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The Number of Limit Cycles Bifurcating from an Elementary Centre of Hamiltonian Differential Systems

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Abstract: This paper studies the number of small limit cycles produced around an elementary center for Hamiltonian differential systems with the elliptic Hamiltonian function $H = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{a}{4}x^4 (a \neq 0)$ under two types of polynomial perturbations of degree m, respectively. It is proved that the Hamiltonian system perturbed in Liénard systems can have at least $[\frac{3m-1}{4}]$ small limit cycles near the center, where $m \leq 101$, and that the related near-Hamiltonian system with general polynomial perturbations can have at least $m + [\frac{m+1}{2}] - 2$ small-amplitude limit cycles, where $m \leq 16$. Furthermore, in any of the cases, the bounds for limit cycles can be reached by studying the isolated zeros of the corresponding first order Melnikov functions and with the help of Maple programs. Here, $[\cdot]$ represents the integer function.

Keywords: Liénard system; near-Hamiltonian system; Hopf bifurcation; elementary center

MSC: 34C07; 34E10; 37G15; 37M20

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1. Introduction and Statement of the Main Results

The famous Hilbert's 16th problem is one of the 23 problems posed by the German mathematician David Hilbert [1] in 1900. The second part of Hilbert's 16th problem is finding the maximum number of limit cycles in a planar polynomial vector field of degree m and investigating their relative positions. Up to now, this problem also remains unsolved, even for $m = 2$. In 1977, Arnold [2] proposed a weaker version on Hilbert's 16th Problem and that can be stated as follows: Consider a polynomial 1-form $\omega := Q(x,y)dx - P(x,y)dy$ with real polynomials $P(x,y)$ and $Q(x,y)$ of degree m in x and y. Then, the problem is finding the maximum number of isolated zeroes of Abelian integrals $\int_{\Gamma_h} \omega$, where Γ_h's are the compact level curves of polynomials with a given degree. The so-called Poincaré–Pontrjagin theorem shows that the number of isolated zeros of the Abelian integrals is a lower bound of the maximum number of limit cycles of a near-Hamiltonian system of the form

$$\dot{x} = H_y(x,y) + \varepsilon P(x,y), \qquad \dot{y} = -H_x + \varepsilon Q(x,y), \tag{1}$$

where the Hamiltonian $H(x,y)$ is a real polynomial of degree $n+1$. The first order Melnikov function (the Abelian integrals) relevant to the perturbed system (1) is

$$M(h) = \int_{\Gamma_h} Q(x,y)dx - P(x,y)dy, \tag{2}$$

where Γ_h is a real oval $H(x,y) = h$. Though finding the number of zeros of the Abelian integrals is a weaker problem of the Hilbert's 16th problem, it is still difficult; see [3].

Smale proposed another restricted version of the Hilbert's 16th problem, which is to study the maximum number of limit cycles in polynomial Liénard system

$$\dot{x} = y, \qquad \dot{y} = -g(x) - yf(x) \tag{3}$$

with $g(x) = x$ and their relative positions, which is called Smale's 13th problem [4]. For the classical Liénard system, Lins et al. [5] showed it has $[\frac{m}{2}]$ limit cycles locally and conjectured that the number is possibly the maximum number for the global case. Later, De Maesschalck and Dumortier [6] proved it has at least $[\frac{m}{2}] + 1$ limit cycles for $m \geq 5$ by a geometric singular perturbation theory.

Smale's 13th problem was also extended to the generalized Liénard system; see [7–12] and the references therein. Most of those cases study the maximum number of small amplitude limit cycles bifurcating from a center or a focus. Llibre et al. [13] exhibited that the generalized Liénard system (3) can have at least $[\frac{m+n-1}{2}]$ small amplitude limit cycles by averaging theory of first, second, and third order. For the system (3) with a quadratic polynomial $g(x)$, the authors in [14,15] proved that the Hopf cyclicity is $[\frac{2m+1}{3}]$. For the system (3) with a cubic polynomial $g(x)$, Christopher and Lynch [14] showed that the Hopf cyclicity is $2[\frac{3(m+2)}{8}]$ for $1 < m \leq 50$ using Lyapunov quantities being the coefficients of the monomials $(x^2 + y^2)^i$ in the total derivative of the Lyapunov function along trajectories associated with system (3).

The origin is *an elementary center* for a Hamiltonian system $\dot{x} = H_y(x,y), \dot{y} = -H_x(x,y)$ if the origin is a singularity and the eigenvalues of the matrix $\frac{\partial(H_y,-H_x)}{\partial(x,y)}(0,0)$ are pure imaginary.

Gavrilov and Iliev [8] studied the limit cycles of polynomial near-Hamiltonian systems (1) with the Hamiltonian

$$H = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{a}{4}x^4, \;\; a \neq 0, \frac{8}{9}, \tag{4}$$

which have four topological structures: $a < 0$ saddle-loop, $0 < a < 1$ eight loop, $a = 1$ cuspidal loop, and $a > 1$ global center. Tian et al. [11] considered the generalized Liénard system (3) where $g(x) = H_x(x,y)$ satisfy (4) with $a = \frac{8}{9}$, and proved that the Hopf cyclicity at the origin is $[\frac{3m+2}{4}]$ using involution.

The paper is concerned about the Hamiltonian system with the Hamiltonian (4) having four topological structures, and finds a lower bound for the maximum number of limit cycles appearing from a center under perturbations using a different method. Consider a generalized Liénard system of the form

$$\dot{x} = y, \qquad \dot{y} = -g(x) + \varepsilon y f(x), \tag{5}$$

where $f(x)$ and $g(x)$ are polynomials of degree m and 3, respectively. Our first result is the following.

Theorem 1. *When $a = -1, \frac{1}{2}, 1$ or 2, there exist Liénard differential systems of the form (5) with $g(x) = x(1 - 2x + ax^2)$, which have $[\frac{3m+2}{4}]$ small amplitude limit cycles bifurcating from the origin with $m = \deg(f) \leq 100$.*

Here, we study a weakened version (5) of the generalized Liénard system (3). The number of limit cycles from a center that we obtain is the same as the Hopf cyclicity in [11] with $a = \frac{8}{9}$, and is attainable in each case of four topological structures of the unperturbed Hamiltonian system. In addition, Wei et al. [12] investigated the Liénard system (5) via the first order Melnikov function, whose the unperturbed system has the Hamiltonian one

$$H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4, \tag{6}$$

including two elementary centers and a double homoclinic loop, and which can have $3[\frac{m-2}{5}] + 1 + \text{mod}(\text{mod}(m-2,5),2)$ limit cycles inside a single homoclinic loop, with $[\frac{m-2}{5}] + \text{mod}(\text{mod}(m-2,5),4)$ small limit cycles near the elementary center inside the loop. The sign $\text{mod}(a,b)$ stands for the remainder of a divided by b. It is easy to see that the number of limit cycles from the center is better than that of [12].

Get back to the weakened Hilbert's 16th problem, and look at Newtonian mechanical problems restricting the Hamiltonian to the form $H(x,y) = y^2 + F_{n+1}(x)$. For the Hamiltonian $H(x,y) = y^2 + x^4 - x^2 - \lambda x$ having two elementary centers and a double loop to a hyperbolic saddle, Petrov [16] in 1990 proved that the number of zeros of elliptic integrals (2) corresponding to the near-Hamiltonian system (1) in a single loop is not more than $m + [\frac{m-3}{2}]$, and system (1) exists such that the integrals have at least $m + [\frac{m-3}{2}]$ zeros. Zhao and Zhang [17] in 1999 provided an upper bound $7m+5$ of the number of isolated zeros of Abelian integrals associated with the system (1) whose Hamiltonian $H(x,y) = y^2 + F_{n+1}(x)$ of degree 4 has at least one center. Liu [18] in 2003 affirmed in the case of the Hamiltonian $H(x,y) = y^2 + x^4 - x^2 - \lambda x$ that the total number of zeros of the integrals over two periodic annuli inside an eight-loop does not exceed $4[\frac{m+1}{2}] - 1$, and that the number in the periodic annulus outside the eight-loop does not exceed $4[\frac{m+1}{2}] + 1$. Tian and Han [19] exhibited that system (1) with the Hamiltonian (6) can have $[\frac{7m-6}{3}]$ limit cycles obtained by studying the isolated zeros of Abelian integrals for $m = 3, 5, 7, 9$, being comprised of $m-1$ limit cycles inside each of a single loop and $[\frac{m}{3}]$ limit cycles outside the double loop. In this paper, we continue to study limit cycles for the cubic Hamiltonian system under m degree polynomial perturbations via zeros of the Abelian integrals. Our second result is the following.

Theorem 2. *When $a = -1, \frac{1}{2}, 1$ or 2, near-Hamiltonian systems of the form (1) exist with the Hamiltonian (4) which have $m + [\frac{m-3}{2}]$ small amplitude limit cycles surrounding the origin with $m \le 16$.*

We note that the number of limit cycles is gained by the isolated zeros of the Abelian integrals in accordance with that of [16]. These limit cycles are located within a homoclinic loop in [16], but in this paper only near an elementary center for each case of four topological structures of the unperturbed Hamiltonian system.

The number of limit cycles in each of the above two theorems is realizable by calculations with the help of Maple programs in Section 3. These programs are theoretically efficient for all m's. Thus, we conjecture that conclusions in Theorems 1 and 2 hold for all m's.

In addition, the Maple programs are operational for all a, and one can obtain the lower bound of the number of limit cycles for the near-Hamiltonian system (1) or the generalized Liénard system (5) with the Hamiltonian (4) provided a.

To study small limit cycles produced around an elementary center under small perturbations, one can use Melnikov functions [20–22], focus values [23,24], or the averaging method [25,26]. In order to prove Theorem 1, we shall study the Melnikov function for system (5) and present another algorithm to compute the coefficients of its corresponding asymptotic expansion, which will be shown in Theorem 2.

This paper is organized as follows: Section 2 shows some preliminary work to prove our main results. In Section 3, we will prove Theorems 1 and 2. And the section will provide two Maple programs used to compute the rank of Jacobian matrices.

2. Preliminaries

Consider an analytic near-Hamiltonian system of the form:

$$\begin{cases} \dot{x} = H_y(x,y) + \varepsilon P_0(x,y,\delta), \\ \dot{y} = -H_x(x,y) + \varepsilon Q_0(x,y,\delta), \end{cases} \tag{7}$$

where $H, P_0, Q_0 \in C^{\omega}(\mathbb{R}^2)$, $\varepsilon \geq 0$ is a small perturbation parameter and $\delta \in D \subset \mathbb{R}^m$ is a vector valued parameter with D a compact subset. Suppose that the unperturbed system (7)$|_{\varepsilon=0}$ has an elementary center C enclosed by a period annulus U. Set $h_c := H(C)$, and $I := \{h = H(x,y) : (x,y) \in U\}$.

It is well known that the first order Melnikov function $M(h,\delta)$ near an elementary center for the system (7) is analytic at the end point h_c, see [27,28]. One can have the following lemma.

Lemma 1. *For the analytic near-Hamiltonian system (7) whose unperturbed system has an elementary center, the first order Melnikov function near the center has the expansion*

$$M(h,\delta) = \sum_{j\geq 0} C_j^0(\delta)(h-h_c)^{j+1}, \qquad h \in U(h_c,\tau) \cap \bar{I}, \tag{8}$$

where $0 < \tau \ll 1$, $U(h_c,\tau)$ stands for a neighborhood of h_c, and

$$C_0^0 = \frac{2\pi}{\beta}\left(\frac{\partial P_0}{\partial x} + \frac{\partial Q_0}{\partial y}\right)\Big|_{\substack{(C,\delta),\\ \varepsilon=0}}$$

with the eigenvalues $\pm i\beta(\beta > 0)$ of the center.

Han et al. [21] in 2009 established an algorithm to compute higher order coefficients C_j^0 in the first order Melnikov function by developing a Maple program. Later, Tian and Han [19] and Wei and Zhang [29] characterized definitely all coefficients with the help of a homoclinic loop. The characteristics still keep in the case of an elementary center without other conditions.

For the analytic near-Hamiltonian system (7) with analytic perturbations P_i and Q_i, $i \in \mathbb{N}$, instead of P_0 and Q_0, one has a new near-Hamiltonian system

$$\begin{cases} \dot{x} = H_y(x,y) + \varepsilon P_i(x,y,\delta), \\ \dot{y} = -H_x(x,y) + \varepsilon Q_i(x,y,\delta). \end{cases}$$

It follows from Lemma 1 that the first order Melnikov function near the center C has the expansion of the form

$$M^i(h,\delta) = \sum_{j\geq 0} C_j^i(\delta)(h-h_c)^{j+1}, \qquad h \in U(h_c,\tau) \cap \bar{I}, \tag{9}$$

with $C_0^i = \frac{2\pi}{\beta}\left(\frac{\partial P_i}{\partial x} + \frac{\partial Q_i}{\partial y}\right)\Big|_{(C,\delta),\varepsilon=0}$.

Set

$$\triangle_l^i := \left\{\delta \in D \mid C_j^i(\delta) = 0, j = 0, 1, \cdots, l\right\}.$$

The feature of the coefficients is stated as follows.

Lemma 2. *For the analytic near-Hamiltonian system (7), whose unperturbed system has an elementary center, assume that there exist analytic functions $P_i(x,y,\delta)$ and $Q_i(x,y,\delta)$ for $i = 1, 2, \cdots, r$, such that, for $\delta \in \Delta_0^{i-1}$, the following equality holds:*

$$\left(\frac{\partial P_{i-1}}{\partial x} + \frac{\partial Q_{i-1}}{\partial y}\right)(x,y,\delta) = \frac{\partial H(x,y)}{\partial x} P_i(x,y,\delta) + \frac{\partial H(x,y)}{\partial y} Q_i(x,y,\delta), \tag{10}$$

in a neighborhood of the center. Then, the coefficients of the first order Melnikov functions (8) *of system* (7) *satisfy*

$$C_i^0\Big|_{\Delta_{i-1}^0} = \frac{1}{(i+1)!}C_0^i,$$

where C_0^i's are given in (9).

In [19], (10) was introduced to study the coefficients in the corresponding expansions of the first order Melnikov functions for bifurcations of limit cycles around an elementary center and a homoclinic loop. This idea was extended to the investigation of limit cycles near a homoclinic loop with a nilpotent singularity in [29] and near a heteroclinic loop in [30], respectively. In Lemma 2, we only focus on finding small-amplitude limit cycles near a center. The next lemma can be found in [27] or Corollary 2.4.1 in [31].

Lemma 3. *For the analytic near-Hamiltonian system* (7) *whose unperturbed system has an elementary center, supposing that a positive integer n and $\delta_0 \in D \subset \mathbb{R}^s$ with D bounded and $s \geq n$ exist, such that*

$$C_i^0(\delta_0) = 0,\ i = 0, \cdots, n-1, \quad C_n^0(\delta_0) \neq 0, \quad \text{and} \quad \det\frac{\partial(C_0^0, C_1^0, C_2^0, \cdots, C_{n-1}^0)}{\partial(\delta_1, \cdots, \delta_n)}(\delta_0) \neq 0,$$

where $\delta = (\delta_1, \delta_2, \cdots, \delta_s)$; then, for any $\varepsilon_0 > 0$ and any neighborhood V of the elementary center, system (7) *in V has precisely n limit cycles for some (ε, δ) satisfying $0 < \varepsilon < \varepsilon_0$ and $|\delta - \delta_0| < \varepsilon_0$.*

Hopf bifurcation by the feature in Lemma 2 follows immediately. Set

$$C_i := C_0^0, \ \text{for } i = 0, \qquad \text{and} \qquad C_i := C_i^0|_{\Delta_{i-1}^0}, \ \text{for } i > 0. \tag{11}$$

Theorem 3. *Let the assumptions in Lemma 2 hold. If $\delta_0 \in \mathbb{R}^s$ exists, and a positive integer n such that*

$$C_i(\delta_0) = 0,\ i = 0, \cdots, n-1, \quad C_n(\delta_0) \neq 0, \quad \text{and} \quad \text{rank}\frac{\partial(C_0, C_1, C_2, \cdots, C_{n-1})}{\partial(\delta_1, \cdots, \delta_s)}(\delta_0) = n,$$

then system (7) *has exactly n limit cycles near an elementary center for some (ε, δ) near $(0, \delta_0)$.*

Proof. To study isolated zeros of the first order Melnikov function (Abelian integrals) (2), we display the relations between C_i and C_i^0, of the coefficients in the Melnikov function.

It follows from (11) that $C_0^0 = C_0$, and consequently for $i \geq 1$

$$C_i^0 = C_i^0\Big|_{\Delta_{i-1,}^0} + O\Big(|C_0^0, C_1^0, \cdots, C_{i-1}^0|\Big) = C_i + O\Big(|C_0^0, C_1^0, \cdots, C_{i-1}^0|\Big).$$

By induction, one obtains that $O(|C_0^0, C_1^0, \cdots, C_{i-1}^0|) = O(|C_0, C_1, \cdots, C_{i-1}|)$ for each $i \geq 1$. Namely,

$$C_i^0 = C_i + O(|C_0, C_1, \cdots, C_{i-1}|),$$

satisfying the conditions in Lemma 3. We finish the proof. □

We note that the number of limit cycles in Theorem 3 is obtained from the perspective of theoretical analysis. The theorem can be applied to the Liénard system (5) or the near-Hamiltonian system (1) with a specific unperturbed Hamiltonian system having an elementary center. However, it is difficult to show definite expressions for the perturbations such that systems (5) or (1) have a certain number of limit cycles, since we can take $C_0, C_1, \cdots, C_{n-1}$ as free parameters such that

$$|C_j| \ll |C_{j+1}| \ll 1 \qquad \text{and} \qquad C_j \cdot C_{j+1} < 0, j = 0, 1, \cdots, n-2.$$

In addition, then

$$|C_j^0| \ll |C_{j+1}^0| \ll 1 \qquad \text{and} \qquad C_j^0 \cdot C_{j+1}^0 < 0, j = 0, 1, \cdots, n-2,$$

hold theoretically. However, in practice, it is not easy to guarantee the relation $|C_j^0| \ll |C_{j+1}^0|$ due to the term $O\big(|C_0, C_1, \cdots, C_{j-1}|\big)$.

To simplify calculations for some specific systems, we show the following corollary further.

Corollary 1. *Let the assumptions in Lemma 2 hold. If there exists a positive integer n such that*

$$\operatorname{rank}\frac{\partial(C_0, C_1, C_2, \cdots, C_n)}{\partial(\delta_1, \cdots, \delta_s)} = n + 1,$$

where $\delta = (\delta_1, \delta_2, \cdots, \delta_s) \in \mathbb{R}^s$, then system (7) has n limit cycles near the elementary center for some (ε, δ).

3. Proofs of Theorems 1 and 2

The Hamiltonian (4) with $a \neq 0$ of the unperturbed system $(5)|_{\varepsilon=0}$ or $(1)|_{\varepsilon=0}$ consists of an elementary center at the origin.

Proof of Theorem 1. Consider the Liénard differential system (5) with a polynomial $f_m = \sum_{i=0}^{m} b_i x^i$. By Lemma 1, one has

$$C_0 = 2\pi b_0. \tag{12}$$

Using Taylor expansion at $x = 0$ yields

$$\frac{1}{1 - 2x + ax^2} = \sum_{j=0}^{\infty} x^j (2 - ax)^j = \sum_{l=0}^{\infty} x^l S_l, \tag{13}$$

where $S_l = \sum\limits_{j=[\frac{l+1}{2}]}^{l} \frac{j!(-a)^{l-j} \cdot 2^{2j-l}}{(l-j)!(2j-l)!}$.

Since $P_0 = 0$ and $Q_0 = f_m(x)y$. In the light of the condition Δ_0^0 in Theorem 2, being $f_m(0) = 0$, the equality (10) holds by taking

$$P_1 = \frac{f_m(x)|_{\Delta_0^0}}{x(1 - 2x + ax^2)} \qquad \text{and} \qquad Q_1 = 0,$$

which gives

$$P_1 = \sum_{l=0}^{\infty} b_l^1 x^l$$

in a neighborhood of zero, where

$$b_l^1 = \begin{cases} \sum_{j=0}^{l} b_{j+1} S_{l-j}, & \text{for} \quad 0 \le l \le m-1, \\ \sum_{j=0}^{m-1} b_{j+1} S_{l-j}, & \text{for} \quad l \ge m, \end{cases}$$

with S_l given in (13).

Set

$$P_{i-1} := \sum_{l=0}^{\infty} b_l^{i-1} x^l \qquad \text{and} \qquad Q_{i-1} := 0$$

for $i \geq 2$ in a neighborhood of zero. It follows from the condition Δ_0^{i-1} and the equality (10) that

$$P_i = \sum_{l=0}^{\infty} b_l^i x^l \qquad \text{and} \qquad Q_i = 0$$

in a neighborhood of zero, where

$$b_l^i = \sum_{j=0}^{l} (j+2) b_{j+2}^{i-1} S_{l-j}$$

for all l's.

By induction, one has for $1 \leq i \leq [\frac{m}{2}]$

$$b_1^i = \prod_{j=1}^{i-1} A_j B_i (b_1, b_2, \cdots, b_{2i})^T, \tag{14}$$

and, for $i > [\frac{m}{2}]$

$$b_1^i = \prod_{j=1}^{i-1} A_j B_i (b_1, b_2, \cdots, b_m)^T, \tag{15}$$

where $A_1 = (2S_1, 3S_0)$, A_j's are the $(2j-1) \times (2j)$ matrices

$$A_j = \begin{pmatrix} 2S_2 & 3S_1 & 4S_0 & 0 & \cdots & 0 \\ 2S_3 & 3S_2 & 4S_1 & 5S_0 & \cdots & 0 \\ 2S_4 & 3S_3 & 4S_2 & 5S_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 2S_{2j-1} & 3S_{2j-2} & 4S_{2j-3} & 5S_{2j-4} & \cdots & (2j+1)S_0 \end{pmatrix},$$

for $2 \leq j \leq i-1$, and $B_1 = (S_1, S_0)$, B_i's are the $(2i-2) \times (2i)$ matrices

$$B_i = \begin{pmatrix} S_2 & S_1 & S_0 & 0 & \cdots & 0 \\ S_3 & S_2 & S_1 & S_0 & \cdots & 0 \\ S_4 & S_3 & S_2 & S_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ S_{2i-1} & S_{2i-2} & S_{2i-3} & S_{2i-4} & \cdots & S_0 \end{pmatrix},$$

for $2 \leq i \leq [\frac{m}{2}]$, and B_i's are the $(2i-2) \times m$ matrices

$$B_i = \begin{pmatrix} S_2 & S_1 & S_0 & 0 & \cdots & 0 \\ S_3 & S_2 & S_1 & S_0 & \cdots & 0 \\ S_4 & S_3 & S_2 & S_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ S_{m-1} & S_{m-2} & S_{m-3} & S_{m-4} & \cdots & S_0 \\ S_m & S_{m-1} & S_{m-2} & S_{m-3} & \cdots & S_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ S_{2i-1} & S_{2i-2} & S_{2i-3} & S_{2i-4} & \cdots & S_{2i-m} \end{pmatrix},$$

for $i > [\frac{m}{2}]$ with S_l given by (13). Noting that the expression $\prod_{j=1}^{i-1} A_j$ in (14) or (15) represents an identity matrix for $i = 1$.

According to Lemma 2 and the expressions (9) and (11), one obtains

$$C_i = \frac{2\pi}{(i+1)!} b_1^i.$$

Substituting (14) and (15) into the last expression, one obtains

$$C_i = \frac{2\pi}{(i+1)!} \prod_{j=1}^{i-1} A_j B_i (b_1, b_2, \cdots, b_{2i})^T, \tag{16}$$

for $1 \le i \le [\frac{m}{2}]$, and

$$C_i = \frac{2\pi}{(i+1)!} \prod_{j=1}^{i-1} A_j B_i (b_1, b_2, \cdots, b_m)^T, \tag{17}$$

for $i > [\frac{m}{2}]$.

On the basis of (12), (16), and (17), it is easy to see that the Jacobian matrix

$$\frac{\partial(C_0, C_1)}{\partial(b_0, b_1, \cdots, b_m)} = \begin{pmatrix} 2\pi & 0 \\ 0 & 2\pi \end{pmatrix},$$

is of full rank for $m = 1$, and we show via the Maple program (see Algorithm 1) that the Jacobian matrices

$$\frac{\partial\left(C_0, C_1, \cdots, C_{[\frac{3m+2}{4}]}\right)}{\partial(b_0, b_1, \cdots, b_m)}, \tag{18}$$

for $2 \le m \le 100$, are of full rank, if $a = -1, \frac{1}{2}, 1$, or 2.

By the arguments on the rank of the Jacobian matrix of the coefficients in the expansion of the first order Melnikov function with respect to the coefficients $b = (b_0, b_1, \ldots, b_m)$ of f_m, one achieves that the C_j's satisfy the conditions of Corollary 1 or Theorem 3 with $n = [\frac{3m+2}{4}]$. Consequently, the Liénard differential system (5) has $[\frac{3m+2}{4}]$ limit cycles for suitable choices of its coefficients.

It completes the proof of the theorem. □

Proof of Theorem 2. Consider the near-Hamiltonian system (1) with the Hamiltonian (4). For simplicity, set

$$F^0(x, y) := \frac{\partial P_0}{\partial x} + \frac{\partial Q_0}{\partial y} = \sum_{i+j=0}^{m-1} d_{ij} x^i y^j.$$

By Lemma 1, we have

$$C_0 = 2\pi d_{00}. \tag{19}$$

The condition Δ_0^0 in Lemma 2 indicates $F^0(0,0) = d_{00} = 0$. Under the condition Δ_0^0, the equality (10) with $i = 1$ holds by taking

$$P_1 = \frac{F^0(x, y)|_{d_{00}=0}}{x(1 - 2x + ax^2)} \quad \text{and} \quad Q_1 = \frac{1}{y}(F^0(x, y) - F^0(x, 0)),$$

that are in a neighborhood of zero

$$P_1(x,y) = \sum_{i=0}^{\infty} a_i^1 x^i \qquad \text{and} \qquad Q_1(x,y) = \sum_{i+j=0}^{m-2} b_{ij}^1 x^i y^j,$$

where $b_{ij}^1 = d_{i,j+1}$, and

$$a_i^1 = \begin{cases} \sum\limits_{k=0}^{i} d_{k+1,0} S_{i-k}, & 0 \le i \le m-2, \\ \sum\limits_{k=0}^{m-2} d_{k+1,0} S_{i-k}, & l \ge m-1. \end{cases}$$

In a neighborhood of zero, let

$$P_{l-1} := \sum_{i=0}^{\infty} a_i^{l-1} x^i \qquad \text{and} \qquad Q_{l-1} := \sum_{i+j=0}^{m-2l+2} b_{ij}^{l-1} x^i y^j$$

for $l \ge 2$. It follows that

$$F^{l-1} := \frac{\partial P_{l-1}}{\partial x} + \frac{\partial Q_{l-1}}{\partial y} = \sum_{i+j=0}^{m-2l+1} d_{ij}^{l-1} x^i y^j + \sum_{i=m-2l+2}^{\infty} d_{i0}^{l-1} x^i,$$

where

$$d_{ij}^{l-1} = \begin{cases} (i+1)a_{i+1}^{l-1} + (j+1)b_{i,j+1}^{l-1}, & j = 0,\ 0 \le i \le m-2l+1, \\ (i+1)a_{i+1}^{l-1}, & j = 0,\ i \ge m-2l+2, \\ (j+1)b_{i,j+1}^{l-1}, & j \ge 1,\ 1 \le i+j \le m-2l+1. \end{cases}$$

As is stated above, under the condition Δ_0^{l-1}, the equality (10) holds by ordering in a neighborhood of $x = 0$

$$P_l = \sum_{i=0}^{\infty} a_i^l x^i, \qquad \text{and} \qquad Q_l = \sum_{i+j=0}^{m-2l} b_{ij}^l x^i y^j,$$

where

$$a_i^l = \sum_{k=0}^{i} d_{k+1,0}^{l-1} S_{i-k} \qquad \text{and} \qquad b_{ij}^l = d_{i,j+1}^{l-1},$$

with S_{i-k} given by (13).

By induction, one has

$$d_{00}^1 = \begin{cases} d_{10} S_1, & m = 2, \\ B_1(d_{10}, d_{20})^T + d_{02}, & m \ge 3, \end{cases} \tag{20}$$

and for $2 \le l \le m + [\frac{m+1}{2}] - 2$,

$$d_{00}^{l} = \sum_{i=1}^{l-1}(2i-1)!!\prod_{j=1}^{l-i}A^{j}(d_{1,2i}, d_{2,2i}, \cdots, d_{2l-2i,2i})^{T} + (2l-1)!!d_{0,2l} \tag{21}$$
$$+\begin{cases} \prod_{j=1}^{l} A_j(d_{10}, d_{20}, \cdots, d_{2l,0})^T, & 2l \le m-1, \\ \prod_{j=1}^{l-1} A_j B_l(d_{10}, d_{20}, \cdots, d_{m-1,0})^T, & 2l \ge m, \end{cases}$$

where $A_1 = B_1 = (S_1, S_0)$, A_j's are the $(2j-1) \times (2j)$ matrices

$$A_j = \begin{pmatrix} 2S_2 & 2S_1 & 2S_0 & 0 & \cdots & 0 \\ 3S_3 & 3S_2 & 3S_1 & 3S_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ (2j-1)S_{2j-1} & (2j-1)S_{2j-2} & (2j-1)S_{2j-3} & (2j-1)S_{2j-4} & \cdots & (2j-1)S_0 \end{pmatrix},$$

for $2 \le j \le l$, and B_l's are the $(2l-2) \times (m-1)$ matrices

$$B_l = \begin{pmatrix} 2S_2 & 2S_1 & 2S_0 & 0 & \cdots & 0 \\ 3S_3 & 3S_2 & 3S_1 & 3S_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ (m-2)S_{m-2} & (m-2)S_{m-3} & (m-2)S_{m-4} & (m-2)S_{m-5} & \cdots & (m-2)S_0 \\ (m-1)S_{m-1} & (m-1)S_{m-2} & (m-1)S_{m-3} & (m-1)S_{m-4} & \cdots & (m-1)S_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ (2l-1)S_{2l-1} & (2l-1)S_{2l-2} & (2l-1)S_{2l-3} & (2l-1)S_{2l-4} & \cdots & (2l-1)S_{2l-m+1} \end{pmatrix},$$

and $d_{ij} = 0$ if $i + j \ge m$.

According to Lemma 2 and the expressions (9) and (11), one has for $l = 1, 2, \cdots, m + [\frac{m+1}{2}] - 2$,

$$C_l = \frac{2\pi}{(l+1)!}d_{00}^{l}. \tag{22}$$

According to the Formulas (19)–(22), we find that C_i's depend solely on the parameters $d_{i,2j}$ in the divergence of the perturbations P_0 and Q_0. Let $E_m := \{d_{i,2j} : i + 2j \le m-1\}$ with $[\frac{m+1}{2}][\frac{m+2}{2}]$ elements. It follows from (19) and (20)–(22) that the Jacobian matrix

$$\frac{\partial(C_0, C_1)}{\partial(d_{00}, d_{10})} = \begin{pmatrix} 2\pi & 0 \\ 0 & 2\pi \end{pmatrix},$$

is of full rank for $m = 2$. By using the Maple program (see Algorithm 2), we have that the Jacobian matrices

$$\frac{\partial\left(C_0, C_1, \cdots, C_{m+[\frac{m+1}{2}]-2}\right)}{\partial E_m}, \tag{23}$$

are of full rank for $3 \le m \le 16$, if $a = -1, \frac{1}{2}, 1$, or 2.

Combing Corollary 1 or Theorem 3 and the rank of the Jacobian matrices, one achieves that the near-Hamiltonian system (5) has $m + [\frac{m+1}{2}] - 2$ limit cycles for suitable choices of the parameters.

It completes the proof of the theorem. □

We note that, in the previous discussions, for any $m \ge 2$, theoretically, it is possible to obtain the rank of the Jacobian matrices with a specific a via Algorithms 1 and 2, only if a computer has a sufficiently strong CPU and the memory ensuring its running for enough time. If we choose $m = 100$, then Algorithm 1 takes 3955 s to run. However, Algorithm 2 runs only to $m = 16$ due to the limitation of the memory of our computer (16 GB).

Algorithm 1 The rank of the Jacobian matrices (18)

```
restart: with(LinearAlgebra):
num := proc (a, m)
local S, F, A, F1, B, BMy, M, c, i;
    S := proc (n)
    local i, sum;
        sum := 0;
        for i from trunc (1/2 * n + 1/2) to n do
            sum := sum + (i)!*((n − i)! * (2i − n)!)^(−1) * (−a)^(n−i) * 2^(2i−n)
        od;
        sum
    end proc;
    F := proc (i, j)→ (j + 1) * S (i + 2 - j);
    A := proc (j)
    local AM;
        if j = 0 then
            AM := Matrix (1, 1, 1)
        else
            if j = 1 then
                AM := Matrix ([4, 3])
            else
                AM := Matrix (2 * j -2, 2 * j, F)
            end if;
        end if;
        AM
    end proc;
    F1 := proc (i, j) → S (i + 2 - j);
    B := proc (i)
    local BM;
        if i = 1 then
            BM := Matrix ([2, 1])
        else
            if i≤ trunc (1/2 * m) then
                BM := Matrix (2 * i -2, 2 * i, F1)
            else
                BM := Matrix (2 * i -2, m, F1)
            end if;
        end if;
        BM
    end proc;
    BMy := proc (i)
    local AM, j;
        AM := Matrix (1, 1, 1);
        for j from 0 to i -1 do
            AM := MatrixMatrixMultiply(AM, A (j))
        od;
        MatrixMatrixMultiply (AM, B (i))
    end proc;
    M := proc (i)
        if i≤trunc (1/2 * m) then
            < <0> | < BMy(i) > | < Matrix(1, m-2 * i,0)> >
        else
            < <0> | < BMy(i) > >
        end if;
    end proc;
    c := < <1> | Matrix (1, m, 0)>;
    for i to trunc (3/4 * m + 1/2) do
        c := < c, M (i)>
    od;
    Rank (c)
end proc;
```

Algorithm 2 The rank of the Jacobian matrices (23)

```
restart: with(LinearAlgebra):
rank := proc (a,m)
local k, PQ, l, i, divPQ, C, diPQ, cof, G, G1, E, e, ser;
    k := trunc (1/2 * m + 1/2) * trunc (1/2 * m + 1);
    PQ := 0;
    for l from 0 to m -1 do
        for i from 0 to l do
            PQ := PQ + c[i, l-i] * x^i * y^(l-i)
        end do;
    end do;
    divPQ[0] := PQ;
    C[0] := subs ([x = 0, y = 0], divPQ[0]);
    ser := proc (l)
        convert (series (1/(1-2 * x + a * x^2), x = 0, l), polynom)
    end proc;
    diPQ := proc (l, F)
    local P, Q;
        P := sort (collect ((subs (y = 0, F)-subs ([x = 0, y = 0], F)) * x^(-1) * ser (2 * k-2 * l + 2), x));
        Q := sort (expand (simplify ((F-subs (y = 0, F)) * y^(-1))), [x, y]);
        sort (diff (P, x) + diff (Q, y), [x, y])
    end proc;
    for i to k do
        divPQ[i] := diPQ (i, divPQ[i -1]);
        C[i] := (factorial (i + 1))^(-1) * subs ([x = 0, y = 0], divPQ[i])
    end do;
    cof := proc (i)
    local f, l, j;
        f := 0;
        for l from 0 to i -1 do
            for j from 0 to trunc (1/2 * l) do
                f := f + c[l-2 * j, 2 * j] * x^(l-2*j) * y^(2*j)
            end do;
        end do;
        coeffs (f, [x, y])
    end proc;
    G := 0;
    for i from 0 to m + trunc (1/2 * (m -1)) -1 do
        G := G + C[i] * x^i
    end do;
    G1 := coeffs (G, x);
    E, e := GenerateMatrix (G1, [cof (m)]);
    Rank (E)
end proc;
```

4. Conclusions

Tian and Han [19] provided a new idea in 2017, which obtains the expressions of the high order coefficients in the asymptotic expansion of the first order Melnikov function (Abelian integrals) near a homoclinic loop under some additional conditions, to obtain more limit cycles near a (double) homoclinic loop. The new idea is to introduce an elementary center. Wei et al. [12] in 2021 further developed their results based on their good idea, and obtained more limit cycles in an $(m+1)$th degree generalized Liénard differential system with the Hamiltonian one of degree 4. Inspired by the new idea, we apply it to Hopf bifurcation near an elementary center in this paper, aiming at finding more limit cycles of specific systems with perturbations of degree large m.

This work is to find small-amplitude limit cycles generated from an elementary center of Hamiltonian systems with the 4-degree Hamiltonian $H = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{a}{4}x^4 (a \neq 0)$, which have four phase portraits, under two types of polynomial perturbations of degree m. The lower bounds of the maximum number of limit cycles are given by Theorems 1 and 2. Obviously, the same number of limit cycles, in the perturbed systems with given m, always appears near an elementary center regardless of phase portraits of the unperturbed Hamiltonian system. To obtain the lower bounds, two Maple Programs are showed in Section 3, and make the lower bounds come true via the first order Melnikov function.

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References

1. Hilbert, D. Mathematical problems. *Bull. Am. Math. Soc.* **1902**, *8*, 437–479. doi: [CrossRef]
2. Arnold, V.I. Loss of stability of self-induced oscillations near resonance, and versal deformations of equivariant vector fields. *Funkcional. Anal. Priložen.* **1977**, *11*, 95.
3. Li, J. Hilbert's 16th problem and bifurcations of planar polynomial vector fields. *Int. J. Bifurc. Chaos Appl. Sci. Eng.* **2003**, *13*, 47–106. [CrossRef]
4. Smale, S. Mathematical problems for the next century. *Math. Intell.* **1998**, *20*, 7–15. doi: [CrossRef]
5. Lins, A.; de Melo, W.; Pugh, C.C. On Liénard's equation. In *Proceedings of the Geometry and Topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976)*; Lecture Notes in Mathematics; Springer: Berlin, Germany, 1977; Volume 597, pp. 335–357.
6. De Maesschalck, P.; Dumortier, F. Classical Liénard equations of degree $n \geq 6$ can have $[\frac{n-1}{2}] + 2$ limit cycles. *J. Differ. Equ.* **2011**, *250*, 2162–2176. [CrossRef]
7. Dumortier, F.; Li, C. On the uniqueness of limit cycles surrounding one or more singularities for Liénard equations. *Nonlinearity* **1996**, *9*, 1489–1500. doi: [CrossRef]
8. Gavrilov, L.; Iliev, I.D. Cubic perturbations of elliptic Hamiltonian vector fields of degree three. *J. Differ. Equ.* **2016**, *260*, 3963–3990. [CrossRef]
9. Lloyd, N.G.; Lynch, S. Small-amplitude limit cycles of certain Liénard systems. *Proc. R. Soc. Lond. Ser. A* **1988**, *418*, 199–208.
10. Sheng, L.; Han, M. Bifurcation of limit cycles from a compound loop with five saddles. *J. Appl. Anal. Comput.* **2019**, *9*, 2482–2495. [CrossRef]
11. Tian, Y.; Han, M.; Xu, F. Bifurcations of small limit cycles in Liénard systems with cubic restoring terms. *J. Differ. Equ.* **2019**, *267*, 1561–1580. [CrossRef]
12. Wei, L.; Zhang, Q.; Zhang, X. On limit cycles near two centers and a double homoclinic loop in Liénard differential system. *J. Differ. Equ.* **2021**, *300*, 226–251. [CrossRef]
13. Llibre, J.; Mereu, A.C.; Teixeira, M.A. Limit cycles of the generalized polynomial Liénard differential equations. *Math. Proc. Camb. Philos. Soc.* **2010**, *148*, 363–383. [CrossRef]
14. Christopher, C.; Lynch, S. Small-amplitude limit cycle bifurcations for Liénard systems with quadratic or cubic damping or restoring forces. *Nonlinearity* **1999**, *12*, 1099–1112. [CrossRef]
15. Han, M. Liapunov constants and Hopf cyclicity of Liénard systems. *Ann. Differ. Equ.* **1999**, *15*, 113–126.
16. Petrov, G.S. Nonoscillation of elliptic integrals. *Funktsional. Anal. i Prilozhen.* **1990**, *24*, 45–50, 96. [CrossRef]
17. Zhao, Y.; Zhang, Z. Linear estimate of the number of zeros of abelian integrals for a kind of quartic Hamiltonians. *J. Differ. Equ.* **1999**, *155*, 73–88. [CrossRef]
18. Liu, C. Estimate of the number of zeros of Abelian integrals for an elliptic Hamiltonian with figure-of-eight loop. *Nonlinearity* **2003**, *16*, 1151–1163. [CrossRef]
19. Tian, Y.; Han, M. Hopf and homoclinic bifurcations for near-Hamiltonian systems. *J. Differ. Equ.* **2017**, *262*, 3214–3234. [CrossRef]
20. Han, M.; Yang, J. The Maximum Number of Zeros of Functions with Parameters and Application to Differential Equations. *J. Nonlinear Model. Anal.* **2021**, *3*, 13–34. [CrossRef]
21. Han, M.; Yang, J.; Yu, P. Hopf bifurcations for near-Hamiltonian systems. *Int. J. Bifurc. Chaos Appl. Sci. Eng.* **2009**, *19*, 4117–4130. [CrossRef]

22. Yu, P.; Han, M. Four limit cycles in quadratic near-integrable systems. *J. Appl. Anal. Comput.* **2011**, *1*, 291–298.
23. Arcet, B.; Romanovski, V.G. On Some Reversible Cubic Systems. *Mathematics* **2021**, *9*, 1446. [CrossRef]
24. Christopher, C. Estimating limit cycle bifurcations from centers. In *Differential Equations with Symbolic Computation*; Trends in Mathematics; Birkhäuser: Basel, Switzerland, 2005; pp. 23–35. [CrossRef]
25. Benterki1, R.; Llibre, J. Periodic Solutions of a Class of Duffing DifferentialEquations. *J. Nonlinear Model. Anal.* **2019**, *1*, 167–177.
26. Buică, A.; Llibre, J. Averaging methods for finding periodic orbits via Brouwer degree. *Bull. Sci. Math.* **2004**, *128*, 7–22. [CrossRef]
27. Han, M. On Hopf cyclicity of planar systems. *J. Math. Anal. Appl.* **2000**, *245*, 404–422. [CrossRef]
28. Li, W.; Llibre, J.; Zhang, X. Melnikov functions for period annulus, nondegenerate centers, heteroclinic and homoclinic cycles. *Pac. J. Math.* **2004**, *213*, 49–77. [CrossRef]
29. Wei, L.; Zhang, X. Limit cycles bifurcating from periodic orbits near a center and a homoclinic loop with a nilpotent singularity of Hamiltonian systems. *Nonlinearity* **2020**, *33*, 2723–2754. [CrossRef]
30. Geng, W.; Tian, Y. Bifurcation of limit cycles near heteroclinic loops in near-Hamiltonian systems. *Commun. Nonlinear Sci. Numer. Simul.* **2021**, *95*, 105666, 10. [CrossRef]
31. Han, M. *Bifurcation Theory of Limit Cycle*; Science Press: Beijing, China, 2013.