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Developable Surfaces Foliated by General Ellipses in Euclidean Space $\mathbf{R}^3$

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Abstract: In this article, we classify the developable surfaces in three-dimensional Euclidean space $\mathbf{R}^3$ that are foliated by general ellipses. We show that the surface has constant Gaussian curvature **(CGC)** and is foliated by general ellipses if and only if the surface is developable, i.e., the Gaussian curvature **G** vanishes everywhere. We characterize all developable surfaces foliated by general ellipses. Some of these surfaces are conical surfaces, and the others are surfaces generated by some special base curves.

Keywords: cyclic surfaces; Gaussian curvature

MSC: 53A05; 53A17

1. Introduction

The study of some classes of surfaces with particular properties in Euclidean space $\mathbf{R}^3$, such as constant angle surfaces, ruled surfaces, canal surfaces, minimal surfaces, cyclic surfaces, and developable surfaces, is one of the major objectives of classical differential geometry [1,2]. A cyclic surface or circular surface is a one-parameter family of regular, fixed-radius circles positioned around a curve that acts as a spine curve [3,4]. Therefore, it is possible that almost all interesting properties had been discovered before the middle of the 20th century. However, this topic has recently attracted attention in several domains (especially architecture, computer-aided design, etc. (see [5,6])). Particular cyclic surfaces have been considered in earlier papers, that is, the canal surface of a space curve, torus, and cylindrical surface are special cyclic surfaces [7,8]. In spatial kinematics, the movement of a one-parameter family of circles with a defined radius generates a cyclic surface, while the movement of a one-parameter family of lines generates a ruled surface [9–11]. The well-known examples of cyclic surfaces are tubes and surfaces of revolution [12]. Nitsche [13] studied cyclic surfaces with nonzero constant mean curvature, and he proved that the only such surfaces are the surfaces of revolution discovered by Delaunay [14].

Let s be an arc-length parameter of the curve $\alpha = \alpha(s)$, which is perpendicular to every s-plane of the foliation. Suppose that the tangent, principal normal and binormal vectors of the curve α are denoted $\{\mathfrak{T}, \mathfrak{N}, \mathfrak{B}\}$ and the planes of the foliation are not parallel. Therefore, we can parameterize the cyclic surface $\Psi(s,t)$ by

$$\mathfrak{X}(s,t) = \beta(s) + \lambda(s)\Big[\mu(s)\cos[t]\,\mathfrak{N} + \nu(s)\sin[t]\,\mathfrak{B}\Big], \quad t \in [0, 2\pi], \tag{1}$$

where $\mu = \mu(s)$, $\nu = \nu(s)$ and $\lambda = \lambda(s) > 0$ are functions of s, $\beta = \beta(s)$ denotes the center of each ellipse of the foliation, and the Frenet equations of the curve α are

$$\mathfrak{T}' = \sigma\,\mathfrak{N}, \qquad \mathfrak{N}' = -\sigma\,\mathfrak{T} + \omega\,\mathfrak{B}, \qquad \mathfrak{B}' = -\omega\,\mathfrak{N}. \tag{2}$$

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where the prime $'$ denotes the derivative with respect to the s-parameter, and σ and ω are the curvature and torsion of α, respectively. We assume that $\sigma \neq 0$ because α is not a straight line, and let

$$\beta'(s) = a\,\mathfrak{T} + b\,\mathfrak{N} + c\,\mathfrak{B} \tag{3}$$

where a, b, and c are smooth functions on s. Without loss of generality, we can assume that $\mu(s) = 1$. Now, we can discuss the following particular cases of the general case:

(1): Lopez [15–17] studied a cyclic surface foliated by a smooth, one-parameter family of circles in three-dimensional Euclidean space $\mathbf{R}^3$, which results directly from our work when $\nu(s) = 1$. In [15], he studied **the CGC** surface in $\mathbf{R}^3$ foliated by circles. In [16,17], he studied surfaces that satisfy a special Linear Weingarten **(LW)** condition of linear type as $\sigma_1 = \epsilon_1\,\sigma_2 + \epsilon_2$ and $\epsilon_3\,\mathbf{M} + \epsilon_4\,G = \epsilon_5$, where ϵ_i, $i = 1, 2, \ldots, 5$ are real numbers, and σ_1 and σ_2 denote the principal curvatures, while $\mathbf{M}$ and $\mathbf{G}$ denote the mean and Gaussian curvatures at each point of the surface. Also, he proved that *A surface of revolution is the only **CGC** cyclic surface* [15].

(2): When $\nu(s) = \epsilon_0 \neq 1$, where ϵ_0 is an arbitrary constant, the surface (1) is a surface foliated by general ellipses, which were studied by Ali and Hamdoon [18]. They proved that, with constant Gaussian curvature $\mathbf{G}$, the following are equivalent:

(a): *The surface foliated by general ellipses is **a CGC** surface.*

(b): *The surface foliated by general ellipses is developable.*

(c): *The surface foliated by general ellipses is a cylindrical surface that is part of a generalized cone or a part of a generalized cylinder.*

In this article, we will discuss and classify the surface foliated by general ellipses in the form (1) such that $\nu(s)$ is not a constant function, and we will show the following main results for zero Gaussian curvatures in $\mathbf{R}^3$:

Theorem 1. *The surface (1) foliated by general ellipses is flat if and only if it is a part of a conical surface or one of the following surfaces: (31), (48), (56), (60), or (64).*

Theorem 2. *The surface foliated by general ellipses is a **CGC** surface (1) if and only if* $\mathbf{G} = 0$.

Theorem 3. *Let Π be a **CGC** surface foliated by pieces of ellipses in parallel planes. Then,*

(1): $\mathbf{G} = 0$.

(2): *Π must be parameterized, up a rigid motion of $\mathbf{R}^3$, as*

$$\mathfrak{X}(s,t) = \Big(\epsilon_1\,s + \epsilon_0, \varepsilon_1\,s + \varepsilon_0, s\Big) + \Big(\zeta_1\,s + \zeta_0\Big)\Big(\cos[t],\ \nu_0\,\sin[t],\ 0\Big), \tag{4}$$

where ϵ_0, ϵ_1, ε_0, ε_1, ζ_0, ζ_1, $\nu_0 \in \mathbf{R}$.

As a corollary of both Theorems 2 and 3, we obtain the following.

Corollary 1. *All surfaces foliated by general ellipses with constant Gauss curvatures must be surfaces of revolution.*

Remark 1. *A conical surface or quadratic surface is a locus of points in the three-dimensional space whose coordinates in a Cartesian coordinate system $X(s,t) = (x,y,z)$ satisfy an algebraic equation of degree two:*

$$\sum_{i,j=1}^{3} a_{ij}\,x^i\,x^j + \sum_{k=0}^{3} a_k\,x^k = 0,$$

where $(x^1, x^2, x^3) = (x,y,z)$, while a_{ij} and a_k are constant coefficients. A conical surface intersects every plane in a (proper or degenerate) conic section. Moreover, the cone consisting of all tangents from a fixed point to a conical surface cuts every plane in a conic section, and the points of contact of this cone with the surface form a conic section [19]. There are 17 standard-form types of conical surfaces. An elliptic paraboloid, generalized cone, ellipsoid, sphere, hyperboloid of one *sheet, hyperboloid of two sheets, and hyperbolic paraboloid are some special conical surfaces. For generalized cylinders, for example, an elliptic cylinder, hyperbolic cylinder, parabolic cylinder, and circular cylinder are also special types of conical surfaces [20].*

Note: The calculations for our problem are very complicated, so *Mathematica* was used for computations.

2. Gaussian Curvatures

Consider Π, a surface in $\mathbf{R}^3$ parameterized by $\mathfrak{X} = \mathfrak{X}(\theta_1, \theta_2)$, and let $\mathfrak{U}$ denote the unit normal vector field on Π. The tangent vectors to the parametric curves of the surface $\mathfrak{X}(\theta_1, \theta_2)$ are

$$\mathfrak{X}_{\theta_1} = \frac{\partial \mathfrak{X}}{\partial \theta_1}, \qquad \mathfrak{X}_{\theta_2} = \frac{\partial \mathfrak{X}}{\partial \theta_2}$$

and the unit normal on this surface is given by

$$\mathfrak{U} = \frac{\mathfrak{X}_{\theta_1} \times \mathfrak{X}_{\theta_2}}{\|\mathfrak{X}_{\theta_1} \times \mathfrak{X}_{\theta_2}\|}$$

where $\times$ refers to the cross-product. The Gaussian curvature $\mathbf{G}$ is

$$G = \frac{\det(\mathfrak{H}_{ij})}{\det(\mathfrak{G}_{ij})}, \tag{5}$$

where

$$\mathfrak{G}_{ij} = \langle \mathfrak{X}_{\theta_i}, \mathfrak{X}_{\theta_j} \rangle, \qquad \mathfrak{H}_{ij} = \langle \mathfrak{X}_{\theta_i \theta_j}, \mathfrak{U} \rangle, \quad i, j = 1, 2. \tag{6}$$

To prove our results, it is necessary to transform the equation $\mathbf{G} =$ constant to an expression as a linear combination of the trigonometric functions $\{\cos[i\,t], \sin[i\,t]\}$, where i is a positive integer. Because the multi-linearity of the determinant shows that the denominator of $\mathfrak{G}$ is a trigonometric polynomial, of the form required by linearization, we can write the above equation in the following interesting form:

$$\sum_{i=0}^{8} \Big(\mathfrak{E}_i(s)\, \cos[i\,t] + \mathfrak{F}_i(s)\, \sin[i\,t] \Big) = 0 \tag{7}$$

where $\mathfrak{E}_i(s)$ and $\mathfrak{F}_i(s)$ are functions of only the variable s. Then, all these coefficients, $\mathfrak{E}_i$ and $\mathfrak{F}_i$, must equal zero. The next step is to calculate the explicitly form of the coefficients $\mathfrak{E}_i$ and $\mathfrak{F}_i$ using a series of operations. Although the scalar curvature $\mathbf{G}$ can be explicitly computed, for instance, using the *Mathematica* program, its expression is somewhat cumbersome. However, the key to our demonstrations is that $\mathbf{G}$ can be written as

$$\mathbf{G} = \frac{P\big(\cos[i\,t], \sin[i\,t]\big)}{Q\big(\cos[i\,t], \sin[i\,t]\big)} = \frac{\sum_{i=0}^{4} \big(\Upsilon_i\, \cos[i\,t] + \Lambda_i\, \sin[i\,t]\big)}{\sum_{i=0}^{8} \big(\Omega_i\, \cos[i\,t] + \Psi_i\, \sin[i\,t]\big)}. \tag{8}$$

Given that the Gaussian curvature $\mathbf{G}$ is assumed to be constant, (8) transforms into

$$P\,(\cos[i\,t], \sin[i\,t]) - G\,Q\,(\cos[i\,t], \sin[i\,t]) = 0. \tag{9}$$

Equation (9) is a linear combination of the functions $\{\cos[i\,t], \sin[i\,t]\}$; then, the corresponding coefficients must vanish. Here, it is not necessary to give the (long) expression of $\mathbf{G}$ but only the coefficients of higher order for the trigonometric functions. Assuming the curvature never vanishes, we can use $\omega(s) = \varpi(s)\,\sigma(s)$ and $b(s) = \xi(s)\,\sigma(s)$, where ϖ and ξ are functions of s.

3. Proof of Theorem 1

In this section, we assume that $\mathbf{G} = 0$ on the surface $\mathfrak{X}(t,s)$. From (9), we have

$$\mathfrak{P}(\cos[i\,t], \sin[i\,t]) = \sum_{i=0}^{4} (\Gamma_i \cos[i\,t] + \Sigma_i \sin[i\,t]) = 0.$$

Explicit computations of the coefficients Γ_i and Σ_i show that the equation $\Sigma_4 = 0$ leads to

$$\varpi' = \frac{\nu^2\,\varpi\left[2\,\nu\,a\,\xi\,\sigma^2 + \nu'\left(2\,a^2 + 3\,\lambda^2\,\sigma^2\right)\right] + 2\left(c + \varpi\,a\right)\left(a\,\nu' - \nu\,\xi\,\sigma^2\right)}{\nu\left(\nu^2 - 1\right)\lambda^2\,\sigma^2}. \tag{10}$$

Equation $\Gamma_4 = 0$ is

$$\begin{aligned}\nu'' = \frac{1}{\nu\,\lambda^2\,\sigma^2}\Bigg(&c\,\sigma^2\left[c - 2\left(\nu^2 - 1\right)a\,\varpi\right] + \nu^2\,\sigma^2\left[\nu^2\,\varpi^2\left(a^2 + \lambda^2\,\sigma^2\right) - 2\,a^2\,\varpi^2\right.\\ &\left.-\sigma^2\,\xi^2 - \lambda^2\,\sigma^2\left(1 + \varpi^2\right)\right] + a^2\left(\varpi^2\,\sigma^2 - \nu'^2\right) + \sigma\,\nu\,\nu'\left(2\,a\,\varpi\,\sigma + r^2\,\sigma'\right)\Bigg).\end{aligned} \tag{11}$$

From the condition $\Sigma_3 = 0$, we obtain

$$\begin{aligned}c' = \frac{1}{\nu\,\lambda^2\,\sigma^2}\Bigg(&\left[c + \varpi\,a\right]\left[\nu\,\sigma\left(4\,a\,\sigma\,\xi + \lambda\left[4\,\sigma\,\lambda' + \lambda\,\sigma'\right]\right)\right.\\ &\left.-2\,\nu'\left(2\,a^2 - \lambda^2\,\sigma^2\right)\right] - \nu\,\varpi\left[a\,\nu\,\nu'\left(3\,\lambda^2\,\sigma^2 - 4\,a^2\right) + \lambda^2\,\sigma^2\,a'\right.\\ &\left.+\nu^2\,\sigma\left(4\,\sigma\,\xi, a^2 - \lambda^2\,\sigma\,a' + \lambda\,a\left[4\,\sigma\,\lambda' + \lambda\,\sigma'\right]\right)\right]\Bigg).\end{aligned} \tag{12}$$

As a result of the equation $\Gamma_3 = 0$, we now have

$$\begin{aligned}\xi' = \frac{1}{\nu^2\,\lambda^2\,\sigma^4}\Bigg(&4\,a^2\,\sigma^2\left[\left(\nu^2 - 1\right)c\,\varpi - \xi\,\nu\,\nu'\right] - 2\,a^3\left[\left(\nu^2 - 1\right)^2\sigma^2\,\varpi^2 - \nu'^2\right]\\ &+\nu\,\lambda\,\sigma^2\left[4\,\nu\,\xi\,\sigma^2\,\lambda' + \lambda\,\nu'\left(\xi\,\sigma^2 + a'\right)\right] - \sigma\,a\left[2\,\sigma\,c^2 + 2\,\sigma\sigma\,\lambda^2\,\nu'^2\right.\\ &\left.-\nu^4\,\sigma^3\,\lambda^2\,\varpi^2 + \nu^2\,\sigma^3\left[\left(\varpi^2 + 1\right)\lambda^2 - 2\,\xi\xi^2\right] + \lambda\,\nu\,\nu'\left(4\,\sigma\,\lambda' + \lambda\,\sigma'\right)\right]\Bigg).\end{aligned} \tag{13}$$

Now, based on the formula $\Gamma_2 = 0$, it leads to

$$\begin{aligned}\lambda'' = \frac{1}{\nu^2\,\lambda^3\,\sigma^4}\Bigg(&3\,a^3\left[\varpi\,\sigma^2\left(\nu^2 - 1\right)\left(2\,c - \varpi\,a\right) - 2\,\nu\,\xi\,\sigma^2\,\nu' + a\,\nu'^2\right]\\ &+\sigma\,a^2\left(3\,\sigma\left(\nu^2\,\sigma^2\,\xi^2 - c^2\right) - 6\,\nu\,\lambda\,\sigma\,\nu'\,\lambda' + \lambda^2\left[\varpi^2\,\sigma^3\left(\nu^2 - 1\right)\left(2\,\nu^2 - 1\right)\right.\right.\\ &\left.\left.-4\,\sigma\,\nu'^2 - \nu\,\nu'\,\sigma'\right]\right) + \lambda^2\,\sigma^3\left(\sigma\,c^2 - \varpi^2\,\lambda^2\,\sigma^3\,\nu^4 + \sigma\,\lambda^2\,\nu'^2 + 2\,\nu\,\lambda\,\sigma\,\nu'\,\lambda'\right.\\ &\left.+\nu^2\left[\lambda^2\,\sigma^3\left(\varpi^2 + 1\right) + \sigma\left(\sigma^2\,\pi^2 + 3\,\lambda'^2 - \xi\,a'\right) + \lambda\,\lambda'\,\sigma'\right]\right)\\ &+\lambda\,a\,\sigma^2\left[6\,\xi\,\nu^2\,\sigma^2\,\lambda' + \lambda\left(\varpi\,c\,\sigma^2\left(2 - 3\,\nu^2\right) + \nu\left[3\,\xi\,\sigma^2\,\nu' + \nu\,\sigma\,\xi\,\sigma' + \nu'\,a'\right]\right)\right]\Bigg).\end{aligned} \tag{14}$$

A straightforward computation for the condition $\Sigma_2 = 0$ leads to the following important condition

$$\nu\,\lambda^2\,\sigma^2\left[\left(\nu^2 - 1\right)\varpi\,a - c\right]a' = \Delta \tag{15}$$

where

$$\begin{aligned} \Delta \;=\; & \nu' \left(a^2 - \sigma^2 \lambda^2\right) \Big[6\, \varpi\, a^2 \left(\nu^2 - 1\right) - \varpi\, \lambda^2 \sigma^2 \nu^2 - 6\, a\, c\Big] \\ & + \nu\, \sigma \left(c + \varpi\, a\right) \Big[\xi\, \sigma \left(\sigma^2 \lambda^2 - 6\, a^2\right) - \lambda\, a \left(6\, \sigma\, \lambda' + \lambda\, \sigma'\right)\Big] \\ & + \varpi\, \sigma\, \nu^3 \Big[2\, \xi\, \sigma\, a \left(3\, a^2 - \sigma^2\right) - \sigma^3 \lambda^3 \lambda' + \lambda\, a^2 \left(6\, \sigma\, \lambda' + \lambda\, \sigma'\right)\Big]. \end{aligned} \tag{16}$$

The condition (15) gives us two possibilities:

3.1. When $c \;\neq\; (\nu^2 - 1)\, \varpi\, a \;\Rightarrow\; a' \;=\; \dfrac{\Delta}{\nu\, \lambda^2 \sigma^2 \Big[(\nu^2 - 1)\, \varpi\, a - c\Big]}$

In this case, the equation $\Sigma_1 \;=\; 0$, becomes

$$\left(a^2 - \lambda^2 \sigma^2\right) \Big[\nu\, \sigma^2 \left(\xi\, a + r\, \lambda'\right) - \left(a^2 - r^2 \sigma^2\right) \nu'\Big] \;=\; 0. \tag{17}$$

Again, from the above condition, we obtain the following two possibilities:

3.1.1. When $a \;\neq\; \pm \sigma\, \lambda \;\Rightarrow\; \nu' \;=\; \dfrac{\nu\, \sigma^2 \left(\xi\, a + \lambda\, \lambda'\right)}{a^2 - \lambda^2 \sigma^2}$

Substituting ν' from the above equation and ν'' from (11), the compatibility condition $\nu'' = \dfrac{d\nu'}{ds}$ implies

$$2\, c \left(3\, a^2 - \sigma^2 \lambda^2\right) + a\, \varpi \Big[(3\, \nu^2 - 2)\, \sigma^2 \lambda^2 - 6\, (\nu^2 - 1)\, a^2\Big] \;=\; 0. \tag{18}$$

The above condition yields two cases:

(3.1.1.1): When $a \;\neq\; \pm \dfrac{\sigma\, \lambda}{\sqrt{3}} \;\Rightarrow\; c \;=\; \dfrac{a\, \varpi \Big[(3\, \nu^2 - 2)\, \lambda^2 \sigma^2 - 6\, (\nu^2 - 1)\, a^2\Big]}{2\, \left(\lambda^2 \sigma^2 - 3\, a^2\right)}$. The computation of the coefficient $\Gamma_0 \;=\; 0$ leads $\varpi\, a \;=\; 0 \;\Rightarrow\; \gamma \;=\; 0$, a contradiction with $c \;\neq\; (\nu^2 - 1)\, \varpi\, a$.

(3.1.1.2): When $a \;=\; \dfrac{\sigma\, \lambda}{\sqrt{3}}$. Again, the condition $a' \;=\; \dfrac{da}{ds}$ gives $\lambda' \;=\; -\sqrt{3}\, \sigma\, \xi$. Substituting in the conditions $\lambda'' \;=\; \dfrac{d\lambda'}{ds}$ and $\nu'' \;=\; \dfrac{d\nu'}{ds}$, we obtain the following conditions, respectively:

$$6\, c^2 + 2\, \sqrt{3}\, (2 - 3\, \nu^2)\, c\, \lambda\, \sigma\, \varpi + (\nu^2 - 1)\, (\nu^2 - 2)\, \lambda^2 \sigma^2 \varpi^2 \;=\; 0,$$

$$6\, c^2 - 4\, \sqrt{3}\, (\nu^2 - 1)\, c\, \lambda\, \sigma\, \varpi + 2\, (\nu^2 - 1)^2 \lambda^2 \sigma^2 \varpi^2 \;=\; 0.$$

The following condition results from subtracting the two conditions above

$$\nu^2 \lambda\, \sigma\, \varpi \left[\sqrt{3}\, c - (\nu^2 - 1)\, \lambda\, \sigma\, \varpi\right] \;=\; 0.$$

Now, there exist two cases:

(a): $\varpi \;=\; 0$. Then $\Gamma_0 \;=\; 0 \;\Rightarrow\; c \;=\; 0$ which contradicts $c \;\neq\; (\nu^2 - 1)\, \varpi\, a$.

(b): $c \;=\; \dfrac{(\nu^2 - 1)\, \lambda\, \sigma\, \varpi}{\sqrt{3}} \;\Rightarrow\; a \;=\; \dfrac{\lambda\, \sigma}{\sqrt{3}}$ which contradicts $c \;\neq\; (\nu^2 - 1)\, \varpi\, c$ again.

3.1.2. $a \;=\; \pm \sigma\, \lambda$

Substituting $a \;=\; \pm \sigma\, \lambda$ and a' from case (3.1), in the condition $a' \;=\; \dfrac{da}{ds}$, we have

$$\left(\sigma\, \xi + r'\right) \Big[(5 - 4\, \nu^2)\, \lambda\, \sigma\, \varpi + 5\, c\Big] \;=\; 0.$$

According to the above condition, let us distinguish the two possibilities:

(3.1.2.1): When $\lambda' = -\sigma\,\xi$. Using λ'' in (14), the compatibility condition $\lambda'' = \dfrac{d\lambda'}{ds}$ results in $c = \dfrac{(3\,\nu^2-4)\,\lambda\,\sigma\,\varpi}{4}$. On the other hand, the condition $c' = \dfrac{dc}{ds}$ gives

$$\nu\,\sigma\,\varpi\left[(5\,\nu^2-6)\,\sigma\,\xi\,\nu-(5\,\nu^2-12)\,\lambda\,\nu'\right] = 0,$$

which implies two cases:

(a): $\varpi = 0 \Rightarrow c = 0$, which contradicts $c = (\nu^2-1)\,\varpi\,a$.

(b): $\nu' = \dfrac{(5\,\nu^2-6)\,\sigma\,\mu\xi\,\nu}{(5\,\nu^2-12)\,\lambda}$. Using ν'' from (11), the compatibility condition $\nu'' = \dfrac{d\nu'}{ds}$ becomes

$$800\,\nu^4\,\xi^2-\lambda^2\,(5\,\nu^2-12)^2\left[32+\left(32-40\,\nu^2+5\,\nu^4\right)\varpi^2\right] = 0,$$

which leads to

$$\xi = \pm\frac{\lambda\,(5\,\nu^2-12)\,\sqrt{\left(40\,\nu^2-32-5\,\nu^4\right)\varpi^2-32}}{20\,\sqrt{2}\,\nu^2}.$$

Now, applying $\xi' = \dfrac{d\xi}{ds}$, we arrive at the following:

$$\left(85\,\nu^4-264\,\nu^2+144\right)\lambda\,\sigma\,\varpi = 0.$$

Because the function $\nu(s)$ is not constant, $\varpi = 0$. Therefore ξ is imaginary, which is a contradiction.

(3.1.2.2): $\lambda' \neq -\sigma\,\xi \Rightarrow c = \dfrac{(4\,\nu^2-5)\,\lambda\,\sigma\,\varpi}{5}$. So $\Gamma_1 = 12\,\nu^2\,\sigma^2\,(\sigma\,\xi+\lambda')\,\lambda^4$, which is impossibly equal to zero.

3.2. When $c = (\nu^2-1)\,\varpi\,a$, Then $\Delta = 0$ and $2\,\nu^2\,\lambda^2\,\sigma\,\varpi\varpi\left[\nu\,\sigma^2\left(\xi\,a+r\,\lambda'\right)-\left(a^2-\lambda^2\,\sigma^2\right)\nu'\right] = 0$

The above condition suggests two possibilities:

3.2.1. $\varpi = 0$

If we substitute $\varpi = 0$ into equation $\Gamma_1 = 0$, one can obtain the following condition:

$$\left[\nu\,\sigma^2\,\xi\,a-a^2\,\nu'+\lambda\,\sigma^2\,(\lambda\,\nu)'\right]\Bigg(4\,a\left(\lambda^2\,\sigma^2-a^2\right)\nu' +\nu\,\sigma\left[4\,a^2\,\sigma\,\xi-\lambda^2\,\sigma\left(\sigma^2\,\xi+a'\right)+\lambda\,a\left(4\,\sigma\,\lambda'+\lambda\,\sigma'\right)\right]\Bigg) = 0, \tag{19}$$

which yields two cases:

(3.2.1.1): $\nu\,\sigma^2\,\xi\,a = a^2\,\nu'-\lambda\,\sigma^2\,(\lambda\,\nu)'$. Again, we consider two subcases:

(I): When $a \neq 0 \Rightarrow \xi = \dfrac{a^2\,\nu'-\lambda\,\sigma^2\,(\lambda\,\nu)'}{\nu\,a\,\sigma^2}$. Now, we rewrite ν'' as follows:

$$\nu'' = \frac{\nu'\,\sigma'}{\sigma}-\nu\,\sigma^2-\frac{\sigma^2\,(\lambda\,\nu)'^2}{\nu\,a^2}. \tag{20}$$

Rewrite again the above equation in the following form:

$$\lambda' = -\frac{\lambda\,\nu'}{\nu}\pm\frac{a}{\sigma}\sqrt{\frac{\nu'\,\sigma'}{\nu\,\sigma}-\sigma^2-\frac{\nu''}{\nu}}. \tag{21}$$

Hence, the compatibility condition $\lambda'' = \dfrac{d\lambda'}{ds}$ gives the following ordinary differential equation (**ODE**):

$$\sigma\, \nu''' + \left(4\,\sigma^3 - \sigma''\right) + \frac{3\,\sigma^3}{\nu}\left(\frac{\nu}{\sigma}\right)'\left(\frac{\nu'}{\sigma}\right)' = 0. \tag{22}$$

If we apply the following transformation,

$$\phi(s) = 2\int \sigma(s)\, ds, \qquad \nu(s) = \sqrt{B(\phi)}$$

Equation (22) becomes

$$\frac{d^3 B(\phi)}{d\phi^3} + \frac{dB(\phi)}{d\phi} = 0. \tag{23}$$

The general solution **(GS)** of the above equation is

$$B(\phi) = \epsilon_1 + \epsilon_2\, \sin[\phi] + +\epsilon_3\, \cos[\phi] \;\Rightarrow\; \nu(s) = \sqrt{\epsilon_1 + \epsilon_2\, \sin[\phi(s)] + +\epsilon_3\, \cos[\phi(s)]}. \tag{24}$$

Solving Equation (21), we obtain

$$a(s) = \frac{\epsilon_1\, \lambda'(s) + \left[\epsilon_2\, \lambda'(s) - \epsilon_3\, \sigma(s)\, \lambda(s)\right]\, \sin[\phi(s)] + \left[\epsilon_3\, \lambda'(s) + \epsilon_2\, \sigma(s)\, \lambda(s)\right]\, \cos[\phi(s)]}{\sqrt{\epsilon_3^2 + \epsilon_2^2 - \epsilon_1^2}}. \tag{25}$$

Then, we have the following solution:

$$b(s) = \frac{\epsilon_1\, \sigma(s)\, \lambda(s) + \left[\epsilon_2\, \lambda'(s) - \epsilon_3\, \sigma(s)\, \lambda(s)\right]\, \cos[\phi(s)] - \left[\epsilon_3\, \lambda'(s) + \epsilon_2\, \sigma(s)\, \lambda(s)\right]\, \sin[\phi(s)]}{\sqrt{\epsilon_3^2 + \epsilon_2^2 - \epsilon_1^2}}, \tag{26}$$

and $c(s) = \omega(s) = 0$, where ϵ_1, ϵ_2, and ϵ_3 are arbitrary constants, while $\sigma(s)$ and $\lambda(s)$ are arbitrary functions of s.

For this solution, the base curve α is a plane curve, and the position vector takes the following form:

$$\alpha(s) = \int \left(\cos\left[\int \sigma[s]\, ds\right], \sin\left[\int \sigma[s]\, ds\right], 0\right) ds. \tag{27}$$

After the computation of the Frenet frame of base curve α, the position vector $\mathfrak{X}_1(s,t) = (x_1, x_2, x_3)$ of the developable surface is given by

$$\begin{cases} x_1 = \lambda(s)\left[\left(\dfrac{\epsilon_2}{\sqrt{\epsilon_3^2 + \epsilon_2^2 - \epsilon_1^2}} - \cos[t]\right)\sin\left[\psi[s]\right] + \dfrac{(\epsilon_1 + \epsilon_3)\, \cos\left[\psi[s]\right]}{\sqrt{\epsilon_3^2 + \epsilon_2^2 - \epsilon_1^2}}\right], \\ x_2 = \lambda(s)\left[\left(\dfrac{\epsilon_2}{\sqrt{\epsilon_3^2 + \epsilon_2^2 - \epsilon_1^2}} + \cos[t]\right)\cos\left[\psi[s]\right] + \dfrac{(\epsilon_1 - \epsilon_3)\, \sin\left[\psi[s]\right]}{\sqrt{\epsilon_3^2 + \epsilon_2^2 - \epsilon_1^2}}\right], \\ x_3 = \lambda(s)\, \sin[t]\, \sqrt{\epsilon_1 + \epsilon_2\, \sin[2\,\psi(s)] + \epsilon_3\, \cos[2\,\psi(s)]}, \end{cases} \tag{28}$$

where $\psi(s) = \int \sigma(s)\, ds$. The above surface, in Cartesian coordinates, is given by:

$$(\epsilon_1 - \epsilon_3)\, x_1^2 - 2\,\epsilon_2\, x_1\, x_2 + (\epsilon_1 + \epsilon_3)\, x_2^2 + x_3^2 = 0.$$

Now, we can write the following Lemma:

Lemma 1. *The developable surface (28) foliated by general ellipses represents a conical surface in Euclidean 3-space.*

(II): When $a(s) = 0$, the condition in the case (3.2.1.1) leads to $\nu(s) = \dfrac{\nu_0}{\lambda(s)}$, where ν_0 is an arbitrary constant. Now, we can rewrite the equation of ξ' as the following:

$$\frac{\xi'}{\xi} = \frac{3\,\lambda'}{\lambda}.$$

The **GS** of the above **ODE** is $\xi(s) = \xi_0\,\lambda^3(s)$, where ξ_0 is an arbitrary constant. Hence, the equation of λ'' becomes

$$\frac{\lambda''}{\lambda'} = \frac{2\,\lambda'}{\lambda} + \frac{\sigma'}{\sigma} + \frac{\sigma^2\,\lambda\,(1+\xi_0^2\,\lambda^4)}{\lambda'}.$$

The **GS** of the above **ODE** is

$$\lambda(s) = \pm 2\left[\frac{\lambda_0 \pm \sqrt{\lambda_0^2 + 4\,\xi_0^2}\,\sin\left[2\int\sigma\,d\sigma\right]}{\left(1+\cos\left[4\int\sigma\,d\sigma\right]\right)\left(\lambda_0^2 - 4\,\xi_0^2\,\tan^2\left[2\int\sigma\,d\sigma\right]\right)}\right]^{1/2}, \tag{29}$$

where λ_0 is an arbitrary constant. Hence, all coefficients Γ_i and Σ_i vanish. In this case, we have:

$$a(s) = c(s) = \omega(s) = 0, \quad b(s) = \xi_0\,\lambda^3(s)\,\sigma(s), \quad \nu(s) = \frac{\nu_0}{\lambda(s)}, \tag{30}$$

where $\lambda(s)$ is given by (29) and $\sigma(s)$ is an arbitrary function of s, while λ_0, ξ_0, and ν_0 are arbitrary constants. Therefore, the parametrization of the tangent vector of the curve $\mathbf{C}(s)$ is given by

$$\beta'(s) = \xi_0\,\lambda^3(s)\,\sigma(s)\,\mathfrak{N}.$$

Hence, there exists $\mathbf{C}_0 \in \mathbf{R}^3$ such that

$$\beta(s) = \mathbf{C}_0 + \xi_0\int\lambda^3(s)\,\sigma(s)\,\mathfrak{N}(s)\,ds,$$

The parametrization of this surface is given by

$$\mathfrak{X}_2(s,t) = \mathbf{C}_0 + \xi_0\int\lambda^3(s)\,\sigma(s)\,\mathfrak{N}\,ds + \lambda(s)\,\cos[t]\,\mathfrak{N} + \nu_0\,\sin[t]\,\mathfrak{B}, \tag{31}$$

$\lambda(s)$ is given by (29), and $\sigma(s)$ is an arbitrary function of s while λ_0, ξ_0 and ν_0 are arbitrary constants.

(3.2.1.2): $4\,a\left(\lambda^2\,\sigma^2 - a^2\right)\nu' + \nu\,\sigma\left[4\,a^2\,\sigma\,\xi - \lambda^2\,\sigma\left(\sigma^2\,\xi + a'\right) + \lambda\,a\left(4\,\sigma\,\lambda' + \lambda\,\sigma'\right)\right] = 0.$ This case splits into two subcases:

(I): When $a \neq \pm\dfrac{\sigma\,\lambda}{2} \Rightarrow \xi = \dfrac{4\,a^3\,\nu' - 4\,\lambda\,\sigma^2\,a\,(\lambda\,\nu)' + \sigma\,\lambda^2\,\nu\,(\sigma\,a' - a\,\sigma')}{\nu\,\sigma^2\,(4\,a^2 - \sigma^2\,\lambda^2)}$. Thus the equation $\Gamma_0 = 0$ becomes

$$\left(a^2 - \sigma^2\,\lambda^2\right)\left[a^2\left(\sigma\,\nu' - \nu\,\sigma'\right) + \nu\,\sigma\,a\,a' - \lambda\,\sigma^3\,(\lambda\,\nu)'\right] = 0.$$

The previous equation has two solutions:

(I-A): $a = \pm\dfrac{\sigma\sqrt{\nu^2\,\lambda^2 + a_0}}{\nu}$ and $a_0 \neq 0$ is an arbitrary constant. Now, we examine the following cases:

(I-A.1): $\nu^2\,\lambda^2 + a_0 \,\neq\, 0$. We have the following conditions:

$$\begin{aligned}
&\sigma\left(a_0 + \nu^2\,\lambda^2\right)\left[\nu\,\sigma + \left(\frac{\nu'}{\sigma}\right)'\right] + \nu\,\left(\lambda\,\nu\right)'^2 \,=\, 0,\\
&\lambda'' \,=\, \frac{\sigma'\,\lambda'}{\sigma\sigma} + \frac{a_0\left(\lambda\,\sigma^2\,\nu^2 - 2\,\lambda\,\nu'^2 - 4\,\nu\,\nu'\,\lambda'\right) + \lambda\,\nu^4\left(\sigma^2\,\lambda^2 - 2\,\lambda'^2\right)}{\nu^2\left(a_0 + \nu^2\,\lambda^2\right)}.
\end{aligned} \tag{32}$$

Now, all coefficients Γ_i and Σ_j are zero and we obtain the following solution:

$$\begin{cases}
a(s) \,=\, \dfrac{\sigma(s)\,\sqrt{\nu^2(s)\,\lambda^2(s) + a_0}}{\nu(s)}, \qquad b(s) \,=\, \dfrac{a_0\,\nu'(s) - \nu^3(s)\,r\lambda(s)\,\lambda'(s)}{\nu^2(s)\,\sqrt{\nu^2(s)\,\lambda^2(s) + a_0}},\\[2ex]
\nu(s) \,=\, \epsilon_1\,\cos[\phi(s)] + \epsilon_2\,\sin[\phi(s)], \qquad \omega(s) \,=\, c(s) \,=\, 0,\\[2ex]
\lambda(s) \,=\, \dfrac{\lambda_1\,\cos[\phi(s)] + \lambda_2\,\sin[\phi(s)]}{\Big(\epsilon_1\,\cos[\phi(s)] + \epsilon_2\,\sin[\phi(s)]\Big)^2 - \lambda_0^2}, \qquad \phi(s) \,=\, \int \sigma(s)\,ds,
\end{cases} \tag{33}$$

where $a_0 \,=\, \dfrac{\left(\lambda_1\,\epsilon_2 - \epsilon_1\,\lambda_2\right)^2 - \lambda_0^2\left(\lambda_1^2 + \lambda_2^2\right)}{\lambda_0^2\left(\epsilon_1^2 + \epsilon_2^2 - \lambda_0^2\right)}$, while $\sigma(s)$ is an arbitrary function of s and λ_1, λ_2, λ_0, ϵ_1, and ϵ_2 are arbitrary constants. Therefore, the parametrization of the tangent vector of the curve $\beta(s)$ is given by

$$\beta'(s) \,=\, \frac{\sqrt{\nu^2\,\lambda^2 + \lambda_0}\,\sigma\,\mathfrak{T}}{\nu} + \frac{\left[\lambda_0\,\nu' - \nu^3\,\lambda\,\lambda'\right]\mathfrak{N}}{\nu^2\,\sqrt{\nu^2\,\lambda^2 + \lambda_0}}.$$

Hence, there exists $\mathbf{C}_0 \in \mathbf{R}^3$ such that

$$\beta(s) \,=\, \mathbf{C}_0 - \frac{\sqrt{\nu^2\,\lambda^2 + \lambda_0}\,\mathfrak{N}}{\nu},$$

The parametrization of this surface is given by

$$\mathfrak{X}_3(s,t) \,=\, \mathbf{C}_0 + \left(\lambda(s)\,\cos[t] - \frac{\sqrt{\nu^2(s)\,\lambda^2(s) + \lambda_0}}{\nu(s)}\right)\mathfrak{N} + \lambda(s)\,\nu(s)\,\sin[t]\,\mathfrak{B}. \tag{34}$$

The above surface can be expressed by the Cartesian equation below:

$$\begin{aligned}
&x_3^2 + \left(\epsilon_2\,x_1 - \epsilon_1\,x_2\right)^2 - \lambda_0^2\left(x_1^2 + x_2^2\right) + \frac{\left(\lambda_1\,\epsilon_2 - \epsilon_1\,\lambda_2\right)^2 - \lambda_0^2\left(\lambda_1^2 + \lambda_2^2\right)}{\lambda_0^2\left(\epsilon_1^2 + \epsilon_2^2 - \lambda_0^2\right)}\\
&= \frac{2\left[\left(\lambda_1\,\epsilon_2 - \epsilon_1\,\lambda_2\right)\left(\epsilon_2\,x_1 - \epsilon_1\,x_2\right) - \lambda_0^2\left(\lambda_1\,x_1 + \lambda_2\,x_2\right)\right]}{\lambda_0\,\sqrt{\epsilon_1^2 + \epsilon_2^2 - \lambda_0^2}}.
\end{aligned} \tag{35}$$

From the above discussion, we proved the following Lemma:

Lemma 2. *The developable surface (34) foliated by general ellipses represents a conical surface in Euclidean 3-space.*

(I-A.2): $\nu^2\,\lambda^2 + \alpha_0 \,=\, 0 \,\Rightarrow\, \nu(s) \,=\, \dfrac{\nu_0}{\lambda(s)}$, where $\alpha_0 \,=\, -\nu_0^2$ is an arbitrary negative constant. This case leads to $a \,=\, b \,=\, c \,=\, 0$, which is a contradiction again.

(I-B): $a(s) = \pm\sigma\,\lambda$. The equation of λ'' can be written as

$$\frac{\lambda''}{\lambda'} = \frac{2\,\lambda'}{\lambda} + \frac{\sigma'}{\sigma} + \frac{\sigma^2\,\lambda}{\lambda'}.$$

The **GS** of the above **ODE** is

$$\lambda(s) = \frac{\lambda_0}{\cos\left[\int \sigma(s)\,ds\right] + \lambda_1\,\sin\left[\int \sigma(s)\,ds\right]}, \tag{36}$$

where λ_0 and λ_1 are arbitrary constants of integration. Therefore, the equation of ν'' becomes

$$\frac{\nu''}{\nu} + \frac{(1+\lambda_2^2)\,\sigma^2\,\lambda^2}{\lambda_1^2} = \frac{\nu'}{\nu}\left[\frac{\sigma'}{\sigma} - \frac{2\,\lambda'}{\lambda} - \frac{\nu'}{\nu}\right]. \tag{37}$$

The **GS** of the above equation is

$$\nu(s) = \nu_0\,\sqrt{\cos\left[\int \sigma(s)\,ds\right] + \nu_1\,\sin\left[\int \sigma(s)\,ds\right]}\,\sqrt{\cos\left[\int \sigma(s)\,ds\right] + \lambda_1\,\sin\left[\int \sigma(s)\,ds\right]}. \tag{38}$$

where ν_0 and ν_1 are arbitrary constants of integration. Now, all coefficients Γ_i and Σ_i are equal to zero. Then, we obtain the following solution:

$$a(s) = \sigma(s)\,\lambda(s), \quad b(s) = -\lambda'(s), \quad c(s) = \omega(s) = 0, \tag{39}$$

where the functions $\lambda(s)$ and $\nu(s)$ are in (36) and (38), respectively, while $\sigma(s)$ is an arbitrary function of s, and λ_0, λ_1, ν_0, and ν_1 are arbitrary constants. Therefore, the parametrization of the tangent vector of the curve $\beta(s)$ is given by

$$\beta'(s) = \lambda(s)\,\sigma(s)\,\mathfrak{T} - \lambda'(s)\,\mathfrak{N}.$$

Since $\omega(s) = 0$, $\alpha(s)$ is a plane curve, and it is easy to prove there exists $\mathbf{C}_0 \in \mathbf{R}^3$ such that

$$\beta(s) = \mathbf{C}_0 - \lambda(s)\,\mathfrak{N}.$$

The explicit parametrization of this surface is given by

$$\mathfrak{X}_4(s,t) = \mathbf{C}_0 + \lambda(s)\,\big(\cos[t]-1\big)\,\mathfrak{N} + \lambda_0\,\nu_0\,\sqrt{\frac{\cos\left[\int \sigma(s)\,ds\right] + \nu_1\,\sin\left[\int \sigma(s)\,ds\right]}{\cos\left[\int \sigma(s)\,ds\right] + \lambda_1\,\sin\left[\int \sigma(s)\,ds\right]}}\,\sin[t]\,\mathbf{b}. \tag{40}$$

The position vector $\mathfrak{X}_4(s,t) = (x_1, x_2, x_3)$ of this developable surface is given by

$$\begin{cases} x_1 = \dfrac{\lambda_0\,\sin\left[\int \sigma(s)\,ds\right]\,\big(1-\cos[t]\big)}{\cos\left[\int \sigma(s)\,ds\right] + \lambda_1\,\sin\left[\int \sigma(s)\,ds\right]}, \\[2ex] x_2 = \dfrac{\lambda_0\,\lambda_1}{\lambda_1 + \cot\left[\int \sigma(s)\,ds\right]} + \dfrac{\lambda_0\,\cos[t]}{1+\lambda_1\,\tan\left[\int \sigma(s)\,ds\right]}, \\[2ex] x_3 = \lambda_0\,\nu_0\,\sqrt{\dfrac{\cos\left[\int \sigma(s)\,ds\right] + \nu_1\,\sin\left[\int \sigma(s)\,ds\right]}{\cos\left[\int \sigma(s)\,ds\right] + \lambda_1\,\sin\left[\int \sigma(s)\,ds\right]}}\,\sin[t]. \end{cases} \tag{41}$$

The above surface satisfies the following equation:

$$\nu_1\,\lambda_1\,x_1^2 + \lambda_0\,(\lambda_1 - \nu_1)\,x_1 - (\lambda_1+\nu_1)\,x_1\,x_2 + x_2^2 + \nu_0^{-2}\,x_3^2 = \lambda_0^2.$$

Then, we have the following Lemma:

Lemma 3. *The developable surface (41) foliated by general ellipses represents a conical surface in Euclidean 3-space.*

(II): When $a = \pm\dfrac{\sigma\,\lambda}{2}$, the condition in the case (3.2.1.2) becomes

$$(\nu\,\lambda)'\left[3\,\lambda\,\nu' + 4\,\nu\,\lambda' + 2\,\sigma\,\nu\,\xi\right] = 0,$$

which yields two cases:

(II-A): $\nu(s) = \dfrac{\nu_0}{\lambda(s)}$, where ν_0 is an arbitrary constant. Now, the condition $\nu'' = \dfrac{d^2\nu}{ds^2}$ yields $\xi = -\dfrac{\lambda'}{2\,\sigma}$. Hence, the equation of λ'' becomes

$$\frac{\lambda''}{\lambda'} = \frac{2\,\lambda'}{\lambda} + \frac{\sigma'}{\sigma} + \frac{\sigma^2\,\lambda}{\lambda'}.$$

The **GS** of the above **ODE** is

$$\lambda(s) = \frac{\lambda_0}{\cos\left[\int \sigma(s)\,ds\right] + \lambda_1\,\sin\left[\int \sigma(s)\,ds\right]}, \tag{42}$$

where λ_0 and λ_1 are arbitrary constants. Hence, all coefficients Γ_i and Σ_i are equal to zero. Therefore, the parametrization of the tangent vector of the curve $\beta(s)$ is given by

$$\beta'(s) = \frac{\sigma\,\lambda\,\mathfrak{T}}{2} - \frac{\lambda'\,\mathfrak{N}}{2}.$$

Since $\omega(s) = 0$, $\alpha(s)$ is a plane curve, and it is easy to prove that there exists $\mathbf{C}_0 \in \mathbf{R}^3$ such that

$$\beta(s) = \mathbf{C}_0 - \frac{\lambda\,\mathfrak{N}}{2}.$$

The parametrization of this surface is given by

$$\mathfrak{X}_5(s,t) = \mathbf{C}_0 + r(s)\left(\cos[t] - \frac{1}{2}\right)\mathfrak{N} + b_0\,\sin[t]\,\mathfrak{B}, \tag{43}$$

where λ_0, λ_1, and ν_0 are arbitrary constants while $\sigma(s)$ is an arbitrary function of s such that $\omega(s) = 0$. The position vector $\mathfrak{X}_5(s,t) = (x_1, x_2, x_3)$ of this developable surface is given by

$$\begin{cases} x_1 = \dfrac{\lambda_0\,\sin\left[\int \sigma(s)\,ds\right](1 - 2\,\cos[t])}{2\left(\cos\left[\int \sigma(s)\,ds\right] + \lambda_1\,\sin\left[\int \sigma(s)\,ds\right]\right)}, \\[2ex] x_2 = \dfrac{\lambda_0\,\lambda_1}{\lambda_1 + \cot\left[\int \sigma(s)\,ds\right]} + \dfrac{\lambda_0\,\cos[t]}{1 + \lambda_1\,\tan\left[\int \sigma(s)\,ds\right]}, \\[2ex] x_3 = \nu_0\,\sin[t]. \end{cases} \tag{44}$$

The above surface satisfies the following equation:

$$\frac{(\lambda_1\,x_1 - x_2)^2}{\lambda_0^2} + \frac{x_3^2}{\lambda_0^2} = 1.$$

Then, the following lemma is proved:

Lemma 4. *The developable surface (44) foliated by general ellipses represents a conical surface in Euclidean 3-space.*

(II-B): $\xi = \dfrac{3\,\lambda\,\nu' + 4\,\nu\,\lambda'}{2\,\sigma\,\nu}$. Now, the equations of ν'' and r'' yields the following conditions:

$$\begin{cases} \dfrac{\nu''}{\nu} + \dfrac{4\,\nu'}{\nu} + \dfrac{8\,\lambda'}{\lambda} - \dfrac{\sigma'}{\sigma} + \dfrac{\nu}{\nu'}\left(\sigma^2 + \dfrac{4\,r\lambda'^2}{\lambda^2}\right), \\ \dfrac{\lambda''}{\lambda} - \dfrac{5\,\lambda'}{\lambda} - \dfrac{6\,\nu'}{\nu} - \dfrac{\sigma'}{\sigma} - \dfrac{\lambda}{\lambda'}\left(\sigma^2 + \dfrac{3\,\nu'^2}{\nu^2}\right). \end{cases} \tag{45}$$

The **GS** of the above equations are

$$\begin{cases} \nu(s) = \sqrt{c_1 + c_2\,\sin\left[2\,\int \sigma(s)\,ds\right] + c_3\,\cos\left[2\,\int \sigma(s)\,ds\right]}, \\ \lambda(s) = \dfrac{c_4}{\nu(s)}\,\exp\left(\dfrac{1}{2}\,\tanh\left[\dfrac{c_2 + (c_1 - c_3)\tan\left[\int \sigma(s)\,ds\right]}{\sqrt{c_3^2 + c_2^2 - c_1^2}}\right]\right). \end{cases} \tag{46}$$

Hence, all coefficients Γ_i and Σ_j are equal to zero. In this case, we have:

$$a(s) = \frac{\sigma(s)\,\lambda(s)}{2}, \quad b(s) = -\left(2\,\lambda'(s) + \frac{3\,\lambda(s)\,\nu'(s)}{2\,\nu(s)}\right), \quad \omega(s) = c(s) = 0, \tag{47}$$

where $\sigma(s)$ is an arbitrary function of s while c_i, $i = 1, \ldots, 4$ are arbitrary constants. Therefore, the parametrization of the tangent vector of the curve $\beta(s)$ is given by

$$\beta'(s) = -\left(\frac{\lambda(s)\,\mathfrak{N}}{2}\right)' - \frac{3\left[\lambda(s)\,\nu(s)\right]'\,\mathfrak{N}}{2\,\nu(s)}.$$

Hence, there exists $\mathbf{C}_0 \in \mathbf{R}^3$ such that

$$\beta(s) = \mathbf{C}_0 - \frac{\lambda(s)\,\mathfrak{N}}{2} - \int \frac{3\left[\lambda(s)\,\nu(s)\right]'\,\mathfrak{N}}{2\,\nu(s)}\,ds.$$

The parametrization of this surface is given by

$$\mathfrak{X}_6(s,t) = \mathbf{C}_0 - \int\left(\frac{3\left[\lambda(s)\,\nu(s)\right]'}{2\,\nu(s)}\right)\mathfrak{N}\,ds + \lambda(s)\left(\cos[t] - \frac{1}{2}\right)\mathfrak{N} + \lambda(s)\,\nu(s)\,\sin[t]\,\mathfrak{B}. \tag{48}$$

3.2.2. When $\varpi \neq 0$, Then $\nu\,\sigma^2\left(\xi\,a + \lambda\,\lambda'\right) - \left(a^2 - \lambda^2\,\sigma^2\right)\nu' = 0$

Let us consider two cases: **(3.2.2.1):** When $a \neq 0$, then $\xi = \dfrac{a^2\,\nu' - \lambda\,\sigma^2\,(\lambda\,\nu)'}{\nu\,a\,\sigma^2}$. If we rewrite the equation of ϖ', then we obtain

$$\frac{\varpi'}{\varpi} = -\frac{3\,\nu\,\nu'}{\nu^2 - 1} \quad \Rightarrow \quad \nu = \frac{\sqrt{b_0 + \varpi^{2/3}}}{\varpi^{1/3}}, \tag{49}$$

where b_0 is an arbitrary non-zero constant. Now, the condition $\nu'' = \dfrac{d\nu'}{ds}$ leads to the following condition:

$$\Omega\, a^2 = \sigma^3 \left[\nu_0\, \lambda\, \varpi' - 3\, \varpi\, \lambda' \left(b_0 + \lambda \varpi^{2/3}\right)\right]^2, \tag{50}$$

where

$$\begin{aligned}\Omega(s) = {} & 9\, \sigma^3\, \varpi^2 \left(b_0 + \varpi^{2/3}\right)^2 \left(b_0\, \varpi^{4/3} - 1\right) \\ & + 3\, b_0\, \varpi \left(b_0 + \varpi^{2/3}\right) [\sigma\, \varpi'' - \sigma'\, \varpi'] - b_0\, \sigma\, \varpi'^2 \left(4\, b_0 + 5\, \varpi^{2/3}\right).\end{aligned} \tag{51}$$

The above equation divides into two cases:

(I): $\Omega(s) \neq 0 \Rightarrow a = \dfrac{\sigma^{3/2}\left[b_0\, \lambda\, \varpi' - 3\, \varpi\, \lambda' \left(b_0 + \varpi^{2/3}\right)\right]}{\sqrt{\Omega(s)}}$. Substituting in $\Gamma_3 = 0$, we obtain the following condition:

$$\begin{aligned}& 9\, \sigma\, \varpi \left[\sigma\, \varpi\, \varpi'' - \left(3\, \varpi\, \sigma' + 5\, \sigma\, \varpi'\right) \varpi''\right] \\ & = \left[9\, \sigma\, \varpi \left(\varpi\, \sigma'' - 5\, \sigma'\, \varpi'\right) - 40\, \sigma^2\, \varpi'^2 + 27\, \varpi^2\, \sigma'^2 + 36\, \sigma^4 \left(\varpi^2 + \varpi^4\right)\right] \varpi'.\end{aligned} \tag{52}$$

Now, all coefficients, Γ_i and Σ_i, are equal to zero. Integrating twice, the above condition gives

$$\varpi' = 3\, \sigma\, \varpi \sqrt{c_1\, \varpi^{4/3} + c_2\, \varpi^{2/3} - \varpi^2 - 1}. \tag{53}$$

Hence, we have

$$\begin{cases} a(s) = \dfrac{b_0\, \lambda(s)\, \varpi'(s) - 3\, \varpi(s)\, \lambda'(s) \left[b_0 + \varpi^{2/3}(s)\right]}{3\, c_0\, \varpi^{5/3}(s)}, \\[2ex] b(s) = \dfrac{b_0\, \lambda'(s)\, \varpi'(s) + 3\, \sigma^2(s)\, \varpi(s) \left[b_0 + (b_0\, c_2 + 1)\, \varpi^{2/3}(s) + b_0^2\, \varpi^{4/3}(s)\right]}{3\, c_0\, \sigma(s)\, \varpi^{5/3}(s)}, \\[2ex] c(s) = b_0\, a(s)\, \varpi^{1/3}(s), \qquad \nu(s) = \sqrt{1 - b_0\, \varpi^{-2/3}(s)}, \end{cases} \tag{54}$$

where $c_0 = \sqrt{b_0^3 + c_1\, b_0^2 - c_2\, b_0 - 1}$, b_0, c_1, and c_2 are arbitrary constants, while $\sigma(s)$ and $\lambda(s)$ are arbitrary functions of s such that the curvature and torsion of the base curve $\alpha(s)$ are related by the following equation:

$$\frac{d}{ds} \ln\left[\frac{\omega(s)}{\sigma(s)}\right] = 3\, \sigma(s) \sqrt{c_1 \left(\frac{\omega(s)}{\sigma(s)}\right)^{4/3} + c_2 \left(\frac{\omega(s)}{\sigma(s)}\right)^{2/3} - \left(\frac{\omega(s)}{\sigma(s)}\right)^2 - 1}. \tag{55}$$

The parametrization of this surface is given by

$$\mathfrak{X}_7(s,t) = \int \left[a(s)\mathfrak{T} + b(s)\mathfrak{N} + b_0 a(s)\varpi^{1/3}(s)\mathfrak{B}\right] ds + \lambda(s)\left[\cos[t]\mathfrak{N} + \sqrt{1 - b_0\, \varpi^{-2/3}(s)}\, \sin[t]\mathfrak{B}\right]. \tag{56}$$

(II): When $\Omega(s) = 0$, the condition (50) leads to the following two conditions:

$$b_0\, \lambda\, \varpi' = 3\, \varpi\, \lambda' \left(b_0 + \varpi^{2/3}\right), \tag{57}$$

$$3\left(\frac{\varpi''}{\varpi'} - \frac{\sigma'}{\sigma}\right) - \left(\frac{4\, b_0 + 5\, \lambda^{2/3}}{b_0 + \lambda^{2/3}}\right) \frac{\varpi'}{\varpi} = \frac{9\, \sigma^2\, \varpi \left(b_0 + \varpi^{2/3}\right) \left(1 - b_0\, \varpi^{4/3}\right)}{b_0\, \varpi'}. \tag{58}$$

Now, all coefficients Γ_i and Σ_i are equal to zero. In this case, we have:

$$\begin{aligned} b(s) &= -\frac{b_0\, a(s)\, \varpi'(s)}{3\,\sigma\,\left[b_0\,\varpi + \varpi^{5/3}(s)\right]}, \qquad c(s) \;=\; b_0\, a(s)\, \varpi^{1/3}(s), \\ \lambda(s) &= \frac{r_0\,\varpi^{1/3}(s)}{\sqrt{b_0 + \varpi^{2/3}(s)}}, \quad \nu(s) \;=\; \frac{\sqrt{b_0 + \varpi^{2/3}(s)}}{\varpi^{1/3}(s)}, \quad \omega(s) \;=\; \sigma(s)\,\varpi(s) \end{aligned} \tag{59}$$

where $a(s)$ and $\sigma(s)$ are arbitrary functions of s and the functions $\varpi(s)$ and $\sigma(s)$ are related by the condition (58), while b_0 and r_0 are arbitrary constants. Therefore, the parametrization of the tangent vector of the curve $\beta(s)$ is given by

$$\beta'(s) \;=\; a(s)\left[\mathfrak{T} - \frac{b_0\,\varpi'(s)\,\mathfrak{N}}{3\,\sigma(s)\,\left[b_0\,\varpi + \lambda^{5/3}(s)\right]} + b_0\,\varpi^{1/3}(s)\,\mathbf{b}\mathfrak{B}\right].$$

Under the condition (58), we can prove that

$$\frac{d}{ds}\left[\frac{\mathfrak{T}}{\lambda(s)} - \frac{b_0\,\varpi'(s)\,\mathfrak{N}}{3\,\sigma(s)\,\lambda(s)\left[b_0\,\varpi + \varpi^{5/3}(s)\right]} + \frac{b_0\,\varpi^{1/3}(s)\,\mathfrak{B}}{\lambda(s)}\right] \;=\; 0.$$

Hence, there exists $\mathbf{C}_0 \in \mathbf{R}^3$ such that

$$\beta(s) \;=\; \mathbf{C}_0 + \eta(s)\left[\mathfrak{T} - \frac{b_0\,\varpi'(s)\,\mathfrak{N}}{3\,\sigma(s)\,\left[b_0\,\varpi + \lambda^{5/3}(s)\right]} + b_0\,\varpi^{1/3}(s)\,\mathfrak{B}\right],$$

where $\lambda(s)\,\eta(s) \;=\; \int\,\lambda(s)\,a(s)\,ds$, and the developable surface is given by

$$\mathfrak{X}_8(s,t) \;=\; \mathbf{C}_0 + \eta(s)\,\mathfrak{T} + \Big(\eta(s)\,b(s) + \lambda(s)\,\cos[t]\Big)\,\mathfrak{N} + \Big(b_0\,\eta(s)\,\varpi^{1/3} + r_0\,\sin[t]\Big)\,\mathfrak{B}, \tag{60}$$

where $\sigma(s)$ and $\omega(s)$ are functions of s satisfying the relation (58) while r_0 and b_0 are arbitrary constants.

(3.2.2.2): $a \;=\; 0 \;\Rightarrow\; \nu \;=\; \dfrac{b_0}{\lambda}$. When we rewrite the equation of ξ', we obtain $\dfrac{\xi'}{\xi} \;=\; \dfrac{3\,\lambda}{\lambda}$. So we have $\xi \;=\; \mu_0\,\lambda^3$, where μ_0 is an arbitrary non-zero constant. Again, rewrite the equation of ϖ', and then obtain the following:

$$\frac{\varpi'}{\varpi} \;=\; \frac{3\,b_0^2\,\lambda'}{\lambda\,(b_0^2 - \lambda^2)}. \tag{61}$$

Therefore, $\lambda \;=\; \dfrac{b_0\,\varpi^{1/3}\,\sqrt{r_0^2 + r_0\,\varpi^{2/3} + \lambda^{4/3}}}{\sqrt{\varpi^2 - r_0^3}}$, where r_0 is an arbitrary constant. Substituting into the equation of $\lambda'' \;=\; \dfrac{\lambda'}{ds}$, we obtain

$$3\left(\frac{\varpi''}{\varpi'} - \frac{\sigma'}{\sigma}\right) - \left(\frac{4\,r_0 - 5\,\varpi^{2/3}}{r_0 - \varpi^{2/3}}\right)\frac{\varpi'}{\varpi} \;=\; \frac{9\,\sigma^2\,\varpi}{\varpi'}\left[\left(1 + r_0\,\varpi^{4/3}\right)\left(r_0 - \varpi^{2/3}\right) + \frac{b_0^4\,\mu_0^2\,\varpi^{4/3}}{r_0 - \varpi^{2/3}}\right]. \tag{62}$$

Now, all coefficients Γ_i and Σ_i are equal to zero. Therefore, we have

$$\begin{aligned} &a(s) \;=\; c(s) \;=\; 0, \quad b(s) \;=\; \mu_0\,\sigma(s)\,\lambda^3(s), \\ &\lambda(s) \;=\; b_0\,\varpi^{1/3})(s)\,\sqrt{\frac{r_0^2 + r_0\,\varpi^{2/3})(s) + \varpi^{4/3})(s)}{\varpi^2(s) - r_0^3}}, \quad \nu(s) \;=\; \frac{b_0}{\lambda(s)}, \end{aligned} \tag{63}$$

where the functions $\sigma(s)$ and $\omega(s)$ are related via the condition (62) while b_0, μ_0, and r_0 are arbitrary constants. Then, the parametrization of the tangent vector of the curve $\beta(s)$ is given by

$$\beta'(s) = \mu_0\,\lambda^3(s)\,\sigma(s)\,\mathfrak{N}.$$

Hence, there exists $\mathbf{C}_0 \in \mathbf{R}^3$ such that

$$\beta(s) \;=\; \mathbf{C}_0 + \mu_0 \int \lambda^3(s)\,\sigma(s)\,\mathfrak{N}(s)\,ds,$$

The explicit parametrization of this surface is given by

$$\mathfrak{X}_9(s,t) \;=\; \mathbf{C}_0 + \mu_0 \int \lambda^3(s)\,\sigma(s)\,\mathfrak{N}\,ds + \lambda(s)\,\cos[t]\;\mathfrak{N} + b_0\,\sin[t]\,\mathfrak{B}, \tag{64}$$

where the functions $\sigma(s)$ and $\omega(s)$ are related by the condition (62) while r_0, μ_0, and b_0 are arbitrary constants. From the above discussion, the main Theorem 1 is proved.

4. Proof of Theorem 2

In this section, we assume that the surface (1) has a non-zero constant Gaussian curvature G_0. In this case, Equation (9) can be written in the form

$$\sum_{i=0}^{8} \Gamma_i(s)\cos[i\,t] + \Sigma_i(s)\sin[i\,t] \;=\; 0$$

One begins to compute the coefficients Γ_i and Σ_i. The first coefficient

$$\Sigma_8 \;=\; 2\,(b^2-1)\,G_0\,\sigma\,\varpi\,\lambda^5\,\nu'\left(\sigma^2\,(\nu^2-1)\left[(\nu^2-1)\,\varpi^2-1\right]-\nu'^2\right).$$

The vanishing of the coefficient Σ_8 yields two possibilities:

4.1. $\varpi(s) \;=\; \pm\dfrac{\sqrt{\nu'^2+\sigma^2\,(\nu^2-1)}}{\sigma\,(\nu^2-1)} \;\neq\; 0$

The computation of the coefficient Γ_8 leads to

$$2\,G_0^2\,\lambda^5\,\nu'^2\left[\nu'^2+\sigma^2\,(\nu^2-1)\right] \;=\; 0,$$

which implies $\nu'^2+\sigma^2\,(\nu^2-1) \;=\; 0$, a contradiction with $\varpi(s) \;\neq\; 0$.

4.2. $\varpi(s) \;=\; 0$

Now, $\Gamma_8 \;=\; -\dfrac{1}{2}\,G_0\,\lambda^5\left[\nu'^2-(1-b\nu^2)\,\sigma^2\right]^2 \;=\; 0$ gives $\nu\nu' \;=\; \pm\sigma\sqrt{1-\nu^2}$, where $|\nu|<1$. In this case, we have

$$\Gamma_6 \;=\; 8\,G_0\,(\nu^2-1)\,\sigma^2\,\lambda^3\left[\left(a\sqrt{1-\nu^2}-\sigma\,\xi\,b\right)^2-c^2\right],$$

$$\Sigma_6 \;=\; 16\,G_0\,(1-\nu^2)\,\sigma^2\,\lambda^3\,\gamma\left(a\sqrt{1-\nu^2}-\sigma\,\xi\,\nu\right).$$

For vanishing coefficients Γ_6 and Σ_6, we obtain the following:

$$c(s) \;=\; 0, \qquad a(s) \;=\; \frac{\sigma\,\xi\,\nu}{\sqrt{1-\nu^2}}.$$

The computation of $\Gamma_4 = 0$ leads to $\lambda' = \dfrac{(2\nu^2-1)\,\sigma\,\lambda}{2\,\nu\sqrt{1-\nu^2}}$. The coefficient $\Gamma_2 = \dfrac{32\,G_0\,\nu^2\,\lambda^3\,\sigma^4\,\xi^2}{\nu^2-1}$ $= 0$ implies $\xi = 0$ and then $(a,\,b,\,c) = (0,\,0,\,0)$, which is a contradiction. Therefore, the proof of the Theorem 2 is completed.

5. Proof of Theorem 3

Let M be a surface in $\mathbf{R}^3$ with zero Gauss curvature G and foliated by a piece of ellipses in parallel planes. Without loss of generality, we assume that the planes of the foliation are parallel to the $(x_1 - x_2)$-plane. Let

$$\mathfrak{X}(s,t) = \Big(f(s) + r(s)\cos[t], g(s) + b(s)\,r(s)\sin[t], s\Big),\ s \in I,\ v \in J, \tag{65}$$

be a local parametrization of M. If we put $G = \dfrac{P}{W}$ in the computations of the Gauss curvature G, it yields

$$P = \sum_{i=0}^{4}(\Gamma_i\cos[i\,t] + \Sigma_i\sin[i\,t]) = 0.$$

A computation yields the following non-zero coefficients :

$$\begin{cases} \Gamma_1 = -4\,b^2\,f'', & \Sigma_1 = -4\,b\,g'', \\ \Gamma_2 = 2\,b\,(2\,b'\,r' + r\,b''), & \Gamma_4 = \dfrac{r\,b'^2}{2}, \\ \Gamma_0 = -\dfrac{r\,(b'^2 + 4\,b\,b'') + 8\,b\,(b\,r')'}{2}. \end{cases} \tag{66}$$

In view of the above expression of $P = 0$, it follows that $b' = 0$, and so $b = b_0$, where b_0 is an arbitrary constant. Then, $\Gamma_0 = -4\,b^2\,r''$. So we must have $r'' = f'' = g'' = 0$. As a consequence, there are constants r_0, r_1, f_0, f_1, g_0, and g_1 such that

$$\begin{cases} r(s) &= r_1\,u + r_0, \\ f(s) &= f_1\,u + f_0, \\ g(s) &= g_1\,u + g_0, \end{cases} \tag{67}$$

that is, the functions f, g, and r are linear on s, and so, the surface is a generalized cone. Therefore, the proof of Theorem 3 is completed.

6. Conclusions

From the above discussion, we have proved the following important theorems:

(1): *The surface (1) foliated by general ellipses is flat if and only if it is a part of a conical surface or it takes one of the following forms: (31), (48), (56), (60), and (64).*

(2): *The surface foliated by general ellipses is a **CGC** surface (1) if and only if* $\mathbf{G} = 0$.

In general, if the surface (1) foliated by general ellipses is flat, then the parameterizations of this surface can take one of the following nine forms: $\mathfrak{X}(s,t) = \mathfrak{X}_i(s,t)$, $i = 1,2,\ldots,9$, where $\mathfrak{X}_i(s,t)$ takes the forms in the Equations (28), (31), (34), (40), (43), (48), (56), (60), and (64) respectively. Four of these surfaces are conical surfaces, as introduced in Lemmas 1–4. The other five surfaces take the forms in the Equations (31), (48), (56), (60) and (64). For the surfaces $\mathfrak{X}(s,t) = \mathfrak{X}_2(s,t)$ and $\mathfrak{X}(s,t) = \mathfrak{X}_6(s,t)$, the base curves are plane curves with arbitrary curvature functions. Furthermore, in the surfaces $\mathfrak{X}(s,t) = \mathfrak{X}_7(s,t)$, $\mathfrak{X}(s,t) = \mathfrak{X}_8(s,t)$, and $\mathfrak{X}(s,t) = \mathfrak{X}_9(s,t)$, the base curves are special types of space curves where the curvatures and torsions are related via the conditions (55), (58) and (62), respectively.

All results introduced by Lopez [15–17] are special cases of our present work when $\nu(s) = 1$. Also, when $\nu(s) = \epsilon_0 \neq 1$, where ϵ_0 is an arbitrary constant, the surface (1) is a surface foliated by general ellipses, which are studied by Ali and Hamdoon [18]. They proved that *The surface foliated by general ellipses is a cylindrical surface that is part of a* *generalized cone or a part of a generalized cylinder*. However, our results are a generalization of these results because a generalized cone and a generalized cylinder are special types of conical surfaces. Recently, many authors considered circular (cyclic) surfaces with a constant radius in Euclidean and Minkowski 3-space. They studied some geometrical properties such as: Singularities and striction curves compared with those of ruled surfaces (see, for example, [7–11]). However, our work is different from these papers in two ways: **(1):** We considered the circular surfaces foliated by general ellipses, which are generalizations of circles. **(2):** We obtained a complete solution of a flat problem of cyclic surfaces foliated by general ellipses. Ali [21] studied the constant mean curvature surfaces foliated by ellipses in three-dimensional Euclidean space $\mathbf{R}^3$. In future work, we hope to study the **CMC** surfaces foliated by general ellipses in Euclidean space $\mathbf{R}^3$ or in Minkowski space $\mathbf{R}^3_1$.

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