theta) > 0, \Re(\gamma) > 0, \Re(\varrho) > 0, \Re(\delta) > 0$ with $\Re(\gamma) + p \geq q$. Also let $a_1, a_2, s, \xi \in \mathbb{R}$ satisfy $0 \leq a_1 < a_2$. If the function $f\colon [a_1, a_2] \to \mathbb{R}$ is a $\mathcal{V}$-fractional differentiable for $\mu \in (0,1]$, then*

$$f(\xi) = \frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} f(s)\mathsf{d}_\mu s + \frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} \omega(\xi,s)\, {}_{\ell}^{\varrho}\mathcal{V}_{\gamma,\theta,\mu}^{\delta,p,q} f(s)\mathsf{d}_\mu s, \quad (17)$$

holds for all $\xi \in [a_1, a_2]$, where $\omega(\xi, s)$ is given as

$$\omega(\xi,s) := \begin{cases} \dfrac{\Gamma(\gamma+\theta)(\delta)_p\left(s^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}, & a_1 \leq s < \xi; \\[2ex] \dfrac{\Gamma(\gamma+\theta)(\delta)_p\left(s^\mu - a_2^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}, & \xi \leq s \leq a_2. \end{cases} \quad (18)$$

Proof. Integrating by parts (see e.g., [8] (Theorem 13)), we have

$$\int_{a_1}^{x} \left(\frac{\Gamma(\gamma+\theta)(\delta)_p\left(s^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q} \right) {}_{\ell}^{\varrho}\mathcal{V}_{\gamma,\theta,\mu}^{\delta,p,q} f(s)\mathsf{d}_\mu s = \frac{\Gamma(\gamma+\theta)(\delta)_p\left(\xi^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q} f(\xi) - \int_{a_1}^{\xi} f(s)\mathsf{d}_\mu s,$$

and

$$\int_{\xi}^{a_2} \left(\frac{\Gamma(\gamma+\theta)(\delta)_p\left(s^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q} \right) {}_{\ell}^{\varrho}\mathcal{V}_{\gamma,\theta,\mu}^{\delta,p,q} f(s)\mathsf{d}_\mu s = \frac{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - \xi^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q} f(\xi) - \int_{\xi}^{a_2} f(s)\mathsf{d}_\mu s.$$

Summing the above inequalities, it yields

$$\frac{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q} f(\xi) = \int_{a_1}^{a_2} f(s)\mathsf{d}_\mu s + \int_{a_1}^{\xi} \left(\frac{\Gamma(\gamma+\theta)(\delta)_p\left(s^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q} \right) {}_{\ell}^{\varrho}\mathcal{V}_{\gamma,\theta,\mu}^{\delta,p,q} f(s)\mathsf{d}_\mu s + \int_{\xi}^{a_2} \left(\frac{\Gamma(\gamma+\theta)(\delta)_p\left(s^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q} \right) {}_{\ell}^{\varrho}\mathcal{V}_{\gamma,\theta,\mu}^{\delta,p,q} f(s)\mathsf{d}_\mu s.$$

Dividing both sides of above equality by the factor $\frac{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}$, we obtain the desired result. □

Using Lemma 3, we get the following Ostrowski inequality involving $\mathcal{V}$-fractional operators.

Theorem 6 (Ostrowski Inequality)**.** *Let $\theta, \gamma, \varrho, \delta \in \mathbb{C}$, where $p, q > 0$ and $\Re(\theta) > 0, \Re(\gamma) > 0, \Re(\varrho) > 0, \Re(\delta) > 0$ with $\Re(\gamma) + p \geq q$. Also let $a_1, a_2, s, \xi \in \mathbb{R}$ satisfy $0 \leq a_1 < a_2$. If the function $f\colon [a_1, a_2] \to \mathbb{R}$ is a $\mathcal{V}$-fractional differentiable, and $\mu \in (0,1]$, then*

$$\begin{aligned}\left| f(\xi) - \frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} f(\xi)\mathsf{d}_\mu\xi \right| \\ \leq \frac{M\,\Gamma(\gamma+\theta)(\delta)_p}{2\mu\,\Gamma(\theta)(\varrho)_q\left(a_2^\mu - a_1^\mu\right)}\left[\left(\xi^\mu - a_1^\mu\right)^2 + \left(a_2^\mu - \xi^\mu\right)^2\right],\end{aligned} \tag{19}$$

where $M := \sup_{\xi\in(a_1,a_2)}\left|{}_{\ell}^{\varrho}\mathcal{V}_{\gamma,\theta,\mu}^{\delta,p,q}f(\xi)\right|$.

Proof. From Lemma 3, we have

$$\begin{aligned}&\left| f(\xi) - \frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} f(s)\mathsf{d}_\mu s \right| \\
&\leq \frac{M\,\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}\left[\int_{a_1}^{\xi}\left|\frac{\Gamma(\gamma+\theta)(\delta)_p\left(s^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}\right|\mathsf{d}_\mu s \right.\\
&\left.+\int_{\xi}^{a_2}\left|\frac{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - s^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}\right|\mathsf{d}_\mu s\right]\\
&= \frac{M\,\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}\left[\int_{a_1}^{\xi}\left(\frac{\Gamma(\gamma+\theta)(\delta)_p\left(s^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}\right)\mathsf{d}_\mu s \right.\\
&\left.+\int_{\xi}^{a_2}\left(\frac{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - s^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}\right)\mathsf{d}_\mu s\right]\end{aligned}$$

$$\begin{aligned}&= \frac{M\,\Gamma(\gamma+\theta)(\delta)_p}{\Gamma(\theta)(\varrho)_q\left(a_2^\mu - a_1^\mu\right)}\left[\int_{a_1}^{\xi}\left(s^\mu - a_1^\mu\right)s^{\mu-1}\mathsf{d}s + \int_{\xi}^{a_2}\left(a_2^\mu - s^\mu\right)s^{\mu-1}\mathsf{d}s\right]\\
&= \frac{M\,\Gamma(\gamma+\theta)(\delta)_p}{\Gamma(\theta)(\varrho)_q\left(a_2^\mu - a_1^\mu\right)}\left[\left(\frac{s^{2\mu} - 2a_1^\mu s^\mu}{2\mu}\right)\Bigg|_{a_1}^{\xi} + \left(\frac{2a_2^\mu s^\mu - s^{2\mu}}{2\mu}\right)\Bigg|_{\xi}^{a_2}\right]\\
&= \frac{M\,\Gamma(\gamma+\theta)(\delta)_p}{2\mu\,\Gamma(\theta)(\varrho)_q\left(a_2^\mu - a_1^\mu\right)}\left[\left(\xi^\mu - a_1^\mu\right)^2 + \left(a_2^\mu - \xi^\mu\right)^2\right],\end{aligned}$$

which completes the proof. □

Especially, if we choose $f(\xi) = \frac{\Gamma(\gamma+\theta)(\delta)_p}{\mu\,\Gamma(\theta)(\varrho)_q}\xi^\mu$ for all $\xi \in [a_1, a_2]$, then from the fact $\omega(\xi, a_1) = 0$ for all $\xi \in [a_1, a_2]$, ${}_{\ell}^{\varrho}\mathcal{V}_{\gamma,\theta,\mu}^{\delta,p,q}f(\xi) = 1$ and $M = 1$, we get

$$\begin{aligned}&\left| f(\xi) - \frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} f(s)\mathsf{d}_\mu s \right|\\
&= \left| \frac{\Gamma(\gamma+\theta)(\delta)_p}{\mu\,\Gamma(\theta)(\varrho)_q}\xi^\mu - \frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(x_2^\mu - x_1^\mu\right)} \int_{x_1}^{x_2} \frac{\Gamma(\gamma+\theta)(\delta)_p}{\mu\,\Gamma(\theta)(\varrho)_q}\xi^\mu\mathsf{d}_\mu\xi \right|\end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{\Gamma(\gamma+\theta)(\delta)_p}{\mu\,\Gamma(\theta)(\varrho)_q} \xi^{\mu} - \left(\frac{\Gamma(\gamma+\theta)(\delta)_p}{\mu\,\Gamma(\theta)(\varrho)_q \left(x_2^{\mu} - x_1^{\mu}\right)} \right) \frac{\xi^{2\mu}}{2} \Bigg|_{x_1}^{x_2} \right| \\
&= \left| \frac{\Gamma(\gamma+\theta)(\delta)_p}{\mu\,\Gamma(\theta)(\varrho)_q} \xi^{\mu} - \left(\frac{\Gamma(\gamma+\theta)(\delta)_p}{\mu\,\Gamma(\theta)(\varrho)_q \left(x_2^{\mu} - x_1^{\mu}\right)} \right) \frac{x_2^{2\mu} - x_1^{2\mu}}{2} \right| \\
&= \frac{\Gamma(\gamma+\theta)(\delta)_p}{\mu\,\Gamma(\theta)(\varrho)_q} \left| x_2^{\mu} - \frac{x_2^{2\mu} - x_1^{2\mu}}{2} \right| \\
&= \frac{\Gamma(\gamma+\theta)(\delta)_p}{\mu\,\Gamma(\theta)(\varrho)_q} \left(x_2^{\mu} - x_1^{\mu}\right),
\end{aligned}
$$

by taking $a_1 = x_1$ and $a_2 = x_2$.

3.4. Grüss Inequality

The main goal of the subsection is to use the Jensen's inequality to explore the Grüss inequality with $\mathcal{V}$-fractional operator, which generalizes the recent results [20].

From Bohner-Peterson [21] (Theorem 6.17) and [22] (Theorem 3.3), the following Jensen inequality holds.

Theorem 7 (Jensen Inequality)**.** *Let $\mu \in (0,1]$ and $a_1, a_2, \xi_1, \xi_2 \in [0,\infty)$ with $0 \le a_1 < a_2$, and let $w\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to (\xi_1, \xi_2)$ be two non-negative and continuous functions with $\int_{a_1}^{a_2} w(\xi)\,\mathrm{d}_\mu\xi > 0$. Assume that $F\colon (\xi_1,\xi_2) \to \mathbb{R}$ is a continuous and a convex function, then*

$$
F\left(\frac{\int_{a_1}^{a_2} w(\xi)\, g(\xi) \mathrm{d}_\mu\xi}{\int_{a_1}^{a_2} w(\xi)\, \mathrm{d}_\mu\xi} \right) \le \frac{\int_{a_1}^{a_2} w(\xi)\, F(g(\xi)) \mathrm{d}_\mu\xi}{\int_{a_1}^{a_2} w(\xi)\, \mathrm{d}_\mu\xi}.
$$

Theorem 8 (Grüss Inequality)**.** *Let $\theta, \gamma, \varrho, \delta \in \mathbb{C}$, where $p, q > 0$ and $\Re(\gamma) > 0, \Re(\theta) > 0$, $\Re(\varrho) > 0, \Re(\delta) > 0$ with $\Re(\gamma) + p \ge q$. Also, let $\mu \in (0,1], a_1, a_2, x \in [0,\infty)$ satisfy $0 \le a_1 < a_2$. Assume that $f, g\colon [a_1, a_2] \to \mathbb{R}$ are continuous functions such that*

$$
m_1 \le f(\xi) \le M_1, \quad m_2 \le g(\xi) \le M_2. \tag{20}
$$

for some $m_1, m_2, M_1, M_2 \in \mathbb{R}$, then

$$
\begin{aligned}
&\Bigg| \frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^{\mu} - a_1^{\mu}\right)} \int_{a_1}^{a_2} f(\xi)\, g(\xi) \mathrm{d}_\mu\xi \\
&\qquad - \left(\frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^{\mu} - a_1^{\mu}\right)} \right)^2 \int_{a_1}^{a_2} f(\xi) \mathrm{d}_\mu\xi \int_{a_1}^{a_2} g(\xi) \mathrm{d}_\mu\xi \Bigg| \\
&\qquad\qquad \le \frac{1}{4}(M_1 - m_1)(M_2 - m_2).
\end{aligned}
$$

Proof. Firstly, we consider the case $f = g$. Let

$$
v(\xi) := \frac{f(\xi) - m_1}{M_1 - m_1} \in [0,1],
$$

i.e., $f(\xi) = m_1 + (M_1 - m_1)v(\xi)$. If we assume that, then

$$
\frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^{\mu} - a_1^{\mu}\right)} \int_{a_1}^{a_2} f(\xi) \mathrm{d}_\mu\xi = 0.
$$

So, we can see that $m_1 \leq 0$ and

$$\int_{a_1}^{a_2} v^2(\xi) \mathsf{d}_\mu \xi \leq \int_{a_1}^{a_2} v(\xi) \mathsf{d}_\mu \xi = \frac{-m_1 \, \Gamma(\gamma+\theta)(\delta)_p \left(a_2^\mu - a_1^\mu\right)}{\mu(M_1 - m_1) \, \Gamma(\theta)(\varrho)_q}.$$

This implies

$$\begin{aligned}
\mathbf{J}(f,f) :=& \frac{\mu \, \Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} f^2(\xi) \mathsf{d}_\mu \xi \\
&- \left(\frac{\mu \, \Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} f(\xi) \mathsf{d}_\mu \xi \int_{a_1}^{a_2} g(\xi) \mathsf{d}_\mu \xi \right)^2 \\
&= \frac{\mu \, \Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} [m_1 + (M_1 - m_1) v(\xi)]^2 \mathsf{d}_\mu \xi \\
&\leq -m_1 \, M_1 = \frac{1}{4}\left[(M_1 - m_1)^2 - (M_1 + m_1)^2 \right] \\
&\leq \frac{1}{4}(M_1 - m_1)^2.
\end{aligned}$$

Additionally, when the case occurs

$$r := \frac{\mu \, \Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} f(\xi) \mathsf{d}_\mu \xi \neq 0.$$

Introduce the function $h(\xi) := f(\xi) - r$. Then, we have that $h(\xi) \in [m_1 - r, M_1 - r]$ and

$$\begin{aligned}
\frac{\mu \, \Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} h(\xi) \mathsf{d}_\mu \xi &= \frac{\mu \, \Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} (f(\xi) - r) \mathsf{d}_\mu \xi \\
&= r - \frac{r \, \mu \, \Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} \mathsf{d}_\mu \xi = 0.
\end{aligned}$$

Consequently, for function h, it has

$$\mathbf{J}(h,h) \leq \frac{1}{4}[M_1 - r - (m_1 - r)]^2 = \frac{1}{4}(M_1 - m_1)^2.$$

However, the facts

$$\begin{aligned}
\mathbf{J}(h,h) &= \frac{\mu \, \Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} (f(\xi) - r)^2 \mathsf{d}_\mu \xi \\
&= \frac{\mu \, \Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p \left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} f^2(\xi) \mathsf{d}_\mu \xi - r^2 = \mathbf{J}(f,f),
\end{aligned}$$

guarantee

$$\mathbf{J}(f,f) = \mathbf{J}(h,h) \leq \frac{1}{4}(M_1 - m_1)^2.$$

Using

$$\mathbf{J}(f,g) := \frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} f(\xi)\,g(\xi)\mathrm{d}_\mu\xi \\ - \left(\frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}\right)^2 \int_{a_1}^{a_2} f(\xi)\mathrm{d}_\mu\xi \int_{a_1}^{a_2} g(\xi)\mathrm{d}_\mu\xi,$$

and the similar proof for the case $\mu = 1$ of [20] (Theorem 3.1), one can easily complete the proof of this case. □

Corollary 3. *Let $\theta, \gamma, \varrho, \delta \in \mathbb{C}$, where $p, q > 0$ and $\Re(\theta) > 0, \Re(\gamma) > 0, \Re(\varrho) > 0, \Re(\delta) > 0$ with $\Re(\gamma) + p \geq q$. Also, let $\mu \in (0,1], a_1, a_2, \xi, s \in [0,\infty)$ satisfy $0 \leq a_1 < a_2$. Assume that the function $f\colon [a_1, a_2] \to \mathbb{R}$ is a $\mathcal{V}$-fractional differentiable and ${}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f$ is continuous such that*

$$m \leq {}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f(\xi) \leq M, \quad \xi \in [a_1, a_2]$$

for some $m, M \in \mathbb{R}$, then

$$\left| f(\xi) - \frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} f(s)\mathrm{d}_\mu s - \left(\frac{2\xi^\mu - a_1^\mu - a_2^\mu}{2\left(a_2^\mu - a_1^\mu\right)}\right)[f(a_2) - f(a_1)] \right| \\ \leq \frac{1}{4}\frac{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}(M - m), \quad (21)$$

holds for all $x \in [a_1, a_2]$.

Proof. Indeed, Lemma 3 implies

$$f(\xi) - \frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} f(s)\mathrm{d}_\mu s \\ = \frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)} \int_{a_1}^{a_2} \omega(\xi,s)\,{}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu} f(s)\mathrm{d}_\mu s, \quad (22)$$

where $\omega(\xi, s)$ is defined in (18). Notice that

$$\frac{\Gamma(\gamma+\theta)(\delta)_p\left(\xi^\mu - a_2^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q} \leq \omega(\xi,s) \leq \frac{\Gamma(\gamma+\theta)(\delta)_p\left(\xi^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}$$

for all $\xi \in [a_1, a_2]$. Applying Theorem 8 to functions $\omega(\xi, s)$ and ${}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu}f$, we obtain

$$\begin{aligned}
&\left|\frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}\int_{a_1}^{a_2}\omega(\xi,s)\,{}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu}f(s)\mathrm{d}_\mu s\right.\\
&\left.-\left(\frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}\right)^2\int_{a_1}^{a_2}\omega(\xi,s)\mathrm{d}_\mu s\int_{a_1}^{a_2}{}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu}f(s)\mathrm{d}_\mu s\right|\\
&\leq \frac{1}{4}\left(\frac{\Gamma(\gamma+\theta)(\delta)_p\left(\xi^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q} - \frac{\Gamma(\gamma+\theta)(\delta)_p\left(\xi^\mu - a_2^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}\right)(M-m)\\
&= \frac{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}{4\mu\,\Gamma(\theta)(\varrho)_q}(M-m).
\end{aligned} \tag{23}$$

The latter together with the facts

$$\begin{aligned}
&\left(\frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}\right)^2\int_{a_1}^{a_2}\omega(\xi,s)\mathrm{d}_\mu s\\
&\qquad= \left(\frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)^2}\right)\int_{a_1}^{\xi}\left(s^\mu - a_1^\mu\right)\mathrm{d}_\mu s\\
&\qquad+ \left(\frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)^2}\right)\int_{\xi}^{a_2}\left(s^\mu - a_2^\mu\right)\mathrm{d}_\mu s\\
&\qquad= \frac{\left(\xi^\mu - a_1^\mu\right)^2 - \left(\xi^\mu - a_2^\mu\right)^2}{2\left(a_2^\mu - a_1^\mu\right)^2} = \frac{2\xi^\mu - a_1^\mu - a_2^\mu}{2\left(a_2^\mu - a_1^\mu\right)},
\end{aligned} \tag{24}$$

and

$$\int_{a_1}^{a_2}{}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu}f(s)\mathrm{d}_\mu s = f(a_2) - f(a_1), \tag{25}$$

concludes the desired inequality (21). □

Corollary 4 (Trapezoidal Inequality)**.** *Let $\theta, \gamma, \varrho, \delta \in \mathbb{C}$, where $p, q > 0$ and $\Re(\gamma) > 0$, $\Re(\theta) > 0$, $\Re(\varrho) > 0$, $\Re(\delta) > 0$ with $\Re(\gamma) + p \geq q$. Also, let $\mu \in (0,1]$, $a_1, a_2, \xi, s \in [0,\infty)$ satisfy $0 \leq a_1 < a_2$. Assume that the function $f\colon [a_1, a_2] \to \mathbb{R}$ is a $\mathcal{V}$-fractional differentiable and ${}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu}f$ is continuous such that*

$$m \leq {}^{\varrho}_{\ell}\mathcal{V}^{\delta,p,q}_{\gamma,\theta,\mu}f(\xi) \leq M, \quad \xi \in [a_1, a_2]$$

for some $m, M \in \mathbb{R}$, then

$$\begin{aligned}
&\left|\frac{f(a_1) + f(a_2)}{2} - \frac{\mu\,\Gamma(\theta)(\varrho)_q}{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}\int_{a_1}^{a_2}f(s)\mathrm{d}_\mu s\right|\\
&\qquad\leq \frac{1}{4}\frac{\Gamma(\gamma+\theta)(\delta)_p\left(a_2^\mu - a_1^\mu\right)}{\mu\,\Gamma(\theta)(\varrho)_q}(M-m).
\end{aligned}$$

Proof. Using Corollary 3 with $\xi = a_2$, we get the desired result. □

4. Conclusions

In this article, we have established the Steffensen–Hayashi inequalities and remainder identity for $\mathcal{V}$-fractional differentiable functions involving the six parameters truncated Mittag–Leffler function and the well-known Gamma function. In addition, we presented some interesting and useful applications from our main results via the frame of $\mathcal{V}$-fractional calculus such that Steffensen, Chebyshev, Grüss, Hermite–Hadamard, and Ostrowski type integral inequalities. In any case, we hope that these results continue to sharpen our understanding of the nature of fractional-type and its affect on the qualitative properties of such $\mathcal{V}$-fractional operators.

Author Contributions: Conceptualization, H.M.S., P.O.M., O.A., A.K.; methodology, P.O.M., O.A., Y.S.H.; software, H.M.S., P.O.M., O.A., A.K.; validation, P.O.M., O.A., Y.S.H.; formal analysis, P.O.M., O.A.; investigation, P.O.M., O.A., A.K.; resources, H.M.S., P.O.M.; data curation, O.A., A.K., Y.S.H.; writing—original draft preparation, P.O.M., O.A.; writing—review and editing, H.M.S., A.K., Y.S.H.; visualization, O.A.; supervision, H.M.S., Y.S.H. All authors have read and agreed to the final version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This work was supported by the Taif University Researchers Supporting Project (No. TURSP-2020/155), Taif University, Taif, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

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