mathematics

Article

On Primal Soft Topology

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Abstract: In a soft environment, we investigated several (classical) structures such as ideals, filters, grills, etc. It is well known that these structures are applied to expand abstract concepts; in addition, some of them offer a vital tool to address some practical issues, especially those related to improving rough approximation operators and accuracy measures. Herein, we contribute to this line of research by presenting a novel type of soft structure, namely "soft primal". We investigate its basic properties and describe its behaviors under soft mappings with the aid of some counterexamples. Then, we introduce three soft operators $(\cdot)^{\diamond}$, $Cl^{\diamond}$ and $(\cdot)^{\square}$ inspired by soft primals and explore their main characterizations. We show that $Cl^{\diamond}$ satisfies the soft Kuratowski closure operator, which means that $Cl^{\diamond}$ generates a unique soft topology we call a primal soft topology. Among other obtained results, we elaborate that the set of primal topologies forms a natural class in the lattice of topologies over a universal set and set forth some descriptions for primal soft topology under specific types of soft primals.

Keywords: soft primal; soft grill; primal soft topology; soft base; soft Kuratowski's closure

MSC: 03E72; 54A05; 54A20

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Citation: Al-shami, T.M.; Ameen, Z.A.; Abu-Gdairi, R.; Mhemdi, A. On Primal Soft Topology. *Mathematics* **2023**, *11*, 2329. https://doi.org/10.3390/math11102329

Academic Editors: Naeem Jan and Jeonghwan Gwak

Received: 27 April 2023
Revised: 11 May 2023
Accepted: 15 May 2023
Published: 16 May 2023

1. Introduction

The initial phase of the soft sets was diagnosed by Molodtsov [1]. The hypothesis of Molodtsov's soft set theory has attracted broad consideration from many distinguished researchers and intellectuals since it overcomes the drawbacks of traditional mathematical tools and has great application superiority in coping with uncertainties [2,3]. After successfully introducing the notion of soft sets, the soft sets were modified and hybridized to fuzzy soft sets, soft rough sets, and recently (a, b)-fuzzy soft sets.

Soft set theory has been applied to a variety of mathematical structures, including soft group theory [4], soft ring theory [5], soft category theory [6], soft algebra [7,8], and so on. In 2011, two methodologies to define soft topology were displayed by Shabir and Naz [9] and Çağman et al. [10]. They differ in the manners of choosing the sets of parameters. Herein, we follow Shabir and Naz's approach, which imposed that a set of parameters must be constant for all soft open sets that produce a soft topology. Following Shabir and Naz's work, many researchers and intellectuals constructed a soft version for the classical topological concepts and notions. For instance, soft separation axioms [11], soft separable spaces [11], soft connected spaces [12], soft compact spaces [13], soft paracompact spaces [12], soft extremally disconnected spaces [14], generalizations of soft open subsets [15,16], Vietoris topology [17], metric spaces [18], and soft Menger spaces [19].

By dropping a certain (part of an) axiom of soft topology, a new (weak) structure can be established. For example, El-Sheikh and Abd El-Latif [20] came up with the idea of supra soft topological spaces by dismissing the finite intersection axiom of a soft topology. Thomas and John [21] weakened supra soft topological spaces by establishing the concept of soft generalized topological spaces, which is the family of soft sets that satisfy an arbitrary union condition of a soft topology. Al-shami [22] introduced the idea of infra soft topological spaces by ignoring the union axiom of a soft topology. Therefore, this route attracted a large number of researchers who investigated important principles in the latter soft structures; for more details, see [23–26].

Another fruitful area of study is how to construct soft topologies over a common universal set. Terepeta [27] provided two exceptional techniques to generate soft topologies from crisp topologies. The soft topology constructed using one of the methods is therefore equal to the enriched soft topology, as demonstrated by Al-shami in [28]. The formulas provided by Terepeta were refined by Alcantud [29] such that it is now possible to generate a soft topology from a system of crisp topologies. Ameen and Al Ghour [30] introduced the so-called soft simple extension of a soft topology. The simple extended soft topology with respect to a soft topology and a soft set is generated by their (soft) union. Kandil et al. [31] introduced the concept of $*$-soft topological spaces, which are a mix of soft topological and soft algebraic structures. The $*$-soft topology is generated by an old soft topology with the help of a soft ideal. The $*$-soft topologies can be called ideal soft topologies. Likewise, the concept of soft topology via soft grills appeared in [32]. In this direction, we define the concept of primal soft topology. This soft topology is constructed via the soft Kuratowski closure operator with respect to a primal soft topological space. Primal soft topologies are a natural generalization of the primal crisp topologies established in [33]. Some operators of primal topological spaces were introduced by Al-Omari et al. [34].

The first reason that we wrote this article is to present a new type of soft structure that enriches studies on soft settings by creating novel frameworks that allow us to establish new soft concepts and properties. Second, we generate a new way to produce soft topology inspired by some soft operators. Finally, we confirm the importance of the soft environments to provide several sorts of analogs for every classical concept. That is, one can exploit the different types of belonging relationships between ordinary points and soft sets to define various kinds of soft operators and then generate some types of soft topologies.

This work is displayed as follows. Following this introduction, we review the definitions and conclusions required to comprehend the information in Section 2. Then, in Section 3, we define the concept of the soft primal and show that soft primals and soft grills are complementary notions. The basic operations on soft primals are studied. In Section 4, we give the definition of a primal soft topological space followed by a soft topological operator $^\diamond$. Then, we study the main properties of $^\diamond$. In addition, we define another soft operator called $Cl^\diamond$ with the help of $^\diamond$ and show that $Cl^\diamond$ is the soft Kuratowski closure operator. This means that $Cl^\diamond$ generates a unique soft topology, which we call a primal soft topology. The fundamental properties of primal soft topologies are investigated. We close Section 4 by defining a soft operator $(\cdot)^\square$ and elucidate its essential features. Finally, in Section 5, we summarize the main contributions and offer some suggestions for the future.

2. Preliminaries

Definition 1 ([1])**.** *A soft set over a nonempty set Y is a set-valued function F from a nonempty set of parameters Δ to the power set 2^Y of Y; it is denoted by the ordered pair (F,Δ). That is, a soft set (F,Δ) over $Y \neq \emptyset$ provides a parameterized collection of subsets of Y, so it may represented as follows:*

$$(F,\Delta) = \{(\lambda, F) : \lambda \in \Delta \text{ and } F(\lambda) \in 2^Y\};$$

where each $F(\lambda)$ is termed a λ-component of (F,Δ). We denote the family of all soft sets over Y with a set of parameters Δ by $S(Y_\Delta)$.

Throughout this manuscript, $(F,\Delta),(G,\Delta)$ denote soft sets over Y.

Definition 2 ([35–37])**.** *A soft set (F,Δ) is as follows:*

(i) *Absolute, symbolized by $\widetilde{Y}$, if $F(\lambda) = Y$ for all $\lambda \in \Delta$.*

(ii) *Null, symbolized by $\widetilde{\phi}$, if $F(\lambda) = \emptyset$ for all $\lambda \in \Delta$.*

(iii) *A soft point if there are $\lambda \in \Delta$ and $y \in Y$ with $F(\lambda) = \{y\}$ and $F(\mu) = \emptyset$ for all $\mu \in \Delta - \{\lambda\}$. A soft point is briefly denoted by y_λ. We write $y_\lambda \in (F, \Delta)$ if $y \in F(\lambda)$.*

Definition 3 ([9,35])**.** *We call (F, Δ) a soft subset of (G, Δ) (or (G, Δ) a soft superset of (G, Δ)), symbolized by $(F, \Delta)\widetilde{\subseteq}(G, \Delta)$ if $F(\lambda) \subseteq G(\lambda)$ for each $\lambda \in \Delta$.*

Definition 4 ([35])**.** *If $G(\lambda) = Y - F(\lambda)$ for all $\lambda \in \Delta$, then we call (G, Δ) a complement of (F, Δ). The complement of (F, Δ) is symbolized by $(F, \Delta)^c = (F^c, \Delta)$.*

Definition 5 ([13,35])**.** *Let (F, Δ) and (G, Δ) be soft sets. Then, the concepts of soft union, soft intersection, soft difference, and soft product are given respectively by*

(i) *$(F, \Delta)\widetilde{\bigcup}(G, \Delta) = (H, \Delta)$, where $H(\lambda) = F(\lambda) \bigcup G(\lambda)$ for all $\lambda \in \Delta$.*

(ii) *$(F, \Delta)\widetilde{\bigcap}(G, \Delta) = (H, \Delta)$, where $H(\lambda) = F(\lambda) \bigcap G(\lambda)$ for all $\lambda \in \Delta$.*

(iii) *$(F, \Delta) \setminus (G, \Delta) = (H, \Delta)$, where $H(\lambda) = F(\lambda) \setminus G(\lambda)$ for all $\lambda \in \Delta$.*

(iv) *$(F, \Delta) \times (G, \Delta) = (H, \Delta)$, where $H(\lambda_1, \lambda_2) = F(\lambda_1) \times G(\lambda_2)$ for all $(\lambda_1, \lambda_2) \in \Delta \times \Delta$.*

Definition 6 ([38])**.** *A soft set (F, Δ) is called finite (or countable) if $F(\lambda)$ is finite (or countable) for each $\lambda \in \Delta$. Otherwise, it is called infinite (or uncountable).*

The adjusted version of the definition of soft functions is given in the following.

Definition 7 ([39])**.** *Let $h : Y \to Z$ and $\pi : \Delta \to P$ be crisp functions. A soft function h_π of $S(Y_\Delta)$ into $S(Z_P)$ is a relation such that each $y_\lambda \in S(Y_\Delta)$ is related to one and only one $z_p \in S(Z_P)$ such that*

$$h_\pi(y_\lambda) = h(y)_{\pi(\lambda)} \text{ for all } y_\lambda \in S(Y_\Delta).$$

In addition, $h_\pi^{-1}(z_p) = \widetilde{\bigcup}_{\substack{y \in h^{-1}(z) \\ \lambda \in \pi^{-1}(p)}} y_\lambda$ *for each* $z_p \in S(Z_P)$.

We describe a soft function as injective (or surjective, bijective) if both of its crisp functions obey this property.

Proposition 1 ([40])**.** *Let $h_\pi : S(Y_\Delta) \to S(Z_P)$ be a soft function and let $(F, \Delta) \in S(Y_\Delta)$ and $(G, \Delta) \in S(Z_P)$. Then*

(i) *$(F, \Delta)\widetilde{\subseteq}h_\pi^{-1}(h_\pi(F, \Delta))$.*

(ii) *If h_π is injective, then $(F, \Delta) = h_\pi^{-1}(h_\pi(F, \Delta))$.*

(iii) *$h_\pi(h_\pi^{-1}(G, P))\widetilde{\subseteq}(G, P)$.*

(iv) *If h_π is surjective, then $h_\pi(h_\pi^{-1}(G, P)) = (G, P)$.*

Definition 8 ([9,10])**.** *A subfamily Θ of $S(Y_\Delta)$ is named a soft topology on Y if it obeys the next stipulations:*

(i) *$\widetilde{Y}$ and $\widetilde{\phi}$ are elements of Θ.*

(ii) *Θ is closed under finite soft intersections.*

(iii) *Θ is closed under arbitrary soft unions.*

The notation (Y, Θ, Δ) is named a soft topological space (briefly, STS). A soft set belonging to Θ is named soft open, and it is named soft closed if its complement is soft open. The family of all soft closed sets in Y is denoted by Θ^c. For $y_\lambda \in \widetilde{Y}$, the family of all members of Θ containing y_λ is denoted by $\Theta(y_\lambda)$.

Definition 9 ([41]). *A soft subset (N,Δ) of an STS (Y,Θ,Δ) is called a soft neighborhood of a soft point y_λ provided that there exists $(G,\Delta)\in\Theta$ such that $y_\lambda\in(G,\Delta)\widetilde{\subseteq}(N,\Delta)$.*

Definition 10 ([42]). *Let $\mathcal{F}\widetilde{\subseteq}S(Y_\Delta)$ and let $\{(Y,\Theta_i,\Delta):i\in I\}$ be an indexed family of STSs on Y with an arbitrary index I such that $\mathcal{F}\widetilde{\subseteq}\Theta_i$. Then, $\widetilde{\bigcap}_{i\in I}\Theta_i$ is called the soft topology on Y generated by $\mathcal{F}$.*

Definition 11 ([9]). *The soft closure of a soft set (B,Δ) in a STS (Y,Θ,Δ) is defined by*

$$Cl(B,\Delta)=\widetilde{\bigcap}\{(F,\Delta):(B,\Delta)\widetilde{\subseteq}(F,\Delta),(F,\Delta)\in\Theta^c\}.$$

Definition 12 ([43]). *A mapping $c:S(Y_\Delta)\to S(Y_\Delta)$ is called a soft (Kuratowski) closure operator on X if it meets the following conditions for any $(F,\Delta),(G,\Delta)\in S(Y_\Delta)$:*

(C1) $c(\widetilde{\phi})=\widetilde{\phi}$.

(C2) $(F,\Delta)\widetilde{\subseteq}c(F,\Delta)$.

(C3) $c(c(F,\Delta))=c(F,\Delta)$.

(C4) $c\big((F,\Delta)\widetilde{\cup}(G,\Delta)\big)=c(F,\Delta)\widetilde{\cup}c(G,\Delta)$.

Definition 13 ([44]). *A subfamily $\mathcal{I}$ of $S(Y_\Delta)$ is said to be a soft ideal on Y if it obeys the following postulates:*

(i) *If $(F,\Delta),(G,\Delta)\in\mathcal{I}$, then $(F,\Delta)\widetilde{\cup}(G,\Delta)\in\mathcal{I}$.*

(ii) *If $(G,\Delta)\in\mathcal{I}$ and $(F,\Delta)\widetilde{\subseteq}(G,\Delta)$, then $(F,\Delta)\in\mathcal{I}$.*

Definition 14 ([32]). *A subfamily $\mathcal{F}$ of $S(Y_\Delta)$ is said to be a soft grill on Y if it satisfies the following postulates:*

(i) $\widetilde{\phi}\notin\mathcal{F}$.

(ii) *If $(G,\Delta)\in\mathcal{F}$ and $(G,\Delta)\widetilde{\subseteq}(H,\Delta)$, then $(H,\Delta)\in\mathcal{F}$. That is, $\mathcal{F}$ is closed under soft superset relation.*

(iii) *If $(G,\Delta)\widetilde{\cup}(H,\Delta)\in\mathcal{F}$, then $(G,\Delta)\in\mathcal{F}$ or $(H,\Delta)\in\mathcal{F}$.*

3. Soft Primal

This segment is allocated to display a novel structure in soft settings, namely soft primal. The basic characteristics of this structure are demonstrated, and its behavior under soft functions is described with an elucidative instance.

Definition 15. *A subfamily $\mathcal{F}$ of $S(Y_\Delta)$ is said to be a soft primal on Y if it satisfies the following postulates:*

(i) $\widetilde{Y}\notin\mathcal{F}$.

(ii) *If $(G,\Delta)\in\mathcal{F}$ and $(H,\Delta)\widetilde{\subseteq}(G,\Delta)$, then $(H,\Delta)\in\mathcal{F}$. That is, $\mathcal{F}$ is closed under soft subset relation.*

(iii) *If $(G,\Delta)\widetilde{\cap}(H,\Delta)\in\mathcal{F}$, then $(G,\Delta)\in\mathcal{F}$ or $(H,\Delta)\in\mathcal{F}$.*

The next result is easy to prove.

Proposition 2. *A subfamily $\mathcal{F}$ of $S(Y_\Delta)$ is a soft primal on Y if and only if the following conditions are satisfied.*

(i) $\widetilde{Y}\notin\mathcal{F}$.

(ii) *If $(G,\Delta)\notin\mathcal{F}$ and $(G,\Delta)\widetilde{\subseteq}(H,\Delta)$, then $(H,\Delta)\notin\mathcal{F}$.*

(iii) *If $(G,\Delta)\notin\mathcal{F}$ and $(H,\Delta)\notin\mathcal{F}$, then $(G,\Delta)\widetilde{\cap}(H,\Delta)\notin\mathcal{F}$.*

Theorem 1. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two soft primals on Y, then $\mathcal{F}_1 \cup \mathcal{F}_2$ is a soft primal on Y.*

Proof. First, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two soft primals on Y. Then $\widetilde{Y} \notin \mathcal{F}_1$ and $\widetilde{Y} \notin \mathcal{F}_2$. So that $\widetilde{Y} \notin \mathcal{F}_1 \cup \mathcal{F}_2$. Second, suppose that $(V,\Delta) \in \mathcal{F}_1 \cup \mathcal{F}_2$ and let $(U,\Delta)\widetilde{\subseteq}(V,\Delta)$. Then, $(V,\Delta) \in \mathcal{F}_1$ or $(V,\Delta) \in \mathcal{F}_2$. This automatically leads to that $(U,\Delta) \in \mathcal{F}_1$ or $(U,\Delta) \in \mathcal{F}_2$. So $(U,\Delta) \in \mathcal{F}_1 \cup \mathcal{F}_2$. Third, let $(U,\Delta),(V,\Delta)$ be soft subsets such that $(U,\Delta)\widetilde{\cap}(V,\Delta) \in \mathcal{F}_1 \cup \mathcal{F}_2$. Then, $(U,\Delta)\widetilde{\cap}(V,\Delta) \in \mathcal{F}_1$ or $(U,\Delta)\widetilde{\cap}(V,\Delta) \in \mathcal{F}_2$. This implies that $(U,\Delta) \in \mathcal{F}_1 \cup \mathcal{F}_2$ or $(V,\Delta) \in \mathcal{F}_1 \cup \mathcal{F}_2$, as required. □

The next example elaborates that the class of soft primals on a set Y is not closed under the intersection operator in general.

Example 1. *Let $Y = \{y\}$ and $\Delta = \{\delta_1, \delta_2\}$. Then $\mathcal{F}_1 = \{\widetilde{\phi}, \{(\delta_1,\emptyset),(\delta_2,\{y\})\}\}$ and $\mathcal{F}_2 = \{\widetilde{\phi}, \{(\delta_1,\{y\}),(\delta_2,\emptyset)\}\}$ are two soft primals on a set Y with Δ. Now, $\mathcal{F}_1 \cap \mathcal{F}_2 = \{\widetilde{\phi}\}$ is not a soft primal because $\{(\delta_1,\emptyset),(\delta_2,\{y\})\}\widetilde{\cap}\{(\delta_1,\{y\}),(\delta_2,\emptyset)\} = \widetilde{\phi} \in \mathcal{F}_1 \cap \mathcal{F}_2$. But neither $\{(\delta_1,\emptyset),(\delta_2,\{y\})\} \in \mathcal{F}_1 \cap \mathcal{F}_2$ nor $\{(\delta_1,\{y\}),(\delta_2,\emptyset)\} \in \mathcal{F}_1 \cap \mathcal{F}_2$.*

Theorem 2. *If $\mathcal{G}$ is a soft grill on Y, then the family $\mathcal{F} = \{(H,\Delta) : (H^c,\Delta) \in \mathcal{G}\}$ is a soft primal on Y.*

Proof. First, it is obvious that $\widetilde{\phi} \notin \mathcal{G}$, so $\widetilde{Y} \notin \mathcal{F}$. Second, let $(V,\Delta) \in \mathcal{F}$, and take any soft subset (U,Δ) of (V,Δ). By the way of building $\mathcal{F}$, we have $(V^c,\Delta) \in \mathcal{G}$. Since $(V^c,\Delta)\widetilde{\subseteq}(U^c,\Delta)$, it follows from the definition of the soft grill that $(U^c,\Delta) \in \mathcal{G}$. This automatically means that $(U,\Delta) \in \mathcal{F}$. Third, let $(U,\Delta),(V,\Delta)$ be soft subsets such that $(U,\Delta)\widetilde{\cap}(V,\Delta) \in \mathcal{F}$. Then, $(U^c,\Delta)\widetilde{\cup}(V^c,\Delta) \in \mathcal{G}$. Therefore, $(U^c,\Delta) \in \mathcal{G}$ or $(V^c,\Delta) \in \mathcal{G}$. Thus, $(U,\Delta) \in \mathcal{F}$ or $(V,\Delta) \in \mathcal{F}$. Hence, we get the desired result. □

Corollary 1. *If $\mathcal{G}$ is a soft primal on Y, then the family $\mathcal{F} = \{(H,\Delta) : (H^c,\Delta) \in \mathcal{G}\}$ is a soft grill on Y.*

Now, we discuss the condition under which primal structures navigate between soft and classical settings.

First, we provide the next example to elucidate that $\mathcal{F}_\delta = \{F(\delta) : (F,\Delta) \in \mathcal{F}\}$, inspired by a soft primal $\mathcal{F}$, does not institute a (crisp) primal for any fixed parameter $\delta \in \Delta$.

Example 2. *Let $Y = \{y,z\}$ and $\Delta = \{\delta_1, \delta_2\}$. Consider the following soft sets:*
$(F_1,\Delta) = \{(\delta_1,\emptyset),(\delta_2,\{y\})\}$;
$(F_2,\Delta) = \{(\delta_1,\emptyset),(\delta_2,\{z\})\}$;
$(F_3,\Delta) = \{(\delta_1,\{y\}),(\delta_2,\emptyset)\}$;
$(F_4,\Delta) = \{(\delta_1,\{z\}),(\delta_2,\emptyset)\}$;
$(F_5,\Delta) = \{(\delta_1,\{z\}),(\delta_2,\{z\})\}$;
$(F_6,\Delta) = \{(\delta_1,\{y\}),(\delta_2,\{z\})\}$;
$(F_7,\Delta) = \{(\delta_1,Y),(\delta_2,\emptyset)\}$;
$(F_8,\Delta) = \{(\delta_1,\emptyset),(\delta_2,Y)\}$;
$(F_9,\Delta) = \{(\delta_1,Y),(\delta_2,\{z\})\}$; *and*
$(F_{10},\Delta) = \{(\delta_1,\{z\}),(\delta_2,\{y\})\}$.
Then, $\mathcal{F} = \{\widetilde{\phi}, (F_i,\Delta) : i = 1,2,\ldots,9\}$ is a soft primal on a set Y with Δ. We obtain $\mathcal{F}_{\delta_1} = \mathcal{F}_{\delta_2} = 2^Y$. Obviously, $\mathcal{F}_{\delta_1}$ and $\mathcal{F}_{\delta_2}$ are not (crisp) primal because Y belongs to both of them.

Theorem 3. *Let $\mathcal{F}$ be a soft primal on a set Y with a set of parameters Ω. Then,*

$$\mathcal{F}_\delta = \{F(\delta) : (F,\Delta) \in \mathcal{F}\} \setminus \{Y\}$$

is a (crisp) primal on Y for any fixed parameter $\delta \in \Delta$.

Proof. It is clear that $Y \notin \mathcal{F}_\delta$. Let $A \in \mathcal{F}_\delta$ and take any subset B of A. Then, there exists a soft subset (G, Δ) in $\mathcal{F}$ such that $G(\delta) = A$. Now, a soft set (H, Δ), given by $H(\delta) = B$ and $H(\delta^*) = \emptyset$ for each $\delta^* \neq \delta$, is a soft subset of (G, Δ), so $(H, \Delta) \in \mathcal{F}$. This means that $H(\delta) = B \in \mathcal{F}_\delta$. Thus, $\mathcal{F}_\delta$ is closed under a subset relation. Finally, let $A \cap B \in \mathcal{F}_\delta$. Then, there exists a soft subset (W, Δ) in $\mathcal{F}$ such that $W(\delta) = A \cap B$. Note that there exist soft subsets (U, Δ) and (V, Δ) such that $U(\delta) = A$, $U(\delta) = B$ and $U(\delta^*) = V(\delta^*) = W(\delta^*)$ for each $\delta^* \neq \delta$. That is, $(W, \Delta) = (U, \Delta)\widetilde{\cap}(V, \Delta)$, so $(U, \Delta) \in \mathcal{F}$ or $(V, \Delta) \in \mathcal{F}$. This automatically leads to $A \in \mathcal{F}_\delta$ or $B \in \mathcal{F}_\delta$. Hence, the proof is complete. □

We close this section by showing how the soft primal behaves under soft mappings.

Remark 1. *Let $h_\pi : S(Y_\Delta) \to S(Z_\Omega)$ be a soft mapping and $\mathcal{G}$ be a soft primal on Z with a set of parameters Ω. The class $\{h_\pi^{-1}(G, \Omega) : (G, \Omega) \in \mathcal{G}\}$ need not be a soft primal on Y with a set of parameters Δ in general. The next example confirms this fact.*

Example 3. *Let $Y = \{y_1, y_2\}$ with a set of parameters $\Delta = \{\delta_1, \delta_2\}$, and $Z = \{z_1, z_2\}$ with a set of parameters $\Omega = \{\omega_1, \omega_2\}$. Now consider the mappings $h : Y \to Z$ and $\pi : \Delta \to \Omega$ are defined as follows*

$$h(y) = z_1 \text{ for each } y \in Y; \text{ and } \pi(\delta_i) = \omega_i \text{ for each } \delta_i \in \Delta$$

Let $\mathcal{G} = S(Z_\Omega) \setminus \{\widetilde{Z}\}$ be a soft primal on Z with Ω. Now, $(F, \Omega) = \{(\omega_1, \{z_1\}), (\omega_2, \{z_1\})\}$ be a soft set in $\mathcal{G}$, so $\{h_\pi^{-1}(G, \Omega) : (G, \Omega) \in \mathcal{G}\}$ is not a soft primal on Y with Δ because $h_\pi^{-1}(F, \Omega) = \widetilde{Y}$, which is not a member of any soft primal on Y with Δ.

Theorem 4. *Let $h_\pi : S(Y_\Delta) \to S(Z_\Omega)$ be an injective soft mapping. If $\mathcal{G}$ is a soft primal on Z with a set of parameters Ω, then the class $\Psi = \{h_\pi^{-1}(G, \Omega) : (G, \Omega) \in \mathcal{G}\} \setminus \{\widetilde{Y}\}$ is a soft primal on Y with a set of parameters Δ.*

Proof. According to the way of building Ψ, $\widetilde{Y} \notin \Psi$, let (W, Δ) be a non-null soft set in Ψ. Then, there exits a soft subset (G, Ω) in $\mathcal{G}$ such that $(W, \Delta) = h_\pi^{-1}(G, \Omega) \neq \widetilde{Y}$. Take any soft subset (U, Δ) of $h_\pi^{-1}(G, \Omega)$, we obtain $h_\pi(U, \Delta)\widetilde{\subseteq}h_\pi(h_\pi^{-1}(G, \Omega))\widetilde{\subseteq}(G, \Omega)$. This means that $h_\pi(U, \Delta) \in \mathcal{G}$, and by the injectiveness of h_π, we get $h_\pi^{-1}(h_\pi(U, \Delta)) = (U, \Delta)$, i.e., (U, Δ) is a proper soft subset of $\widetilde{Y}$. Thus, $(U, \Delta) \in \Psi$, which means that Ψ is closed under subset relation. Finally, let $(U, \Delta)\widetilde{\cap}(V, \Delta)$ be an element of Ψ. This implies that there exists $(H, \Omega) \in \mathcal{G}$ such that $(U, \Delta)\widetilde{\cap}(V, \Delta) = h_\pi^{-1}(H, \Omega) \neq \widetilde{Y}$. By the injectiveness of h_π, we get $h_\pi(U, \Delta)\widetilde{\cap}h_\pi(V, \Delta) = h_\pi(h_\pi^{-1}(H_1, \Omega))\widetilde{\subseteq}(H, \Omega)$, which automatically means that $h_\pi(U, \Delta)\widetilde{\cap}h_\pi(V, \Delta) \in \mathcal{G}$. By the third condition of the soft primal, $h_\pi(U, \Delta) \in \mathcal{G}$ or $h_\pi(V, \Delta) \in \mathcal{G}$. Again, by the injectiveness of h_π we get $h_\pi^{-1}(h_\pi(U, \Delta)) = (U, \Delta) \neq \widetilde{Y}$ or $h_\pi^{-1}(h_\pi(V, \Delta)) = (V, \Delta) \neq \widetilde{Y}$. Thus, $h_\pi^{-1}(h_\pi(U, \Delta)) \in \Psi$ or $h_\pi^{-1}(h_\pi(V, \Delta)) \in \Psi$, which ends the proof. □

Corollary 2. *Let $h_\pi : S(Y_\Delta) \to S(Z_\Omega)$ be a bijective soft mapping. If $\mathcal{G}$ is a soft primal on Z with a set of parameters Ω, then the class $\Psi = \{h_\pi^{-1}(G, \Omega) : (G, \Omega) \in \mathcal{G}\}$ is a soft primal on Y with a set of parameters Δ.*

Proof. Since h_π is surjective, $h_\pi^{-1}(H, \Omega) = \widetilde{Y}$ iff $(H, \Omega) = \widetilde{Z}$. This means that $\widetilde{Z} \notin \mathcal{G}$, so $\widetilde{Y} \notin \Psi$. The second and third stipulations of the soft primal are derived following a similar argument to the above proof. □

It is easy to prove the proposition below.

Proposition 3. *Let $h_\pi : S(Y_\Delta) \to S(Z_\Omega)$ be a bijective soft mapping. If $\mathcal{G}$ is a soft primal on Y with a set of parameters Δ, then the class $\Psi = \{h_\pi(G, \Omega) : (G, \Omega) \in \mathcal{G}\}$ is a soft primal on Z with a set of parameters Ω.*

4. Primal Soft Topology

Herein, we first initiate the concept of primal soft topological spaces. Then, we define a soft operator $(\cdot)^{\diamond}$ using the elements of the soft topology and soft primal. We scrutinize its essential characterizations and infer some of its relationships associated with soft topological closure operators. Afterwards, we introduce a soft operator $Cl^{\diamond}$ inspired by the previous soft operator $(\cdot)^{\diamond}$ and apply this to produce a new soft topology (called primal soft topology) finer than the original soft topology. Finally, we display a soft operator $(\cdot)^{\square}$ and elucidate its essential features.

Definition 16. *The quadruple $(Y, \Theta, \Delta, \mathcal{F})$ is said to be a primal soft topological space (briefly, PSTS), where (Y, Θ, Δ) is a soft topological space and $\mathcal{F}$ is a soft primal on Y.*

Definition 17. *Let $(Y, \Theta, \Delta, \mathcal{F})$ be a PSTS. Then, a soft mapping $(\cdot)^{\diamond} : S(Y_{\Delta}) \to S(Y_{\Delta})$ is defined as follows, $(G, \Delta)^{\diamond}(Y, \Theta, \Delta, \mathcal{F}) = \{y_{\lambda} \in \widetilde{Y} : (G^c, \Delta)\widetilde{\cup}(U^c, \Delta) \in \mathcal{F}$ for each $(U, \Delta) \in \Theta(y_{\lambda})\}$ for each soft subset (G, Δ). In brief, we write $(G, \Delta)^{\diamond}$ or $(G, \Delta)^{\diamond}_{\mathcal{F}}$ instead of $(G, \Delta)^{\diamond}(Y, \Theta, \Delta, \mathcal{F})$.*

The next example elucidates that the properties $(G, \Delta)^{\diamond}\widetilde{\subseteq}(G, \Delta)$ and $(G, \Delta)\widetilde{\subseteq}(G, \Delta)^{\diamond}$ are false in general.

Example 4. *Take a soft primal $\mathcal{F}$ displayed in Example 2 and let $\Theta = \{\widetilde{\phi}, \widetilde{Y}, (F_6, \Delta), (F_{10}, \Delta)\}$ be a soft topology on a set Y with Δ. One can check that $(F_6, \Delta)\widetilde{\nsubseteq}(F_6, \Delta)^{\diamond} = \widetilde{\phi}$. On the other hand, $(F_1, \Delta)^{\diamond} = (F_{10}, \Delta)\widetilde{\nsubseteq}(F_1, \Delta)$.*

In the following theorem, we provide the main properties of a soft mapping $(\cdot)^{\diamond}$, which will be helpful to prove some results given later.

Theorem 5. *Let (F, Δ) and (G, Δ) be soft subsets of a PSTS $(Y, \Theta, \Delta, \mathcal{F})$. Then, the next statements hold true.*

(i) $\widetilde{\phi}^{\diamond} = \widetilde{\phi}$.

(ii) $Cl((F, \Delta)^{\diamond}) = (F, \Delta)^{\diamond}$.

(iii) *If $(F^c\Delta) \notin \mathcal{F}$, then $(F, \Delta)^{\diamond} = \widetilde{\phi}$.*

(iv) *If $(F, \Delta) \in \Theta^c$, then $(F, \Delta)^{\diamond}\widetilde{\subseteq}(F, \Delta)$.*

(v) *If $(F, \Delta)\widetilde{\subseteq}(G, \Delta)$, then $(F, \Delta)^{\diamond}\widetilde{\subseteq}(G, \Delta)^{\diamond}$.*

(vi) $((F, \Delta)^{\diamond})^{\diamond}\widetilde{\subseteq}(F, \Delta)^{\diamond}$.

(vii) $[(F, \Delta)\widetilde{\cup}(G, \Delta)]^{\diamond} = (F, \Delta)^{\diamond}\widetilde{\cup}(G, \Delta)^{\diamond}$.

(viii) $[(F, \Delta)\widetilde{\cap}(G, \Delta)]^{\diamond}\widetilde{\subseteq}(F, \Delta)^{\diamond}\widetilde{\cap}(G, \Delta)^{\diamond}$.

Proof.

(i) Since $\widetilde{\phi}^c\widetilde{\cup}(F, \Delta) = \widetilde{Y}$ for any soft set (F, Δ) and $\widetilde{Y} \notin \mathcal{F}$, so $\widetilde{\phi}^{\diamond}$ shall be null.

(ii) If $y_{\lambda} \in Cl((F, \Delta)^{\diamond})$ and any $(H, \Delta) \in \Theta(y_{\lambda})$, then $(F, \Delta)^{\diamond}\widetilde{\cap}(H, \Delta) \neq \widetilde{\phi}$. One find $z_{\mu} \in \widetilde{Y}$ such that $z_{\mu} \in (F, \Delta)^{\diamond}\widetilde{\cap}(H, \Delta)$. Therefore, $(F^c, \Delta)\widetilde{\cup}(W^c, \Delta) \in \mathcal{F}$ for all $(W, \Delta) \in \Theta(z_{\mu})$. This means that $(F^c, \Delta)\widetilde{\cup}(H^c, \Delta) \in \mathcal{F}$ and so $y_{\lambda} \in (F, \Delta)^{\diamond}$. Hence, $Cl((F, \Delta)^{\diamond})\widetilde{\subseteq}(F, \Delta)^{\diamond}$. The reverse of the inclusion is always true. Thus, $Cl((F, \Delta)^{\diamond}) = (F, \Delta)^{\diamond}$.

(iii) Suppose otherwise that there exists $y_{\lambda} \in \widetilde{Y}$ such that $y_{\lambda} \in (F, \Delta)^{\diamond}$. Then, $(F^c, \Delta)\widetilde{\cup}(H^c, \Delta) \in \mathcal{F}$ for each $(H, \Delta) \in \Theta$. However, since $(F^c\Delta) \notin \mathcal{F}$, by Proposition 2, $(F^c, \Delta)\widetilde{\cup}(H^c, \Delta) \in \mathcal{F}$ for some $(H, \Delta) \in \Theta$—a contradiction. Thus, $(F, \Delta)^{\diamond} = \widetilde{\phi}$.

(iv) Suppose that $(F, \Delta) \in \Theta^c$. Let $y_{\lambda} \in (F, \Delta)^{\diamond}$. Assume $y_{\lambda} \notin (F, \Delta)$. Then $(F^c, \Delta) \in \Theta(y_{\lambda})$. Since $y_{\lambda} \in (F, \Delta)^{\diamond}$, so $(F^c, \Delta)\widetilde{\cup}(H^c, \Delta) \in \mathcal{F}$ for each $(H, \Delta) \in \Theta(y_{\lambda})$. This concludes that $\widetilde{Y} = (F, \Delta)\widetilde{\cup}(F^c, \Delta) = ((F^c)^c, \Delta)\widetilde{\cup}(F^c, \Delta) \in \mathcal{F}$, a contradiction. Thus, $(F, \Delta)^{\diamond}\widetilde{\subseteq}(F, \Delta)$.

(v) Assume that $(F,\Delta)\widetilde{\subseteq}(G,\Delta)$. If $y_\lambda \in (F,\Delta)^\diamond$, then $(F^c,\Delta)\widetilde{\cup}(H^c,\Delta) \in \mathcal{F}$ for all $(H,\Delta) \in \Theta(y_\lambda)$. Since $(F,\Delta)\widetilde{\subseteq}(G,\Delta)$, so $(G^c,\Delta)\widetilde{\cup}(H^c,\Delta) \in \mathcal{F}$. Hence, $y_\lambda \in (G,\Delta)^\diamond$ and thus, $(F,\Delta)^\diamond\widetilde{\subseteq}(G,\Delta)^\diamond$.

(vi) By (ii), $(F,\Delta)^\diamond \in \Theta^c$, so $((F,\Delta)^\diamond)^c \in \Theta$. Therefore, $((F,\Delta)^\diamond)^\diamond\widetilde{\subseteq}(F,\Delta)^\diamond$.

(vii) Since $(F,\Delta)\widetilde{\subseteq}(F,\Delta)\widetilde{\cup}(G,\Delta)$ and $(G,\Delta)\widetilde{\subseteq}(F,\Delta)\widetilde{\cup}(G,\Delta)$, then, by (iv), $(F,\Delta)^\diamond\widetilde{\subseteq}[(F,\Delta)\widetilde{\cup}(G,\Delta)]^\diamond$ and $(G,\Delta)^\diamond\widetilde{\subseteq}[(F,\Delta)\widetilde{\cup}(G,\Delta)]^\diamond$. It follows that $(F,\Delta)^\diamond\widetilde{\cup}(G,\Delta)^\diamond\widetilde{\subseteq}[(F,\Delta)\widetilde{\cup}(G,\Delta)]^\diamond$. For the converse of the inclusion, if $y_\lambda \notin (F,\Delta)^\diamond\widetilde{\cup}(G,\Delta)^\diamond$, then $y_\lambda \notin (F,\Delta)^\diamond$ and $y_\lambda \notin (G,\Delta)^\diamond$. This implies that there exist $(H,\Delta),(W,\Delta) \in \Theta(y_\lambda)$ such that $(F^c,\Delta)\widetilde{\cup}(H^c,\Delta) \notin \mathcal{F}$ and $(G^c,\Delta)\widetilde{\cup}(W^c,\Delta) \notin \mathcal{F}$. Set $(R,\Delta) = (H,\Delta)\widetilde{\cap}(W,\Delta)$. Then $(R,\Delta) \in \Theta(y_\lambda)$ for which $(F^c,\Delta)\widetilde{\cup}(R^c,\Delta) \notin \mathcal{F}$ and $(G^c,\Delta)\widetilde{\cup}(R^c,\Delta) \notin \mathcal{F}$ (from Proposition 2). Since $\mathcal{F}$ is soft primal, we get that $[(F,\Delta)\widetilde{\cup}(G,\Delta)]^c\widetilde{\cup}(R^c,\Delta) = (F^c,\Delta)\widetilde{\cap}(G^c,\Delta)\ \widetilde{\cup}(R^c,\Delta) = (F^c,\Delta)\widetilde{\cup}(R^c,\Delta)\widetilde{\bigcap}(G^c,\Delta)\widetilde{\cup}(R^c,\Delta) \notin \mathcal{F}$. Thus, $y_\lambda \notin [(F,\Delta)\widetilde{\cup}(G,\Delta)]^\diamond$. Consequently, $[(F,\Delta)\widetilde{\cup}(G,\Delta)]^\diamond = (F,\Delta)^\diamond\widetilde{\cup}(G,\Delta)^\diamond$.

(viii) Since $(F,\Delta)\widetilde{\cap}(G,\Delta)\widetilde{\subseteq}(F,\Delta)$ and $(F,\Delta)\widetilde{\cap}(G,\Delta)\widetilde{\subseteq}(G,\Delta)$, then, by (iv), $[(F,\Delta)\widetilde{\cap}\,(G,\Delta)]^\diamond\widetilde{\subseteq}(F,\Delta)^\diamond$ and $[(F,\Delta)\widetilde{\cap}(G,\Delta)]^\diamond\widetilde{\subseteq}(G,\Delta)^\diamond$. Therefore, $[(F,\Delta)\widetilde{\cap}(G,\Delta)]^\diamond\widetilde{\subseteq}(F,\Delta)^\diamond\widetilde{\cap}(G,\Delta)^\diamond$.

□

As illustrated below, it may not always be possible to achieve the equality of (viii) in Theorem 5.

Example 5. *Consider the $Y,\Delta,\mathcal{F},(F_1,\Delta)$, and (F_9,Δ) given in Example 2. Let $\Theta = \{\widetilde{\phi},\widetilde{Y}\}$. Obviously, $[(F_1,\Delta)\widetilde{\cap}(F_9,\Delta)]^\diamond = \widetilde{\phi}$. On the other hand, $(F_1,\Delta)^\diamond\widetilde{\cap}(F_9,\Delta)^\diamond = \widetilde{Y}$.*

Theorem 6. *Let (F,Δ) and (G,Δ) be soft subsets of a PSTS $(Y,\Theta,\Delta,\mathcal{F})$ such that (F,Δ) is soft open. Then, $(F,\Delta)\widetilde{\cap}(G,\Delta)^\diamond\widetilde{\subseteq}[(F,\Delta)\widetilde{\cap}(G,\Delta)]^\diamond$.*

Proof. Given $(F,\Delta),(G,\Delta) \in S(Y_\Delta)$ such that $(F,\Delta) \in \Theta$. If $y_\lambda \in (F,\Delta)\widetilde{\cap}(G,\Delta)^\diamond$, then $y_\lambda \in (F,\Delta)$ and $y_\lambda \in (G,\Delta)^\diamond$, and so $(G^c,\Delta)\widetilde{\cup}(H^c,\Delta) \in \mathcal{F}$ for all $(H,\Delta) \in \Theta(y_\lambda)$. Since $(F,\Delta) \in \Theta(y_\lambda)$, then $(G^c,\Delta)\widetilde{\cup}[(H,\Delta)\widetilde{\cap}(F,\Delta)]^c \in \mathcal{F}$. However, $(G^c,\Delta)\widetilde{\cup}[(H,\Delta)\widetilde{\cap}(F,\Delta)]^c = [(F,\Delta)\widetilde{\cap}(G,\Delta)]^c\widetilde{\cup}(H^c,\Delta)$, so it implies that $y_\lambda \in [(F,\Delta)\widetilde{\cap}(G,\Delta)]^\diamond$. Hence, $(F,\Delta)\widetilde{\cap}(G,\Delta)^\diamond\widetilde{\subseteq}[(F,\Delta)\widetilde{\cap}(G,\Delta)]^\diamond$. □

Theorem 7. *Let $(Y,\Theta,\Delta,\mathcal{F})$ be a PSTS. If $\Theta^c\widetilde{\subseteq}\mathcal{F}$, then $(G,\Delta)\widetilde{\subseteq}(G,\Delta)^\diamond$ for each $(G,\Delta) \in \Theta$.*

Proof. Since $\widetilde{\phi}^\diamond = \widetilde{\phi}$, clearly $\widetilde{\phi}\widetilde{\subseteq}\widetilde{\phi}^\diamond$. Next, we need to find $\widetilde{Y}^\diamond$. Since $\Theta^c\widetilde{\subseteq}\mathcal{F}$, then we must have $\widetilde{Y} = \widetilde{Y}^\diamond$. Indeed, if for some $y_\lambda \in \widetilde{Y}, y_\lambda \notin \widetilde{Y}^\diamond$. Therefore, there exists $(H,\Delta) \in \Theta(x_\lambda)$ such that $(H^c,\Delta) = (H^c,\Delta)\widetilde{\cup}\widetilde{Y}^c \notin \mathcal{F}$, a contradiction. If $(G,\Delta) \in \Theta$, by Theorem 6, $(G,\Delta) = (G,\Delta)\widetilde{\cap}\widetilde{Y}^\diamond\widetilde{\subseteq}[(G,\Delta)\widetilde{\cap}\widetilde{Y}]^\diamond = (G,\Delta)^\diamond$. This leads to the result. □

Theorem 8. *Let (F,Δ) and (G,Δ) be soft subsets of a PSTS $(Y,\Theta,\Delta,\mathcal{F})$. Then, $(F,\Delta)^\diamond \setminus (G,\Delta)^\diamond = [(F,\Delta) \setminus (G,\Delta)]^\diamond \setminus (G,\Delta)^\diamond$.*

Proof. Consider the decomposition

$$(F,\Delta) = [(F,\Delta) \setminus (G,\Delta)]\widetilde{\bigcup}[(F,\Delta)\widetilde{\cap}(G,\Delta)].$$

Then applying Theorem 5 (v) and (vii) to it, we obtain

$$\begin{aligned}(F,\Delta)^\diamond &= [(F,\Delta) \setminus (G,\Delta)]^\diamond\widetilde{\bigcup}[(F,\Delta)\widetilde{\cap}(G,\Delta)]^\diamond \\ &\widetilde{\subseteq} [(F,\Delta) \setminus (G,\Delta)]^\diamond\widetilde{\bigcup}(G,\Delta)^\diamond.\end{aligned}$$

Therefore, $(F,\Delta)^\diamond \setminus (G,\Delta)^\diamond\widetilde{\subseteq}[(F,\Delta) \setminus (G,\Delta)]^\diamond \setminus (G,\Delta)^\diamond$.

On the other hand, since $(F,\Delta)\setminus(G,\Delta)\widetilde{\subseteq}(F,\Delta)$, by Theorem 5 (v), $[(F,\Delta)\setminus(G,\Delta)]^{\diamond}\widetilde{\subseteq}(F,\Delta)^{\diamond}$ and so, $[(F,\Delta)\setminus(G,\Delta)]^{\diamond}\setminus(G,\Delta)^{\diamond}\widetilde{\subseteq}(F,\Delta)^{\diamond}\setminus(G,\Delta)^{\diamond}$. Summing up the obtained inclusions, we get $(F,\Delta)^{\diamond}\setminus(G,\Delta)^{\diamond}=[(F,\Delta)\setminus(G,\Delta)]^{\diamond}\setminus(G,\Delta)^{\diamond}$. □

Theorem 9. *Let (F,Δ) and (G,Δ) be soft subsets of a PSTS $(Y,\Theta,\Delta,\mathcal{F})$ such that $(G^c\Delta)\notin\mathcal{F}$. Then, $[(F,\Delta)\widetilde{\cup}(G,\Delta)]^{\diamond}=(F,\Delta)^{\diamond}=[(F,\Delta)\setminus(G,\Delta)]^{\diamond}$.*

Proof. This follows from Theorems 5 and 8. □

Definition 18. *Let $(Y,\Theta,\Delta,\mathcal{F})$ be a PSTS. Then, a soft mapping $Cl^{\diamond}:S(Y_{\Delta})\to S(Y_{\Delta})$ is defined as follows $Cl^{\diamond}(G,\Delta)=(G,\Delta)\widetilde{\cup}(G,\Delta)^{\diamond}$ for any soft subset (G,Δ).*

Theorem 10. *Let (F,Δ) and (G,Δ) be soft subsets of a PSTS $(Y,\Theta,\Delta,\mathcal{F})$. Then, the next statements hold true.*

(i) $Cl^{\diamond}(\widetilde{\phi})=\widetilde{\phi}$.
(ii) $Cl^{\diamond}(\widetilde{Y})=\widetilde{Y}$.
(iii) $(F,\Delta)\widetilde{\subseteq}Cl^{\diamond}(F,\Delta)$.
(iv) *If $(F,\Delta)\widetilde{\subseteq}(G,\Delta)$, then $Cl^{\diamond}(F,\Delta)\widetilde{\subseteq}Cl^{\diamond}(G,\Delta)$.*
(v) $Cl^{\diamond}[(F,\Delta)\widetilde{\cup}(G,\Delta)]=Cl^{\diamond}(F,\Delta)\widetilde{\cup}Cl^{\diamond}(G,\Delta)$.
(vi) $Cl^{\diamond}(Cl^{\diamond}(F,\Delta)=Cl^{\diamond}(F,\Delta)$.

Proof.

(i) Since $\widetilde{\phi}^{\diamond}=\widetilde{\phi}$, so $Cl^{\diamond}(\widetilde{\phi})=\widetilde{\phi}\widetilde{\cup}\widetilde{\phi}^{\diamond}=\widetilde{\phi}$.
(ii) This is clear as $Cl^{\diamond}(\widetilde{Y})=\widetilde{Y}\widetilde{\cup}\widetilde{Y}^{\diamond}=\widetilde{Y}$.
(iii) This is also easy as $(F,\Delta)\widetilde{\subseteq}(F,\Delta)\widetilde{\cup}(F,\Delta)^{\diamond}=Cl^{\diamond}(F,\Delta)$.
(iv) Suppose $(F,\Delta),(G,\Delta)\in S(Y_{\Delta})$ with $(F,\Delta)\widetilde{\subseteq}(G,\Delta)$. By Theorem 5 (v), $(F,\Delta)^{\diamond}\widetilde{\subseteq}(G,\Delta)^{\diamond}$ and so $(F,\Delta)\widetilde{\cup}(F,\Delta)^{\diamond}\widetilde{\subseteq}(G,\Delta)\widetilde{\cup}(G,\Delta)^{\diamond}$. Thus, $Cl^{\diamond}(F,\Delta)\widetilde{\subseteq}Cl^{\diamond}(G,\Delta)$.
(v) By the same technique used in (iv) and applying Theorem 5 (vii), one can easily conclude that $Cl^{\diamond}[(F,\Delta)\widetilde{\cup}(G,\Delta)]=Cl^{\diamond}(F,\Delta)\widetilde{\cup}Cl^{\diamond}(G,\Delta)$.
(vi) To show that $Cl^{\diamond}(Cl^{\diamond}(F,\Delta)\widetilde{\subseteq}Cl^{\diamond}(F,\Delta)$, we implicitly use multiple statements of Theorem 5. Now,

$$\begin{aligned}
Cl^{\diamond}(Cl^{\diamond}(F,\Delta) &= Cl^{\diamond}(F,\Delta)\widetilde{\bigcup}(Cl^{\diamond}(F,\Delta))^{\diamond}\\
&= Cl^{\diamond}(F,\Delta)\widetilde{\bigcup}[(F,\Delta)\widetilde{\cup}(F,\Delta)^{\diamond}]^{\diamond}\\
&= Cl^{\diamond}(F,\Delta)\widetilde{\bigcup}(F,\Delta)^{\diamond}\widetilde{\bigcup}((F,\Delta)^{\diamond})^{\diamond}\\
&\widetilde{\subseteq} Cl^{\diamond}(F,\Delta)\widetilde{\bigcup}(F,\Delta)^{\diamond}\widetilde{\bigcup}(F,\Delta)^{\diamond}\\
&= Cl^{\diamond}(F,\Delta).
\end{aligned}$$

The converse is always true by using (iii). Therefore, $Cl^{\diamond}(Cl^{\diamond}(F,\Delta)=Cl^{\diamond}(F,\Delta)$. □

Theorem 11. *Let (F,Δ) be a soft subset of a PSTS $(Y,\Theta,\Delta,\mathcal{F})$. If $(F,\Delta)\widetilde{\subseteq}(F,\Delta)^{\diamond}$, then $Cl(F,\Delta)=Cl^{\diamond}(F,\Delta)=Cl((F,\Delta)^{\diamond})=(F,\Delta)^{\diamond}$.*

Proof. We first prove that $Cl(F,\Delta)=Cl^{\diamond}(F,\Delta)$. Since by Theorem 14, $\Theta\widetilde{\subseteq}\Theta^{\diamond}$, then $Cl^{\diamond}(F,\Delta)\widetilde{\subseteq}Cl(F,\Delta)$. Let $y_{\lambda}\notin Cl^{\diamond}(F,\Delta)$. Then, we can find $(H,\Delta)\in\Theta(y_{\lambda})$ and $(R,\Delta)\in\mathcal{F}$ containing y_{λ} such that $[(H,\Delta)\widetilde{\cap}(R,\Delta)]\widetilde{\cap}(F,\Delta)=\widetilde{\phi}$. Then, $[[(H,\Delta)\widetilde{\cap}(R,\Delta)]\widetilde{\cap}(F,\Delta)]^{\diamond}=\widetilde{\phi}$ and so, $[[(H,\Delta)\widetilde{\cap}(F,\Delta)]\setminus(R^c,\Delta)]^{\diamond}=\widetilde{\phi}$. By Theorem 9, $[(H,\Delta)\widetilde{\cap}(F,\Delta)]^{\diamond}=\widetilde{\phi}$, and hence by Theorem 6, $(H,\Delta)\widetilde{\cap}(F,\Delta)^{\diamond}=\widetilde{\phi}$. Since $(F,\Delta)\widetilde{\subseteq}(F,\Delta)^{\diamond}$, then $(H,\Delta)\widetilde{\cap}(F,\Delta)=\widetilde{\phi}$ implies

$y_\lambda \notin Cl(F,\Delta)$. This proves that $Cl(F,\Delta) = Cl^\diamond(F,\Delta)$. Then, Theorem 5 (ii) shows that $Cl((F,\Delta)^\diamond) = (F,\Delta)^\diamond$. Now, if $y_\lambda \notin Cl(F,\Delta)$, then one can find $(W,\Delta) \in \Theta(y_\lambda)$ such that $(F,\Delta)\widetilde{\cap}(W,\Delta) = \widetilde{\phi}$. This means that $[(F,\Delta)\widetilde{\cap}(W,\Delta)]^c = \widetilde{Y} \notin \mathcal{F}$ and thus, $y_\lambda \notin (F,\Delta)^\diamond$. Hence, $(F,\Delta)^\diamond \widetilde{\subseteq} Cl(F,\Delta)$ and so, $Cl((F,\Delta)^\diamond)\widetilde{\subseteq} Cl(Cl(F,\Delta)) = Cl(F,\Delta)$. On the other hand, since $(F,\Delta)\widetilde{\subseteq}(F,\Delta)^\diamond$, then $Cl(F,\Delta)\widetilde{\subseteq} Cl((F,\Delta)^\diamond)$. Therefore, $Cl(F,\Delta) = Cl((F,\Delta)^\diamond) = (F,\Delta)^\diamond$. □

Theorem 12. *Let $(Y,\Theta,\Delta,\mathcal{F})$ be a PSTS. Then, a soft mapping $Cl^\diamond : S(Y_\Delta) \to S(Y_\Delta)$ given by $Cl^\diamond(G,\Delta)^\diamond = (G,\Delta)\widetilde{\cup}(G,\Delta)^\diamond$ for any soft subset (G,Δ) is a Kuratowski's soft closure operator.*

Proof. Theorem 10 guarantees that $Cl^\diamond$ satisfies all the postulates in Definition 12. Thus, $Cl^\diamond$ is a Kuratowski's soft closure operator. □

Theorem 13. *Let $(Y,\Theta,\Delta,\mathcal{F})$ be a PSTS. Then, the family $\Theta^\diamond = \{(G,\Delta)\widetilde{\subseteq}\widetilde{Y} : Cl^\diamond(G^c,\Delta) = (G^c,\Delta)\}$ forms a soft topology on Y.*

Proof. It follows from Theorem 1 in [43]. □

Definition 19. *We call a soft topology $\Theta^\diamond$ produced by the above theorem a primal soft topology. If it is necessary, we write $\Theta^\diamond_{\mathcal{F}}$ instead of $\Theta^\diamond$.*

The following examples demonstrate that the set of primal topologies forms a natural class in the lattice of topologies over a universal set.

Example 6. *Let (Y,Θ_{ind},Δ) be the indiscrete soft topological space, where Y is any set containing more than one point and Δ is any set of parameters, and let $y_{\lambda_0} \in \widetilde{Y}$. Suppose $\mathcal{F} = \{(F,\Delta) : (F,\Delta) \in S(Y_\Delta), y_{\lambda_0} \notin (F,\Delta)\}$. Then, $\mathcal{F}$ meets all the axioms mentioned in Definition 15, so it is a soft primal. Given any $(R,\Delta) \in S(Y_\Delta)$. If $(R,\Delta) \in \mathcal{F}$, then $(R,\Delta)^\diamond = \widetilde{\phi}$. If $(R,\Delta) \notin \mathcal{F}$, then $(R,\Delta)^\diamond = \widetilde{Y}$. Therefore, each soft set excluding y_{λ_0} is a soft $\Theta^\diamond$-closed set together with $\widetilde{Y}$. Therefore, $\Theta^\diamond = \Theta_{inc}$, where $\Theta_{inc} = \{(G,\Delta) : (G,\Delta) \in S(Y_\Delta), y_{\lambda_0} \in (G,\Delta)\}\widetilde{\cup}\{\widetilde{\phi}\}$ (it is called included soft point topology in Example 2 in [42]).*

Example 7. *Let Δ be a set of parameters. If Θ is the soft topology on the set of real numbers $\mathbb{R}$ generated by*

$$\{((p,q),\Delta); p,q \in \mathbb{R}; p < q\}$$

, let $\mathcal{F}$ be the family of countable soft subsets of $\widetilde{\mathbb{R}}$. Obviously, $\mathcal{F}$ is a soft primal. For any soft set (R,Δ) over Y, if (R,Δ) is in $\mathcal{F}$ or not, then one can easily check that $(R,\Delta)^\diamond = \widetilde{\phi}$. Therefore, all soft subsets of $\widetilde{Y}$ are soft $\Theta^\diamond$-closed sets. Hence, $\Theta^\diamond = S(Y_\Delta)$ is the discrete soft topology.

Theorem 14. *Let $(Y,\Theta,\Delta,\mathcal{F})$ be a PSTS. Then, a primal soft topology $\Theta^\diamond$ is finer than a soft topology Θ.*

Proof. If $(G,\Delta) \in \Theta$, then $(G^c,\Delta) \in \Theta^c$. By Theorem 5 (iv), $(G^c,\Delta)^\diamond \widetilde{\subseteq} (G^c,\Delta)$. Therefore, $Cl^\diamond(G^c,\Delta) = (G^c,\Delta)\widetilde{\cup}(G^c,\Delta)^\diamond \widetilde{\subseteq} (G^c,\Delta)$. On the other hand, $(G^c,\Delta)\widetilde{\subseteq} Cl^\diamond(G^c,\Delta)$ is always correct. Consequently, $Cl^\diamond(G^c,\Delta) = (G^c,\Delta)$ and so, $(G,\Delta) \in \Theta^\diamond$. Hence, $\Theta\widetilde{\subseteq}\Theta^\diamond$. □

Theorem 15. *For any PSTS $(Y,\Theta,\Delta,\mathcal{F})$, the next results hold.*

(i) *If $\mathcal{F} = \widetilde{\phi}$, then $\Theta^\diamond = S(Y_\Delta)$.*

(ii) *If $\mathcal{F} = S(Y_\Delta) \setminus \{\widetilde{Y}\}$, then $\Theta = \Theta^\diamond$.*

Proof. **(i)** Let $\mathcal{F} = \widetilde{\phi}$. It suffices to show that $S(Y_\Delta)\widetilde{\subseteq}\Theta^\diamond$. Indeed, the converse is correct. Let $(G,\Delta) \in S(Y_\Delta)$. By assumption, $(G,\Delta)^\diamond = \widetilde{\phi}$ for all $(G,\Delta) \in S(Y_\Delta)$. Hence, $Cl^\diamond(G^c\Delta) = (G^c\Delta)$. By Theorem 13, $(G^c,\Delta) \in \Theta^\diamond$. This proves that $\Theta^\diamond = S(Y_\Delta)$.

(ii) We only need to show that $\Theta^{\diamond}\widetilde{\subseteq}\Theta$. The reverse of the inclusion follows from Theorem 14. Let $(G,\Delta)\in\Theta^{\diamond}$. Then $(G^c,\Delta)=Cl^{\diamond}(G^c,\Delta)=(G^c,\Delta)\widetilde{\cup}(G^c,\Delta)^{\diamond}$. Therefore, $(G^c,\Delta)^{\diamond}\widetilde{\subseteq}(G^c,\Delta)$. Let $y_\lambda\in\widetilde{Y}$ such that $y_\lambda\notin(G^c,\Delta)$. Clearly, $y_\lambda\notin(G^c,\Delta)^{\diamond}$ and so, there is a $(H,\Delta)\in\Theta(y_\lambda)$ such that $(H^c,\Delta)\widetilde{\cup}((G^c)^c,\Delta)=(H^c,\Delta)\widetilde{\cup}(G,\Delta)\notin\mathcal{F}$. Since $\mathcal{F}=S(Y_\Delta)\setminus\{\widetilde{Y}\}$, so $(H^c,\Delta)\widetilde{\cup}(G,\Delta)$ must equal $\widetilde{Y}$ and hence, $(H,\Delta)\widetilde{\cap}(G^c,\Delta)=\widetilde{\phi}$. This implies that $y_\lambda\notin Cl(G^c,\Delta)$. Therefore, we obtain that $Cl(G^c,\Delta)\widetilde{\subseteq}(G^c,\Delta)$. This proves that $(G^c,\Delta)\in\Theta^c$. Thus, $(G,\Delta)\in\Theta$ and so $\Theta^{\diamond}=\Theta$.
□

Theorem 16. *Let (G,Δ) be a soft subset of a PSTS $(Y,\Theta,\Delta,\mathcal{F})$. Then, the next results hold.*

(i) $(G,\Delta)\in\Theta^{\diamond}$ *iff for any* $y_\lambda\in(G,\Delta)$, *there exists* $(H,\Delta)\in\Theta(y_\lambda)$ *such that* $(H^c,\Delta)\widetilde{\cup}(G,\Delta)\notin\mathcal{F}$.

(ii) *If* $(G,\Delta)\notin\mathcal{F}$, *then* $(G,\Delta)\in\Theta^{\diamond}$.

Proof.

(i) If $(G,\Delta)\in\Theta^{\diamond}$, then $(G^c,\Delta)=Cl^{\diamond}(G^c,\Delta)=(G^c,\Delta)\widetilde{\cup}(G^c,\Delta)^{\diamond}$, and so $(G^c,\Delta)^{\diamond}\widetilde{\subseteq}(G^c,\Delta)$. Therefore, $(G,\Delta)\widetilde{\subseteq}((G^c,\Delta)^{\diamond})^c$. This means that for any $y_\lambda\in(G,\Delta)$, $y_\lambda\notin(G^c,\Delta)^{\diamond}$, and so there exists $(H,\Delta)\in\Theta(y_\lambda)$ such that $(H^c,\Delta)\widetilde{\cup}((G^c)^c,\Delta)=(H^c,\Delta)\widetilde{\cup}(G,\Delta)\notin\mathcal{F}$. The claim follows.

The converse can be concluded by reversing the above steps.

(ii) Let $(G,\Delta)\notin\mathcal{F}$ and let $y_\lambda\in(G,\Delta)$. Then, there exists always the soft open set $\widetilde{Y}$ containing y_λ such that $(G,\Delta)\widetilde{\cup}\widetilde{Y}^c=(G,\Delta)\notin\mathcal{F}$. By (i), we obtain that $(G,\Delta)\in\Theta^{\diamond}$.

□

Theorem 17. *Let $(Y,\Theta,\Delta,\mathcal{F})$ be a PSTS. Then, the family*

$$\mathcal{B}_{\mathcal{F}}=\{(G,\Delta)\widetilde{\cap}(F,\Delta):(G,\Delta)\in\Theta \text{ and } (F,\Delta)\notin\mathcal{F}\}$$

is a soft base for the primal soft topology $\Theta^{\diamond}$ on Y.

Proof. We first need to check that $\mathcal{B}_{\mathcal{F}}\widetilde{\subseteq}\Theta^{\diamond}$. If $(B,\Delta)\in\mathcal{B}_{\mathcal{F}}$, then $(B,\Delta)=(G,\Delta)\widetilde{\cap}(F,\Delta)$ for some $(G,\Delta)\in\Theta$ and $(F,\Delta)\notin\mathcal{F}$. Since, by Theorem 14, $\Theta\widetilde{\subseteq}\Theta^{\diamond}$, so $(G,\Delta)\in\Theta^{\diamond}$. By Theorem 16 (ii), $(F,\Delta)\in\Theta^{\diamond}$ and therefore, $(B,\Delta)\in\Theta^{\diamond}$. We now show that $\mathcal{B}_{\mathcal{F}}$ is a soft base. Let $y_\lambda\in\widetilde{Y}$ and $(G,\Delta)\in\Theta^{\diamond}(y_\lambda)$. By Theorem 16 (i), there exists $(H,\Delta)\in\Theta(y_\lambda)$ such that $(H^c,\Delta)\widetilde{\cup}(G,\Delta)\notin\mathcal{F}$. Set $(B,\Delta)=(H,\Delta)\widetilde{\cap}[(H^c,\Delta)\widetilde{\cup}(G,\Delta)]$. Indeed, $y_\lambda\in(B,\Delta)\widetilde{\subseteq}(G,\Delta)$. Thus, $\mathcal{B}_{\mathcal{F}}$ is a soft base for $\Theta^{\diamond}$. □

Theorem 18. *Let $(Y,\Theta,\Delta,\mathcal{F})$ and $(Y,\Theta,\Delta,\mathcal{G})$ be two PSTSs such that $\mathcal{F}\widetilde{\subseteq}\mathcal{G}$. Then $\Theta^{\diamond}_{\mathcal{G}}\subseteq\Theta^{\diamond}_{\mathcal{F}}$.*

Proof. If $(F,\Delta)\in\Theta^{\diamond}_{\mathcal{G}}$, then $(F^c,\Delta)=(F^c,\Delta)\widetilde{\cup}(F^c,\Delta)^{\diamond}_{\mathcal{G}}$ implies $(F^c,\Delta)^{\diamond}_{\mathcal{G}}\widetilde{\subseteq}(F^c,\Delta)$. Assume that $y_\lambda\in\widetilde{Y}$ such that $y_\lambda\notin(F^c,\Delta)$. Then $y_\lambda\notin(F^c,\Delta)^{\diamond}_{\mathcal{G}}$ and hence, there exists $(H,\Delta)\in\Theta(y_\lambda)$ such that $(H^c,\Delta)\widetilde{\cup}(F,\Delta)\notin\mathcal{G}$. Since $\mathcal{F}\widetilde{\subseteq}\mathcal{G}$, so $(H^c,\Delta)\widetilde{\cup}(F,\Delta)\notin\mathcal{F}$ and hence, $y_\lambda\notin(F^c,\Delta)^{\diamond}_{\mathcal{F}}$. Therefore, $(F^c,\Delta)^{\diamond}_{\mathcal{F}}\widetilde{\subseteq}(F^c,\Delta)$. This implies that $Cl^{\diamond}(F^c,\Delta)=(F^c,\Delta)\widetilde{\cup}(F^c,\Delta)^{\diamond}_{\mathcal{F}}=(F^c,\Delta)$ and thus, $(F,\Delta)\in\Theta^{\diamond}_{\mathcal{F}}$. This proves that $\Theta^{\diamond}_{\mathcal{G}}\subseteq\Theta^{\diamond}_{\mathcal{F}}$. □

Now, we look at a new operator and explore its major properties.

Definition 20. *Let $(Y,\Theta,\Delta,\mathcal{F})$ be a PSTS. Then, a soft mapping $(\cdot)^{\square}:S(Y_\Delta)\to S(Y_\Delta)$ is defined as follows $(F,\Delta)^{\square}(Y,\Theta,\Delta,\mathcal{F})=\{y_\lambda\in\widetilde{Y}:[(G,\Delta)\setminus(F,\Delta)]^c\notin\mathcal{F}$ for some $(G,\Delta)\in\Theta(y_\lambda)\}$ for each soft subset (F,Δ) over Y. In brief, we write $(F,\Delta)^{\square}$ or $(F,\Delta)^{\square}_{\mathcal{F}}$ instead of $(F,\Delta)^{\square}(Y,\Theta,\Delta,\mathcal{F})$.*

We shall remark that neither $(G,\Delta)^{\square}\widetilde{\subseteq}(G,\Delta)$ nor $(G,\Delta)\widetilde{\subseteq}(G,\Delta)^{\square}$ are generally correct, and counterexamples are not difficult to obtain.

The following conclusions cover a number of fundamental characteristics of how the operator $^{\square}$ behaves.

Theorem 19. *Let (F,Δ) be a soft subset of a PSTS $(Y,\Theta,\Delta,\mathcal{F})$. Then*

$$(F,\Delta)^{\square} = \widetilde{Y} \setminus (F^c,\Delta)^{\diamond}.$$

Proof. If $y_\lambda \in (F,\Delta)^{\square}$, then there exists $(H,\Delta) \in \Theta$ such that $[(H,\Delta) \setminus (F,\Delta)]^c \notin \mathcal{F}$. But $[(H,\Delta) \setminus (F,\Delta)]^c = [(H,\Delta)\widetilde{\cap}(F^c,\Delta)]^c = (H^c,\Delta)\widetilde{\cup}(F,\Delta)$ implies $y_\lambda \notin (F^c,\Delta)^{\diamond}$. Therefore, $y_\lambda \in \widetilde{Y} \setminus (F^c,\Delta)^{\diamond}$.

The converse can be followed by reversing the earlier steps. □

The next consequence is a direct outcome of the preceding conclusion.

Corollary 3. *For any soft subset (F,Δ) of a PSTS $(Y,\Theta,\Delta,\mathcal{F})$, we have*

(i) $(F^c,\Delta)^{\square} = [(F,\Delta)^{\diamond}]^c$.
(ii) $[(F,\Delta)^{\square}]^c = (F^c,\Delta)^{\diamond}$.
(iii) $(F^c,\Delta)^{\square\square} = [(F,\Delta)^{\diamond\diamond}]^c$.
(iv) $[(F,\Delta)^{\square\square}]^c = (F^c,\Delta)^{\diamond\diamond}$.

Lemma 1. *For a soft subset (F,Δ) of a PSTS $(Y,\Theta,\Delta,\mathcal{F})$, $(F,\Delta)^{\square} \in \Theta$.*

Proof. Let (F,Δ) be a soft subset of a PSTS $(Y,\Theta,\Delta,\mathcal{F})$. By Theorems 5 (ii) and 19, $(F,\Delta)^{\square} = \widetilde{Y} \setminus (F^c,\Delta)^{\diamond} = \widetilde{Y} \setminus Cl((F^c,\Delta)^{\diamond})$. Since $Cl((F^c,\Delta)^{\diamond})$ is soft Θ-closed, therefore $(F,\Delta)^{\square} \in \Theta$. □

Theorem 20. *Let $(F,\Delta),(G,\Delta)$ be soft subset of a PSTS $(Y,\Theta,\Delta,\mathcal{F})$. Then,*

(i) $\widetilde{Y}^{\square} = \widetilde{Y}$.
(ii) $(F,\Delta)^{\square}\widetilde{\subseteq}(F,\Delta)^{\square\square}$.
(iii) *If $(F,\Delta)\widetilde{\subseteq}(G,\Delta)$, then $(F,\Delta)^{\square}\widetilde{\subseteq}(G,\Delta)^{\square}$.*
(iv) $[(F,\Delta)\widetilde{\cap}(G,\Delta)]^{\square} = (F,\Delta)^{\square}\widetilde{\cap}(G,\Delta)^{\square}$.
(v) $(F,\Delta)^{\square}\widetilde{\cup}(G,\Delta)^{\square}\widetilde{\subseteq}[(F,\Delta)\widetilde{\cup}(G,\Delta)]^{\square}$.

Proof.

(i) Applying Theorem 19 to $\widetilde{Y}$, we have $\widetilde{Y}^{\square} = \widetilde{Y} \setminus [\widetilde{Y}^c]^{\diamond} = \widetilde{Y} \setminus [\widetilde{\phi}]^{\diamond} = \widetilde{Y} \setminus \widetilde{\phi} = \widetilde{Y}$.

(ii) Let $y_\lambda \in (F,\Delta)^{\square}$. By Theorem 19, $y_\lambda \notin (F^c,\Delta)^{\diamond}$ and then, by Theorem 5 (vi), $y_\lambda \notin (F^c,\Delta)^{\diamond\diamond}$. From (iv) in Corollary 3, we obtain that $y_\lambda \notin [(F,\Delta)^{\square\square}]^c$ implies $y_\lambda \in (F,\Delta)^{\square\square}$. Thus, $(F,\Delta)^{\square}\widetilde{\subseteq}(F,\Delta)^{\square\square}$.

(iii) Let $y_\lambda \in (F,\Delta)^{\square}$. Then $[(H,\Delta) \setminus (F,\Delta)]^c \notin \mathcal{F}$ for some $(H,\Delta) \in \Theta$. Since $(F,\Delta)\widetilde{\subseteq}(G,\Delta)$, then $[(H,\Delta) \setminus (F,\Delta)]\widetilde{\supseteq}[(H,\Delta) \setminus (G,\Delta)]$ and so $[(H,\Delta) \setminus (F,\Delta)]^c\widetilde{\subseteq}[(H,\Delta) \setminus (G,\Delta)]^c$. By Proposition 2, $[(H,\Delta) \setminus (G,\Delta)]^c \notin \mathcal{F}$. This means that $y_\lambda \in (G,\Delta)^{\square}$. Thus, $(F,\Delta)^{\square}\widetilde{\subseteq}(G,\Delta)^{\square}$.

(iv) Since $(F,\Delta)\widetilde{\cap}(G,\Delta)\widetilde{\subseteq}(F,\Delta)$ and $(F,\Delta)\widetilde{\cap}(G,\Delta)\widetilde{\subseteq}(G,\Delta)$, then by (iii), $[(F,\Delta)\widetilde{\cap}(G,\Delta)]^{\square}\widetilde{\subseteq}(F,\Delta)^{\square}$ and $[(F,\Delta)\widetilde{\cap}(G,\Delta)]^{\square}\widetilde{\subseteq}(G,\Delta)^{\square}$. Therefore, $[(F,\Delta)\widetilde{\cap}(G,\Delta)]^{\square}\widetilde{\subseteq}(F,\Delta)^{\square}\widetilde{\cap}(G,\Delta)^{\square}$. To prove the converse, we let $y_\lambda \in (F,\Delta)^{\square}\widetilde{\cap}(G,\Delta)^{\square}$. Then $[(H,\Delta) \setminus (F,\Delta)]^c \notin \mathcal{F}$ and $[(W,\Delta) \setminus (G,\Delta)]^c \notin \mathcal{F}$ for some $(H,\Delta),(W,\Delta) \in \Theta(y_\lambda)$. Set $(R,\Delta) = (H,\Delta)\widetilde{\cap}(W,\Delta)$. Then $(R,\Delta) \in \Theta(y_\lambda)$ and, by Proposition 2, we have $[(R,\Delta) \setminus (F,\Delta)]^c \notin \mathcal{F}$ and $[(R,\Delta) \setminus (G,\Delta)]^c \notin \mathcal{F}$. This implies that $\big((R,\Delta) \setminus [(F,\Delta)\widetilde{\cap}$

$(G,\Delta)])^c = [(R,\Delta) \setminus (F,\Delta)]^c \widetilde{\cap} [(R,\Delta) \setminus (G,\Delta)]^c \notin \mathcal{F}$. Thus, $y_\lambda \in [(F,\Delta)\widetilde{\cap}(G,\Delta)]^\square$ and hence, $(G,\Delta)^\square \widetilde{\cap} (G,\Delta)^\square \widetilde{\subseteq} [(F,\Delta)\widetilde{\cap}(G,\Delta)]^\square$. This completes the proof.

(v) This matches the first part of the proof of (iv).

□

Theorem 21. *Let $(F,\Delta),(G,\Delta)$ be soft subsets of a PSTS $(Y,\Theta,\Delta,\mathcal{F})$ such that $(F^c,\Delta) \notin \mathcal{F}$. The following conclusions hold true:*

(i) $(F,\Delta)^\square = \widetilde{Y} \setminus \widetilde{Y}^\diamond$.

(ii) $[(G,\Delta) \setminus (F,\Delta)]^\square = (G,\Delta)^\square$.

(iii) $[(G,\Delta)\widetilde{\cup}(F,\Delta)]^\square = (G,\Delta)^\square$.

Proof.

(i) Since $(F^c,\Delta) \notin \mathcal{F}$, by Theorem 9, $(F^c,\Delta)^\diamond = \widetilde{Y}^\diamond$. Therefore, $(F,\Delta)^\square = \widetilde{Y} \setminus (F^c,\Delta)^\diamond = \widetilde{Y} \setminus \widetilde{Y}^\diamond$.

(ii) Now, by the use of Theorem 9, we can get $[(G,\Delta) \setminus (F,\Delta)]^\square = \widetilde{Y} \setminus ([(G,\Delta) \setminus (F,\Delta)]^c)^\diamond = \widetilde{Y} \setminus [(G^c,\Delta)\widetilde{\cup}(F,\Delta)]^\diamond = \widetilde{Y} \setminus (G^c,\Delta)^\diamond = (G,\Delta)^\square$.

(iii) Similar to (ii).

□

Definition 21. *Let (G,Δ) be a soft subset of a PSTS $(Y,\Theta,\Delta,\mathcal{F})$ and let $\Theta^\diamond$ be the primal soft topology on Y. A soft point y_λ is called a soft $\Theta^\diamond$-interior point of (G,Δ) if there exists $(H,\Delta) \in \Theta^\diamond$ such that $y_\lambda \in (H,\Delta)\widetilde{\subseteq}(G,\Delta)$. The set of all soft $\Theta^\diamond$-interior points of (G,Δ) is symbolized by $Int^\diamond(G,\Delta)$.*

Theorem 22. *Let (F,Δ) be a soft subset of a PSTS $(Y,\Theta,\Delta,\mathcal{F})$. Then $(F,\Delta)\widetilde{\cap}(F,\Delta)^\square = Int^\diamond(F,\Delta)$.*

Proof. Let $y_\lambda \in (F,\Delta)\widetilde{\cap}(F,\Delta)^\square$. Then $y_\lambda \in (F,\Delta)$ and $[(H,\Delta) \setminus (F,\Delta)]^c \notin \mathcal{F}$ for some $(H,\Delta) \in \Theta(y_\lambda)$. By Theorem 17, $(D,\Delta) = (H,\Delta)\widetilde{\cap}[(H,\Delta) \setminus (F,\Delta)]^c \in \Theta^\diamond$ such that $y_\lambda \in (D,\Delta)\widetilde{\subseteq}(F,\Delta)$. Thus, $y_\lambda \in Int^\diamond(F,\Delta)$.

On the other hand, suppose $y_\lambda \in Int^\diamond(F,\Delta)$. Then, there exists a basic soft $\Theta^\diamond$-open set $(W,\Delta)\widetilde{\cap}(R,\Delta)$ containing y_λ, where $(W,\Delta) \in \Theta(y_\lambda)$ and $(R,\Delta) \notin \mathcal{F}$, such that $y_\lambda \in (W,\Delta)\widetilde{\cap}(R,\Delta)\widetilde{\subseteq}(F,\Delta)$. This implies that $(R,\Delta)\widetilde{\subseteq}[(W,\Delta) \setminus (F,\Delta)]^c$. By Proposition 2, $[(W,\Delta) \setminus (F,\Delta)]^c \notin \mathcal{F}$. Therefore, $y_\lambda \in (F,\Delta)\widetilde{\cap}(F,\Delta)^\square$. □

Theorem 23. *Let $(Y,\Theta,\Delta,\mathcal{F})$ be a PSTS. The family*

$$\Omega = \{(F,\Delta) : (F,\Delta) \in S(Y_\Delta), (F,\Delta)\widetilde{\subseteq}(F,\Delta)^\square\}$$

is a soft topology on Y. Furthermore, $\Omega = \Theta^\diamond$.

Proof. Suppose $\Omega = \{(F,\Delta) : (F,\Delta) \in S(Y_\Delta), (F,\Delta)\widetilde{\subseteq}(F,\Delta)^\square\}$. We need first to prove that Ω is a soft topology. Clearly, $\widetilde{\phi}, \widetilde{Y} \in \Omega$ as $\widetilde{\phi}\widetilde{\subseteq}\widetilde{\phi}^\square$ and $\widetilde{Y}\widetilde{\subseteq}\widetilde{Y}^\square = \widetilde{Y}$. Let $(F,\Delta),(G,\Delta) \in \Omega$. By Theorem 20, $(F,\Delta)\widetilde{\cap}(G,\Delta)\widetilde{\subseteq}(F,\Delta)^\square\widetilde{\cap}(G,\Delta)^\square = [(F,\Delta)\widetilde{\cap}(G,\Delta)]^\square$. Hence, $(F,\Delta)\widetilde{\cap}(G,\Delta) \in \Omega$. Let $\{(F_i,\Delta) : i \in I\}\widetilde{\subseteq}\Omega$. Again by Theorem 20, $(F_i,\Delta)\widetilde{\subseteq}(F_i,\Delta)^\square\widetilde{\subseteq}(\widetilde{\bigcup}_{i\in I}(F_i,\Delta))^\square$ for each i. Therefore, $\widetilde{\bigcup}_{i\in I}(F_i,\Delta)\widetilde{\subseteq}[\widetilde{\bigcup}_{i\in I}(F_i,\Delta)]^\square$ and so $\widetilde{\bigcup}_{i\in I}(F_i,\Delta) \in \Omega$.

We now show that $\Omega = \Theta^\diamond$. If $(H,\Delta) \in \Omega$, then $(H,\Delta)\widetilde{\subseteq}(H,\Delta)^\square$ and so, by Theorem 19, $(H,\Delta)\widetilde{\subseteq}\widetilde{Y} \setminus (H^c,\Delta)^\diamond$ implies $(H^c,\Delta)^\diamond\widetilde{\subseteq}(H^c,\Delta)$. This means that (H^c,Δ) is soft $\Theta^\diamond$-closed, and thus $(H,\Delta) \in \Theta^\diamond$. Now, let $(H,\Delta) \in \Theta^\diamond$ and $y_\lambda \in (H,\Delta)$. By Theorem 17, there exists $(W,\Delta) \in \Theta(y_\lambda)$ and $(F,\Delta) \notin \mathcal{F}$ such that $y_\lambda \in (W,\Delta)\widetilde{\cap}(F,\Delta)\widetilde{\subseteq}(H,\Delta)$. Evidently, $(F,\Delta)\widetilde{\subseteq}[(W,\Delta) \setminus (H,\Delta)]^c$ and then $[(W,\Delta) \setminus (H,\Delta)]^c \notin \mathcal{F}$. Therefore, $y_\lambda \in (H,\Delta)^\square$ and so $\Theta^\diamond\widetilde{\subseteq}\Omega$. The conclusion follows. □

This concludes that

Corollary 4. *For a soft subset* (F, Δ) *of a PSTS* $(Y, \Theta, \Delta, \mathcal{F})$, $(F, \Delta)\widetilde{\subseteq}(F, \Delta)^{\square}$ *whenever* $(F, \Delta) \in \Theta$.

5. Conclusions and Future Work

Shabir and Naz [9] and Çağman et al. [10] demonstrated, separately, the concept of a soft topology on a universal set, which is an extension of the classical (crisp) topology. This topological generalization has grown to be an interesting area of study. Various methods of constructing soft topologies have appeared in the literature. We have made a new contribution to the field of soft topology by studying the concept of primal soft topology. This research is based on the soft primal, which is a complementary notion of a soft grill. Soft primals can be considered a generalization of soft ideals. We have discussed some basic operations on soft primals. A primal soft topological space is defined as a soft topological space along with a soft primal. Then, we have defined and investigated a soft operator, symbolized by $^{\diamond}$, with respect to a soft topological space. The operator $^{\diamond}$ is used to define another soft topological operator called $Cl^{\diamond}$. Various properties of $Cl^{\diamond}$ have been discussed. Among the properties, we have seen that $Cl^{\diamond}$ conforms to all the axioms of the soft Kuratowski's closure operator, so it naturally generates a soft topology called a primal soft topology. The uniqueness of the primal soft topology is guaranteed by Theorem 1 in [43]. Some examples have been offered to illustrate that primal soft topologies are natural (non-trivial) soft topologies. The primal soft topology is finer than the original soft topology. In addition, we have established the fundamental properties of primal soft topologies.

The results obtained in this paper are preliminary, and future research could give more insights by exploring further properties of the primal soft topology, such as primal soft interior, primal soft closure, primal soft limit points, etc. Additionally, the separation axioms, compactness, and connectedness of primal soft topologies are also possible lines of research on this topic. On the other hand, we applied ideal structures to set up some generalized rough approximation spaces for the purpose of improving approximation operators (lower and upper) and increasing the value of accuracy of the decision made; see [45]. This work opens up the door for possible contributions to this trend by combining primal structures with generalized rough approximation spaces in classical and soft settings.

Author Contributions: Conceptualization, T.M.A.-s.; methodology, T.M.A.-s. and Z.A.A.; formal analysis, T.M.A.-s., Z.A.A., R.A.-G. and A.M.; investigation, T.M.A.-s., Z.A.A., R.A.-G. and A.M.; writing—original draft preparation, T.M.A.-s. and Z.A.A.; writing—review and editing, T.M.A.-s. and Z.A.A.; funding acquisition, R.A.-G. and A.M. All authors have read and agreed to the published version of the manuscript.

Funding: This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444).

Data Availability Statement: No data were used to support this study.

Conflicts of Interest: The authors declare no conflict of interest.

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