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Some Further Coefficient Bounds on a New Subclass of Analytic Functions

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Abstract: The coefficient problem is an essential topic in the theory of univalent functions theory. In the present paper, we consider a new subclass \mathcal{SQ} of analytic functions with $f'(z)$ subordinated to $1/(1-z)^2$ in the open unit disk. This class was introduced and studied by Răducanu. Our main aim is to give the sharp upper bounds of the second Hankel determinant $\mathcal{H}_{2,3}(f)$ and the third Hankel determinant $\mathcal{H}_{3,1}(f)$ for $f \in \mathcal{SQ}$. This may help to understand more properties of functions in this class and inspire further investigations on higher Hankel determinants for this or other popular sub-classes of univalent functions.

Keywords: subordination; analytic functions; Hankel determinant; coefficient bounds

MSC: 30C45; 30C80



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1. Introduction and Definitions

We first give some basic concepts of analytic functions that are necessary for better understanding our further discussions in this article. Let \mathcal{A} denote the normalized analytic functions defined in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the series expansion of the form

$$f(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad z \in \mathbb{D}. \quad (1)$$

We say f is univalent, if for $z_1, z_2 \in \mathbb{D}$, $z_1 \neq z_2$ implies that $f(z_1) \neq f(z_2)$. Assuming that $\mathcal{S} \subseteq \mathcal{A}$ is the collection of the univalent functions in \mathbb{D} . \mathcal{P} is often used to denote the Carathéodory functions, which are analytic in \mathbb{D} with positive real part and normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in \mathbb{D}. \quad (2)$$

In the past years, various classes of univalent functions are intensively studied. The representative examples are the star-like functions \mathcal{S}^* , convex functions \mathcal{C} and bounded turning functions \mathcal{R} . They are defined, respectively, by

$$\mathcal{S}^* := \left\{ \Re \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D} \right\}, \tag{3}$$

$$\mathcal{C} := \left\{ \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in \mathbb{D} \right\}, \tag{4}$$

$$\mathcal{R} := \{ \Re(f'(z)) > 0, \quad z \in \mathbb{D} \}. \tag{5}$$

The relationship between the class \mathcal{C} and \mathcal{S}^* is that $f(z) \in \mathcal{C}$ if and only if $zf'(z) \in \mathcal{S}^*$, see [1]. We emphasize that the class \mathcal{R} is not a subset of \mathcal{S}^* . Additionally, \mathcal{R} does not contain \mathcal{S}^* , see [2].

It is said that g_1 is subordinate to g_2 in \mathbb{D} if there is an analytic function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $g_1(z) = g_2(\omega(z))$. We denote that g_1 is subordinate to g_2 by the notation $g_1 \prec g_2$ and the function ω is said to be a Schwarz function. In geometry, $g_1 \prec g_2$ in \mathbb{D} means that $g_1(\mathbb{D}) \subset g_2(\mathbb{D})$. In case g_2 is univalent in \mathbb{D} , the subordination $g_1 \prec g_2$ is equivalent to

$$g_1(0) = g_2(0) \quad \text{and} \quad g_1(\mathbb{D}) \subset g_2(\mathbb{D}). \tag{6}$$

Let φ be a univalent function with $\varphi'(0) > 0$ and $\Re\varphi > 0$. Suppose also that $\varphi(\mathbb{D})$ is star-like with respect to the point $\varphi(0) = 1$ and symmetric along the real line axis. Using the function φ and subordination, one can define a general class $\mathcal{S}^*(\varphi)$ by setting

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z), \quad z \in \mathbb{D} \right\}. \tag{7}$$

This class was introduced by Ma and Minda [3]. Taking $\varphi(z) = \frac{1+z}{1-z}$, $\mathcal{S}^*(\varphi)$ is the class of star-like functions \mathcal{S}^* . It was extensively investigated by many researchers through some particular choices of φ , see for example [4–11].

For $f \in \mathcal{S}$, the Hankel determinant $\mathcal{H}_{m,n}(f)$ defined by

$$\mathcal{H}_{m,n}(f) := \begin{vmatrix} b_n & b_{n+1} & \dots & b_{n+m-1} \\ b_{n+1} & b_{n+2} & \dots & b_{n+m} \\ \vdots & \vdots & \dots & \vdots \\ b_{n+m-1} & b_{n+m} & \dots & b_{n+2m-2} \end{vmatrix} \tag{8}$$

was introduced and studied by Pommerenke [12,13] early in 1966, where $m, n \in \mathbb{N}$ and $b_1 = 1$. It was shown to be an effective tool in the study of power series with integral coefficients and singularities, also in pure mathematics and applied mathematics, see for instance [14–22].

To obtain the sharp upper bound of the third Hankel determinant

$$\mathcal{H}_{3,1}(f) = 2b_2b_3b_4 - b_3^3 - b_4^2 + b_3b_5 - b_2^2b_5 \tag{9}$$

is not an easy thing. We note that the sharp bound of $|\mathcal{H}_{3,1}(f)|$ for star-like functions was just proved by Kowalczyk et al. [23] in 2022. The exact bound is $\frac{4}{9}$. Before it was solved, there are many works investigated this problem, see [24,25] and the references. For other advances in finding the bounds of the third Hankel determinant for sub-classes of univalent functions or p -valent functions, we refer to [26–35].

From the definition, we know

$$\mathcal{H}_{2,3}(f) = b_3b_5 - b_4^2. \tag{10}$$

Although it seems more simple to calculate the sharp upper bounds of $|\mathcal{H}_{2,3}(f)|$, the results on $|\mathcal{H}_{2,3}(f)|$ for star-like functions and convex functions are still not proved as we know.

In [36], Răducanu introduced a new class of analytic functions \mathcal{SQ} satisfying the condition

$$\Re \sqrt{f'(z)} > \frac{1}{2}, \quad z \in \mathbb{D}, \tag{11}$$

or in terms of subordination

$$f'(z) \prec \frac{1}{(1-z)^2}, \quad z \in \mathbb{D}. \tag{12}$$

For the functions in the class \mathcal{SQ} , the upper bounds of some initial coefficients, the second Hankel determinant $\mathcal{H}_{2,2}(f)$, and the Zalcman functional were investigated.

In the present paper, we aim to give the sharp upper bounds of the second Hankel determinant $\mathcal{H}_{2,3}(f)$ and the third Hankel determinant $\mathcal{H}_{3,1}(f)$ for functions in this class.

2. A set of Lemmas

To prove our main results, we need the following lemmas. The first lemma is often used to connect the coefficients of the proposed function class and the Carathéodory functions.

Lemma 1 (see [37]). *Let $p \in \mathcal{P}$ be of the form (2). Then,*

$$2p_2 = p_1^2 + \zeta(4 - p_1^2), \tag{13}$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1\zeta - p_1(4 - p_1^2)\zeta^2 + 2(4 - p_1^2)(1 - |\zeta|^2)\delta, \tag{14}$$

$$8p_4 = p_1^4 + (4 - p_1^2)\zeta [p_1^2(\zeta^2 - 3\zeta + 3) + 4\zeta] - 4(4 - p_1^2)(1 - |\zeta|^2) [p_1(\zeta - 1)\delta + \bar{\zeta}\delta^2 - (1 - |\delta|^2)\rho] \tag{15}$$

for some $\zeta, \delta, \rho \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

We will use the following results to prove that the maximum value of our obtained three variables function is achieved on one face of its defined domain.

Lemma 2. *For all $(p, q) \in [0, 2) \times [\frac{1}{2}, 1]$, we have*

$$(2 + p)^2(2 - p)q^2 + (2 + p)(3p^2 + 74p - 64)q + 6p^3 - 78p^2 + 120 \geq 0. \tag{16}$$

Proof. Let

$$F_1(p, q) := (2 + p)^2(2 - p)q^2 + (2 + p)(3p^2 + 74p - 64)q + 6p^3 - 78p^2 + 120. \tag{17}$$

It is noted that

$$\begin{aligned} F_1(p, q) &\geq 2(2 + p)(2 - p)q^2 + (2 + p)(3p^2 + 74p - 64)q + 6p^3 - 78p^2 + 120 \\ &= (2 + p)[2(2 - p)q^2 + (3p^2 + 74p - 64)q] + 6p^3 - 78p^2 + 120 \\ &=: (2 + p)F_2(p, q) + 6p^3 - 78p^2 + 120, \end{aligned}$$

where

$$F_2(p, q) = 2(2 - p)q^2 + (3p^2 + 74p - 64)q. \tag{18}$$

Let $\tilde{p}_0 = \frac{\sqrt{1561}-37}{3} \approx 0.8365$ be the only positive root of the equation $3p^2 + 74p - 64 = 0$ lies on the interval $[0, 2)$. If $p \geq \tilde{p}_0$, we obtain $3p^2 + 74p - 64 \geq 0$ and thus $F_2(p, q) \geq F_2\left(p, \frac{1}{2}\right)$. Then,

$$F_1(p, q) \geq \frac{15}{2}p^3 - \frac{77}{2}p^2 + 42p + 58 =: \omega(p), \quad p \in [\tilde{p}_0, 2). \tag{19}$$

Since $\omega'(p) = \frac{45}{2}p^2 - 77p + 42 = 0$ has no positive roots lie on $[\tilde{p}_0, 2)$, we find that $\omega(p) \geq \omega(2) = 48 > 0$. Hence, we find that $F_1(p, q) \geq 0$ on $[\tilde{p}, 2) \times \left[\frac{1}{2}, 1\right]$.

Fix $p \in [0, \tilde{p})$, let us take F_2 as a quadratic polynomial with respect to q . Then, the symmetric axis of F_2 is defined by

$$q_0 = \frac{64 - 74p - 3p^2}{4(2 - p)} > 0. \tag{20}$$

Let $\tilde{p}_1 = \frac{\sqrt{1393}-5}{3} \approx 0.7743$ be the only positive root of the equation $3p^2 + 70p - 56 = 0$. If $p \leq \tilde{p}_1$, we have $q_0 \geq 1$. Then, $F_2(p, q) \geq F_2(p, 1)$, which induces to

$$F_1(p, q) \geq (2 + p)F_2(p, 1) + 6p^3 - 78p^2 + 120 = 3p(28 + 3p^2) \geq 0. \tag{21}$$

If $p \in (\tilde{p}_1, \tilde{p}_0)$, from $\frac{1}{2} \leq q \leq 1$ it is found that

$$\begin{aligned} F_1(p, q) &\geq (2 + p)^2(2 - p) \cdot \frac{1}{4} + (2 + p)(3p^2 + 74p - 64) \cdot 1 + 6p^3 - 78p^2 + 120 \\ &= \frac{35}{4}p^3 + \frac{3}{2}p^2 + 85p - 6 \geq 85p - 6 > 0. \end{aligned}$$

Hence, $F_1(p, q) \geq 0$ for all $(p, q) \in [0, \tilde{p}_0) \times \left[\frac{1}{2}, 1\right]$. Now, we can conclude that $F_1(p, q) \geq 0$ on $[0, 2) \times \left[\frac{1}{2}, 1\right]$. The assertion in Lemma 2 thus follows. \square

Lemma 3. For all $(p, q) \in [0, 2) \times \left[\frac{1}{2}, 1\right]$, we have

$$(1 + p)(4 - p^2)q^2 + (3p^3 + 40p^2 + 84p - 64)q + 6p^3 - 39p^2 + 60 \geq 0. \tag{22}$$

Proof. Let

$$F_3(p, q) := (1 + p)(4 - p^2)q^2 + (3p^3 + 40p^2 + 84p - 64)q + 6p^3 - 39p^2 + 60. \tag{23}$$

By the basic fact that $-q^2 + 40q - 39 \geq -\frac{75}{4}$ and $4q^2 - 64q + 60 \geq 0$, we have

$$\begin{aligned} F_3(p, q) &\geq (4 - p^2)q^2 + (40p^2 + 84q - 64)q - 39p^2 + 60 \\ &= (-q^2 + 40q - 39)p^2 + (84q)p + 4q^2 - 64q + 60 \\ &\geq -\frac{75}{4}p^2 + 42p \geq 0. \end{aligned}$$

Then, we obtain the inequality in Lemma 3. \square

3. Main Results

The sharp upper bounds of the second Hankel determinant $|\mathcal{H}_{2,2}(f)|$ for $f \in \mathcal{SQ}$ was obtained in [36], we further consider the sharp upper bounds of $|\mathcal{H}_{2,3}(f)|$ for functions in this class.

Theorem 1. Suppose that $f \in \mathcal{SQ}$. Then,

$$|\mathcal{H}_{2,3}(f)| \leq \frac{2}{5}. \tag{24}$$

The bound is sharp with the extremal function given by

$$f_1(z) = \frac{1}{2} \arctan(z) + \frac{z}{2(1-z^2)} = z + \frac{2}{3}z^3 + \frac{3}{5}z^5 + \frac{4}{7}z^7 + \dots, \quad z \in \mathbb{D}. \tag{25}$$

Proof. Let $f \in \mathcal{SQ}$. Using subordination principal, a Schwarz function ω exists so that

$$f'(z) = \frac{1}{(1-\omega(z))^2}, \quad z \in \mathbb{D}. \tag{26}$$

Suppose that

$$\chi(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1z + p_2z^2 + p_3z^3 + p_4z^4 + \dots, \quad z \in \mathbb{D}, \tag{27}$$

we note that $\chi \in \mathcal{P}$ and

$$\omega(z) = \frac{\chi(z)-1}{\chi(z)+1} = \frac{p_1z + p_2z^2 + p_3z^3 + p_4z^4 + \dots}{2 + p_1z + p_2z^2 + p_3z^3 + p_4z^4 + \dots}, \quad z \in \mathbb{D}. \tag{28}$$

Using (1), we obtain

$$f'(z) = 1 + 2b_2z + 3b_3z^2 + 4b_4z^3 + 5b_4z^4 + \dots, \quad z \in \mathbb{D}. \tag{29}$$

Using (28), it is observed that

$$\begin{aligned} \frac{1}{(1-\omega(z))^2} &= 1 + p_1z + \left(p_2 + \frac{1}{4}p_1^2\right)z^2 + \left(p_3 + \frac{1}{2}p_1p_2\right)z^3 \\ &+ \left(p_4 + \frac{1}{2}p_1p_3 + \frac{1}{4}p_2^2\right)z^4 + \dots, \quad z \in \mathbb{D}. \end{aligned} \tag{30}$$

Comparing the coefficients of (29) and (30), we have

$$b_2 = \frac{1}{2}p_1, \tag{31}$$

$$b_3 = \frac{1}{3}\left(p_2 + \frac{1}{4}p_1^2\right), \tag{32}$$

$$b_4 = \frac{1}{4}\left(p_3 + \frac{1}{2}p_1p_2\right), \tag{33}$$

$$b_5 = \frac{1}{5}\left(p_4 + \frac{1}{2}p_1p_3 + \frac{1}{4}p_2^2\right). \tag{34}$$

Let $f \in \mathcal{SQ}$ and $f_\beta(z) = e^{-i\beta}f(e^{i\beta}z)$, $\beta \in \mathbb{R}$. Then,

$$\Re \sqrt{f'_\beta(z)} = \Re \sqrt{f'(e^{i\beta}z)} > \frac{1}{2}, \quad z \in \mathbb{D}. \tag{35}$$

Thus, $f_\beta \in \mathcal{SQ}$. As

$$\mathcal{H}_{2,3}(f_\beta) = e^{6i\beta}(b_3b_5 - b_4^2) = e^{6i\beta}\mathcal{H}_{2,3}(f), \tag{36}$$

we know $|\mathcal{H}_{2,3}(f_\beta)| = |\mathcal{H}_{2,3}(f)|$ for every $\beta \in \mathbb{R}$. This makes it possible to assume that when estimating $|\mathcal{H}_{2,3}(f)|$, one selected coefficient of f is a non-negative real number, see [38]. Assume b_2 is real and $p_1 := p \in [0, 2]$. Substituting (31)–(34) into (10), we have

$$\mathcal{H}_{2,3}(f) = \frac{1}{960} \left(8p^3 p_3 + 16p_2^3 - 28pp_2p_3 + 64p_2p_4 + 16p^2p_4 - 60p_3^2 - 11p^2p_2^2 \right). \tag{37}$$

Let $\alpha = 4 - p^2$. By applying Lemma 1 and inserting the formulae of p_2, p_3 and p_4 into (37), we obtain

$$\begin{aligned} \mathcal{H}_{2,3}(f) = \frac{1}{960} \{ & 16\bar{\zeta}^3\alpha^2 + 2\bar{\zeta}^3\alpha^3 + 24p^2\bar{\zeta}^2\alpha - 9p^4\bar{\zeta}^2\alpha + p^4\bar{\zeta}\alpha + 6p^4\bar{\zeta}^3\alpha \\ & - \frac{27}{4}p^2\bar{\zeta}^2\alpha^2 + \frac{13}{2}p^2\bar{\zeta}^3\alpha^2 + \frac{1}{4}p^2\bar{\zeta}^4\alpha^2 - 15\alpha^2(1 - |\bar{\zeta}|^2)^2\delta^2 \\ & + 6p^3\alpha(1 - |\bar{\zeta}|^2)\delta - 16|\bar{\zeta}|^2\alpha^2(1 - |\bar{\zeta}|^2)\delta^2 - p\bar{\zeta}^2\alpha^2(1 - |\bar{\zeta}|^2)\delta \\ & - 21p\bar{\zeta}\alpha^2(1 - |\bar{\zeta}|^2)\delta - 24p^3\bar{\zeta}\alpha(1 - |\bar{\zeta}|^2)\delta - 24p^2\bar{\zeta}\alpha(1 - |\bar{\zeta}|^2)\delta^2 \\ & + 24p^2\alpha(1 - |\bar{\zeta}|^2)(1 - |\delta|^2)\rho + 16\bar{\zeta}\alpha^2(1 - |\bar{\zeta}|^2)(1 - |\delta|^2)\rho \}, \end{aligned}$$

where $\bar{\zeta}, \delta, \rho$ satisfying $|\bar{\zeta}| \leq 1, |\delta| \leq 1$ and $|\rho| \leq 1$. After rearrangements, we can put $\mathcal{H}_{2,3}(f)$ in the form of

$$\mathcal{H}_{2,3}(f) = \frac{1}{960} [u_1(p, \bar{\zeta}) + u_2(p, \bar{\zeta})\delta + u_3(p, \bar{\zeta})\delta^2 + \Phi(p, \bar{\zeta}, \delta)\rho], \tag{38}$$

where

$$\begin{aligned} u_1(p, \bar{\zeta}) &= \frac{1}{4} (4 - p^2) \bar{\zeta} [p^2 (4 - p^2) \bar{\zeta}^3 + 6(p^4 - 4p^2 + 64) \bar{\zeta}^2 \\ &\quad - 3p^2 (4 + 3p^2) \bar{\zeta} + 4p^4], \\ u_2(p, \bar{\zeta}) &= (4 - p^2) (1 - |\bar{\zeta}|^2) p [-(4 - p^2) \bar{\zeta}^2 - (3p^2 + 84) \bar{\zeta} + 6p^2], \\ u_3(p, \bar{\zeta}) &= (4 - p^2) (1 - |\bar{\zeta}|^2) [(4 - p^2) (-|\bar{\zeta}|^2 - 15) - 24p^2 \bar{\zeta}], \\ \Phi(p, \bar{\zeta}, \delta) &= 8(4 - p^2) (1 - |\bar{\zeta}|^2) (1 - |\delta|^2) [2(4 - p^2) \bar{\zeta} + 3p^2]. \end{aligned}$$

Let $|\bar{\zeta}| =: q$ and $|\delta| =: y$. By $|\rho| \leq 1$, it follows that

$$\begin{aligned} |\mathcal{H}_{2,3}(f)| &\leq \frac{1}{960} [|u_1(p, \bar{\zeta})| + |u_2(p, \bar{\zeta})|y + |u_3(p, \bar{\zeta})|y^2 + |\Phi(p, \bar{\zeta}, \delta)|] \\ &\leq \frac{1}{960} [\Gamma(p, q, y)], \end{aligned} \tag{39}$$

where

$$\Gamma(p, q, y) = \sigma_1(p, q) + \sigma_2(p, q)y + \sigma_3(p, q)y^2 + \sigma_4(p, q)(1 - y^2), \tag{40}$$

with

$$\begin{aligned} \sigma_1(p, q) &= \frac{1}{4} (4 - p^2) q [p^2 (4 - p^2) q^3 + 6(p^4 - 4p^2 + 64) q^2 \\ &\quad + 3p^2 (4 + 3p^2) q + 4p^4], \\ \sigma_2(p, q) &= (4 - p^2) (1 - q^2) p [(4 - p^2) q^2 + (84 + 3p^2) q + 6p^2], \\ \sigma_3(p, q) &= (4 - p^2) (1 - q^2) [(4 - p^2) (q^2 + 15) + 24p^2 q], \\ \sigma_4(p, q) &= 8(4 - p^2) (1 - q^2) [2(4 - p^2) q + 3p^2]. \end{aligned}$$

The inequality (39) was obtained from the fact that $|u_j(p, \xi)| \leq \sigma_j(p, q)$ for $j = 1, 2, 3$ and $|\Phi(p, \xi, \delta)| \leq \sigma_4(p, q)(1 - |\delta|^2)$. Here, $|u_1(p, \xi)| \leq \sigma_1(p, q)$ follows from $4 - p^2 \geq 0$ and $p^4 - 4p^2 + 64 \geq 0$ on $[0, 2]$.

Now, the main work is turning to find the maximum value of Γ in the closed cuboid $\Omega := [0, 2] \times [0, 1] \times [0, 1]$. In virtue of $\Gamma(0, 1, 1) = 384$, we have $\max_{(p,q,y) \in \Omega} \Gamma(p, q, y) \geq 384$. In the following, it is shown that $\max_{(p,q,y) \in \Omega} \Gamma(p, q, y) = 384$.

Setting $p = 2$, $\Gamma(2, q, y) \equiv 0$. When $q = 1$, we have

$$\Gamma(p, 1, y) = -\frac{9}{2}p^6 + 20p^4 - 104p^2 + 384 =: r_1(p). \tag{41}$$

According to the observation of $-\frac{9}{2}p^6 + 20p^4 - 104p^2 \leq 0$ for $p \in [0, 2]$, it is found that r_1 has a maximum value 384 achieved at $p = 0$. Then, without loss of generality, we may choose $p < 2$ and $q < 1$ to illustrate that the maximum value of Γ is less than or equal to 384.

Take $(p, q, y) \in [0, 2) \times [0, 1) \times (0, 1)$. By differentiating partially of Γ with respect to y , we know

$$\frac{\partial \Gamma}{\partial y} = \sigma_2(p, q) + 2[\sigma_3(p, q) - \sigma_4(p, q)]y. \tag{42}$$

Let $\frac{\partial \Gamma}{\partial y} = 0$. Then, the critical point \tilde{y}_0 is given by

$$\tilde{y}_0 = \frac{(4 - p^2)pq^2 + (84 + 3p^2)pq + 6p^3}{2(1 - q)[(4 - p^2)q + 39p^2 - 60]}. \tag{43}$$

Since we have $\tilde{y}_0 \in (0, 1)$, the following two inequalities must be satisfied simultaneously:

$$(2 + p)^2(2 - p)q^2 + (2 + p)(3p^2 + 74p - 64)q + 6p^3 - 78p^2 + 120 < 0 \tag{44}$$

and

$$p^2 > \frac{4(15 - q)}{39 - q}. \tag{45}$$

Now we have to obtain the solutions fulfilling inequalities (44) and (45) to guarantee the existence of critical points with $\tilde{y}_0 \in (0, 1)$. From Lemma 2, it is noted that the inequality (44) is impossible to hold for $q \in [\frac{1}{2}, 1)$. For any critical points $(\hat{p}, \hat{q}, \hat{y})$, we know $\hat{q} < \frac{1}{2}$ provided that $\hat{y} \in (0, 1)$.

If we take $\varrho(t) = \frac{4(5-t)}{39-t}$, it is seen that ϱ is decreasing over $[0, 1]$ in view of $\varrho'(t) < 0$ in $[0, 1]$. From (45), it follows that $\hat{p}^2 \geq \varrho(\frac{1}{2}) = \frac{36}{77}$. Now we consider $(p, q, y) \in [\sqrt{\frac{36}{77}}, 2) \times [0, \frac{1}{2}) \times (0, 1)$. From $1 - q^2 \leq 1$ and $q < \frac{1}{2}$, we know

$$\sigma_1(p, q) \leq \sigma_1\left(p, \frac{1}{2}\right) =: \tau_1(p) \tag{46}$$

and

$$\sigma_j(p, q) \leq \frac{4}{3}\sigma_j\left(p, \frac{1}{2}\right) =: \tau_j(p), \quad j = 2, 3, 4. \tag{47}$$

Then, it is not hard to find that

$$\Gamma(p, q, y) \leq \tau_1(p) + \tau_2(p)y + \tau_3(p)y^2 + \tau_4(p)(1 - y^2) =: \Xi(p, y). \tag{48}$$

Because of $\frac{\partial \Xi}{\partial y} = \tau_2(p) + 2[\tau_3(p) - \tau_4(p)]y$, it is easy to check that

$$\frac{\partial \Xi}{\partial y} \Big|_{y=1} = \tau_2(p) + 2[\tau_3(p) - \tau_4(p)] = \frac{1}{4}(4 - p^2)r_2(p), \tag{49}$$

where $r_2(p) := 29p^3 - 154p^2 + 172p + 232$. As $r_2(p) \geq 0$ on $\left[\sqrt{\frac{36}{77}}, 2\right)$, we find that $\frac{\partial \Xi}{\partial y} \Big|_{y=1} \geq 0$. Combining the fact that $\frac{\partial \Xi}{\partial y} \Big|_{y=0} = \tau_2(p) \geq 0$ and $\frac{\partial \Xi}{\partial y}$ is linear and continuous with respect to y , we conclude that

$$\frac{\partial \Xi}{\partial y} \geq \min \left\{ \min \frac{\partial \Xi}{\partial y} \Big|_{y=0}, \min \frac{\partial \Xi}{\partial y} \Big|_{y=1} \right\} \geq 0, \quad y \in (0, 1). \tag{50}$$

This leads to $\Xi(p, y) \leq \Xi(p, 1) = \tau_1(p) + \tau_2(p) + \tau_3(p) =: r_3(p)$. It is an easy task to check that

$$r_3(p) = -\frac{79}{64}p^6 - \frac{29}{4}p^5 + \frac{65}{8}p^4 - 14p^3 - \frac{343}{4}p^2 + 172p + 292. \tag{51}$$

For $0 \leq p \leq 2$, we have

$$\begin{aligned} r_3(p) &\leq \frac{65}{8}p^4 - 14p^3 - \frac{343}{4}p^2 + 172p + 292 \\ &\leq \frac{65}{4}p^3 - 14p^3 - \frac{343}{4}p^2 + 172p + 292 \\ &= \frac{9}{4}p^3 - \frac{343}{4}p^2 + 172p + 292 \\ &\leq \frac{9}{2}p^2 - \frac{343}{4}p^2 + 172p + 292 \\ &= -\frac{325}{4}p^2 + 172p + 292 \leq 384. \end{aligned}$$

Thus, $\Gamma(p, q, y) < 384$ on $\left[\sqrt{\frac{36}{77}}, 2\right) \times \left[0, \frac{1}{2}\right) \times (0, 1)$, which further gives that $\Gamma(\hat{p}, \hat{q}, \hat{y}) < 384$. Therefore, it is left to discuss the boundary points $\partial\Omega$ to find the maximum value of Γ .

If we set $y = 0$ and $y = 1$, it is seen that

$$\Gamma(p, q, 0) = \sigma_1(p, q) + \sigma_4(p, q) \tag{52}$$

and

$$\Gamma(p, q, 1) = \sigma_1(p, q) + \sigma_2(p, q) + \sigma_3(p, q). \tag{53}$$

Then, we have

$$\begin{aligned} \Gamma(p, q, 1) - \Gamma(p, q, 0) &= \sigma_2(p, q) + \sigma_3(p, q) - \sigma_4(p, q) \\ &= (4 - p^2)(1 - q^2)\Lambda(p, q), \end{aligned}$$

where

$$\Lambda(p, q) = (1 + p)(4 - p^2)q^2 + (3p^3 + 40p^2 + 84p - 64)q + 6p^3 - 39p^2 + 60. \tag{54}$$

From Lemma 3, we see $\Lambda(p, q) \geq 0$ on $[0, 2) \times \left[\frac{1}{2}, 1\right)$, which induces to

$$\Gamma(p, q, 0) \leq \Gamma(p, q, 1), \quad (p, q) \in [0, 2) \times \left[\frac{1}{2}, 1\right). \tag{55}$$

When $q < \frac{1}{2}$, then from (46) and (47), we obtain

$$\Gamma(p, q, 0) \leq \tau_1(p) + \tau_4(p) =: \eta(p). \tag{56}$$

Using some basic calculations, it is found that

$$\eta(p) = -\frac{79}{64}p^6 - \frac{89}{8}p^4 + \frac{81}{4}p^2 + 176, \tag{57}$$

which has a maximum value of about 184.4481 attained at $p \approx 0.8960$. Then, we can say that

$$\Gamma(p, q, 0) < 384, \quad [0, 2) \times \left[0, \frac{1}{2}\right). \tag{58}$$

Based on both (55) and (58), it remains to find the maximum value of Γ on the face $y = 1$ of Ω .

When $y = 1$, we have

$$\Gamma(p, q, 1) = (4 - p^2) \left[v_4(p)q^4 + v_3(p)q^3 + v_2(p)q^2 + v_1(p)q + v_0(p) \right] =: Q(p, q), \tag{59}$$

where

$$\begin{aligned} v_4(p) &= \frac{1}{4}(4 - p^2)(p^2 - 4p - 4), \\ v_3(p) &= \frac{3}{2}p^4 - 3p^3 - 30p^2 - 84p + 96, \\ v_2(p) &= \frac{9}{4}p^4 - 7p^3 + 17p^2 + 4p - 56, \\ v_1(p) &= p(p^3 + 3p^2 + 24p + 84), \\ v_0(p) &= 6p^3 - 15p^2 + 60. \end{aligned}$$

The last work is to calculate the maximum value of Q on $[0, 2] \times [0, 1]$. On the vertices $(0, 0)$, $(2, 0)$, $(0, 1)$ and $(2, 1)$, we have $Q(2, 0) = Q(2, 1) = 0$, $Q(0, 0) = 240$ and $Q(0, 1) = 384$.

If we take the sides of $[0, 2] \times [0, 1]$, we have

$$Q(0, q) = -16q^4 + 384q^3 - 224q^2 + 240 =: s(q). \tag{60}$$

As $s'(q) = -64q^3 + 1152q^2 - 448q = 0$ has only one positive root $\hat{q}_0 = 9 - \sqrt{74} \approx 0.3977$, we know the maximum value of s is 384 attained at $q = 1$. When $p = 2$, then $Q(2, q) \equiv 0$ on $[0, 1]$.

For the case of $(0, 2) \times (0, 1)$, we determine the critical points of Q by solving the system of equations

$$\frac{\partial Q}{\partial q} = (4 - p^2) \left[v_1(p) + 2v_2(p)q + 3v_3(p)q^2 + 4v_4(p)q^3 \right] = 0 \tag{61}$$

and

$$\begin{aligned} \frac{\partial Q}{\partial p} &= (4 - p^2) \left[v'_0(p) + v'_1(p)q + v'_2(p)q^2 + v'_3(p)q^3 + v'_4(p)q^4 \right] \\ &\quad - 2p \left[v_0(p) + v_1(p)q + v_2(p)q^2 + v_3(p)q^3 + v_4(p)q^4 \right] = 0, \end{aligned}$$

it is got that there are no critical points lie in $(0, 2) \times (0, 1)$. According to all the above discussions, we say $Q(p, q) \leq 384$ on $[0, 2] \times [0, 1]$. That is, $\Gamma(p, q, y) \leq 384$ for all $(p, q) \in [0, 2] \times [0, 1] \times [0, 1]$, which gives the conclusion that

$$|\mathcal{H}_{2,3}(f)| \leq \frac{384}{960} = \frac{2}{5}. \tag{62}$$

For the sharpness, it is noted that for the function f_1 defined in (25), we have

$$f_1'(z) = \frac{1}{(1 - z^2)^2}, \quad z \in \mathbb{D}. \tag{63}$$

Thus, $f_1 \in \mathcal{SQ}$ according to the definition where the Schwarz function can be chosen as $\omega(z) = z^2$. Additionally, $\mathcal{H}_{2,3}(f_1) = \frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}$. The proof of Theorem 1 is then completed. \square

The third Hankel determinant was widely studied for various interesting sub-classes of univalent functions. In the following, we give the sharp bounds of $|\mathcal{H}_{3,1}(f)|$ for our considered function class.

Theorem 2. *Suppose that $f \in \mathcal{SQ}$. Then,*

$$|\mathcal{H}_{3,1}(f)| \leq \frac{1}{4}. \tag{64}$$

The bound is sharp with the extremal function given by

$$f_2(z) = \int_0^z \frac{1}{(1 - t^3)^2} dt = z + \frac{1}{2}z^4 + \frac{3}{7}z^7 + \frac{2}{5}z^{10} + \dots, \quad z \in \mathbb{D}. \tag{65}$$

Proof. From the definition, the third Hankel determinant is determined by

$$\mathcal{H}_{3,1}(f) = 2b_2b_3b_4 - b_4^2 - b_2^2b_5 - b_3^3 + b_3b_5. \tag{66}$$

Taking $f \in \mathcal{SQ}$ and $f_\beta(z) = e^{-i\beta}f(e^{i\beta}z)$, $\beta \in \mathbb{R}$, we have $f_\beta \in \mathcal{SQ}$ and

$$\mathcal{H}_{3,1}(f_\beta) = e^{6i\beta}\mathcal{H}_{3,1}(f). \tag{67}$$

That is to say, $|\mathcal{H}_{3,1}(f_\beta)| = |\mathcal{H}_{3,1}(f)|$ for every $\beta \in \mathbb{R}$. It allows us to choose b_2 of f to be real when estimating $|\mathcal{H}_{3,1}(f)|$. From (31) we may assume $p_1 = p \in [0, 2]$. Substituting (31)–(34) into (66), the result is given by

$$\begin{aligned} \mathcal{H}_{3,1}(f) = & \frac{1}{8640} \left(36p^3p_3 - 5p^6 - 176p_2^3 + 30p^4p_2 + 468pp_2p_3 \right. \\ & \left. + 576p_2p_4 - 288p^2p_4 - 540p_3^2 - 87p^2p_2^2 \right). \end{aligned} \tag{68}$$

Let $\alpha = 4 - p^2$. An application of Lemma 1 shows that

$$\begin{aligned} \mathcal{H}_{3,1}(f) = & \frac{1}{8640} \left\{ -22\zeta^3\alpha^3 + 144\zeta^3\alpha^2 + \frac{9}{4}p^2\zeta^2\alpha^2 - \frac{63}{2}p^2\zeta^3\alpha^2 + \frac{9}{4}p^2\zeta^4\alpha^2 \right. \\ & - 135\alpha^2(1 - |\zeta|^2)^2\delta^2 - 9p\zeta^2\alpha^2(1 - |\zeta|^2)\delta - 9p\zeta\alpha^2(1 - |\zeta|^2)\delta \\ & \left. - 144|\zeta|^2\alpha^2(1 - |\zeta|^2)\delta^2 + 144\alpha^2\zeta(1 - |\zeta|^2)(1 - |\delta|^2)\rho \right\}, \end{aligned}$$

where $\xi, \delta, \rho \in \mathbb{D}$. From the above expression, we can write $\mathcal{H}_{3,1}(f)$ in the following form:

$$\mathcal{H}_{3,1}(f) = \frac{1}{8640} \left[v_1(p, \xi) + v_2(p, \xi)\delta + v_3(p, \xi)\delta^2 + \tilde{\Phi}(p, \xi, \delta)\rho \right], \tag{69}$$

where

$$\begin{aligned} v_1(p, \xi) &= \frac{1}{4} (4 - p^2)^2 \xi^2 \left[9p^2 \xi^2 + 2(112 - 19p^2)\xi + 9p^2 \right] \\ v_2(p, \xi) &= -9(4 - p^2)^2 (1 - |\xi|^2) p \xi ((1 + \xi)), \\ v_3(p, \xi) &= -9(4 - p^2)^2 (1 - |\xi|^2) (|\xi|^2 + 15), \\ \tilde{\Phi}(p, \xi, \delta) &= 144(4 - p^2)^2 (1 - |\xi|^2) (1 - |\delta|^2) \xi. \end{aligned}$$

Set $|\xi| =: q$ and $|\delta| =: y$. From $|\rho| \leq 1$, it induces that

$$\begin{aligned} |\mathcal{H}_{3,1}(f)| &\leq \frac{1}{8640} \left[|v_1(p, \xi)| + |v_2(p, \xi)|y + |v_3(p, \xi)|y^2 + |\tilde{\Phi}(p, \xi, \delta)| \right] \\ &\leq \frac{1}{8640} [\Theta(p, q, y)], \end{aligned} \tag{70}$$

where

$$\Theta(p, q, y) = \zeta_1(p, q) + \zeta_2(p, q)y + \zeta_3(p, q)y^2 + \zeta_4(p, q)(1 - y^2), \tag{71}$$

with

$$\begin{aligned} \zeta_1(p, q) &= \frac{1}{4} (4 - p^2)^2 \left[9p^2 q^4 + 2(112 - 19p^2)q^3 + 9p^2 q^2 \right], \\ \zeta_2(p, q) &= 9(4 - p^2)^2 (1 - q^2) pq(1 + q), \\ \zeta_3(p, q) &= 9(4 - p^2)^2 (1 - q^2) (q^2 + 15), \\ \zeta_4(p, q) &= 144(4 - p^2)^2 (1 - q^2) q. \end{aligned}$$

Here, we use the inequality $|v_1(p, \xi)| \leq \zeta_1(p, q)$, which holds on the condition that $112p^4 - 19p^2 \geq 0$ for all $p \in [0, 2]$.

Now, the problem reduces to find the maximum value of Θ in the same domain Ω . In view of

$$\zeta_3(p, q) - \zeta_4(p, q) = 9(4 - p^2)^2 (1 - q^2) (q^2 - 16q + 15) \geq 0 \tag{72}$$

on $[0, 2] \times [0, 1]$, we observe that

$$\frac{\partial \Theta}{\partial y} = \zeta_2(p, q) + 2[\zeta_3(p, q) - \zeta_4(p, q)]y \geq 0. \tag{73}$$

This gives the fact that $\Theta(p, q, y) \leq \Theta(p, q, 1)$. Then, it still needs to find the maximum value of Θ on the face $y = 1$ of Ω .

When we choose $y = 1$, it is found that

$$\begin{aligned} \Theta(p, q, 1) &= (4 - p^2)^2 \left[\frac{9}{4} (p^2 - 4p - 4)q^4 - (19p^2 + 18p - 112)q^3 \right. \\ &\quad \left. + 9(p^2 + 4p - 56)q^2 + 9pq + 135 \right] \\ &=: K_1(p, q). \end{aligned}$$

From $p^2 - 4p - 4 \leq 0$ on $[0, 2]$, it is found that

$$\begin{aligned} K_1(p, q) &\leq (4 - p^2)^2 \left[- (19p^2 + 18p - 112)q^3 + 9(p^2 + 4p - 56)q^2 \right. \\ &\quad \left. + 9pq + 135 \right] \\ &=: K_2(p, q). \end{aligned}$$

From the fact that $-19p^2 - 18p - 112 \geq 0$ on $[0, 2]$ and $q^3 \leq q^2$, it further leads to

$$\begin{aligned} K_2(p, q) &\leq (4 - p^2)^2 \left[- (19p^2 + 18p - 112)q^2 + 9(p^2 + 4p - 56)q^2 \right. \\ &\quad \left. + 9pq + 135 \right] \\ &= (4 - p^2)^2 \left[- (10p^2 - 18p + 392)q^2 + 9pq + 135 \right] \\ &=: K_3(p, q). \end{aligned}$$

By fixing p in K_3 , one can obtain the quadratic polynomial with respect to q , the coefficient of q^2 is $-(4 - p^2)^2(10p^2 - 18p + 392) \leq 0$ and the symmetric axis is defined by

$$\bar{q}_0 = \frac{9p}{2(10p^2 - 18p + 392)}. \tag{74}$$

It is easy to check that $\bar{q}_0 \in [0, 1)$ and $10p^2 - 18p + 396 \geq 360$. Thus, we obtain

$$\begin{aligned} K_3(p, q) &\leq (4 - p^2)^2 \left[135 + \frac{81p^2}{4} \cdot \frac{1}{(10p^2 - 18p + 392)} \right] \\ &\leq (4 - p^2)^2 \left(135 + \frac{9p^2}{160} \right) \\ &=: l(p). \end{aligned}$$

It is not hard to see that l has a maximum value 2160 achieved at $p = 0$. This shows $K_1(p, q) \leq 2160$ on $[0, 2] \times [0, 1]$, which provides the fact that $\Theta(p, q, y) \leq 2160$ for all $(p, q, y) \in [0, 2] \times [0, 1] \times [0, 1]$. Hence, we have

$$|\mathcal{H}_{3,1}(f)| \leq \frac{2160}{8640} = \frac{1}{4}. \tag{75}$$

The bound is sharp with the equality obtained by the function f_2 defined in (65). Clearly,

$$f_2'(z) = \frac{1}{(1 - z^3)^2}, \quad z \in \mathbb{D}. \tag{76}$$

Taking $\omega(z) = z^3$, it is known that $f_2'(z) \prec \frac{1}{(1 - z)^2}$ and thus $f_2 \in \mathcal{SQ}$. It is verified that $\mathcal{H}_{3,1}(f_2) = -\frac{1}{4}$. The proof of Theorem 2 is thus completed. \square

4. Conclusions

The coefficient problem is basic and essential in the theory of univalent functions. In this paper, we calculate the sharp bounds of the second and third Hankel determinant for a new class \mathcal{SQ} of analytic functions introduced by Răducanu. For functions in this class, it satisfies that $f'(z)$ subordinated to $1/(1 - z)^2$ in the open unit disk \mathbb{D} . We may expect that functions in \mathcal{SQ} are univalent. However, it is not proven yet. It is an interesting topic to investigate the univalence or the non-univalence and higher order Hankel determinants for functions in this class.

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