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Fixed Point Results in $\mathfrak{F}$-Bipolar Metric Spaces with Applications

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Abstract: The aim of this research article is to obtain fixed point results in the context of $\mathfrak{F}$-bipolar metric spaces. The obtained results extend some fixed point theorems in the existing literature. We also provide a non-trivial example to validate our claims. The existence and uniqueness of the solution of the integral equation are proved as applications of our leading results. Furthermore, the existence of the unique solution in homotopy theory is also investigated.

Keywords: fixed point; $\mathfrak{F}$-bipolar metric space; generalized contraction; integral equation; homotopy

MSC: 47H10; 46S40; 54H25

1. Introduction

Fixed point theory is one of the most glorious and prominent theories in functional analysis and has extensive applications in other fields. Additionally, the basic and elementary theorem in this theory is Banach contraction principle [1], in which the contractive mapping is defined on a complete metric space. The self-contractive mapping given in this principle is naturally continuous although it is not useful if the mapping is discontinuous. The crucial disadvantage of this theorem is how we apply self-contractive mapping if it is discontinuous. This problem was resolved by Kannan [2] in the past, where a fixed point theorem without continuity was proved. Later on, Reich [3] combined Banach contraction and Kannan's contraction and presented a result in 1971. Fisher [4] initiated rational expressions in contractive inequality and presented a result in the background of complete metric spaces. For more features in this way, we mention the researchers in [5–7].

In all of the above results, the concept of metric space plays a substantial and significant role, which was instinctively initiated by M. Frechet [8] in 1906. Thereafter, many authors have generalized the concept of metric by either weakening the metric axioms or altering the domain and range of it. Czerwik [9] weakened the triangular inequality of metric space by putting a non-negative constant $s \geq 1$ on the right hand side of it and gave the idea of b-metric space. In [10], Branciari initiated the idea of rectangular metric space and extended the conception of metric space by putting rectangular inequality on the place of the triangle inequality. The rectangular inequality associates the distance of four elements. Jleli et al. [11] introduced a new and fascinating space, which is famous as an $\mathfrak{F}$-metric space in which the triangle inequality is satisfied inside a continuous function. $\mathfrak{F}$-metric space is a generalization of classical metric space, b-metric space and Branciari metric space. Subsequently, Al-Mazrooei et al. [12] used the notion of $\mathfrak{F}$-metric space and proved some results for rational inequality that includes some non-negative constants.

In these extensions of metric space, we take the distance between elements of one set. So, a question arises naturally of how the distance between points of two distinct sets can be discussed. Such problems of computing distance can be confronted in different fields of the mathematics. In spite of that fact, Mutlu et al. [13] gave the notion of bipolar metric space to resolve such issues. Additionally, this updated notion of bipolar metric space leads to the development and progress of fixed point results in fixed point theory. However, a lot of decisive work has been investigated the existence for fixed points of self- and multivalued

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mappings in the setting of bipolar metric space (see [14–24]) and references therein). Very recently, Rawat et al. [25] unified the above two innovative concepts, namely $\mathfrak{F}$-metric space and bipolar metric space, and introduced the concept of $\mathfrak{F}$-bipolar metric space and proved the Banach contraction principle in this newly introduced metric space.

In this research work, we utilize the notion of $\mathfrak{F}$-bipolar metric space, which is more general metric space than $\mathfrak{F}$-metric space and bipolar metric space and establish fixed point results for Reich and Fisher type contractions. In this way, we generalize the main results of Rawat et al. [25] in $\mathfrak{F}$-bipolar metric space, Mutlu et al. [13] in bipolar metric space, and Jleli et al. [11] and Al-Mazrooei et al. [12] in $\mathfrak{F}$-metric space. As applications of our leading results, we study conditions for the existence and uniqueness of an integral equation. Moreover, we apply our result to investigate the existence of the unique solution in homotopy theory.

2. Preliminaries

An outstanding Banach fixed point theorem [1] is stated in the following manner.

Theorem 1 ([1])**.** *Let $(\mathfrak{W}, \varsigma)$ be a complete metric space and let $\Im : \mathfrak{W} \to \mathfrak{W}$. If there exists some non-negative real number $\lambda \in [0,1)$ such that*

$$\varsigma(\Im\ell, \Im\hbar) \leq \lambda\varsigma(\ell, \hbar),$$

for all $\ell, \hbar \in \mathfrak{W}$, then the mapping $\Im$ has a unique fixed point.

Kannan [2] presented the following theorem in which the given mapping is not necessarily continuous.

Theorem 2 ([2])**.** *Let $(\mathfrak{W}, \varsigma)$ be a complete metric space and let $\Im : \mathfrak{W} \to \mathfrak{W}$. If there exists some non-negative real number $\lambda \in [0, \frac{1}{2})$ such that*

$$\varsigma(\Im\ell, \Im\hbar) \leq \lambda(\varsigma(\ell, \Im\ell) + \varsigma(\hbar, \Im\hbar)),$$

for all $\ell, \hbar \in \mathfrak{W}$, then the mapping $\Im$ has a unique fixed point.

In 1971, Reich [3] combined the Banach contraction principle and Kannan fixed point theorem as follows.

Theorem 3 ([3])**.** *Let $(\mathfrak{W}, \varsigma)$ be a complete metric space and let $\Im : \mathfrak{W} \to \mathfrak{W}$. If there exist some non-negative real numbers $\lambda_1, \lambda_2 \in [0,1)$ such that $\lambda_1 + 2\lambda_2 < 1$ and*

$$\varsigma(\Im\ell, \Im\hbar) \leq \lambda_1\varsigma(\ell, \hbar) + \lambda_2(\varsigma(\ell, \Im\ell) + \varsigma(\hbar, \Im\hbar)),$$

for all $\ell, \hbar \in \mathfrak{W}$, then the mapping $\Im$ has a unique fixed point.

In [4], Fisher gave a result for contractive inequality consisting of rational expression as follows:

Theorem 4 ([4])**.** *Let $(\mathfrak{W}, \varsigma)$ be a complete metric and let $\Im : \mathfrak{W} \to \mathfrak{W}$. If there exist some non-negative real numbers $\lambda_1, \lambda_2 \in [0,1)$ such that $\lambda_1 + \lambda_2 < 1$ and*

$$\varsigma(\Im\ell, \Im\hbar) \leq \lambda_1\varsigma(\ell, \hbar) + \lambda_2 \frac{\varsigma(\ell, \Im\ell)\varsigma(\Im\hbar, \hbar)}{1 + \varsigma(\ell, \hbar)},$$

for all $\ell, \hbar \in \mathfrak{W}$, then there exists a unique point $\ell^* \in \mathfrak{W}$ such that $\Im\ell^* = \ell^*$.

In 2018, Jleli et al. [11] gave an absorbing extension of a metric space in the following fashion.

Let $\mathcal{F}$ be a family of continuous functions $f : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:

($\mathfrak{F}_1$) f is non-decreasing,
($\mathfrak{F}_2$) for each sequence $\{t_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} f(\alpha_n) = -\infty$.

Definition 1 ([11]). *Let $\mathfrak{W} \neq \emptyset$ and let $\varsigma : \mathfrak{W}\times\mathfrak{W} \to [0, +\infty)$. Assume that there exists some $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ in such a way*

(i) $\varsigma(\ell, \hbar) = 0$ *if and only if* $\ell = \hbar$,
(ii) $\varsigma(\ell, \hbar) = \varsigma(\hbar, \ell)$,
(iii) *for every* $p \in \mathbb{N}$, $p \geq 2$, *and for every* $(u_n)_{n=1}^{p} \subset \mathfrak{W}$ *with* $(u_1, u_p) = (\ell, \hbar)$, *we have*

$$\varsigma(\ell, \hbar) > 0 \Rightarrow f(\varsigma(\ell, \hbar)) \leq f\left(\sum_{n=1}^{p-1} \varsigma(u_n, u_{n+1})\right) + \alpha.$$

Then, $(\mathfrak{W}, \varsigma)$ is alleged to be an $\mathfrak{F}$-metric space.

Example 1 ([11]). *Let $\mathfrak{W} = \mathbb{R}$. Define $\varsigma : \mathfrak{W}\times\mathfrak{W} \to [0, +\infty)$ by*

$$\varsigma(\ell, \hbar) = \begin{cases} (\ell - \hbar)^2 & \text{if } (\ell, \hbar) \in [0, 3] \times [0, 3] \\ |\ell - \hbar| & \text{if } (\ell, \hbar) \notin [0, 3] \times [0, 3], \end{cases}$$

with $f(t) = \ln(t)$ and $\alpha = \ln(3)$, then $(\mathfrak{W},\varsigma)$ is an $\mathfrak{F}$-metric space.

On the other hand, Mutlu et al. [13] gave the conception of bipolar metric space as follows.

Definition 2 ([13]). *Let $\mathfrak{W}$ and $\mathfrak{Q}$ be nonempty sets and let $\varsigma : \mathfrak{W} \times \mathfrak{Q} \to [0, +\infty)$ be a given function. If the function ς verifies*

(bi$_1$) $\varsigma(\ell, \hbar) = 0$ *if and only if* $\ell = \hbar$,
(bi$_2$) $\varsigma(\ell, \hbar) = \varsigma(\hbar, \ell)$, *if* $\ell, \hbar \in \mathfrak{W} \cap \mathfrak{Q}$,
(bi$_3$) $\varsigma(\ell, \hbar) \leq \varsigma(\ell, \hbar^{/}) + \varsigma(\ell^{/}, \hbar^{/}) + \varsigma(\ell^{/}, \hbar)$;

for all $(\ell, \hbar), (\ell^{/}, \hbar^{/}) \in \mathfrak{W} \times \mathfrak{Q}$. Then, the triple $(\mathfrak{W},\mathfrak{Q}, \varsigma)$ is said to be a bipolar metric space.

Example 2 ([13]). *Let $\mathfrak{W}$ and $\mathfrak{Q}$ be the family of all singleton and compact subsets of $\mathbb{R}$ respectively. Define $\varsigma : \mathfrak{W} \times \mathfrak{Q} \to [0, +\infty)$ by*

$$\varsigma(\ell, \Xi) = |\ell - \inf(\Xi)| + |\ell - \sup(\Xi)|$$

for $\{\ell\} \subseteq \mathfrak{W}$ and $\Xi \subseteq \mathfrak{Q}$, then $(\mathfrak{W},\mathfrak{Q}, \varsigma)$ is a bipolar metric space.

Definition 3 ([13]). *Let $(\mathfrak{W}_1, \mathfrak{Q}_1, \varsigma_1)$ and $(\mathfrak{W}_2, \mathfrak{Q}_2, \varsigma_2)$ be two bipolar metric spaces. A mapping $\Im : \mathfrak{W}_1 \cup \mathfrak{Q}_1 \rightrightarrows \mathfrak{W}_2 \cup \mathfrak{Q}_2$ is said to be a covariant mapping, if $\Im(\mathfrak{W}_1) \subseteq \mathfrak{W}_2$ and $\Im(\mathfrak{Q}_1) \subseteq \mathfrak{Q}_2$. Similarly, a mapping $\Im : \mathfrak{W}_1 \cup \mathfrak{Q}_1 \rightrightarrows \mathfrak{W}_2 \cup \mathfrak{Q}_2$ is said to be a contravariant mapping, if $\Im(\mathfrak{W}_1) \subseteq \mathfrak{Q}_2$ and $\Im(\mathfrak{W}_2) \subseteq \mathfrak{Q}_1$.*

To make distinction between these mappings, we will represent covariant mapping as $\Im : (\mathfrak{W}_1, \mathfrak{Q}_1) \rightrightarrows (\mathfrak{W}_2, \mathfrak{Q}_2)$ and contravariant mapping as $\Im : (\mathfrak{W}_1, \mathfrak{Q}_1) \rightleftarrows (\mathfrak{W}_2, \mathfrak{Q}_2)$.

Very recently, Rawat et al. [25] unified the above two innovative conceptions, specifically $\mathfrak{F}$-metric space and bipolar metric space, and provided the idea of $\mathfrak{F}$-bipolar metric space in this way.

Definition 4 ([25]). *Let $\mathfrak{W}$ and $\mathfrak{Q}$ be nonempty sets and let $\varsigma : \mathfrak{W} \times \mathfrak{Q} \to [0, +\infty)$. Assume that there exists some $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ such that*

(D_1) *$\varsigma(\ell,\hbar) = 0$ if and only if $\ell = \hbar$,*
(D_2) *$\varsigma(\ell,\hbar) = \varsigma(\hbar,\ell)$, if $\ell,\hbar \in \mathfrak{W} \cap \mathfrak{Q}$,*
(D_3) *for every $p \in \mathbb{N}$, $p \geq 2$, and for every $(u_n)_{n=1}^{p} \subset \mathfrak{W}$ and $(v_n)_{n=1}^{p} \subset \mathfrak{Q}$ with $(u_1, v_p) = (\ell,\hbar)$, we have*

$$\varsigma(\ell,\hbar) > 0 \Rightarrow f(\varsigma(\ell,\hbar)) \leq f\left(\sum_{n=1}^{p-1} \varsigma(u_{n+1}, v_n) + \sum_{n=1}^{p} \varsigma(u_n, v_n)\right) + \alpha.$$

Then $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is called an $\mathfrak{F}$-bipolar metric space.

Example 3. *Let $\mathfrak{W}$=$\{1,2\}$ and $\mathfrak{Q} = \{2,7\}$. Define $\varsigma : \mathfrak{W} \times \mathfrak{Q} \to [0,+\infty)$ by*

$$\varsigma(1,2) = 6,\ \varsigma(1,7) = 10, \varsigma(2,7) = 2,\ \varsigma(2,2) = 0.$$

Now since

$$10 = \varsigma(1,7) > \varsigma(1,2) + \varsigma(2,7) = 6 + 2 = 8,$$

so the triangle inequality of bipolar metric space is not satisfied and thus, $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is not a bipolar metric space. Now it can be easily seen that ς satisfies the first two axioms (D_1 and D_2) of $\mathfrak{F}$-bipolar metric space. We only satisfy the third axiom (D_3).

Case 1. $\varsigma(1,2) > 0$ implies

$$\ln(6) = \ln(\varsigma(1,2)) \leq \ln(\varsigma(1,7) + \varsigma(2,7) + \varsigma(2,2)) = \ln(12),$$

Case 2. $\varsigma(2,7) > 0$ implies

$$\ln(2) = \ln(\varsigma(2,7)) \leq \ln(\varsigma(2,2) + \varsigma(1,2) + \varsigma(1,7)) = \ln(16),$$

thus, the axiom (D_3) is satisfied in both Case 1 and Case 2 with $f(t) = \ln(t) \in \mathcal{F}$ and $\alpha = 0$.
Case 3. $\varsigma(1,7) > 0$ implies

$$\ln(10) = \ln(\varsigma(1,7)) \leq \ln(\varsigma(1,2) + \varsigma(2,2) + \varsigma(2,7)) + \alpha = \ln(8) + \alpha,$$

thus the axiom (D_3) is satisfied with $f(t) = \ln(t) \in \mathcal{F}$ and $\alpha > 1$.

Thus, all the conditions of an $\mathfrak{F}$-bipolar metric space are satisfied and $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is an $\mathfrak{F}$-bipolar metric space.

Remark 1 ([25]). *Taking $\mathfrak{Q} = \mathfrak{W}$, $p = 2n$, $v_\jmath = v_{2\jmath-1}$ and $v_\jmath = u_{2\jmath}$ in the above Definition 4, we establish a sequence $(v_\jmath)_{\jmath=1}^{2n} \in \mathfrak{W}$ with $(v_1, v_{2n}) = (\ell,\hbar)$ such that assertion (iii) of Definition 1 is satisfied. Hence, every $\mathfrak{F}$-metric space is an $\mathfrak{F}$-bipolar metric space, but the converse is not true in general.*

Definition 5 ([25]). *Let $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be an $\mathfrak{F}$-bipolar metric space.*

(i) *An element $\ell \in \mathfrak{W} \cup \mathfrak{Q}$ is said to be a right point if $\ell \in \mathfrak{Q}$ and a left point if $\ell \in \mathfrak{W}$. Additionally, ℓ is said to be a central point if it is both a right and left point.*
(ii) *A sequence $\hbar_n$ on the set $\mathfrak{Q}$ is said to be a right sequence and a sequence (ℓ_n) on $\mathfrak{W}$ is called a left sequence. In an $\mathfrak{F}$-bipolar metric space, a right or a left sequence is said to be a sequence.*
(iii) *A sequence (ℓ_n) is said to converge to an element ℓ, if and only if (ℓ_n) is a right sequence, ℓ is a left point and $\lim_{n\to\infty} \varsigma(\ell,\ell_n) = 0$, or (ℓ_n) is a left sequence, ℓ is a right point and $\lim_{n\to\infty} \varsigma(\ell_n,\ell) = 0$. A bisequence $(\ell_n,\hbar_n)$ on $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is a sequence on the set $\mathfrak{W} \times \mathfrak{Q}$. If the sequences (ℓ_n) and $(\hbar_n)$ are convergent, then the bisequence $(\ell_n,\hbar_n)$ is also convergent, and if (ℓ_n) and $(\hbar_n)$ converge to a common element, then the bisequence $(\ell_n,\hbar_n)$ is called biconvergent.*
(iv) *A bisequence $(\ell_n,\hbar_n)$ in an $\mathfrak{F}$-bipolar metric space $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is called a Cauchy bisequence, if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $\varsigma(\ell_n,\hbar_p) < \epsilon$, for all $n, p \geq n_0$.*

Definition 6 ([25])**.** *An $\mathfrak{F}$-bipolar metric space $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is said to be complete, if every Cauchy bisequence in $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is convergent.*

3. Main Results

Definition 7. *Let $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be an $\mathfrak{F}$-bipolar metric space. A mapping $\Im : (\mathfrak{W}, \mathfrak{Q}, \varsigma) \rightleftarrows (\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is said to be Reich type contraction if there exist some constants $\lambda_1, \lambda_2, \lambda_3 \in [0, 1)$ such that $\lambda_1 + \lambda_2 + \lambda_3 < 1$ and*

$$\varsigma(\Im\hbar, \Im\ell) \leq \lambda_1\varsigma(\ell, \hbar) + \lambda_2\varsigma(\ell, \Im\ell) + \lambda_3\varsigma(\Im\hbar, \hbar), \tag{1}$$

for all $(\ell, \hbar) \in \mathfrak{W} \times \mathfrak{Q}$.

Theorem 5. *Let $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be a complete $\mathfrak{F}$-bipolar metric space and let $\Im : (\mathfrak{W}, \mathfrak{Q}, \varsigma) \rightleftarrows (\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be contravariant Reich type contraction, then the mapping $\Im : \mathfrak{W} \cup \mathfrak{Q} \to \mathfrak{W} \cup \mathfrak{Q}$ has a unique fixed point, provided that the mapping $\Im : (\mathfrak{W}, \mathfrak{Q}, \varsigma) \rightleftarrows (\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is continuous.*

Proof. Let ℓ_0 be an arbitrary point in $\mathfrak{W}$. Define the bisequence $(\ell_n, \hbar_n)$ in $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ by

$$\hbar_n = \Im\ell_n \text{ and } \ell_{n+1} = \Im\hbar_n$$

for all $n = 1, 2, \ldots$ Now by (1), we have

$$\begin{aligned} \varsigma(\ell_n, \hbar_n) &= \varsigma(\Im\hbar_{n-1}, \Im\ell_n,) \\ &\leq \lambda_1\varsigma(\ell_n, \hbar_{n-1}) + \lambda_2\varsigma(\ell_n, \Im\ell_n) + \lambda_3\varsigma(\Im\hbar_{n-1}, \hbar_{n-1}) \\ &= \lambda_1\varsigma(\ell_n, \hbar_{n-1}) + \lambda_2\varsigma(\ell_n, \hbar_n) + \lambda_3\varsigma(\ell_n, \hbar_{n-1}), \end{aligned}$$

which implies that

$$\varsigma(\ell_n, \hbar_n) \leq \frac{\lambda_1 + \lambda_3}{1 - \lambda_2}\varsigma(\ell_n, \hbar_{n-1}). \tag{2}$$

Moreover,

$$\begin{aligned} \varsigma(\ell_n, \hbar_{n-1}) &= \varsigma(\Im\hbar_{n-1}, \Im\ell_{n-1},) \\ &\leq \lambda_1\varsigma(\ell_{n-1}, \hbar_{n-1}) + \lambda_2\varsigma(\ell_{n-1}, \Im\ell_{n-1}) + \lambda_3\varsigma(\Im\hbar_{n-1}, \hbar_{n-1}) \\ &= \lambda_1\varsigma(\ell_{n-1}, \hbar_{n-1}) + \lambda_2\varsigma(\ell_{n-1}, \hbar_{n-1}) + \lambda_3\varsigma(\ell_n, \hbar_{n-1}), \end{aligned}$$

which implies that

$$\varsigma(\ell_n, \hbar_{n-1}) \leq \frac{\lambda_1 + \lambda_2}{1 - \lambda_3}\varsigma(\ell_{n-1}, \hbar_{n-1}). \tag{3}$$

Setting $\vartheta = \max\left\{\frac{\lambda_1+\lambda_3}{1-\lambda_2}, \frac{\lambda_1+\lambda_2}{1-\lambda_3}\right\} < 1$. Then by (2) and (3), it is easy to see that

$$\varsigma(\ell_n, \hbar_n) \leq \vartheta^{2n}\varsigma(\ell_0, \hbar_0). \tag{4}$$

Similarly, we have

$$\varsigma(\ell_{n+1}, \hbar_n) \leq \vartheta^{2n+1}\varsigma(\ell_0, \hbar_0), \tag{5}$$

for all $n = 1, 2, \ldots$ Let $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathfrak{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \text{ implies} f(t) < f(\epsilon) - \alpha. \tag{6}$$

Now, from (4) and (5), we obtain

$$\begin{aligned}
&\sum_{j=n}^{p-1} \varsigma(\ell_{j+1}, \hbar_j) + \sum_{j=n}^{p} \varsigma(\ell_j, \hbar_j) \\
\leq\ & \left(\vartheta^{2n} + \vartheta^{2n+2} + \ldots + \vartheta^{2p}\right)\varsigma(\ell_0, \hbar_0) + \left(\vartheta^{2n+1} + \vartheta^{2n+3} + \ldots + \vartheta^{2p-1}\right)\varsigma(\ell_0, \hbar_0) \\
\leq\ & \vartheta^{2n} \sum_{n=0}^{\infty} \vartheta^n \varsigma(\ell_0, \hbar_0) = \frac{\vartheta^{2n}}{1-\vartheta}\varsigma(\ell_0, \hbar_0),
\end{aligned}$$

for $p > n$. Since $\lim_{n\to\infty} \frac{\vartheta^{2n}}{1-\vartheta}\varsigma(\ell_0, \hbar_0) = 0$, so there exists $n_0 \in \mathbb{N}$, such that

$$0 < \frac{\vartheta^{2n}}{1-\vartheta}\varsigma(\ell_0, \hbar_0) < \delta,$$

for $n \geq n_0$. Hence, for $p > n \geq n_0$, using $(\mathfrak{F}_1)$ and inequality (6), we have

$$f\left(\sum_{j=n}^{p-1} \varsigma(\ell_{j+1}, \hbar_j) + \sum_{j=n}^{p} \varsigma(\ell_j, \hbar_j)\right) \leq f\left(\frac{\vartheta^{2n}}{1-\vartheta}\varsigma(\ell_0, \hbar_0)\right) < f(\epsilon) - \alpha. \tag{7}$$

From (D_3) and inequality (7), we find that $\varsigma(\ell_n, \hbar_p) > 0$ implies

$$f(\varsigma(\ell_n, \hbar_p)) \leq f\left(\sum_{j=n}^{p-1} \varsigma(\ell_{j+1}, \hbar_j) + \sum_{j=n}^{p} \varsigma(\ell_j, \hbar_j)\right) + \alpha < f(\epsilon).$$

Similarly, for $n > p \geq n_0$, $\varsigma(\ell_n, \hbar_p) > 0$ implies

$$f(\varsigma(\ell_n, \hbar_p)) \leq f\left(\sum_{j=p}^{n-1} \varsigma(\ell_{j+1}, \hbar_j) + \sum_{j=p}^{n} \varsigma(\ell_j, \hbar_j)\right) + \alpha < f(\epsilon).$$

Then, by $(\mathfrak{F}_1)$, $\varsigma(\ell_n, \hbar_p) < \epsilon$, for all $p, n \geq n_0$. Thus, $(\ell_n, \hbar_n)$ is a Cauchy bisequence in $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$. As $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is complete, so $(\ell_n, \hbar_n)$ biconverges to a point $\omega \in \mathfrak{W} \cap \mathfrak{Q}$. Thus, $(\ell_n) \to \omega$, $(\hbar_n) \to \omega$. Moreover, since the contravariant mapping $\Im$ is continuous, so we have

$$(\ell_n) \to \omega \text{ implies that } (\hbar_n) = (\Im\ell_n) \to \Im\omega.$$

Additionally, since $(\hbar_n)$ has a limit ω in $\mathfrak{W} \cap \mathfrak{Q}$ and the limit is unique. Thus, $\Im\omega = \omega$. So, $\Im$ has a fixed point. □

Now, if ϖ is another and distinct fixed point of $\Im$, then $\Im\varpi = \varpi$ yields that $\varpi \in \mathfrak{W} \cap \mathfrak{Q}$. Then,

$$\begin{aligned}
\varsigma(\omega, \varpi) &= \varsigma(\Im\omega, \Im\varpi) \leq \lambda_1\varsigma(\varpi, \omega) + \lambda_2\varsigma(\varpi, \Im\varpi) + \lambda_3\varsigma(\Im\omega, \omega) \\
&= \lambda_1\varsigma(\varpi, \omega),
\end{aligned}$$

which is a contradiction, except $\omega = \varpi$.

Corollary 1 ([25]). *Let $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be a complete $\mathfrak{F}$-bipolar metric space and let $\Im : (\mathfrak{W}, \mathfrak{Q}, \varsigma) \rightleftarrows (\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be a contravariant mapping. If there exists some constant $\lambda \in [0, 1)$ and*

$$\varsigma(\Im\hbar, \Im\ell) \leq \lambda\varsigma(\ell, \hbar),$$

for all $(\ell, \hbar) \in \mathfrak{W} \times \mathfrak{Q}$, then the mapping $\Im : \mathfrak{W} \cup \mathfrak{Q} \to \mathfrak{W} \cup \mathfrak{Q}$ has a unique fixed point, provided that the mapping $\Im : (\mathfrak{W}, \mathfrak{Q}, \varsigma) \rightleftarrows (\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is continuous.

Proof. Take $\lambda_1 = \lambda$, $\lambda_2 = \lambda_3 = 0$ in Theorem 5. □

Remark 2. *If $f(t) = \ln(t)$, for $t > 0$ and $\alpha = 0$ in the above Corollary, then $\mathfrak{F}$-bipolar metric space reduced to bipolar metric space and we derive main result of Mutlu et al. [13] as a direct consequence.*

Remark 3. *If $\mathfrak{W} = \mathfrak{Q}$ in the above Corollary, then $\mathfrak{F}$-bipolar metric space reduced to $\mathfrak{F}$-metric space and we derive the main result of Jleli et al. [11] as a direct consequence.*

Corollary 2. *Let $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be a complete $\mathfrak{F}$-bipolar metric space and let $\Im : (\mathfrak{W}, \mathfrak{Q}, \varsigma) \rightleftarrows (\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be a contravariant mapping. If there exists some constant $\eta < \frac{1}{3}$ and*

$$\varsigma(\Im\hbar, \Im\ell) \leq \eta(\varsigma(\ell, \hbar) + \varsigma(\ell, \Im\ell) + \varsigma(\Im\hbar, \hbar)),$$

for all $(\ell, \hbar) \in \mathfrak{W} \times \mathfrak{Q}$, then the mapping $\Im : \mathfrak{W} \cup \mathfrak{Q} \to \mathfrak{W} \cup \mathfrak{Q}$ has a unique fixed point, provided that the mapping $\Im : (\mathfrak{W}, \mathfrak{Q}, \varsigma) \rightleftarrows (\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is continuous.

Proof. Take $\lambda_1 = \lambda_2 = \lambda_3 = \eta$ in Theorem 5. □

Corollary 3. *Let $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be a complete $\mathfrak{F}$-bipolar metric space and let $\Im : (\mathfrak{W}, \mathfrak{Q}, \varsigma) \rightleftarrows (\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be a contravariant mapping. If there exists some constant $\eta < \frac{1}{2}$ and*

$$\varsigma(\Im\hbar, \Im\ell) \leq \eta(\varsigma(\ell, \Im\ell) + \varsigma(\Im\hbar, \hbar)),$$

for all $(\ell, \hbar) \in \mathfrak{W} \times \mathfrak{Q}$, then the mapping $\Im : \mathfrak{W} \cup \mathfrak{Q} \to \mathfrak{W} \cup \mathfrak{Q}$ has a unique fixed point, provided that the mapping $\Im : (\mathfrak{W}, \mathfrak{Q}, \varsigma) \rightleftarrows (\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is continuous.

Proof. Take $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = \eta$ in Theorem 5. □

Now, we state a theorem that is a natural extension of Theorem 5 in this way.

Theorem 6. *Let $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be a complete $\mathfrak{F}$-bipolar metric space and let $\Im : (\mathfrak{W}, \mathfrak{Q}, \varsigma) \rightleftarrows (\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be a contravariant mapping. If there exists some constant $0 < \eta < 1$ such that*

$$\varsigma(\Im\hbar, \Im\ell) \leq \eta \max\{\varsigma(\ell, \hbar), \varsigma(\ell, \Im\ell), \varsigma(\Im\hbar, \hbar)\}, \tag{8}$$

for all $(\ell, \hbar) \in \mathfrak{W} \times \mathfrak{Q}$, then the mapping $\Im : \mathfrak{W} \cup \mathfrak{Q} \to \mathfrak{W} \cup \mathfrak{Q}$ has a unique fixed point, provided that the mapping $\Im : (\mathfrak{W}, \mathfrak{Q}, \varsigma) \rightleftarrows (\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is continuous.

Proof. Let ℓ_0 be an arbitrary point in $\mathfrak{W}$. Define the bisequence $(\ell_n, \hbar_n)$ in $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ by

$$\hbar_n = \Im\ell_n \text{ and } \ell_{n+1} = \Im\hbar_n$$

for all $n = 1, 2, \ldots$ Now, by (8), we have

$$\begin{aligned} \varsigma(\ell_n, \hbar_n) &= \varsigma(\Im\hbar_{n-1}, \Im\ell_n,) \\ &\leq \eta \max\{\varsigma(\ell_n, \hbar_{n-1}), \varsigma(\ell_n, \Im\ell_n), \varsigma(\Im\hbar_{n-1}, \hbar_{n-1})\} \\ &= \eta \max\{\varsigma(\ell_n, \hbar_{n-1}), \varsigma(\ell_n, \hbar_n), \varsigma(\ell_n, \hbar_{n-1})\} \\ &= \eta \max\{\varsigma(\ell_n, \hbar_{n-1}), \varsigma(\ell_n, \hbar_n)\}. \end{aligned} \tag{9}$$

If $\max\{\varsigma(\ell_n, \hbar_{n-1}), \varsigma(\ell_n, \hbar_n)\} = \varsigma(\ell_n, \hbar_n)$, then we have

$$\varsigma(\ell_n, \hbar_n) \leq \eta\varsigma(\ell_n, \hbar_n),$$

which is a contradiction to the fact that $0 < \eta < 1$. Thus, $\max\{\varsigma(\ell_n, \hbar_{n-1}), \varsigma(\ell_n, \hbar_n)\} = \varsigma(\ell_n, \hbar_{n-1})$. Hence, by (9), we have

$$\varsigma(\ell_n, \hbar_n) \leq \eta\varsigma(\ell_n, \hbar_{n-1}). \tag{10}$$

Likewise,

$$\begin{aligned}
\varsigma(\ell_n, \hbar_{n-1}) &= \varsigma(\Im\hbar_{n-1}, \Im\ell_{n-1},) \\
&\leq \eta \max\{\varsigma(\ell_{n-1}, \hbar_{n-1}), \varsigma(\ell_{n-1}, \Im\ell_{n-1}), \varsigma(\Im\hbar_{n-1}, \hbar_{n-1},)\} \\
&= \eta \max\{\varsigma(\ell_{n-1}, \hbar_{n-1}), \varsigma(\ell_{n-1}, \hbar_{n-1}), \varsigma(\ell_n, \hbar_{n-1})\} \\
&= \eta \max\{\varsigma(\ell_{n-1}, \hbar_{n-1}), \varsigma(\ell_n, \hbar_{n-1})\}.
\end{aligned}$$

If $\max\{\varsigma(\ell_{n-1}, \hbar_{n-1}), \varsigma(\ell_n, \hbar_{n-1})\} = \varsigma(\ell_n, \hbar_{n-1})$, then we have

$$\varsigma(\ell_n, \hbar_{n-1}) \leq \eta\varsigma(\ell_n, \hbar_{n-1}),$$

which is a contradiction to the fact that $0 < \eta < 1$. Thus, $\max\{\varsigma(\ell_{n-1}, \hbar_{n-1}), \varsigma(\ell_n, \hbar_{n-1})\} = \varsigma(\ell_{n-1}, \hbar_{n-1})$. Hence, by (9), we have

$$\varsigma(\ell_n, \hbar_{n-1}) \leq \eta\varsigma(\ell_{n-1}, \hbar_{n-1}). \tag{11}$$

Now, by (10) and (11), it is easy to see that

$$\varsigma(\ell_n, \hbar_n) \leq \eta^{2n}\varsigma(\ell_0, \hbar_0).$$

Similarly,

$$\varsigma(\ell_{n+1}, \hbar_n) \leq \eta^{2n+1}\varsigma(\ell_0, \hbar_0).$$

The remaining part of the proof is the same as of Theorem 5. □

Definition 8. *Let $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be an $\mathfrak{F}$-bipolar metric space. A mapping $\Im : (\mathfrak{W}, \mathfrak{Q}, \varsigma) \rightleftarrows (\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is said to be rational contraction if there exist some constants $\lambda_1, \lambda_2 \in [0, 1)$ such that $\lambda_1 + \lambda_2 < 1$ and*

$$\varsigma(\Im\hbar, \Im\ell) \leq \lambda_1\varsigma(\ell, \hbar) + \lambda_2 \frac{\varsigma(\ell, \Im\ell)\varsigma(\Im\hbar, \hbar)}{1 + \varsigma(\ell, \hbar)}, \tag{12}$$

for all $(\ell, \hbar) \in \mathfrak{W} \times \mathfrak{Q}$.

Theorem 7. *Let $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be a complete $\mathfrak{F}$-bipolar metric space and let $\Im : (\mathfrak{W}, \mathfrak{Q}, \varsigma) \rightleftarrows (\mathfrak{W}, \mathfrak{Q}, \varsigma)$ be contravariant rational contraction, then the mapping $\Im : \mathfrak{W} \cup \mathfrak{Q} \to \mathfrak{W} \cup \mathfrak{Q}$ has a unique fixed point, provided that the mapping $\Im : (\mathfrak{W}, \mathfrak{Q}, \varsigma) \rightleftarrows (\mathfrak{W}, \mathfrak{Q}, \varsigma)$ is continuous.*

Proof. Let ℓ_0 and $\hbar_0$ be arbitrary points in $\mathfrak{W}$ and $\mathfrak{Q}$, respectively. Define the bisequence $(\ell_n, \hbar_n)$ in $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ by

$$\hbar_n = \Im\ell_n \text{ and } \ell_{n+1} = \Im\hbar_n$$

for all $n = 1, 2, \ldots$ Now, by (12), we have

$$\begin{aligned}
\varsigma(\ell_n, \hbar_n) &= \varsigma(\Im\hbar_{n-1}, \Im\ell_n,) \\
&\leq \lambda_1\varsigma(\ell_n, \hbar_{n-1}) \\
&\quad + \lambda_2 \frac{\varsigma(\ell_n, \Im\ell_n)\varsigma(\Im\hbar_{n-1}, \hbar_{n-1})}{1 + \varsigma(\ell_n, \hbar_{n-1})} \\
&= \lambda_1\varsigma(\ell_n, \hbar_{n-1}) + \lambda_2 \frac{\varsigma(\ell_n, \hbar_n)\varsigma(\ell_n, \hbar_{n-1})}{1 + \varsigma(\ell_n, \hbar_{n-1})} \\
&\leq \lambda_1\varsigma(\ell_n, \hbar_{n-1}) + \lambda_2\varsigma(\ell_n, \hbar_n),
\end{aligned}$$

which implies that

$$\varsigma(\ell_n, \hbar_n) \leq \frac{\lambda_1}{1 - \lambda_2}\varsigma(\ell_n, \hbar_{n-1}). \tag{13}$$

Moreover,

$$\begin{aligned}
\varsigma(\ell_n, \hbar_{n-1}) &= \varsigma(\Im\hbar_{n-1}, \Im\ell_{n-1},) \\
&\leq \lambda_1 \varsigma(\ell_{n-1}, \hbar_{n-1}) \\
&\quad + \lambda_2 \frac{\varsigma(\ell_{n-1}, \Im\ell_{n-1})\varsigma(\Im\hbar_{n-1}, \hbar_{n-1},)}{1 + \varsigma(\ell_{n-1}, \hbar_{n-1})} \\
&= \lambda_1 \varsigma(\ell_{n-1}, \hbar_{n-1}) + \lambda_2 \frac{\varsigma(\ell_{n-1}, \hbar_{n-1})\varsigma(\ell_n, \hbar_{n-1})}{1 + \varsigma(\ell_{n-1}, \hbar_{n-1})} \\
&\leq \lambda_1 \varsigma(\ell_{n-1}, \hbar_{n-1}) + \lambda_2 \varsigma(\hbar_{n-1}, \ell_n),
\end{aligned}$$

which implies that

$$\varsigma(\ell_n, \hbar_{n-1}) \leq \frac{\lambda_1}{1-\lambda_2} \varsigma(\ell_{n-1}, \hbar_{n-1}). \tag{14}$$

Now, if we take $\frac{\lambda_1}{1-\lambda_2} = \vartheta$, then (13) and (14) become

$$\varsigma(\ell_n, \hbar_n) \leq \vartheta \varsigma(\ell_n, \hbar_{n-1}) \tag{15}$$

and

$$\varsigma(\ell_n, \hbar_{n-1}) \leq \vartheta \varsigma(\ell_{n-1}, \hbar_{n-1}). \tag{16}$$

Thus, by (15) and (16), we have

$$\varsigma(\ell_n, \hbar_n) \leq \vartheta^{2n} \varsigma(\ell_0, \hbar_0). \tag{17}$$

Similarly,

$$\varsigma(\ell_{n+1}, \hbar_n) \leq \vartheta^{2n+1} \varsigma(\ell_0, \hbar_0), \tag{18}$$

for all $n = 1, 2, \ldots$ Let $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathfrak{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \text{ implies } f(t) < f(\epsilon) - \alpha. \tag{19}$$

From (17) and (18), we obtain

$$\begin{aligned}
&\sum_{j=n}^{p-1} \varsigma(\ell_{j+1}, \hbar_j) + \sum_{j=n}^{p} \varsigma(\ell_j, \hbar_j) \\
\leq\ & \left(\vartheta^{2n} + \vartheta^{2n+2} + \ldots + \vartheta^{2p}\right)\varsigma(\ell_0, \hbar_0) + \left(\vartheta^{2n+1} + \vartheta^{2n+3} + \ldots + \vartheta^{2p-1}\right)\varsigma(\ell_0, \hbar_0) \\
\leq\ & \vartheta^{2n} \sum_{n=0}^{\infty} \vartheta^n \varsigma(\ell_0, \hbar_0) = \frac{\vartheta^{2n}}{1-\vartheta} \varsigma(\ell_0, \hbar_0),
\end{aligned}$$

for $p > n$. Since $\lim_{n\to\infty} \frac{\vartheta^{2n}}{1-\vartheta} \varsigma(\ell_0, \hbar_0) = 0$, so there exists $n_0 \in \mathbb{N}$, such that

$$0 < \frac{\vartheta^{2n}}{1-\vartheta} \varsigma(\ell_0, \hbar_0) < \delta,$$

for $n \geq n_0$. Hence, for $p > n \geq n_0$, using $(\mathfrak{F}_1)$ and inequality (19), we have

$$f\left(\sum_{j=n}^{p-1} \varsigma(\ell_{j+1}, \hbar_j) + \sum_{j=n}^{p} \varsigma(\ell_j, \hbar_j)\right) \leq f\left(\frac{\vartheta^{2n}}{1-\vartheta} \varsigma(\ell_0, \hbar_0)\right) < f(\epsilon) - \alpha. \tag{20}$$

By (D_3) and inequality (20), we find that $\varsigma(\ell_n, \hbar_p) > 0$ implies

$$f\big(\varsigma\big(\ell_n,\hbar_p\big)\big) \leq f\left(\sum_{j=n}^{p-1}\varsigma\big(\ell_{j+1},\hbar_j\big)+\sum_{j=n}^{p}\varsigma\big(\ell_j,\hbar_j\big)\right)+\alpha < f(\epsilon).$$

Similarly, for $n > p \geq n_0$, $\varsigma\big(\ell_n,\hbar_p\big) > 0$ implies

$$f\big(\varsigma\big(\ell_n,\hbar_p\big)\big) \leq f\left(\sum_{j=p}^{n-1}\varsigma\big(\ell_{j+1},\hbar_j\big)+\sum_{j=p}^{n}\varsigma\big(\ell_j,\hbar_j\big)\right)+\alpha < f(\epsilon).$$

Then, by $(\mathfrak{F}_1)$, $\varsigma\big(\ell_n,\hbar_p\big) < \epsilon$, for all $p,n \geq n_0$. Thus, $(\ell_n,\hbar_n)$ is a Cauchy bisequence in $(\mathfrak{W},\mathfrak{Q},\varsigma)$. As $(\mathfrak{W},\mathfrak{Q},\varsigma)$ is complete, so $(\ell_n,\hbar_n)$ biconverges to a point $\omega \in \mathfrak{W}\cap\mathfrak{Q}$. So $(\ell_n) \to \omega, (\hbar_n) \to \omega$. Additionally, since the contravariant mapping $\Im$ is continuous, we have

$$(\ell_n) \to \omega \text{ implies that } (\hbar_n) = (\Im\ell_n) \to \Im\omega.$$

Moreover, since $(\hbar_n)$ has a limit ω in $\mathfrak{W}\cap\mathfrak{Q}$ and the limit is unique. Thus, $\Im\omega = \omega$. So, $\Im$ has a fixed point. □

Now, if ϖ is another distinct fixed point of $\Im$, then $\Im\varpi = \varpi$ yields that $\varpi \in \mathfrak{W}\cap\mathfrak{Q}$. Then,

$$\begin{aligned}\varsigma(\omega,\varpi) &= \varsigma(\Im\omega,\Im\varpi) \leq \lambda_1\varsigma(\varpi,\omega)+\lambda_2\frac{\varsigma(\varpi,\Im\varpi)\varsigma(\Im\omega,\omega)}{1+\varsigma(\varpi,\omega)}\\ &= \lambda_1\varsigma(\varpi,\omega),\end{aligned}$$

which is a contradiction, except $\omega = \varpi$.

Remark 4. *If $\mathfrak{W} = \mathfrak{Q}$ in the above theorem, then $\mathfrak{F}$-bipolar metric space reduced to $\mathfrak{F}$-metric space and we derive the main result of Al-Mazrooei et al. [12] as a direct consequence.*

Example 4. *Let $\mathfrak{W} = \{9,10,18,20\}$ and $\mathfrak{Q} = \{3,5,11,18\}$. Define the usual metric $\varsigma\colon \mathfrak{W}\times\mathfrak{Q} \to [0,\infty)$ by*

$$\varsigma(\ell,\hbar) = |\ell-\hbar|.$$

Then, $(\mathfrak{W},\mathfrak{Q},\varsigma)$ is a complete $\mathfrak{F}$-bipolar metric space. Define the contravariant mapping $\Im : \mathfrak{W}\cup\mathfrak{Q} \rightleftarrows \mathfrak{W}\cup\mathfrak{Q}$ by

$$\Im(\ell) = \begin{cases} 18, & \text{if } \ell \in \mathfrak{W}\cup\{11\} \\ 9, & \text{otherwise.}\end{cases}$$

Then, all the conditions of Theorem 7 are satisfied with $\lambda_1 = \frac{4}{5}$ and $\lambda_2 = \frac{1}{6}$. Hence, by Theorem 7, $\Im$ has a unique fixed point, which is $18 \in \mathfrak{W}\cap\mathfrak{Q}$.

4. Application

4.1. Integral Equations

In this section, we study conditions for the existence and uniqueness of an integral equation.

Theorem 8. *Let us consider the integral equation*

$$\varphi(\ell) = g(\ell) + \int_{\mathfrak{W}\cup\mathfrak{Q}} K(\ell,\hbar,\varphi(\ell))d\hbar,$$

where $\mathfrak{W}\cup\mathfrak{Q}$ is a Lebesgue measurable set. Assume that

(i) $K : \big(\mathfrak{W}^2\cup\mathfrak{Q}^2\big)\times[0,\infty) \to [0,\infty)$ *and* $f \in \mathfrak{L}^\infty(\mathfrak{W})\cup\mathfrak{L}^\infty(\mathfrak{Q})$,

(ii) There is a continuous function $\Upsilon : \mathfrak{W}^2\cup\mathfrak{Q}^2 \to [0,\infty)$ *such that*

$$|K(\ell,\hbar,\varphi(\hbar)) - K(\ell,\hbar,\phi(\hbar))| \leq \Upsilon(\ell,\hbar)\left\{\begin{array}{c}\lambda_1|\phi(\hbar)-\varphi(\hbar)| \\ +\lambda_2\frac{|\phi(\hbar)-I\phi(\hbar)||I\varphi(\hbar)-\varphi(\hbar)|}{1+|\phi(\hbar)-\varphi(\hbar)|}\end{array}\right\},$$

for all $\ell, \hbar \in (\mathfrak{W}^2 \cup \mathfrak{Q}^2)$, *and* $I : \mathfrak{L}^\infty(\mathfrak{W}) \cup \mathfrak{L}^\infty(\mathfrak{Q}) \to \mathfrak{L}^\infty(\mathfrak{W}) \cup \mathfrak{L}^\infty(\mathfrak{Q})$

(iii) $\left\| \int_{\mathfrak{W}\cup\mathfrak{Q}} \Upsilon(\ell,\hbar)\varsigma\hbar \right\| \leq 1$, *i.e.*, $\sup_{\ell\in\mathfrak{W}\cup\mathfrak{Q}} \int_{\mathfrak{W}\cup\mathfrak{Q}} |\Upsilon(\ell,\hbar)|\varsigma\hbar \leq 1$.

Then, the integral equation has a unique solution in $\mathfrak{L}^\infty(\mathfrak{W}) \cup \mathfrak{L}^\infty(\mathfrak{Q})$.

Proof. Let $\Xi = \mathfrak{L}^\infty(\mathfrak{W})$ and $\Theta = \mathfrak{L}^\infty(\mathfrak{Q})$ be two normed linear spaces, where $\mathfrak{W}$ and $\mathfrak{Q}$ are Lebesgue measurable sets and $m(\mathfrak{W}\cup\mathfrak{Q}) < \infty$. Let $d : \Xi \times \Theta \to [0,\infty)$ be given as

$$d(\xi,\zeta) = \|\xi - \zeta\|_\infty,$$

for all $\xi, \zeta \in \Xi \times \Theta$. Then, (Ξ,Θ,d) is a complete $\mathfrak{F}$-bipolar metric space. Define $I : \Xi \cup \Theta \to \Xi \cup \Theta$ by

$$I(\varphi(\ell)) = g(\ell) + \int_{\mathfrak{W}\cup\mathfrak{Q}} K(\ell,\hbar,\varphi(\ell))d\hbar,$$

for $\ell \in \mathfrak{W}\cup\mathfrak{Q}$. Now, we have

$$\begin{aligned}
d(I(\varphi(\ell)), I(\phi(\ell))) &= \|I(\varphi(\ell)) - I(\phi(\ell))\| \\
&= \left| \int_{\mathfrak{W}\cup\mathfrak{Q}} K(\ell,\hbar,\varphi(\ell))d\hbar - \int_{\mathfrak{W}\cup\mathfrak{Q}} K(\ell,\hbar,\phi(\ell))d\hbar \right| \\
&\leq \int_{\mathfrak{W}\cup\mathfrak{Q}} |K(\ell,\hbar,\varphi(\ell)) - K(\ell,\hbar,\phi(\ell))|d\hbar \\
&\leq \int_{\mathfrak{W}\cup\mathfrak{Q}} \Upsilon(\ell,\hbar) \left\{ \begin{array}{c} \lambda_1|\phi(\hbar) - \varphi(\hbar)| \\ +\lambda_2 \frac{|\phi(\hbar) - I(\phi(\hbar))||I(\varphi(\hbar)) - \varphi(\hbar)|}{1+|\phi(\hbar)-\varphi(\hbar)|} \end{array} \right\} d\hbar \\
&\leq \left\{ \begin{array}{c} \lambda_1\|\phi(\hbar) - \varphi(\hbar)\| \\ +\lambda_2 \frac{\|\phi(\hbar) - I(\phi(\hbar))\|\|I(\varphi(\hbar)) - \varphi(\hbar)\|}{1+\|\phi(\hbar)-\varphi(\hbar)\|} \end{array} \right\} \int_{\mathfrak{W}\cup\mathfrak{Q}} |\Upsilon(\ell,\hbar)|d\hbar \\
&\leq \left\{ \begin{array}{c} \lambda_1\|\phi - \varphi\| \\ +\lambda_2 \frac{\|\phi - I(\phi)\|\|I(\varphi) - \varphi\|}{1+\|\phi-\varphi\|} \end{array} \right\} \sup_{\ell\in\mathfrak{W}\cup\mathfrak{Q}} \int_{\mathfrak{W}\cup\mathfrak{Q}} |\Upsilon(\ell,\hbar)|d\hbar \\
&\leq \lambda_1\|\phi - \varphi\| + \lambda_2 \frac{\|\phi - I(\phi)\|\|I(\varphi) - \varphi\|}{1+\|\phi-\varphi\|} \\
&= \lambda_1 d(\phi,\varphi) + \lambda_2 \frac{d(\phi, I(\phi))d(I(\varphi),\varphi)}{1+d(\phi,\varphi)}.
\end{aligned}$$

Thus, by Theorem 7, I has a unique fixed point in $\Xi \cup \Theta$. □

4.2. Homotopy

Theorem 9. *Let* $(\mathfrak{W}, \mathfrak{Q}, \varsigma)$ *be a complete* $\mathfrak{F}$*-bipolar metric space and let* (Ξ,Θ) *be an open subset of* $(\mathfrak{W},\mathfrak{Q})$ *and* $(\overline{\Xi},\overline{\Theta})$ *be a closed subset of* $(\mathfrak{W},\mathfrak{Q})$ *and* $(\Xi,\Theta) \subseteq (\overline{\Xi},\overline{\Theta})$. *Suppose* $\mathfrak{L} : (\overline{\Xi}\cup\overline{\Theta}) \times [0,1] \to \mathfrak{W}\cup\mathfrak{Q}$ *is a mapping satisfying the assertions:*

(hom1) $\ell \neq \mathfrak{L}(\ell,q)$ *for each* $\ell \in \partial\Xi \cup \partial\Theta$ *and* $q \in [0,1]$, *where* $\partial\Xi$ *and* $\partial\Theta$ *represent the differential of* Ξ *and* Θ, *respectively,*

(hom2) *for all* $\ell \in \overline{\Xi}$, $\hbar \in \overline{\Theta}$ *and* $q \in [0,1]$

$$\varsigma(\mathfrak{L}(\hbar,q), \mathfrak{L}(\ell,q)) \leq \lambda_1\varsigma(\ell,\hbar) + \lambda_2 \frac{\varsigma(\ell,\mathfrak{L}(\ell,q))\varsigma(\mathfrak{L}(\hbar,q),\hbar)}{1+\varsigma(\ell,\hbar)},$$

where $0 \leq \lambda_1 + \lambda_2 < 1$,

(hom3) *there exists* $M \geq 0$ *such that*

$$\varsigma(\mathfrak{L}(\ell,r), \mathfrak{L}(\hbar,o)) \leq M|r - o|,$$

for all $\ell \in \overline{\Xi}$, $\hbar \in \overline{\Theta}$ *and* $r, o \in [0,1]$.

Then, $\mathfrak{L}(\cdot,0)$ *has a fixed point if and only if* $\mathfrak{L}(\cdot,1)$ *has a fixed point.*

Proof. Let $\Re_1 = \{\tau \in [0,1] : \ell = \mathfrak{L}(\ell,\tau), \ell \in \Xi\}$ and $\Re_1 = \{o \in [0,1] : \hbar = \mathfrak{L}(\hbar,o), \hbar \in \Theta\}$. Since $\mathfrak{L}(\cdot,0)$ has a fixed point in $\Xi \cup \Theta$, then we find $0 \in \Re_1 \cap \Re_2$. Thus, $\Re_1 \cap \Re_2 \neq \emptyset$. Now, we shall prove that $\Re_1 \cap \Re_2$ is both open and closed in $[0,1]$ and so, by connectedness, $\Re_1 = \Re_2 = [0,1]$. Let $\left(\{\tau_n\}_{n=1}^{\infty}\right), \left(\{o_n\}_{n=1}^{\infty}\right) \subseteq (\Re_1, \Re_2)$ with $(\tau_n, o_n) \to (\rho,\rho) \in [0,1]$ as $n \to \infty$. We also claim that $\rho \in \Re_1 \cap \Re_2$. Since $(\tau_n, o_n) \in \Re_1 \cap \Re_2$, for $n \in \mathbb{N} \cup \{0\}$. Hence, there exists a bisequence $(\ell_n, \hbar_n) \in (\Xi, \Theta)$ such that $\hbar_n = \mathfrak{L}(\ell_n, \tau_n)$ and $\ell_{n+1} = \mathfrak{L}(\hbar_n, o_n)$. Additionally, we obtain

$$\begin{aligned}
\varsigma(\ell_{n+1},\hbar_n) &= \varsigma(\mathfrak{L}(\hbar_n,o_n),\mathfrak{L}(\ell_n,\tau_n)) \\
&\leq \lambda_1\varsigma(\ell_n,\hbar_n) + \lambda_2\frac{\varsigma(\ell_n,\mathfrak{L}(\ell_n,\tau_n))\varsigma(\mathfrak{L}(\hbar_n,o_n),\hbar_n)}{1+\varsigma(\ell_n,\hbar_n)} \\
&= \lambda_1\varsigma(\ell_n,\hbar_n) + \lambda_2\frac{\varsigma(\ell_n,\hbar_n)\varsigma(\ell_{n+1},\hbar_n)}{1+\varsigma(\ell_n,\hbar_n)} \\
&\leq \lambda_1\varsigma(\ell_n,\hbar_n) + \lambda_2\varsigma(\ell_{n+1},\hbar_n),
\end{aligned}$$

which implies that

$$\varsigma(\ell_{n+1},\hbar_n) \leq \frac{\lambda_1}{1-\lambda_2}\varsigma(\ell_n,\hbar_n).$$

Additionally,

$$\begin{aligned}
\varsigma(\ell_n,\hbar_n) &= \varsigma(\mathfrak{L}(\hbar_{n-1},o_{n-1}),\mathfrak{L}(\ell_n,\tau_n)) \\
&\leq \lambda_1\varsigma(\ell_n,\hbar_{n-1}) + \lambda_2\frac{\varsigma(\ell_n,\mathfrak{L}(\ell_n,\tau_n))\varsigma(\mathfrak{L}(\hbar_{n-1},o_{n-1}),\hbar_{n-1})}{1+\varsigma(\ell_n,\hbar_{n-1})} \\
&= \lambda_1\varsigma(\ell_n,\hbar_{n-1}) + \lambda_2\frac{\varsigma(\ell_n,\hbar_n)\varsigma(\ell_n,\hbar_{n-1})}{1+\varsigma(\ell_n,\hbar_{n-1})} \\
&\leq \lambda_1\varsigma(\ell_n,\hbar_{n-1}) + \lambda_2\varsigma(\ell_n,\hbar_n),
\end{aligned}$$

which implies that

$$\varsigma(\ell_n,\hbar_n) \leq \frac{\lambda_1}{1-\lambda_2}\varsigma(\ell_n,\hbar_{n-1}).$$

Doing the same procedure as performed in Theorem 7, one can simply prove that $(\ell_n, \hbar_n)$ is a Cauchy bisequence in (Ξ,Θ). As (Ξ,Θ) is complete, so there exists $\rho_1 \in \Xi \cap \Theta$ such that $\lim_{n\to\infty}(\ell_n) = \lim_{n\to\infty}(\hbar_n) = \rho_1$. Now, we have

$$\begin{aligned}
\varsigma(\mathfrak{L}(\rho_1,o),\hbar_n) &= \varsigma(\mathfrak{L}(\rho_1,o),\mathfrak{L}(\ell_n,\tau_n)) \\
&\leq \lambda_1\varsigma(\ell_n,\rho_1) + \lambda_2\frac{\varsigma(\ell_n,\mathfrak{L}(\ell_n,\tau_n))\varsigma(\mathfrak{L}(\rho_1,o),\rho_1)}{1+\varsigma(\ell_n,\lambda_1)} \\
&= \lambda_1\varsigma(\ell_n,\rho_1) + \lambda_2\frac{\varsigma(\ell_n,\hbar_n)\varsigma(\mathfrak{L}(\rho_1,o),\rho_1)}{1+\varsigma(\ell_n,\rho_1)}.
\end{aligned}$$

Applying the limit as $n \to \infty$, we obtain $\varsigma(\mathfrak{L}(\rho_1,o),\rho_1) = 0$, which implies that $\mathfrak{L}(\rho_1,o) = \rho_1$. Similarly, $\mathfrak{L}(\rho_1,\tau) = v_1$. Thus, $\tau = o \in \Re_1 \cap \Re_2$, and evidently $\Re_1 \cap \Re_2$ is a closed set in $[0,1]$. □

Next, we have to prove that $\Re_1 \cap \Re_2$ is open in $[0,1]$. Suppose $(\tau_0, o_0) \in (\Re_1, \Re_2)$, then there is a bisequence $(\ell_0, \hbar_0)$ so that $\ell_0 = \mathfrak{L}(\ell_0, \tau_0)$, $\hbar_0 = \mathfrak{L}(\hbar_0, o_0)$. Since $\Xi \cup \Theta$ is open, there exists some $r > 0$ such that $B_\varsigma(\ell_0, r) \subseteq \Xi \cup \Theta$ and $B_\varsigma(r, \hbar_0) \subseteq \Xi \cup \Theta$, where $B_\varsigma(\ell_0, r)$ and $B_\varsigma(r, \hbar_0)$ represent the open balls with centers ℓ_0 and $\hbar_0$, respectively, and radius r. Choose $\tau \in (o_0 - \epsilon, o_0 + \epsilon)$ and $o \in (\tau_0 - \epsilon, \tau_0 + \epsilon)$ such that

$$|\tau - o_0| \leq \frac{1}{M^n} < \frac{\epsilon}{2},$$

$$|o - \tau_0| \leq \frac{1}{M^n} < \frac{\epsilon}{2},$$

and

$$|\tau_0 - o_0| \leq \frac{1}{M^n} < \frac{\epsilon}{2}.$$

Hence, we have

$$\hbar \in \overline{B_{\Re_1 \cup \Re_2}(\ell_0, r)} = \{\hbar : \hbar_0 \in \Theta | \varsigma(\ell_0, \hbar) \leq r + \varsigma(\ell_0, \hbar_0)\},$$

and

$$\ell \in \overline{B_{\Re_1 \cup \Re_2}(r, \hbar_0)} = \{\ell : \ell_0 \in \Xi | \varsigma(\ell, \hbar_0) \leq r + \varsigma(\ell_0, \hbar_0)\}.$$

Moreover, we have

$$\begin{aligned}
\varsigma(\mathfrak{L}(\ell,\tau),\hbar_0) &= \varsigma(\mathfrak{L}(\ell,\tau),\mathfrak{L}(\hbar_0,o_0)) \\
&\leq \varsigma(\mathfrak{L}(\ell,\tau),\mathfrak{L}(\hbar,o_0)) + \varsigma(\mathfrak{L}(\ell_0,\tau),\mathfrak{L}(\hbar,o_0)) + \varsigma(\mathfrak{L}(\ell_0,\tau),\mathfrak{L}(\hbar_0,o_0)) \\
&\leq 2M|\tau - o_0| + \varsigma(\mathfrak{L}(\ell_0,\tau),\mathfrak{L}(\hbar,o_0)) \\
&\leq \frac{2}{M^n - 1} + \lambda_1 \varsigma(\ell_0,\hbar) + \lambda_2 \frac{\varsigma(\ell_0,\mathfrak{L}(\ell_0,\tau))\varsigma(\mathfrak{L}(\hbar,o_0),\hbar)}{1 + \varsigma(\ell_0,\hbar)} \\
&= \frac{2}{M^n - 1} + \lambda_1 \varsigma(\ell_0,\hbar) + \lambda_2 \frac{\varsigma(\ell_0,\ell_0)\varsigma(\hbar,\hbar)}{1 + \varsigma(\ell_0,\hbar)} \\
&= \frac{2}{M^n - 1} + \lambda_1 \varsigma(\ell_0,\hbar) \\
&\leq \frac{2}{M^n - 1} + \varsigma(\ell_0,\hbar).
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\varsigma(\mathfrak{L}(\ell,\tau),\hbar_0) \leq \varsigma(\ell_0,\hbar) \leq r + \varsigma(\ell_0,\hbar_0).$$

By corresponding fashion, we obtain

$$\varsigma(\ell_0,\mathfrak{L}(\hbar,o)) \leq \varsigma(\ell,\hbar_0) \leq r + \varsigma(\ell_0,\hbar_0).$$

However,

$$\varsigma(\ell_0,\hbar_0) = \varsigma(\mathfrak{L}(\ell_0,\tau_0),\mathfrak{L}(\hbar_0,o_0)) \leq M|\tau_0 - o_0| \leq \frac{1}{M^{n-1}} \to 0,$$

as $n \to \infty$, which implies that $\ell_0 = \hbar_0$. Therefore, for each fixed o, $o = \tau \in (o_0 - \epsilon, o_0 + \epsilon)$ and $\mathfrak{L}(\cdot,\tau) : \overline{B_{\Re_1 \cup \Re_2}(\ell_0, r)} \to \overline{B_{\Re_1 \cup \Re_2}(\ell_0, r)}$. As all the hypothesis of Theorem 7 hold, $\mathfrak{L}(\cdot,\tau)$ has a fixed point in $\overline{\Xi} \cap \overline{\Theta}$, which must be in $\Xi \cap \Theta$. Then, $\tau = o \in \Re_1 \cap \Re_2$ for each $o \in (o_0 - \epsilon, o_0 + \epsilon)$. Hence, $(o_0 - \epsilon, o_0 + \epsilon) \in \Re_1 \cap \Re_2$, which gives $\Re_1 \cap \Re_2$ as open in $[0,1]$. The converse can be proved by using the same process.

5. Conclusions

In this research paper, we have used the conception of $\mathfrak{F}$-bipolar metric space and established some theorems for Reich- and Fisher-type contractions. We have derived certain fixed point results of self-mappings in the background of $\mathfrak{F}$-bipolar metric space and bipolar metric space as outcomes of our main results. The existence and uniqueness of the solution of the integral equation is proved as applications of our leading results. Furthermore, the existence of a unique solution in homotopy theory is also investigated.

The established theorems in this paper can be expanded to fuzzy and multivalued mappings in the setting of $\mathfrak{F}$-bipolar metric spaces for future work. Furthermore, one can obtain common fixed point theorems for these contractions. As applications of the above-mentioned outlines in the foundation of $\mathfrak{F}$-bipolar metric space, certain integral and differential inclusions can be solved.

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