*mathematics*

MDPI

*Article*

# Optimizing Pricing, Pre-Sale Incentive, and Inventory Decisions with Advance Sales and Trade Credit under Carbon Tax Policy

**Mei-Chuan Cheng [1], Hui-Chiung Lo [2] and Chih-Te Yang [2,*]**

[1] Department of Information Management, Fu Jen Catholic University, New Taipei City 242, Taiwan; 155379@mail.fju.edu.tw
[2] Department of Business Administration, Tamkang University, New Taipei City 251, Taiwan; hclo@mail.tku.edu.tw
* Correspondence: 122300@mail.tku.edu.tw

**Abstract:** This paper aims to propose a comprehensive inventory model including pricing, pre-sale incentives, advance sales, trade credit, and carbon tax policies. The novelty of this study lies in its holistic approach to addressing these relevant and practical issues. The major purpose is to determine the optimal pricing, pre-order discount, and replenishment decisions to maximize the total profit under carbon tax policy. Through theoretical analysis, this study develops several theorems to demonstrate properties of optimal solutions and an easy-to-use algorithm to derive optimal solutions. Further, several numerical examples are provided to demonstrate the solution process for different scenarios and the effects of various parameters on optimal alternatives and solutions. This study provides companies management implications to address the challenges posed by the global movement to reduce carbon emissions while maintaining their profitability.

**Keywords:** inventory system; trade credit; advance sales; discount rate; carbon emission schemes; carbon tax

**MSC:** 90B05



**Citation:** Cheng, M.-C.; Lo, H.-C.; Yang, C.-T. Optimizing Pricing, Pre-Sale Incentive, and Inventory Decisions with Advance Sales and Trade Credit under Carbon Tax Policy. *Mathematics* **2023**, *11*, 2534. https://doi.org/10.3390/math11112534

Academic Editors: Ripon Kumar Chakrabortty and María del Carmen Valls Martínez

Received: 16 April 2023
Revised: 20 May 2023
Accepted: 29 May 2023
Published: 31 May 2023



## 1. Introduction

According to The Global Risks Report released by the World Economic Forum (WEF) in 2022, four of the top five most likely risks in the world are related to global warming and extreme climate. It is agreed that carbon dioxide emission is the main driver of global climate change, and its threat is increasing year by year. In the past 100 years, the concentration of carbon dioxide in the atmosphere has increased by 29% [1]. More and more countries have successively announced the so-called carbon emission reduction or carbon neutrality goal. For example, the United States pledged to reduce its net carbon emissions by 52% by 2030 compared to 2005; Japan has pledged to reduce net carbon emissions by 46% from 2013 levels by 2030; and China is also set to peak its carbon emissions by 2030 and has pledged to achieve carbon neutrality by 2060. To achieve these goals, countries are actively planning and formulating relevant policies and measures to curb the continuous increase in greenhouse gas emissions, including the development of new alternative or renewable energy sources, the formulation of energy-saving and carbon-reduction regulations, and the promotion of carbon trading markets, carbon tax collection, carbon compensation mechanisms. These policies and specific measures will inevitably affect the business operations of enterprises. If enterprises do not take the impact of carbon emissions into consideration when making relevant operational decisions, the survival and development of enterprises may be affected due to neglected external costs.

Along with market competitiveness enhancement, increasing market share is the most important issue for companies. Many companies provide advance sales to attract more customers, and advance sales policies are widely used by retailers such as Amazon, Toys R Us, Maxim's Bakery, Electronics Boutique, and Eslite Bookstore. In the real market, suppliers often allow retailers a fixed time period to settle their owed amount. This delay is an effective way to boost sales for the supplier and reduce purchase costs for the retailer. The World Trade Organization found that trade credit finance plays a crucial role, as 80% to 90% of world trade relies on it [2]. Offering a permissible delay in payment encourages customers to "buy now and pay later," which could attract more customers who may view this as a form of discount. Thus, trade credit financing significantly impacts both the buyer's order amount and the seller's profitability, making it an essential and relevant factor.

## 2. Literature Review

In recent years, carbon emission reduction has become an important issue in inventory management, and production and inventory models with carbon emission reduction policies have also gained a lot of development. For example, based on the Economic Order Quantity (EOQ) model, He et al. [3], Konur [4], and Battini et al. [5] tried to establish sustainable inventory models under various carbon emission policies including carbon cap, carbon trade, carbon offset, and carbon tax. Daryanto and Wee [6,7], Taleizadeh and Soleymanfar [8], and Zhang and Liu [9] developed Economic Production Quantity (EPQ) models considering carbon emission reduction policies. Xu et al. [10] and Tiwari et al. [11] began to establish a production inventory model with carbon emission reduction policies from the perspective of supply chain integration. Mishra et al. [12] have further taken deteriorating items into account and developed a production and inventory model of carbon emission reduction technology. Recently, Shen et al. [13] proposed a supply chain inventory model for high-carbon-emitting enterprises for hybrid carbon emission reduction policies based on the goal of carbon neutrality. Other related studies include Sadigh et al. [14], Qi et al. [15], Halat and Hafezalkotob [16], Huang et al. [17], Chang et al. [18], and so on.

Tsao [19] examined a retailer's promotion and replenishment policies with advance sales discounts, considering trade credits for both the supplier and retailer, and presented an algorithm to determine the optimal promotion effort and replenishment cycle time. Chen and Cheng [20] proposed an inventory model for retailers who receive a permissible delay in payments from suppliers while also offering advance sales to customers. Incorporating advance sales not only reduces financial risks, but it also increases interest earned from payments received from committed orders prior to the regular sale season. Dye and Hsieh [21] established an advance sales system for deteriorating items, dividing each sales cycle into advance and spot sales periods, where customers must make reservations during the advance sales period, and cancellations are allowed. Cheng and Ouyang [22] developed an inventory model with price-dependent demand for a retailer who simultaneously receives trade credit from a supplier and offers advance sales and an appreciation period to customers. Youjun et al. [23] provided a non-multicycle inventory model for deteriorating items with advance sales, in which the demand rate is time-varying and price-dependent, and cancellations are allowed but require payment of a cancellation fee. Seref et al. [24] studied a retailer's inventory and pricing decisions in an advance selling scenario that involves strategic consumers who consider possible inventory unavailability during spot sales. Cheng et al. [25] established an inventory model for deteriorating items with a return period and price-dependent demand, where the retailer offers two-phase advance sales to customers. Recently, Duary et al. [26] developed an inventory model that considers both advance and delayed payments, price discounts, deteriorating items, capacity constraints, and partially backlogged shortages.

Lou and Wang [27] expanded the EPQ model to include defective items and a two-level trade credit. Tsao [28] further built a decision model that integrated inventory, location, and preservation considerations for perishable items that are not immediately sold and allowed for a permissible delay in payment. Zhong et al. [29] analyzed the effects of trade credit on warehouse location selection and warehouse–retailer assignments, using both analytical and numerical methods. Chang et al. [30] conducted a study on optimal pricing and lot-sizing strategies for perishable goods, taking into account different payment terms such as advance, cash, and credit. Meanwhile, Li et al. [31] explored a three-tier supplier–retailer–consumer supply chain for perishable products, wherein the retailer obtains full trade credit from the supplier but provides partial trade credit to high-risk customers, with demand factoring in selling price, expiration date, and credit period. Tsao [32] examined the impact of default risk when offering trade credit to buyers in a supplier–wholesaler channel and developed coordinated contracts to manage such risk. Shi et al. [33] developed an EOQ for perishable products, taking into account cash, advance, and credit payments, as well as carbon tax regulations. Mallick et al. [34] developed an inventory model that considers stochastic lead time demand with lead time crashing cost, a lead time-dependent credit period, and partial backorders. Li et al. [35] developed a supplier–retailer–customer chain where the retailer offers a downstream cash–credit payment (partially in cash and partially in credit) to customers and receives an upstream advance-cash–credit (ACC) payment from the supplier. The demand is affected by the combined effect of selling price and stock age, and the deterioration rate is time-varying. Additional articles related to this topic from the last decade can be found in the works of various authors, including Li et al. [36], Chung et al. [37], Jani et al. [38], and others. A comparison of the present paper with the relevant literature is provided in Table 1 to help the readers understand the contributions of our model. Motivated by previous studies and investigations, this study aims to address existing research gaps by developing an inventory model that links a firm's profit by incorporating the effects of the optimal prior-sales discount rate, the optimal unit spot sales price, and the selling period. This study endeavors to address this gap by making contributions to three streams of the literature: (1) carbon emission reduction policies, (2) advance sales and trade credit, and (3) the inventory model with price-dependent demand.

**Table 1.** A comparison between the present model and related previous research.

| Authors (Year) | Model Type | Advance Sales | Trade Credit | Carbon Tax | Price-Dependent Demand Rate |
|---|---|---|---|---|---|
| Battini et al. [5] | EOQ | | | V | |
| Daryanto & Wee [7] | EPQ | | | V | |
| Tiwari et al. [11] | Production–inventory | | | V | |
| Mishra et al. [12] | EPQ | | | V | |
| Shen et al. [13] | Production–inventory | | | V | |
| Cheng & Ouyang [22] | EOQ | V | V | | V |
| Cheng et al. [25] | EOQ | V | | | V |
| Duary et al. [26] | EOQ | | V | V | V |
| Li et al. [35] | EOQ | | V | | V |
| Jani et al. [38] | EOQ | | V | | V |
| Present model | EOQ | V | V | V | V |

In essence, this study introduces a model for managing inventory over a single period in situations where the supplier extends trade credit and the retailer offers advance sales, at a discounted price, to lure in more customers during the advance selling period. To commit to the advance sale, customers must pre-pay a deposit when placing their order, whereas those who opt out of this offer may still purchase the product during the spot selling period at its regular price. Since the discount can sway undecided customers, retailers can expect to generate more demand during the advance selling period by offering advance sales. Additionally, the advance sales system reduces financial risks and increases interest earned since partial payments are received from pre-committed orders ahead of the selling season. By imposing a carbon tax, governments aim to incentivize industries and individuals to reduce their carbon emissions. The tax provides a financial incentive for businesses to adopt cleaner technologies, invest in renewable energy sources, and improve energy efficiency.

We aim to maximize the total profit by determining the optimal prior-sales discount rate, the optimal unit spot sales price, and the selling period.

## 3. Notation and Assumptions

Before developing the mathematical model, this section first lists the notation used and the assumptions required for the proposed model. It is hereby stated as follows:

**Notation:**

| | |
|---|---|
| $c$ | Unit purchasing cost. |
| $c_r$ | Unit carbon tax. |
| $\hat{c}$ | Amount of carbon emissions per unit product purchased by the retailer. |
| $h$ | Unit holding cost per unit time excluding the interest charges. |
| $\hat{h}$ | Amount of carbon emissions per unit per unit of time stored by the retailer. |
| $S$ | Cost of placing an order. |
| $\hat{S}$ | Amount of carbon emissions generated by the retailer per order. |
| $I_c$ | Interest charges per TWD investment in stocks per unit time. |
| $I_e$ | Interest earned per TWD per unit time. |
| $M$ | Trade credit period. |
| $t_p$ | Advance selling period. |
| $\theta$ | The order cancellation rate, where $0 \le \theta < 1$. |
| $\beta$ | The prepaid deposit rate, where $0 < \beta \le 1$. |
| $\delta$ | Advance sales discount rate, i.e., $(1-\delta)p$, is the unit advance sales price, where $\delta < 1 - c/p$, a decision variable. |
| $p$ | Unit spot sales price, a decision variable. |
| $D(p)$ | The demand rate, is a function of $p$. |
| $T$ | Spot selling period, a decision variable. |
| $Q$ | The order quantity. |
| $p^*$ | The optimal unit spot sales price. |
| $\delta^*$ | The optimal advance sales discount rate. |
| $T^*$ | The optimal spot selling period. |
| $Q^*$ | The optimal order quantity. |
| $Z(T,\delta,p)$ | Total profit, which is a function of $T$, $\delta$, and $p$. |
| $Z^*$ | Maximum total profit, i.e., $Z^* = (T^*,\ \delta^*,\ p^*)$. |

**Assumptions:**

1. The inventory system here is for a single item in a single season.
2. The replenishment occurs instantaneously at an infinite rate.
3. Customers who accept an advance sales offer must pre-pay a deposit for the pre-committed orders. For those who cancel their pre-committed orders, no refund is permitted.
4. The carbon emissions generated by the retailer mainly come from operational activities such as ordering, purchase, and storage.
5. The demand rate, $D$, is linearly dependent on the selling price, $p$, and can be expressed as $D(p) = a - bp$, where a and b are positive constants. We also assume that the demand rate is always positive. That is, $p < a/b$.
6. The retailer offers an advance sale to its customers with respective discount rate $\delta$.
7. Shortages are not allowed.

## 4. Model Formulation

The proposed model incorporates the common phenomenon in the real market that the supplier allows the retailer a fixed time period to settle the total account, while the retailer provides the advance sales that motivate customers to commit to their orders at a discounted price during the advance sales period. Figure 1 displays the behavior of inventory levels with advance sales. During the advance selling period $[0, t_p]$, all customers are offered a unit advance sales price $p_a = (1-\delta)p$ for their purchases and are required to pre-pay a deposit with the rate $\beta$. In addition, the order cancellation rate $\theta$ is given. Hence, the sales volume during the pre-sale period is $(1-\theta)D(p_a)t_p$. At time $t_p$, the quantity

$Q$ arrives, and the inventory level not only meets the pre-sale order but then begins to decrease in response to market demand until time $T$.

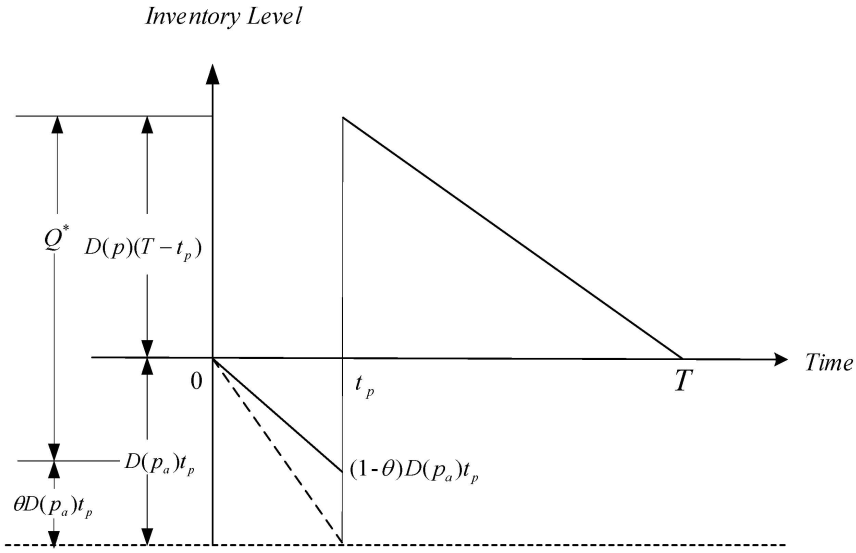


**Figure 1.** Diagram of changes in the retailer's inventory level with advance sales.

The objective here is to maximize the retailer's total profit, which comprises the following components:

(a) The sales revenue is $(1-\theta)p_aD(p_a)t_p + pD(p)\left(T-t_p\right)$.
(b) The deposit income arising from orders canceled is $\beta\theta p_aD(p_a)t_p$.
(c) The cost of placing an order is $S$.
(d) The cost of purchasing is $c\left[(1-\theta)D(p_a)t_p + D(p)\left(T-t_p\right)\right]$.
(e) The cost of carrying inventory (excluding interest payable) is $\frac{hD(p)(T-t_p)^2}{2}$.
(f) The carbon tax is $c_r\left\{\hat{S} + \hat{c}\left[(1-\theta)D(p_a)t_p + D(p)\left(T-t_p\right)\right] + \frac{\hat{h}D(p)\left(T-t_p\right)^2}{2}\right\}$.
(g) The interest payable and interest earned.

To calculate the interest payable and interest earned, based on whether the payment is made before or after the end of the spot selling period, the retailer has the following two alternatives: (i) $T \leq t_p + M$ and (ii) $T \geq t_p + M$. Figure 2 displays the cumulative quantity to earn interest and to incur interest charges in these two situations. When $T \leq t_p + M$ (please refer to Figure 2a), it means that the time to pay for the goods is later than the order cycle, so the retailer will not have to pay interest and can earn interest through the income during the pre-sale and sales period. On the contrary, if $T \geq t_p + M$ (please refer to Figure 2b), the retailer will have to pay interest after time point $t_p + M$. The detailed calculations of interest payable and earned for the two alternatives are described below.

Alternative 1:

In this scenario, the permissible payment time ends either during or after the spot selling period. As a result, the retailer does not incur any interest charges for the items kept in stock. Moreover, the retailer leverages the sales revenue to earn interest at the rate of $I_e$ during the period $\left[0, t_p + M\right]$. First, during the advance selling period [0, $t_p$], all customers are offered a unit advance sales price $(1-\delta)p$ for their purchases and are required to pay a deposit $\beta(1-\delta)p$ for a pre-committed order. Once the advance selling period is over (i.e., the beginning of the spot selling period), the customer will receive the item he/she pre-ordered and has to pay the remaining balance to the retailer. By using the deposit income, the retailer can earn interest, $\frac{p_aI_e\beta D(p_a){t_p}^2}{2} + p_aI_e\beta D(p_a)t_pM$.

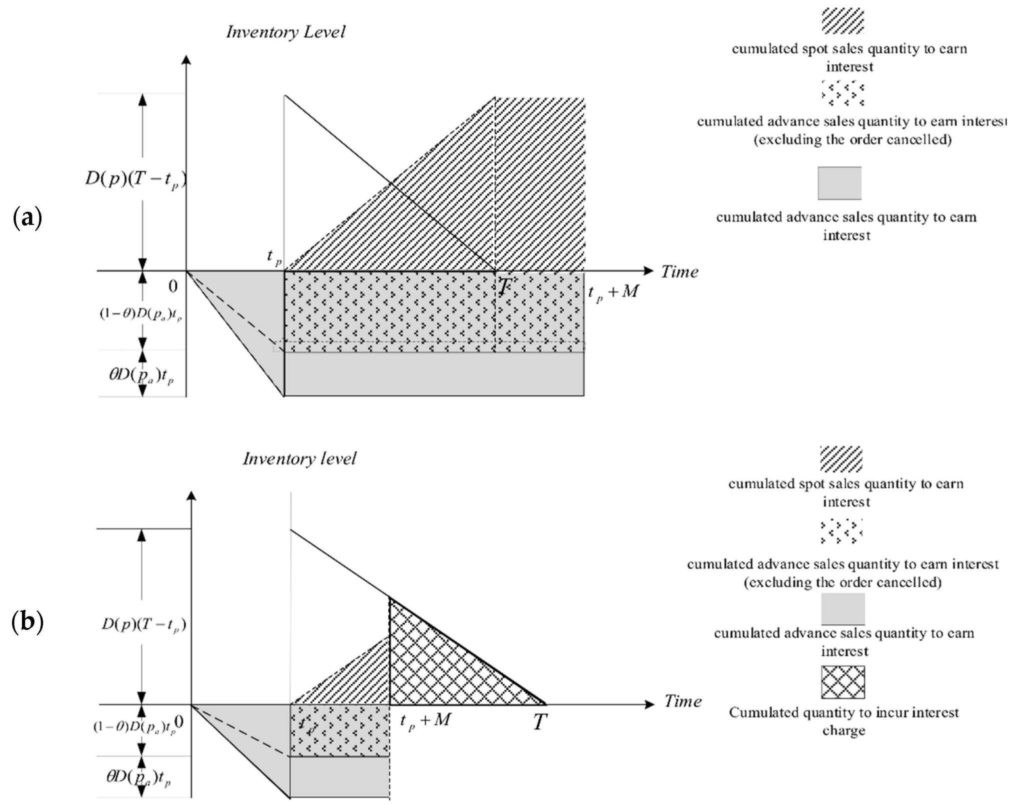


**Figure 2.** The retailer's cumulated quantity to earn interest and to incur interest charges (**a**) Case 1. $T \leq t_p + M$; (**b**) Case 2. $T \geq t_p + M$.

Further, at time $t_p$, the customers who does not cancel their orders will receive their purchased items and pay the retailer the remaining. Therefore, during the trade credit period $[t_p, t_p + M]$, the retailer uses this amount to earn interest $p_a I_e(1-\beta)(1-\theta)D(p_a)t_p M$. In addition, during the spot selling period $[t_p, T]$, the retailer sells the products and uses the sales revenue to earn interest. Therefore, the interest earned during $[t_p, t_p + M]$ is $\frac{pI_eD(p)(T-t_p)^2}{2} + pI_eD(p)(T-t_p)(t_p + M - T)$. Thus, the interest earned during this sales season, including both advance sales and spot sales, is as follows:

$$
\begin{aligned}
&\frac{p_a I_e \beta D(p_a){t_p}^2}{2} + p_a I_e \beta D(p_a) t_p M + p_a I_e (1-\beta)(1-\theta) D(p_a) t_p M \\
&\quad + \frac{pI_eD(p)(T-t_p)^2}{2} + pI_eD(p)(T-t_p)(t_p+M-T) \\
&= p_a I_e D(p_a) t_p \left[\frac{\beta t_p}{2} + (1-\theta+\beta\theta)M\right] + pI_eD(p)(T-t_p)\left(M - \frac{T-t_p}{2}\right)
\end{aligned}
$$

Alternative 2:

In this case, the permissible payment time ends on or prior to the termination of the spot selling period. The interest payable is $\frac{cI_cD(p)(T-t_p-M)^2}{2}$. Similar to the situation in Alternative 1, by using the deposit income, the retailer can earn interest $\frac{p_a I_e \beta D(p_a){t_p}^2}{2}$ $+ p_a I_e \beta D(p_a) t_p M$. During the trade credit period $[t_p, t_p + M]$, the retailer uses the remaining amount of the advance sales revenue (excluding the deposit income) to earn interest $p_a I_e(1-\beta)(1-\theta)D(p_a)t_p M$. In addition, during the spot selling period, the retailer sells the products and earns interest $\frac{pI_eD(p)M^2}{2}$ using the revenue generated from sales. Thus, the interest earned during this sales season including advance sales and spot sales is as follows:

$$\begin{aligned}\frac{p_aI_e\beta D(p_a){t_p}^2}{2} &+ p_aI_e\beta D(p_a)t_pM + p_aI_e(1-\beta)(1-\theta)D(p_a)t_pM + \frac{pI_eD(p)M^2}{2} \\ &= p_aI_eD(p_a)t_p\left[\frac{\beta t_p}{2} + (1-\theta+\beta\theta)M\right] + \frac{pI_eD(p)M^2}{2}\end{aligned}$$

Therefore, from the above and substituting $p_a$ for $(1-\delta)p$, it can calculate the retailer's total profit as:

$$Z(T,\delta,p) = \begin{cases} Z_1(T,\delta,p), & if\ T \le t_p + M, \\ Z_2(T,\delta,p), & if\ T \ge t_p + M, \end{cases} \tag{1}$$

where:

$$\begin{aligned}Z_1(T,\delta,\ p) = &(1-\theta+\beta\theta)(1-\delta)p[a-b(1-\delta)p]t_p + p(a-bp)\left(T-t_p\right) - \left(S+c_r\hat{S}\right) \\ &-(c+c_r\hat{c})\left\{(1-\theta)[a-b(1-\delta)p]t_p + (a-bp)\left(T-t_p\right)\right\} \\ &-\frac{\left(h+c_r\hat{h}\right)(a-bp)\left(T-t_p\right)^2}{2} + (1-\delta)pI_e[a-b(1-\delta)p]t_p \\ &\times\left[\frac{\beta t_p}{2} + M(1-\theta+\beta\theta)\right] + pI_e(a-bp)\left(T-t_p\right)\left(M - \frac{T-t_p}{2}\right)\end{aligned} \tag{2}$$

and:

$$\begin{aligned}Z_2(T,\delta,\ p) = &(1-\theta+\beta\theta)(1-\delta)p[a-b(1-\delta)p]t_p + p(a-bp)\left(T-t_p\right) - \left(S+c_r\hat{S}\right) \\ &-(c+c_r\hat{c})\left\{(1-\theta)[a-b(1-\delta)p]t_p + (a-bp)\left(T-t_p\right)\right\} \\ &-\frac{\left(h+c_r\hat{h}\right)(a-bp)\left(T-t_p\right)^2}{2} - \frac{cI_c(a-bp)\left(T-t_p-M\right)^2}{2} \\ &+(1-\delta)pI_e[a-b(1-\delta)p]t_p\left[\frac{\beta t_p}{2} + M(1-\theta+\beta\theta)\right] + \frac{pI_e(a-bp)M^2}{2}\end{aligned} \tag{3}$$

Note that $Z_1\left(t_p+M,\delta,p\right) = Z_2\left(t_p+M,\delta,p\right)$. Hence, for fixed $\delta$ and p, $Z(T,\delta,p)$ is continuous at point $T = t_p + M$.

## 5. Model Solution and Theorical Results

This section that follows presents the solution procedure and identifies the optimal solution for the two cases previously mentioned. The purpose is to determine $T^*$, $\delta^*$, and $p^*$, which maximize the total profit $Z(T,\delta,p)$ shown in Equation (1).

First, for given $T$ and $\delta$, the necessary condition for the total profit in Equations (2) and (3) are $\frac{dZ_1(T,\delta,\ p)}{dp} = 0$ and $\frac{dZ_2(T,\delta,\ p)}{dp} = 0$, which give:

$$\begin{aligned}\frac{dZ_1(T,\delta,\ p)}{dp} = &(1-\theta+\beta\theta)(1-\delta)[a-2b(1-\delta)p]t_p + (a-2bp)\left(T-t_p\right) \\ &+(c+c_r\hat{c})b\left[(1-\theta)(1-\delta)t_p + \left(T-t_p\right)\right] + \frac{\left(h+c_r\hat{h}\right)b\left(T-t_p\right)^2}{2} \\ &+(1-\delta)I_e[a-2b(1-\delta)p]t_p\left[\frac{\beta t_p}{2} + (1-\theta+\beta\theta)M\right] \\ &+I_e(a-2bp)\left(T-t_p\right)\left(M - \frac{T-t_p}{2}\right) = 0,\end{aligned} \tag{4}$$

and:

$$\begin{aligned}\frac{dZ_2(T,\delta,\ p)}{dp} &= (1-\theta+\beta\theta)(1-\delta)[a-2b(1-\delta)p]t_p + (a-2bp)\left(T-t_p\right) \\ &+(c+c_r\hat{c})b\left[(1-\theta)(1-\delta)t_p + \left(T-t_p\right)\right] + \frac{\left(h+c_r\hat{h}\right)b\left(T-t_p\right)^2}{2} \\ &+\frac{cI_c b\left(T-t_p-M\right)^2}{2} + (1-\delta)I_e[a-2b(1-\delta)p]t_p\left[\frac{\beta t_p}{2}+(1-\theta+\beta\theta)M\right] \\ &+\frac{I_e(a-2bp)M^2}{2} = 0,\end{aligned} \tag{5}$$

Then, the following theorem is given.

**Theorem 1.** *For given $T$ and $\delta$, the total profits $Z_1(T,\delta,p)$ and $Z_2(T,\delta,p)$ have unique global maximum values at the points $p=p_1$ and $p_2$, respectively, where $p_1$ and $p_2$ can be solved by Equations* (4) *and* (5).

**Proof.** Please see Appendix A. □

Next, for given $p$, taking the first-order and second-order partial derivatives of $Z_1(T,\delta,\ p)$ with respect to $T$ and $\delta$, we have:

$$\begin{aligned}\frac{\partial Z_1(T,\delta,\ p)}{\partial T} &= (a-bp)\left[p-(c+c_r\hat{c})-\left(h+c_r\hat{h}\right)\left(T-t_p\right)+pI_e\left(t_p+M-T\right)\right] \\ \frac{\partial Z_1(T,\delta,\ p)}{\partial \delta} &= -pt_p\{(1-\theta+\beta\theta)[a-2b(1-\delta)p]+(c+c_r\hat{c})(1-\theta)b \\ &+I_e[a-2b(1-\delta)p]\left[\frac{\beta t_p}{2}+(1-\theta+\beta\theta)M\right]\Bigg\} \\ \frac{\partial^2 Z_1(T,\delta,\ p)}{\partial T^2} &= -(a-bp)\left[\left(h+c_r\hat{h}\right)-pI_e\right]\end{aligned} \tag{6}$$

and:

$$\frac{\partial^2 Z_1(T,\delta,\ p)}{\partial \delta^2} = -2bp^2t_p\left\{(1-\theta+\beta\theta)-I_e\left[\frac{\beta t_p}{2}+(1-\theta+\beta\theta)M\right]\right\} < 0 \tag{7}$$

Furthermore, for given $p$, the following can be obtained:

$$\frac{\partial^2 Z_1(T,\delta,\ p)}{\partial T\partial\delta} = \frac{\partial^2 Z_1(T,\delta,\ p)}{\partial\delta\partial T} = 0 \tag{8}$$

Therefore, for given $p$, the determinant of the Hessian matrix at the point $(T,\delta)$ is:

$$\begin{vmatrix}\frac{\partial^2 Z_1(T,\delta,p)}{\partial T^2}, & \frac{\partial^2 Z_1(T,\delta,p)}{\partial T\partial\delta} \\ \frac{\partial^2 Z_1(T,\delta,p)}{\partial\delta\partial T}, & \frac{\partial^2 Z_1(T,\delta,p)}{\partial\delta^2}\end{vmatrix} = 2bp^2t_p(a-bp)\left(h+c_r\hat{h}-pI_e\right)\left\{(1-\theta+\beta\theta)-I_e\left[\frac{\beta t_p}{2}+M(1-\theta+\beta\theta)\right]\right\} > 0 \tag{9}$$

Hence, for given $p$, the Hessian matrix is a negative definite at the point $(T,\delta)$. Consequently, the optimal solution occurs at the point $(T,\delta)$, which satisfies $\frac{\partial Z_1(T,\delta,p)}{\partial T}=0$ and $\frac{\partial Z_1(T,\delta,p)}{\partial\delta}=0$, simultaneously.

Setting Equations (6) and (7) to zero and then solving for $T$ and $\delta$, we obtain the optimal solutions of $T$ and $\delta$ (denoted by $T_1$ and $\delta_1$, respectively) as follows:

$$T_1 = t_p + \frac{p-(c+c_r\hat{c})+pI_eM}{h+c_r\hat{h}+pI_e} \tag{10}$$

and:

$$\delta_1 = \frac{2bp - a}{2bp} - \frac{(c + c_r\hat{c})(1-\theta)}{2p\left[(1-\theta+\beta\theta)(1+I_eM) + \frac{I_e\beta t_p}{2}\right]} \tag{11}$$

Similarly, for given $p$, taking the first-order and second-order partial derivatives of $Z_2(T, \delta, p)$ with respect to $T$ and $\delta$, it obtains:

$$\frac{\partial Z_2(T,\delta,p)}{\partial T} = (a-bp)\left[p - (c+c_r\hat{c}) - \left(h + c_r\hat{h}\right)(T-t_p) + cI_c(T - t_p - M)\right] \tag{12}$$

$$\begin{aligned}\frac{\partial Z_2(T,\delta,\ p)}{\partial \delta} = -pt_p\{&(1-\theta+\beta\theta)[a-2b(1-\delta)p] + (c+c_r\hat{c})(1-\theta)b \\ &+I_e[a-2b(1-\delta)p]\left[\frac{\beta t_p}{2} + M(1-\theta+\beta\theta)\right]\right\}\end{aligned} \tag{13}$$

$$\frac{\partial^2 Z_2(T,\delta,\ p)}{\partial T^2} = -(a-bp)\left[\left(h + c_r\hat{h}\right) + cI_c\right] < 0 \tag{14}$$

and:

$$\frac{\partial^2 Z_2(T,\delta,p)}{\partial \delta^2} = -2bp^2t_p\left\{(1-\theta+\beta\theta) - I_e\left[\frac{\beta t_p}{2} + M(1-\theta+\beta\theta)\right]\right\} < 0 \tag{15}$$

Similarly, for given $p$, the following can be obtained:

$$\frac{\partial^2 Z_2(T,\delta,p)}{\partial T\partial\delta} = \frac{\partial^2 Z_2(T,\delta,p)}{\partial\delta\partial T} = 0 \tag{16}$$

Therefore, for given $p$, the determinant of the Hessian matrix at the point $(T, \delta)$ is:

$$\begin{vmatrix} \frac{\partial^2 Z_2(T,\delta,p)}{\partial T^2}, & \frac{\partial^2 Z_2(T,\delta,p)}{\partial T\partial\delta} \\ \frac{\partial^2 Z_2(T,\delta,p)}{\partial\delta\partial T}, & \frac{\partial^2 Z_2(T,\delta,p)}{\partial\delta^2} \end{vmatrix} = 2bp^2t_p(a-bp)\left(h + c_r\hat{h} + cI_c\right)\left\{(1-\theta+\beta\theta) - I_e\left[\frac{\beta t_p}{2} + M(1-\theta+\beta\theta)\right]\right\} > 0 \tag{17}$$

Hence, the Hessian matrix is a negative definite at the point $(T, \delta)$. Consequently, the optimal solution occurs at the point $(T, \delta)$, which satisfies $\frac{\partial Z_2(T,\delta,p)}{\partial T} = 0$ and $\frac{\partial Z_2(T,\delta,p)}{\partial\delta} = 0$, simultaneously.

Setting Equations (12) and (13) to zero and then solving for $T$ and $\delta$ , we obtain the optimal solutions of $T$ and $\delta$ (denoted by $T_2$ and $\delta_2$, respectively) as follows:

$$T_2 = t_p + \frac{p - (c + c_r\hat{c}) + cI_cM}{\left(h + c_r\hat{h}\right) + cI_c} \tag{18}$$

and:

$$\delta_2 = \frac{2bp - a}{2bp} - \frac{(c + c_r\hat{c})(1-\theta)}{2p\left[(1-\theta+\beta\theta)(1+I_eM) + \frac{I_e\beta t_p}{2}\right]}. \tag{19}$$

Summarizing the above results, we establish the following theorem to help the retailer to obtain the optimal ordering policy when he/she is provided the permissible delay by the supplier; meanwhile, he/she provides the customers advance sales.

**Theorem 2.** *For given p and* $\Delta_1 < 0 < \Delta_2$*, we have:*

*(a)* *If* $p - (c + c_r\hat{c}) \le \left(h + c_r\hat{h}\right)M$*, the optimal solution is* $T = T_1$ *and* $\delta = \delta_1$ *given in (12) and (13), respectively.*

*(b)* *If* $p - (c + c_r\hat{c}) \ge \left(h + c_r\hat{h}\right)M$*, the optimal solution is* $T = T_2$ *and* $\delta = \delta_2$ *given in (20) and (21), respectively.*

**Proof.** Please see Appendix A. □

The management interpretation in Theorem 2 is that if the unit holding cost for time $M$, $\left(h + c_r\hat{h}\right)M$, is higher than or equal to the gross margin, $p - (c + c_r\hat{c})$, the optimal spot selling period $T^*$ will be shorter than the trade credit period $M$. Otherwise, the optimal spot selling period $T^*$ will be longer than or equal to the trade credit period $M$.

Now, the following algorithm can be established to obtain the optimal solution $(T^*, \delta^*, p^*)$ of the entire problem. The convergence of the procedure can easily be proved by adopting a similar graphical technique used in Ouyang et al. [39]). The flowchart of the algorithm is also shown as in Figure 3.

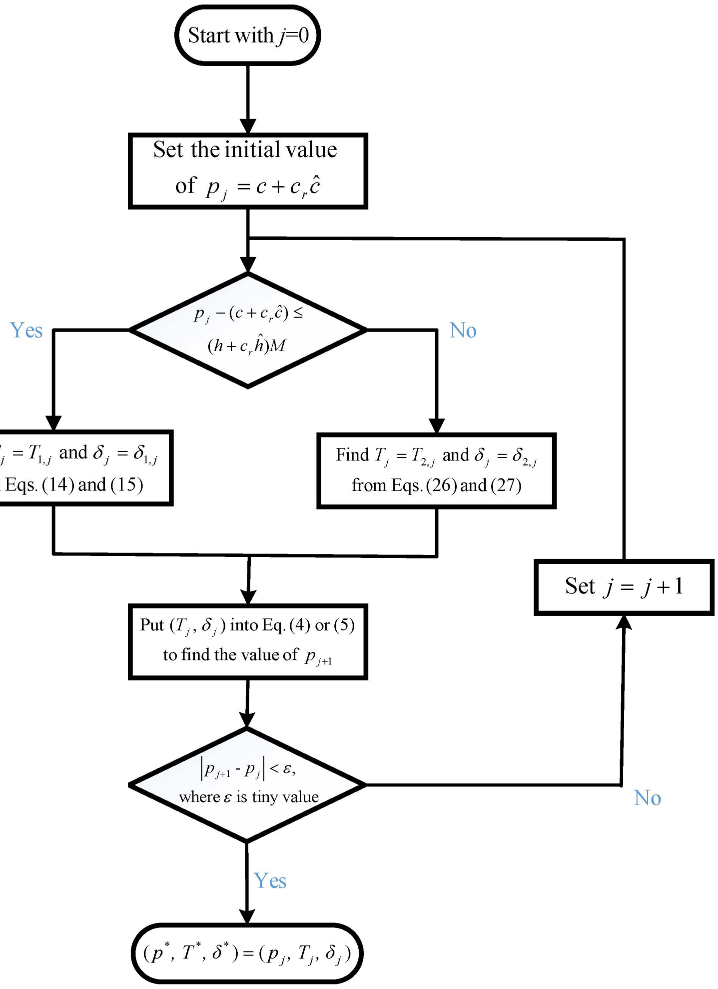


**Figure 3.** The flowchart of solution process for the proposed model.

*Algorithm*

**Step 1.** Start with $j = 0$ and the initial value of $p_j = c + c_r\hat{c}$.

**Step 2.** Check the values of $p_j - (c + c_r\hat{c})$ and $\left(h + c_r\hat{h}\right)M$.

**Step 2-1.** If $p_j - (c + c_r\hat{c}) \le \left(h + c_r\hat{h}\right)M$, calculate the values of $T_j = T_{1,j}$ and $\delta_j = \delta_{1,j}$, put $(T_j, \delta_j)$ into (4) to solve the value of $p_{j+1} = p_{1,\, j+1}$, and go to Step 3.

**Step 2-2.** If $p_j - (c + c_r\hat{c}) \ge \left(h + c_r\hat{h}\right)M$, calculate the values of $T_j = T_{2,j}$ and $\delta_j = \delta_{2,j}$, put $(T_j, \delta_j)$ into (5) to solve the value of $p_{j+1} = p_{2,\, j+1}$, and go to Step 3.

**Step 3.** If the difference between $p_{j+1}$ and $p_j$ is tiny, set $p^* = p_{j+1}$, $T^* = T_{j+1}$, and $\delta^* = \delta_{j+1}$, and $(p^*, T^*, \delta^*)$ is the optimal solution. Otherwise, set $j = j+1$ and go back to Step 2.

**Step 4.** Stop.

Once the optimal solution $(p^*, T^*, \delta^*)$ is obtained, the optimal order quantity $Q^* = [a - b(1-\delta^*)p^*]t_p + (a - bp^*)(T^* - t_p)$ and the total amount of carbon emissions (denoted by $CE^*$) $CE^* = \hat{S} + \hat{c}\{(1-\theta)[a - b(1-\delta^*)p^*]t_p + (a - bp^*)(T^* - t_p)\} + \frac{\hat{h}(a-bp^*)(T^*-t_p)^2}{2}$ and the maximum total profit $Z^* = Z(T^*, \delta^*, p^*)$ follow.

## 6. Numerical Examples

In this section, we first provide two numerical examples for different alternatives according to the algorithm described in the previous section. Following, based on Examples 1 and 2, we explore the impact of changes in parameters in the proposed models on the optimal solutions, total carbon emissions, and total profits.

**Example 1.** *Supposing a books retailer in Taiwan provides some real data about one of their items as follows. The supplier offers a permissible delay if the payment is made within 60 days (i.e., $M = 2$ month) , while the retailer offers customers 30 days to place their orders in advance (i.e., $t_p = 1$ month). The interest earned per TWD per year is 12% (i.e., interest rate per month, $I_e = 0.01$) and the interest charge per TWD investment in stocks per year is 18% (i.e., interest rate per month, $I_c = 0.015$). In addition, c = TWD 182/unit, a = 800, b = 2.5, $\theta = 0.2$, $\beta = 0.5$, $S =$ TWD 500, $h$ = TWD 50/unit/month, $\hat{S} = 50$ kg/order, $\hat{c} = 1.5$ kg/unit, $\hat{h} = 0.2$ kg/unit/month and $c_r$ = TWD 0.5/kg. Based on the above given parameters, it is found that the optimal decision happens in Alternative 1 ($T \le t_p + M$), and the optimal solution is $(T^*, \delta^*, p^*) = (T_1, \delta_1, p_1) = (2.7990, 0.1232, 272.057)$ by the algorithm in Figure 3. Therefore, the optimal amounts of carbon emissions can be obtained as $CE^* = 656.599$ kg, the optimal order quantity is $Q^* = 378.54$ units, and the total profit is $Z^* = Z_1(T_1, \delta_1, p_1)$ = TWD 25,874.*

**Example 2.** *Consider the same data as in Example 1, except the holding cost h = TWD 30/unit/month, it is found that the optimal decision happens in Alternative 2 ($T > t_p + M$), and the optimal solutions is $(T^*, \delta^*, p^*) = (T_2, \delta_2, p_2) = (3.9204, 0.1241, 272.329)$. The optimal amount of carbon emissions is $CE^* = 918.094$ kg, the optimal order quantity is $Q^* = 510.968$ units, and the total profit is $Z^* = Z_2(T_2, \delta_2, p_2)$ = TWD 32,128. By comparing Example 1, it can be seen that the holding cost has a significant impact on the alternation of the optimal solution.*

**Example 3.** *In this example, we explore the effect of different values of $c_r$, M, and $t_p$ on the optimal solutions. The data used are the same as in Example 2, and the computational results are shown in Table 2. From the results shown in Table 2, the following observations can be made:*

*(1) When the value of M is higher (for example, $M \ge 2$ in Table 2), the retailer is more likely to choose Alternative 1, which implies the length of the retailer's inventory period, ($T - t_p$), will be less than the length of the trade credit period, M.*

*(2) As the length of advance selling period $t_p$ increases, the optimal spot selling period $T^*$, the optimal advance sales discount rate $\delta^*$, and the optimal spot selling price $p^*$ increase. When Alternative 1 is the optimal decision, the optimal amounts of carbon emissions, order quantity, and total profit increase as the length of advance selling period $t_p$ increases. In contrast, when Alternative 2 is the optimal decision, the optimal amounts of carbon emissions, order quantity, and total profit first increase and then decrease once the length of the advance selling period $t_p$ increases.*

*(3) With the increase in the length of trade credit period M, all the optimal spot selling period $T^*$, the optimal advance sales discount rate $\delta^*$, the amounts of carbon emissions, order quantity, and total profit increase while the optimal spot selling price $p^*$ decreases.*

*(4) As the carbon tax $c_r$ increases, all the optimal spot selling period $T^*$, the optimal advance sales discount rate $\delta^*$, amounts of carbon emissions, order quantity, and total profit decrease while*

*the optimal spot selling price $p^*$ increases. Further, when considering the scenario where $c_r$ is 0, the proposed model can be simplified to a special model without carbon policy, which is simpler to Cheng and Ouyang [22].*

**Table 2.** The optimal solutions under different values of $c_r$, $M$, and $T_p$..

| $c_r$ | $M$ | $t_p$ | Alternatives | $T^*$ | $\delta^*$ | $p^*$ | $Q^*$ | $CE^*$ | $Z^*$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | Alternative2 | 2.7932 | 0.1235 | 273.101 | 371.507 | 644.960 | 24,760 |
| | | 2 | Alternative2 | 3.7932 | 0.1247 | 273.101 | 534.064 | 888.796 | 40,032 |
| | | 3 | Alternative2 | 7.4251 | 0.3791 | 333.755 | 524.444 | 769.328 | 32,909 |
| | 2 | 1 | Alternative1 | 2.8116 | 0.1236 | 271.809 | 381.817 | 662.263 | 26,204 |
| | | 2 | Alternative1 | 3.8116 | 0.1247 | 271.809 | 546.640 | 909.497 | 42,133 |
| | | 3 | Alternative1 | 4.8116 | 0.1259 | 271.809 | 712.703 | 1158.590 | 58,429 |
| | 3 | 1 | Alternative1 | 2.8708 | 0.1249 | 270.949 | 395.212 | 685.736 | 27,775 |
| | | 2 | Alternative1 | 3.8708 | 0.1261 | 270.948 | 562.236 | 936.271 | 44,368 |
| | | 3 | Alternative1 | 4.8708 | 0.1272 | 270.948 | 730.465 | 1188.610 | 61,331 |
| 0.5 | 1 | 1 | Alternative2 | 2.7800 | 0.1231 | 273.347 | 368.214 | 639.275 | 24,439 |
| | | 2 | Alternative2 | 3.7800 | 0.1243 | 273.347 | 530.123 | 882.137 | 39,589 |
| | | 3 | Alternative2 | 7.5381 | 0.3873 | 334.090 | 532.016 | 775.482 | 31,480 |
| | 2 | 1 | Alternative1 | 2.7990 | 0.1232 | 272.057 | 378.540 | 656.599 | 25,874 |
| | | 2 | Alternative1 | 3.7990 | 0.1244 | 272.057 | 542.723 | 902.873 | 41,680 |
| | | 3 | Alternative1 | 4.7990 | 0.1255 | 272.057 | 708.152 | 1151.020 | 57,852 |
| | 3 | 1 | Alternative1 | 2.8581 | 0.1246 | 271.193 | 391.883 | 679.949 | 27,434 |
| | | 2 | Alternative1 | 3.8581 | 0.1257 | 271.193 | 558.276 | 929.539 | 43,902 |
| | | 3 | Alternative1 | 4.8581 | 0.1268 | 271.193 | 725.880 | 1180.940 | 60,739 |
| 1 | 1 | 1 | Alternative2 | 2.7669 | 0.1227 | 273.594 | 364.943 | 633.634 | 24,121 |
| | | 2 | Alternative2 | 3.7669 | 0.1239 | 273.594 | 526.203 | 875.524 | 39,150 |
| | | 3 | Alternative2 | 7.6498 | 0.3955 | 334.394 | 539.784 | 781.877 | 30,003 |
| | 2 | 1 | Alternative1 | 2.7864 | 0.1228 | 272.305 | 375.284 | 650.978 | 25,547 |
| | | 2 | Alternative1 | 3.7864 | 0.1240 | 272.305 | 538.828 | 896.293 | 41,230 |
| | | 3 | Alternative1 | 4.7864 | 0.1251 | 272.305 | 703.622 | 1143.480 | 57,278 |
| | 3 | 1 | Alternative1 | 2.8454 | 0.1242 | 271.437 | 388.575 | 674.207 | 27,095 |
| | | 2 | Alternative1 | 3.8454 | 0.1253 | 271.437 | 554.338 | 922.851 | 43,439 |
| | | 3 | Alternative1 | 4.8454 | 0.1264 | 271.437 | 721.316 | 1173.320 | 60,150 |

**Example 4.** *This example conducts a sensitivity analysis on the parameters of the proposed model. For convenience, the impacts of parameters on the optimal solutions on the basis of the numerical values in Example 1 are discussed. Each parameter is increased or decreased by 25% or 50%, and the remaining parameters are held constant. The results of the sensitivity analysis are presented in Table 3 and Figure 4.*

**Table 3.** Impacts of major parameters on the optimal solutions.

| Parameters | Values | $T^*$ | $\delta^*$ | $p^*$ | $Q^*$ | $CE^*$ | $Z^*$ |
|---|---|---|---|---|---|---|---|
| $a$ | 640 | 2.6021 | 0.1005 | 229.604 | 204.648 | 373.910 | 8458 |
| | 720 | 3.2485 | 0.1130 | 250.881 | 339.574 | 606.275 | 17,744 |
| | 800 | 3.9204 | 0.1241 | 272.329 | 510.968 | 918.094 | 32,128 |
| | 880 | 4.5910 | 0.1334 | 293.734 | 718.001 | 1314.84 | 52,759 |
| | 960 | 5.2609 | 0.1415 | 315.117 | 960.676 | 1803.66 | 80,779 |
| $b$ | 2 | 5.5957 | 0.1451 | 325.803 | 876.304 | 1677.86 | 78,318 |
| | 2.25 | 4.6655 | 0.1344 | 296.111 | 668.884 | 1233.03 | 49,931 |
| | 2.5 | 3.9204 | 0.1241 | 272.329 | 510.968 | 918.094 | 32,128 |
| | 2.75 | 3.3096 | 0.1141 | 252.834 | 389.051 | 689.432 | 20,714 |
| | 3 | 2.8130 | 0.1047 | 236.656 | 294.731 | 521.688 | 13,287 |
| $S$ | 40 | 3.9204 | 0.1241 | 272.329 | 510.968 | 918.094 | 32,138 |
| | 45 | 3.9204 | 0.1241 | 272.329 | 510.968 | 918.094 | 32,133 |
| | 50 | 3.9204 | 0.1241 | 272.329 | 510.968 | 918.094 | 32,128 |
| | 55 | 3.9204 | 0.1241 | 272.329 | 510.968 | 918.094 | 32,123 |
| | 60 | 3.9204 | 0.1241 | 272.329 | 510.968 | 918.094 | 32,118 |
| $h$ | 24 | 4.6015 | 0.1245 | 272.461 | 590.950 | 1090.58 | 35,883 |
| | 27 | 4.2256 | 0.1243 | 272.395 | 546.811 | 994.045 | 33,811 |
| | 30 | 3.9204 | 0.1241 | 272.329 | 510.968 | 918.094 | 32,128 |
| | 33 | 3.6676 | 0.1239 | 272.263 | 481.282 | 856.849 | 30,734 |
| | 36 | 3.4548 | 0.1237 | 272.197 | 456.292 | 806.456 | 29,561 |
| $c$ | 145.6 | 4.7123 | 0.1446 | 260.585 | 745.632 | 1373.16 | 54,893 |
| | 163.8 | 4.3144 | 0.1342 | 266.465 | 622.149 | 1130.24 | 42,436 |
| | 182 | 3.9204 | 0.1241 | 272.329 | 510.968 | 918.094 | 32,128 |
| | 200.2 | 3.5303 | 0.1143 | 278.169 | 411.877 | 734.762 | 23,743 |
| | 218.4 | 3.1432 | 0.1049 | 283.972 | 324.670 | 578.376 | 17,057 |
| $\hat{S}$ | 40 | 3.9204 | 0.1241 | 272.329 | 510.968 | 908.094 | 32,133 |
| | 45 | 3.9204 | 0.1241 | 272.329 | 510.968 | 913.094 | 32,131 |
| | 50 | 3.9204 | 0.1241 | 272.329 | 510.968 | 918.094 | 32,128 |
| | 55 | 3.9204 | 0.1241 | 272.329 | 510.968 | 923.094 | 32,126 |
| | 60 | 3.9204 | 0.1241 | 272.329 | 510.968 | 928.094 | 32,123 |
| $\hat{h}$ | 0.16 | 3.9222 | 0.124080 | 272.329 | 511.184 | 898.193 | 32,138 |
| | 0.18 | 3.9213 | 0.124079 | 272.329 | 511.076 | 908.150 | 32,133 |
| | 0.2 | 3.9204 | 0.124079 | 272.329 | 510.968 | 918.094 | 32,128 |
| | 0.22 | 3.9195 | 0.124078 | 272.329 | 510.859 | 928.026 | 32,123 |
| | 0.24 | 3.9186 | 0.124077 | 272.329 | 510.751 | 937.946 | 32,118 |
| $\hat{c}$ | 1.2 | 3.9236 | 0.12416 | 272.280 | 511.830 | 766.164 | 32,205 |
| | 1.35 | 3.9220 | 0.12412 | 272.305 | 511.399 | 842.194 | 32,166 |
| | 1.5 | 3.9204 | 0.12408 | 272.329 | 510.968 | 918.094 | 32,128 |
| | 1.65 | 3.9188 | 0.12404 | 272.353 | 510.536 | 993.866 | 32,090 |
| | 1.8 | 3.9172 | 0.12400 | 272.378 | 510.106 | 1069.51 | 32,052 |

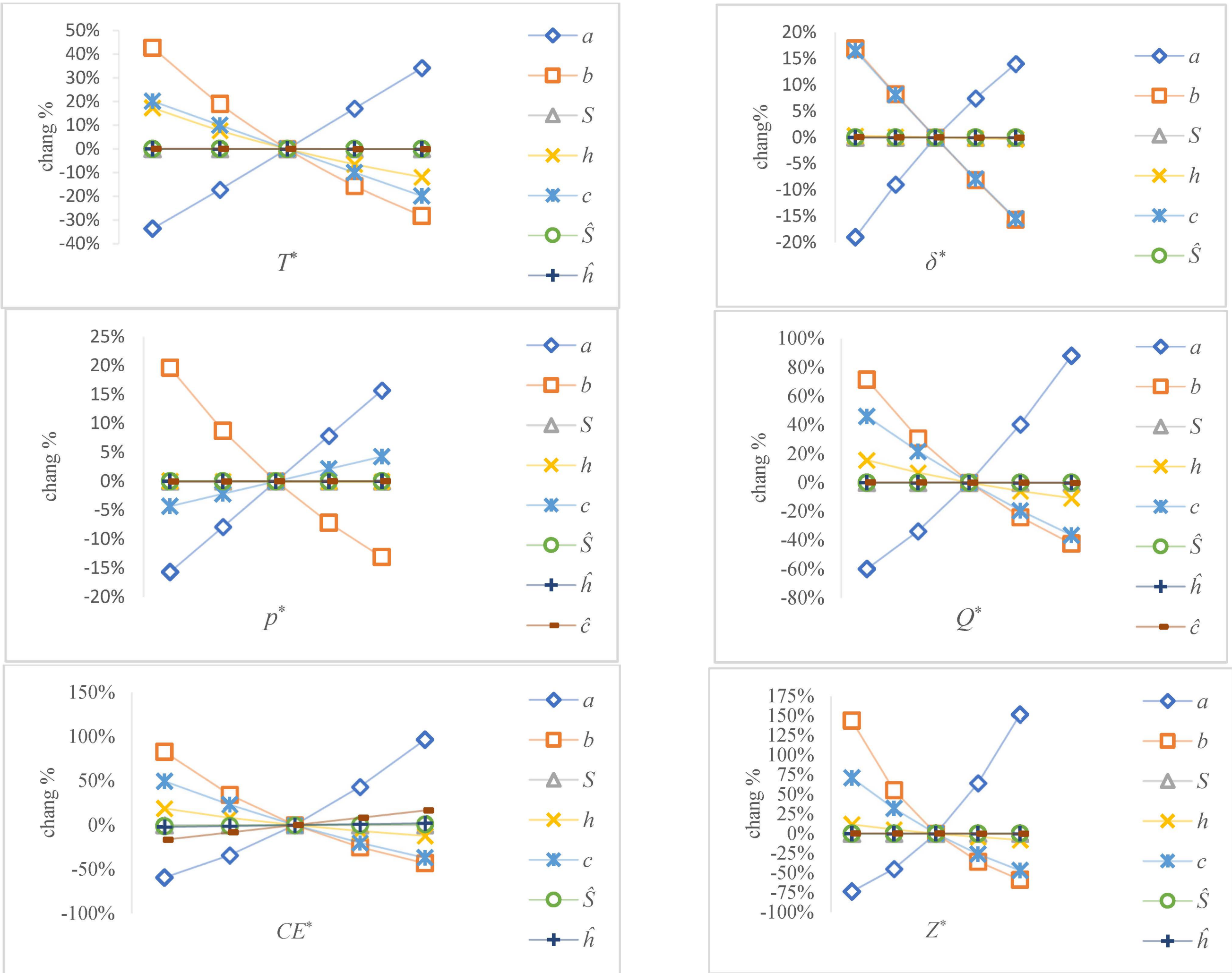


**Figure 4.** The changes in the optimal solutions for various values of major parameters.

From Table 3 and Figure 4, the following observations can be found:

(1) An increase in autonomous consumption causes an increase in the retailer's selling price, advance sales discount rate, order quantity, and total profit. In contrast, as induced consumption increases, the retailer's selling price, advance sales discount rate, order quantity, and total profit decrease.

(2) Regardless of the increase in fixed cost or amount of carbon emissions generated by the retailer per order, it will not affect the optimal solutions, but the total profit will increase accordingly.

(3) The increase in holding cost leads to the decrease in the retailer's selling price, advance sales discount rate, order quantity, and total profit.

(4) Similar to the holding cost, an increase in purchase cost results in decreases in the retailer's advance sales discount rate, order quantity, and total profit. The difference is that the increase in purchase cost will be reflected in the selling price increase.

(5) If the unit carbon emissions resulting from the retailer's purchase or storage of products increase, the retailer's selling price, advance sales discount rate, order quantity, and total profit will decrease.

## 7. Conclusions

This study developed a pricing, pre-sale incentive, and inventory model with advance sales and trade credit where the customers can commit their orders at a discounted price prior to the beginning of the selling season and the supplier allows a specified credit period to pay back the cost of goods bought without paying any interest. In order to meet the goals of sustainable development, the proposed model also takes carbon tax policy into consideration. The purpose is to obtain the optimal pricing, pre-order discount, and replenishment decisions under carbon tax policy, so as to maximize the total profit. Based on whether the payment is made before or after the end of the spot selling period, there are two alternatives considered in the proposed model. In theoretical analysis, two theorems are developed to determine the optimal alternative and solution, the amount of carbon emissions and the total profit. In the rest of this section, we propose some managerial implications from the results of the numerical analysis and then state the research limitations as well as future research directions.

**Management implications**

Numerical examples in the previous section demonstrated the solution procedures and sensitivity analyses of the optimal solutions with respect to major parameters and identified the following managerial implications:

(1) The holding cost and trade credit period have significant impacts on the alternation of the optimal solution. In particular, when facing high holding cost or long trade credit period offered by the supplier, the retailer keeps the length of inventory period as short as possible to enjoy the benefits of delayed payments. These are similar to the results of Chen and Cheng [20], Cheng and Ouyang [22], and Li et al. [31].
(2) It is known from previous studies on sustainable inventory models that all the optimal decisions, the amount of carbon emission, and total profit will decrease as the tax rate increases. However, what has not been mentioned in the previous literature is that the increase in carbon tax will make the retailer lower the advance sales discount rate to reduce the willingness of customers to pre-order.
(3) An increase in autonomous consumption leads to an increase in the retailer's selling price, advance sales discount rate, order quantity, the amount of carbon emission, and total profit, while induced consumption has the opposite effect. From an economic perspective, consumers' spontaneous consumption may gradually decrease under the trend of rising environmental protection awareness. For the retailer, it can respond by lowering price, reducing order quantity, and reducing advance sales discount. In addition, it can also develop towards green products to increase spontaneous consumption.
(4) Although an increase in fixed carbon emissions generated per order does not affect the optimal solutions, it will increase the total profit. On the other hand, an increase in unit carbon emissions from the retailer's purchasing or holding of products will lead to a decrease in the retailer's selling price, advance sales discount rate, order quantity, and total profit.
(5) The retailer can suffer negative impacts on its profitability due to an increase in holding or purchase costs, causing it to modify selling prices, advance sales discounts, order quantities, and total profit. Furthermore, while purchase cost increases lead to a rise in selling prices, holding cost increments do not.

Overall, this study provided management implications and suggestions for companies seeking to incorporate carbon emission reduction-based pricing, pre-order discounts, and replenishment policies. According to the analysis and numerical results, companies can benefit from these policies by attracting new customers, extending the selling period, and maximizing profits while contributing to global efforts to reduce carbon emissions.

**Limitations and future research**

In order to facilitate the development of the proposed model, this study has several research limitations, of course, and it also implies the following directions for future research:

(1) First, alongside carbon tax, several other carbon emission policies exist, such as carbon cap, carbon cap-and-trade, and carbon offset [5,17]. That is to say, considering different carbon emission policies is a feasible research direction.

(2) Furthermore, this study investigates a given advance selling period and trade credit period. It could be of interest to consider the situation in which the retailer determines when to start the advance sales system or faces a conditional trade credit.
(3) Finally, in order to make the model proposed in this study more general, it can also be extended to deteriorating items [25,30], allowing shortages [12], imperfect production systems [1], etc.

**Author Contributions:** Conceptualization, methodology, formal analysis, software, writing—original draft, M.-C.C.; conceptualization, formal analysis, writing—original draft, H.-C.L.; formal analysis, writing—original draft, C.-T.Y. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work was partially supported by the National Science and Technology Council, Taiwan, under grant numbers NSC 110-2410-H-231-001-MY2 and NSC 100-2410-H-570-001.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The authors would like to thank the editor and anonymous reviewers for their valuable and constructive comments, which have led to a significant improvement in the manuscript.

**Conflicts of Interest:** The authors declare no conflict of interest.

## Appendix A

**Proof of Theorem 1:** For given $T$ and $\delta$, taking the second derivatives for $Z_1(T,\delta,\ p)$ and $Z_2(T,\delta,\ p)$ with respect to $p$, it is found that:

$$\begin{aligned}\frac{d^2Z_1(T,\delta,\ p)}{dp^2} &= -2b\Big\{(1-\theta+\beta\theta)(1-\delta)^2t_p+\left(T-t_p\right)+I_e(1-\delta)^2t_p \\ &\times\left[\frac{\beta t_p}{2}+M(1-\theta+\beta\theta)\right]+I_e\left(T-t_p\right)\left(M-\frac{T-t_p}{2}\right)\Big\}<0\end{aligned} \tag{A1}$$

$$\begin{aligned}\frac{d^2Z_2(T,\delta,\ p)}{dp^2} &= -2b\Big\{(1-\theta+\beta\theta)(1-\delta)^2t_p+\left(T-t_p\right)+I_e(1-\delta)^2t_p \\ &\times\left[\frac{\beta t_p}{2}+M(1-\theta+\beta\theta)\right]+I_eM^2\Big\}<0\end{aligned} \tag{A2}$$

Therefore, $Z_1(T,\delta,\ p)$ and $Z_2(T,\delta,\ p)$ are concave functions of $p_1$ and $p_2$, where $p_1$ and $p_2$ can be solved by (4) and (5). This completes the proof. □

**Proof of Theorem 2:** To gain profit, the unit advance sales price must be higher than the unit purchasing cost, that is, $(1-\delta_1)p>(c+c_r\hat{c})$. In addition, to attract more customers with a lower price in the advance selling period, the advance sales discount rate $\delta_1$ must be greater than zero. Thus, the inequality $0<\delta_1<1-\frac{c+c_r\hat{c}}{p}$ must be satisfied. We substituted (13) into this inequality and obtained:

If $\Delta_1<0<\Delta_2$, then

$$0<\delta_1<1-\frac{c+c_r\hat{c}}{p} \tag{A3}$$

where:

$$\Delta_1\equiv[2b(c+c_r\hat{c})-a]\left[(1-\theta+\beta\theta)(1+I_eM)+\frac{I_e\beta t_p}{2}\right]-b(c+c_r\hat{c})(1-\theta) \tag{A4}$$

and:

$$\Delta_2\equiv[2bp-a]\left[(1-\theta+\beta\theta)(1+I_eM)+\frac{I_e\beta t_p}{2}\right]-b(c+c_r\hat{c})(1-\theta) \tag{A5}$$

Moreover, to ensure $T_1\le t_p+M$, we substituted (12) into the inequality $T_1\le t_p+M$ and obtained:

If $p - (c + c_r\hat{c}) \leq \left(h + c_r\hat{h}\right)M$, then

$$T_1 \leq t_p + M \tag{A6}$$

Likewise, to ensure $T_2 \geq t_p + M$, we substituted (20) into the inequality $T_2 \geq t_p + M$ and obtained:

$$\text{if } p - (c + c_r\hat{c}) \geq \left(h + c_r\hat{h}\right)M, \text{ then } T_2 \geq t_p + M. \tag{A7}$$

Following (A3), (A6), and (A7), this completes the proof. □

## References

1. Rout, C.; Paul, A.; Kumar, R.S.; Chakraborty, D.; Goswami, A. Cooperative sustainable supply chain for deteriorating item and imperfect production under different carbon emission regulations. *J. Clean. Prod.* **2020**, *272*, 122170. [CrossRef]
2. Li, R.; Teng, J.T.; Zheng, Y. Optimal credit term, order quantity and pricing policies for perishable products when demand depends on price, expiration date, and credit period. *Ann. Oper. Res.* **2019**, *280*, 377–405. [CrossRef]
3. He, P.; Zhang, W.; Xu, X.; Bian, Y. Production lot-sizing and carbon emissions under cap-and-trade and carbon tax regulations carbon tax regulations. *J. Clean. Prod.* **2014**, *103*, 241–248. [CrossRef]
4. Konur, D. Carbon constrained integrated inventory control and truckload transportation with heterogeneous freight trucks. *Int. J. Prod. Econ.* **2014**, *153*, 268–279. [CrossRef]
5. Battini, D.; Persona, A.; Sgarbossa, F. A sustainable EOQ model: Theoretical formulation and applications. *Int. J. Prod. Econ.* **2014**, *149*, 145–153. [CrossRef]
6. Daryanto, Y.; Wee, H.M. Sustainable Economic Production Quantity Models: An Approach toward a Cleaner Production. *Int. J. Adv. Manag. Sci.* **2018**, *6*, 206–212. [CrossRef]
7. Daryanto, Y.; Wee, H. Low Carbon Economic Production Quantity Model for Imperfect Quality Deteriorating Items. *Int. J. Ind. Eng. Eng. Manag.* **2019**, *1*, 1–8. [CrossRef]
8. Taleizadeh, A.A.; Soleymanfar, V.R.; Govindan, K. Sustainable economic production quantity models for inventory systems with shortage. *J. Clean. Prod.* **2018**, *174*, 1011–1020. [CrossRef]
9. Zhang, R.Y.; Liu, Q. Low carbon constrained EPQ model and computing. In Proceedings of the 2018 Eighth International Conference on Instrumentation & Measurement, Computer, Communication and Control, Harbin, China, 19–21 July 2018.
10. Xu, X.; He, P.; Xu, H.; Zhang, Q. Supply chain coordination with green technology under cap-and-trade regulation. *Int. J. Prod. Econ.* **2017**, *183*, 433–442. [CrossRef]
11. Tiwari, S.; Daryanto, Y.; Wee, H.M. Sustainable inventory management with deteriorating and imperfect quality items considering carbon emission. *J. Clean. Prod.* **2018**, *192*, 281–292. [CrossRef]
12. Mishra, U.; Wu, J.-Z.; Sarkar, B. A sustainable production-inventory model for a controllable carbon emissions rate under shortages. *J. Clean. Prod.* **2020**, *256*, 120268. [CrossRef]
13. Shen, L.; Lin, F.; Cheng, T.C.E. Low-Carbon Transition Models of High Carbon Supply Chains under the Mixed Carbon Cap-and-Trade and Carbon Tax Policy in the Carbon Neutrality Era. *Int. J. Environ. Res. Public Health* **2022**, *19*, 11150. [CrossRef] [PubMed]
14. Sadigh, A.N.; Chaharsooghi, S.K.; Sheikhmohammady, M. A game theoretic approach to coordination of pricing, advertising, and inventory decisions in a competitive supply chain. *J. Ind. Manag. Optim.* **2016**, *12*, 337–355. [CrossRef]
15. Qi, Q.; Wang, J.; Bai, Q. Pricing decision of a two-echelon supply chain with one supplier and two retailers under a carbon cap regulation. *J. Clean. Prod.* **2017**, *151*, 286–302. [CrossRef]
16. Halat, K.; Hafezalkotob, A. Modeling carbon regulation policies in inventory decisions of a multi-stage green supply chain: A game theory approach. *Comput. Ind. Eng.* **2019**, *128*, 807–830. [CrossRef]
17. Huang, Y.-S.; Fang, C.-C.; Lin, Y.-A. Inventory management in supply chains with consideration of Logistics, green investment and different carbon emissions policies. *Comput. Ind. Eng.* **2020**, *139*, 106207. [CrossRef]
18. Chang, C.-C.; Lu, C.-J.; Yang, C.-T. Multistage supply chain production–inventory model with collaborative preservation technology investment. *Sci. Iran.* **2022**, *29*, 2099–2114. [CrossRef]
19. Tsao, Y.-C. Retailer's optimal ordering and discounting policies under advance sales discount and trade credits. *Comput. Ind. Eng.* **2009**, *56*, 208–215. [CrossRef]
20. Chen, M.L.; Cheng, M.C. Optimal order quantity under advance sales and permissible delays in payments. *Afr. J. Bus. Manag.* **2011**, *5*, 7325–7334.
21. Dye, C.-Y.; Hsieh, T.-P. Joint pricing and ordering policy for an advance booking system with partial order cancellations. *Appl. Math. Model.* **2013**, *37*, 3645–3659. [CrossRef]
22. Cheng, M.C.; Ouyang, L.Y. Advance sales system with price-dependent demand and an appreciation period under trade credit. *Int. J. Inf. Manag. Sci.* **2014**, *25*, 251–262.
23. Youjun, Z.; Yan, Z.; Liang, C. An inventory model with advance sales and demand rate dependent on price. In Proceedings of the 2015 12th International Conference on Service Systems and Service Management, Guangzhou, China, 22–24 June 2015.

24. Seref, M.M.H.; Seref, O.; Alptekinoglu, A.; Erengüç, S.S. Advance selling to strategic consumers. *Comput. Manag. Sci.* **2016**, *13*, 597–626.
25. Cheng, M.-C.; Hsieh, T.-P.; Lee, H.-M.; Ouyang, L.-Y. Optimal ordering policies for deteriorating items with a return period and price-dependent demand under two-phase advance sales. *Oper. Res.* **2020**, *20*, 585–604. [CrossRef]
26. Duary, A.; Das, S.; Arif, M.G.; Abualnaja, K.M.; Khan, M.A.A.; Zakarya, M.; Shaikh, A.A. Advance and delay in payments with the price-discount inventory model for deteriorating items under capacity constraint and partially back-logged shortages. *Alex. Eng. J.* **2022**, *61*, 1735–1745. [CrossRef]
27. Lou, K.-R.; Wang, L. Optimal lot-sizing policy for a manufacturer with defective items in a supply chain with up-stream and down-stream trade credits. *Comput. Ind. Eng.* **2013**, *66*, 1125–1130. [CrossRef]
28. Tsao, Y.-C. Joint location, inventory, and preservation decisions for non-instantaneous deterioration items under delay in payments. *Int. J. Syst. Sci.* **2016**, *47*, 572–585. [CrossRef]
29. Zhong, Y.; Shu, J.; Xie, W.; Zhou, Y.-W. Optimal trade credit and replenishment policies for supply chain network design. *Omega* **2018**, *81*, 26–37. [CrossRef]
30. Chang, C.T.; Ouyang, L.Y.; Teng, J.T.; Lai, K.K.; Cárdenas-Barrón, L.E. Manufacturer's pricing and lot-sizing decisions for perishable goods under various payment terms by a discounted cash flow analysis. *Int. J. Prod. Econ.* **2019**, *218*, 83–95. [CrossRef]
31. Li, R.; Liu, Y.; Teng, J.-T.; Tsao, Y.-C. Optimal pricing, lot-sizing and backordering decisions when a seller demands an advance-cash-credit payment scheme. *Eur. J. Oper. Res.* **2019**, *278*, 283–295. [CrossRef]
32. Tsao, Y.-C. Coordinating contracts under default risk control-based trade credit. *Int. J. Prod. Econ.* **2019**, *212*, 168–175. [CrossRef]
33. Shi, Y.; Zhang, Z.; Chen, S.C.; Cárdenas Barrón, L.E.; Skouri, K. Optimal replenishment decisions for perishable products under cash, advance, and credit payments considering carbon tax regulations. *Int. J. Prod. Econ.* **2020**, *223*, 107514. [CrossRef]
34. Mallick, R.K.; Patra, K.; Mondal, S.K. Mixture inventory model of lost sale and back-order with stochastic lead time demand on permissible delay in payments. *Ann. Oper. Res.* **2020**, *292*, 341–369. [CrossRef]
35. Li, R.; Yang, H.-L.; Shi, Y.; Teng, J.-T.; Lai, K.-K. EOQ-based pricing and customer credit decisions under general supplier payments. *Eur. J. Oper. Res.* **2021**, *289*, 652–665. [CrossRef]
36. Li, R.; Skouri, K.; Teng, J.T.; Yang, W.G. Seller's optimal replenishment policy and payment term among advance, cash, and credit payments. *Int. J. Prod. Econ.* **2018**, *197*, 35–42. [CrossRef]
37. Chung, K.J.; Liao, J.J.; Srivastava, H.M.; Lee, S.F.; Lin, S.D. The EOQ model for deteriorating items with a conditional trade credit linked to order quantity in a supply chain system. *Mathematics* **2021**, *9*, 2311. [CrossRef]
38. Jani, M.Y.; Betheja, M.R.; Bhadoriya, A.; Chaudhari, U.; Abbas, M.; Alqahtani, M.S. Optimal Pricing Policies with an Allowable Discount for Perishable Items under Time-Dependent Sales Price and Trade Credit. *Mathematics* **2022**, *10*, 1948. [CrossRef]
39. Ouyang, L.Y.; Wu, K.S.; Yang, C.T. Coordinating replenishment and pricing policies for non-instantaneous deteriorating items with price sensitive demand. *Int. J. Syst. Sci.* **2009**, *40*, 1273–1281. [CrossRef]