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Homogenization of Smoluchowski Equations in Thin Heterogeneous Porous Domains

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Abstract: In a thin heterogeneous porous layer, we carry out a multiscale analysis of Smoluchowski's discrete diffusion–coagulation equations describing the evolution density of diffusing particles that are subject to coagulation in pairs. Assuming that the thin heterogeneous layer is made up of microstructures that are uniformly distributed inside, we obtain in the limit an upscaled model in the lower space dimension. We also prove a corrector-type result very useful in numerical computations. In view of the thin structure of the domain, we appeal to a concept of two-scale convergence adapted to thin heterogeneous media to achieve our goal.

Keywords: homogenization; Smoluchowski equation; two-scale convergence; thin domains

MSC: 35B27; 35F61; 92B05

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1. Introduction and the Main Results

The use of the Smoluchowski equation has proved very efficient in modeling several natural and physical phenomena in chemistry, astrophysics, aerosol science, physics, engineering and biological sciences, just to cite a few. Some applications arise in the modeling of polymerization in chemistry, the motion of a system of particles that are suspended in a gas, the behavior of fuel mixtures in engines (in engineering science), the formation of stars and planets (in physics) and red blood cell aggregation. In this work, we are particularly interested in its application to the aggregation and diffusion of particles.

More precisely, we are concerned with the application of the Smoluchowski equation in the modeling of Alzheimer's disease (AD), as it is a system of partial differential equations aimed at describing the evolving densities of diffusing particles subject to coagulation in pairs. Recently, the crucial role of the Smoluchowski equations in the multiscale modeling of the evolution of AD at different scales has been considered in [1–4], where the authors proposed a suitable mathematical model for the aggregation and diffusion of β-amyloid (Aβ) in the brain affected by AD at the micro-scale (that is, at the size of a single neuron) and at the primary step of the disease when small amyloid fibrils are free to move and merge. We also refer to [5–8] for some other works in the same direction. In the model considered in [2], a tiny part of cerebral tissue is viewed as a bounded domain $\Omega \subset \mathbb{R}^3$, which is perforated by removing from it a set of periodically distributed holes of size ε (the neurons). Moreover, the production of Aβ in monomeric form at the level of neuron membranes is modeled using a non-homogeneous Neumann condition on the boundary of the porosities.

In the current work, we consider the model stated in [2] but, this time, in a thin porous layer. This is motivated by the fact that Alzheimer's disease particularly affects the cerebral cortex (responsible for language and information processing) and hippocampus (essential for memory), which represent very thin layers of brain tissue and contain thousands and millions of neurons. Here, we describe a very small layer of brain tissue by using a highly heterogeneous thin porous layer in which the heterogeneities are due to the number of millions of neurons that the brain tissue can contain. To be more precise, our model problem at the micro-level is stated below.

Let Ω be a bounded open Lipschitz connected subset in $\mathbb{R}^2$. For $0 < \varepsilon < 1$ to be freely fixed, we set

$$\Omega_\varepsilon = \Omega \times (-\varepsilon, \varepsilon) = \left\{ (\overline{x}, x_3) \in \mathbb{R}^3 : \overline{x} \in \Omega \text{ and } -\varepsilon < x_3 < \varepsilon \right\}.$$

We denote by $Z = Y \times I$ the reference layer cell, where $Y = (0,1)^2$ and $I = (-1,1)$. Let $Z_f \subset Z$ be a compact set in Z with a smooth boundary, which represents a generic neuron, and let $Z_s = Z \backslash Z_f$ be the supporting cerebral tissue (often called the solid part in the literature of porous media).

Let us set a notation that is used throughout this work. Let $0 < \varepsilon \leq 1$. For any set $S \subset \mathbb{R}^3$ and any $k \in \mathbb{Z}^3$ (with $\mathbb{Z}$ denoting the integers), we set

$$S^{\varepsilon,k} = \left\{ x \in \mathbb{R}^3 : x = \varepsilon(k+y) \text{ for } y \in S \right\}.$$

With this in mind, let $K_\varepsilon = \{k \in \mathbb{Z}^2 \times \{0\} : Z^{\varepsilon,k} \subset \Omega_\varepsilon\}$, and set $T^\varepsilon = \cup_{k \in K_\varepsilon} Z_f^{\varepsilon,k}$. We define the thin porous layer by

$$\Omega^\varepsilon = \Omega_\varepsilon \backslash T^\varepsilon \text{ (points in } \Omega_\varepsilon \text{ lying off } T^\varepsilon).$$

The boundary of Ω^ε is divided into two parts: the outer boundary $\partial_D \Omega^\varepsilon = \partial\Omega_\varepsilon$ and the inner boundary $\Gamma^\varepsilon = \partial T^\varepsilon$. We also denote by $\Gamma = \partial Z_f$ so that $\Gamma^\varepsilon = \cup_{k \in K_\varepsilon} \Gamma^{\varepsilon,k}$. Finally we denote by ν the outward unit normal to Γ^ε. We assume that Ω^ε is connected and that $|Z_s| > 0$, where $|Z_s|$ stands for the Lebesgue measure of Z_s in $\mathbb{R}^3$. The ε-model reads as follows: for $m = 1$, u_1^ε solves the PDE

$$\begin{cases} \frac{\partial u_1^\varepsilon}{\partial t} - div(d_1 \nabla u_1^\varepsilon) + u_1^\varepsilon \sum\limits_{j=1}^{M} a_{1,j} u_j^\varepsilon = 0 \text{ in } Q_\varepsilon = (0,T) \times \Omega^\varepsilon \\ \frac{\partial u_1^\varepsilon}{\partial \nu} = 0 \text{ on } (0,T) \times \partial\Omega_\varepsilon \\ \frac{\partial u_1^\varepsilon}{\partial \nu} = \varepsilon \psi^\varepsilon \text{ on } (0,T) \times \Gamma^\varepsilon \\ u_1^\varepsilon(0,x) = 0 \text{ in } \Omega^\varepsilon; \end{cases} \tag{1}$$

for $1 < m < M$, u_m^ε solves the PDE

$$\begin{cases} \frac{\partial u_m^\varepsilon}{\partial t} - div(d_m \nabla u_m^\varepsilon) + u_m^\varepsilon \sum\limits_{j=1}^{M} a_{m,j} u_j^\varepsilon = f_m^\varepsilon \text{ in } Q_\varepsilon \\ \frac{\partial u_m^\varepsilon}{\partial \nu} = 0 \text{ on } (0,T) \times \partial\Omega^\varepsilon \\ u_m^\varepsilon(0,x) = 0 \text{ in } \Omega^\varepsilon; \end{cases} \tag{2}$$

and for $m = M$, u_M^ε solves the equation

$$\begin{cases} \frac{\partial u_M^\varepsilon}{\partial t} - div(d_M \nabla u_M^\varepsilon) = g_\varepsilon \text{ in } Q_\varepsilon \\ \frac{\partial u_M^\varepsilon}{\partial \nu} = 0 \text{ on } (0,T) \times \partial\Omega^\varepsilon \\ u_M^\varepsilon(0,x) = 0 \text{ in } \Omega^\varepsilon, \end{cases} \tag{3}$$

where

$$f_m^\varepsilon = \frac{1}{2} \sum_{j=1}^{m-1} a_{j,m-j} u_j^\varepsilon u_{m-j}^\varepsilon,\ g_\varepsilon = \frac{1}{2} \sum_{\substack{j+k \geq M \\ j<M,\ k<M}} a_{j,k} u_j^\varepsilon u_k^\varepsilon \text{ and } \psi^\varepsilon(t,x) = \psi(t, \overline{x}, \frac{x}{\varepsilon}) \ ((t,x) \in Q_\varepsilon). \tag{4}$$

We assume the following:

Hypothesis 1 (H1). *The coefficients $a_{i,j}$ are positive constants and satisfy $a_{i,j} = a_{j,i}$ $(1 \leq i, j \leq M)$ with $a_{M,M} = 0$, and the diffusion coefficients d_i are positive constants that become smaller as j becomes large;*

Hypothesis 2 (H2). *The function ψ^ε is defined by $\psi^\varepsilon(t,x) = \psi(t, \overline{x}, \frac{x}{\varepsilon})$ $((t,x) \in Q_\varepsilon)$, where $\psi \in \mathcal{C}^1([0,T]; \mathcal{C}^1(\overline{\Omega}; \mathcal{C}^1_{per}(Y; \mathcal{C}^1(I))))$ with $0 \leq \psi \leq 1$ and $\psi(0, \overline{x}, y) = 0$ for $(\overline{x}, y) \in \Omega \times Z$.*

In (H2), $\mathcal{C}^1_{per}(Y; \mathcal{C}^1(I))$ denotes the space of functions in $\mathcal{C}^1_{loc}(\mathbb{R}^2; \mathcal{C}^1(I))$ that are Y-periodic. In (1)–(3), ∇ stands for the usual gradient operator, while *div* denotes the divergence operator with respect to the variable x; T is a positive number representing the final time. The unknowns are the vector value functions $\boldsymbol{u}^\varepsilon : Q_\varepsilon \to \mathbb{R}^M$, $\boldsymbol{u}^\varepsilon = (u_1^\varepsilon, \ldots, u_M^\varepsilon)$, where the coordinate $u_m^\varepsilon \geq 0$ $(1 \leq m < M)$ stands for the concentration of m-clusters, that is, the clusters made of m identical elementary particles, while u_M^ε takes into account the aggregation of more than $M - 1$ monomers. It is worth noting that the meaning of u_M^ε is different from that of u_m^ε $(m < M)$, as it aims to describe the sum of densities of all the large assemblies. It is assumed that the large assemblies exhibit all the same coagulation properties and do not coagulate with each other. We also assume that the only reaction allowing clusters to form large clusters is a binary coagulation mechanism, while the movement of clusters leading to aggregation arises only from a diffusion process described by the constant diffusion coefficient d_m $(1 \leq m \leq M)$. The kinetic coefficient $a_{i,j}$ arises from a reaction in which an $(i+j)$-cluster is formed from an i-cluster and a j-cluster. Therefore, they can be interpreted as coagulation rates. Finally, f_m^ε $(1 < m < M)$ represents the formation of m-clusters via the coalescence of smaller clusters, and g_ε accounts for the formation of large clusters via the coalescence of other large ones that have the same coagulation properties.

Our main aim in this work is to investigate the limiting behavior as $\varepsilon \to 0$ of the solution $\boldsymbol{u}^\varepsilon$ to (1)–(3) under the assumptions (H1)–(H2). This falls within the scope of a multiscale analysis through the homogenization theory in thin porous domains.

Most structures in nature exhibit multiscale features both in space and time. In biological sciences, modeling and simulation have proven to be useful and necessary in describing and explaining many biological processes. To meet the challenge of their complexity, and in order to numerically model such features and capture these multiscale phenomena as correct as possible, mathematical modeling and theoretical concepts combined with the development of efficient algorithms and simulation tools must be emphasized and promoted. One such mathematical concept that has seen tremendous development during the past 50 years is the theory of *homogenization*. Roughly speaking, homogenization consists of replacing the generally complicated study of heterogeneous and composite phenomena, often modeled using (nonlinear) partial differential equations (PDEs) with variable coefficients, by the study of equivalent homogeneous phenomena with the same overall properties but modeled using PDEs with non-oscillating coefficients, which is ideal for numerical analyses, interpretation and predictions, hence the important role of this step. Homogenization offers a rigorous mathematical framework allowing for the modeling and analysis of composites in various environments. This is especially the case when the environment is represented by a domain that is the union (or the complement of the union) of subdomains of a very small size, say, a domain containing infinitely many holes such as the one under consideration in this work. That is why the macroscopic model that is derived in this work is more relevant in practice than the microscopic one.

There is a huge literature on homogenization in fixed or porous media. A few works deal with the homogenization theory in thin heterogeneous domains; see, e.g., [9–15]. As for homogenization in thin heterogeneous porous media, very few results are known up to now. We may cite [9–12,14]. Concerning the Smoluchowski equation as stated in this work, to the best of our knowledge, the only work dealing with its homogenization is the study in [2], in which the considered domain is a uniformly perforated one that is not thin. Our contribution in this work is twofold: (1) The domain Ω^ε is a thin heterogeneous porous layer. This renders the homogenization procedure not easy to handle. Indeed, to achieve our goal in Theorem 1 below, we make use of the partial mean integral operator M_ε (see below for its definition) associated with the extension operator, while in [2], even the extension operator is not used. (2) We prove in Theorem 2 a corrector-type result allowing us to approximate each u_m^ε by a function of the form $v_m^\varepsilon(t,x) = u_m(t,\overline{x}) + \varepsilon u_m^1(t,\overline{x},x/\varepsilon)$, where the functions u_m and u_m^1 do not depend on ε. We summarize our main results below.

Theorem 1. *Assume that* (H1)–(H2) *hold. For any* $\varepsilon > 0$, *let* $\boldsymbol{u}^\varepsilon = (u_m^\varepsilon)_{1\le m\le M}$ *be the unique solution of* (1)–(3) *in the class* $(\mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}(Q_\varepsilon))^M$, *(*$\alpha \in (0,1)$*). Let also* M_ε *and* E_ε *respectively denote the partial mean integral operator and the extension operator defined by* (37) *(see Section 3) and in Lemma 1 (see Section 2). Then, as* $\varepsilon \to 0$, *one has, for any* $1 \le m \le M$,

$$M_\varepsilon E_\varepsilon u_m^\varepsilon \to u_m \text{ in } L^2(Q)\text{-strong}, \tag{5}$$

$$M_\varepsilon \nabla E_\varepsilon u_m^\varepsilon \to \nabla_{\overline{x}} u_m \text{ in } L^2(Q)^2\text{-weak}, \tag{6}$$

$$M_\varepsilon E_\varepsilon \frac{\partial u_m^\varepsilon}{\partial t} \to \frac{\partial u_m}{\partial t} \text{ in } L^2(Q)\text{-weak}, \tag{7}$$

where $\boldsymbol{u} = (u_m)_{1\le m\le M} \in [L^\infty(Q) \cap L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))]^M$ *is the unique solution of system* (8)–(10) *below:*

$$\begin{cases} \theta\frac{\partial u_1}{\partial t} - div_{\overline{x}}(d_1 A\nabla_{\overline{x}} u_1) + \theta u_1 \sum\limits_{j=1}^{M} a_{1,j}u_j = d_1\widetilde{\psi} \text{ in } Q = (0,T)\times\Omega \\ A\nabla_{\overline{x}} u_1 \cdot n = 0 \text{ on } (0,T)\times\partial\Omega \\ u_1(0,\overline{x}) = 0 \text{ in } \Omega; \end{cases} \tag{8}$$

If $1 < m < M$,

$$\begin{cases} \theta\frac{\partial u_m}{\partial t} - div_{\overline{x}}(d_m A\nabla_{\overline{x}} u_m) + \theta u_m \sum\limits_{j=1}^{M} a_{m,j}u_j - \frac{\theta}{2}\sum\limits_{j=1}^{M} a_{j,m-j}u_j u_{m-j} = 0 \text{ in } Q \\ A\nabla_{\overline{x}} u_m \cdot n = 0 \text{ on } (0,T)\times\partial\Omega \\ u_m(0,\overline{x}) = 0 \text{ in } \Omega; \end{cases} \tag{9}$$

and

$$\begin{cases} \theta\frac{\partial u_M}{\partial t} - div_{\overline{x}}(d_M A\nabla_{\overline{x}} u_m) - \frac{\theta}{2}\sum\limits_{\substack{j+k\ge M \\ j<M,\, k<M}} a_{j,k}u_j u_k = 0 \text{ in } Q \\ A\nabla_{\overline{x}} u_M \cdot n = 0 \text{ on } (0,T)\times\partial\Omega \\ u_M(0,\overline{x}) = 0 \text{ in } \Omega. \end{cases} \tag{10}$$

Moreover, $\boldsymbol{u} \in (\mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}(Q))^M$ *and is such that*

$$u_m > 0 \text{ in } Q,\ m = 1,\dots,M. \tag{11}$$

In (8)–(10), n *is the outward unit normal to* $\partial\Omega$ *and the matrix* $A = I_2 + \nabla_{\overline{y}}\omega$, *where* I_2 *is the* 2×2 *identity matrix and* $\omega = (\omega_i)_{i=1,2}$, *with* ω_i *being the unique solution (up to the addition of function* $v_i \in H^1_\#(Y;H^1(I))$ *such that* $v_i = 0$ *in* Z_s*) in* $H^1_\#(Y;H^1(I)) = \{u \in H^1_{per}(Y;H^1(I)) : \int_{Z_s} u dy = 0\}$ *of the cell problem*

$$\begin{cases} div_y(e_i + \nabla_y\omega_i) = 0 \text{ in } Z_s, \quad (e_i + \nabla_y\omega_i)\cdot\nu = 0 \text{ on } \Gamma, \\ \omega_i(.,y_3) \text{ is } Y\text{-periodic}, \end{cases} \tag{12}$$

where, here, ν *stands for the outward unit normal to* Γ, *and* e_i *is the ith vector of the canonical basis in* $\mathbb{R}^3$; *the functions* $\widetilde{\psi}$ *and* θ *are respectively defined by* $\widetilde{\psi}(t,\overline{x}) = \int_\Gamma \psi(t,\overline{x},y)d\sigma(y)$, $(t,\overline{x}) \in Q$ *and* $\theta = |Z_s|$ *(the Lebesgue measure of* Z_s *in* $\mathbb{R}^3$*).*

The partial mean integral M_ε considered in Theorem 1 is defined, for function ϕ, by

$$M_\varepsilon \phi(t,\overline{x}) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \phi(t,\overline{x},\zeta)d\zeta \text{ for } (t,\overline{x}) \in Q.$$

System (8)–(10) is the upscaled model arising from the ε-model (1)–(3). It is posed in a two-dimensional space, leading to an expected dimension reduction problem, as usually is the case for the homogenization theory in thin domains. Moreover, the Neumann boundary behavior in (1) now plays the role (in the upscaled model) of the source term in the leading equation in (8) so that, in the case of (1), the limiting equation does not have the same form as the original equation posed in the ε-model. For (9) and (10), apart from the diffusion term, they are similar to the ε-equations in (2) and (3).

Now, let ω_i ($i = 1, 2$) and u_m ($1 \leq m \leq M$) be as in Theorem 1. We set

$$u_m^1(t,\overline{x},y) = \sum_{j=1}^{2} \omega_j(y)\frac{\partial u_m}{\partial x_j}(t,\overline{x}) \equiv \omega(y)\cdot\nabla_{\overline{x}}u_1(t,\overline{x}) \text{ for } (t,\overline{x},y) \in Q\times Z, \tag{13}$$

where $\omega = (\omega_1, \omega_2)$. We have $u_m^1 \in L^2(Q) \otimes H^1_\#(Y; H^1(I))$, where $H^1_\#(Y; H^1(I))$ stands for the space of u functions in $H^1_{loc}(\mathbb{R}^2; H^1(I))$ that are Y-periodic and satisfy $\int_{Z_s} u(y)dy = 0$.

With this in mind, the second main result is a corrector-type result and reads as follows:

Theorem 2. *For each $1 \leq m \leq M$, assume that u_m^1 defined by (13) belongs to $L^2(0,T;H^1(\Omega)) \otimes \mathcal{C}^1_\#(Y;H^1(I))$, where $\mathcal{C}^1_\#(Y;H^1(I)) = \{u \in \mathcal{C}^1_{loc}(\mathbb{R}^2;H^1(I)) : u$ is Y-periodic and $\int_{Z_s} udy = 0\}$. Then, as $\varepsilon \to 0$, one has*

$$\varepsilon^{-\frac{1}{2}}\left\| u_m^\varepsilon - u_m - \varepsilon(u_m^1)^\varepsilon \right\|_{L^2(0,T;H^1(\Omega^\varepsilon))} \to 0 \tag{14}$$

where $(u_m^1)^\varepsilon(t,x) = u_m^1(t,\overline{x},x/\varepsilon)$ for $(t,x) \in Q_\varepsilon$.

The result in Theorem 2 allows us to approximate u_m^ε in Q_ε by function v_m^ε of the form $v_m^\varepsilon(t,x) = u_m(t,\overline{x}) + u_m^1(t,\overline{x},x/\varepsilon)$ for $(t,x) \in Q_\varepsilon$. Theorem 2 is new in the literature of the homogenization of the Smoluchowski equation and is very important as far as the quantitative homogenization theory of such kind of equations is concerned.

The plan of this work is as follows: In Section 2, we investigate the well posedness of (1)–(3) and provide useful uniform estimates. Section 3 deals with the treatment of the concept of the two-scale convergence of thin heterogeneous domains. We prove therein some compactness results that are used in the homogenization process. With the help of the results obtained in Section 3, we pass to the limit in (1)–(3) in Section 4, where we prove the first main result of the work, viz., Theorem 1. We also prove Theorem 2 in the same section, and we close the work with a conclusion.

2. Well Posedness and Uniform Estimates

The current section deals with the existence and uniqueness of the solution to (1)–(3), along with some useful a priori estimates. We begin with the following theorem:

Theorem 3. *Assume that* (H1)–(H2) *hold true. For any $\varepsilon > 0$, system* (1)–(3) *possesses a unique weak solution $\boldsymbol{u}^\varepsilon = (u_m^\varepsilon)_{1\leq m\leq M} \in (\mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}(Q_\varepsilon))^M$ ($\alpha \in (0,1)$ be fixed) such that*

$$u_m^\varepsilon(t,x) > 0 \text{ for } (t,x) \in Q_\varepsilon,\ m = 1,\ldots,M.$$

Furthermore, there exists $\varepsilon_0 > 0$ such that, for all $1 \leq m \leq M$,

$$\|u_m^\varepsilon\|_{L^\infty(Q_\varepsilon)} \leq C, \tag{15}$$

$$\|\nabla u_m^\varepsilon\|_{L^2(Q_\varepsilon)} \le C\varepsilon^{\frac{1}{2}}, \tag{16}$$

$$\left\|\frac{\partial u_m^\varepsilon}{\partial t}\right\|_{L^2(Q_\varepsilon)} \le C\varepsilon^{\frac{1}{2}}, \tag{17}$$

and

$$\|\psi^\varepsilon\|_{L^2((0,T)\times\Gamma^\varepsilon)} \le C\|\psi\|_{L^2(0,T;\mathcal{C}(\overline{\Omega}\times\Gamma))}, \tag{18}$$

for all $0 < \varepsilon \le \varepsilon_0$, *where* $C > 0$ *is independent of* m *and* ε.

Proof. The well posedness of (1)–(3) has been addressed in [1,2,4,16]. We are concerned here only with the uniform estimates (15)–(17), with estimate (18) being a classical result arising from the trace result. We just emphasize that, since $|\Gamma^\varepsilon| = O(1)$ ($|\Gamma^\varepsilon|$ stands for the Lebesgue measure of Γ^ε), no scaling is needed in the left-hand side of (18). Now, for (15), we follow exactly the same lines of reasoning as in [2] to obtain it. Both (16) and (17) remain to be checked. We first consider (16). We distinguish the cases $m = 1$ and $1 < m \le M$.

We start with $m = 1$. By multiplying $(1)_1$ by u_1^ε and integrating over Ω^ε, followed by the use the divergence theorem, we obtain

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\|u_1^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + d_1\|\nabla u_1^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \textstyle\int_{\Omega^\varepsilon}\left(|u_1^\varepsilon|^2\sum_{j=1}^M a_{1,j}u_j^\varepsilon\right)dx \\
&= \varepsilon d_1 \textstyle\int_{\Gamma^\varepsilon}\psi(t,\overline{x},\tfrac{x}{\varepsilon})u_1^\varepsilon(t,x)d\sigma_\varepsilon(x) \\
&\le \tfrac{\varepsilon d_1}{2}\|\psi^\varepsilon(t)\|^2_{L^2(\Gamma^\varepsilon)} + \tfrac{\varepsilon d_1}{2}\|u_1^\varepsilon(t)\|^2_{L^2(\Gamma^\varepsilon)},
\end{aligned} \tag{19}$$

where the last inequality above stems from Hölder's and Young's inequalities. We use a well-known trace inequality to deduce the existence of a positive constant C_1 independent of ε such that

$$\varepsilon\|u_1^\varepsilon(t)\|^2_{L^2(\Gamma^\varepsilon)} \le C_1\left(\int_{\Omega^\varepsilon}|u_1^\varepsilon(t)|^2dx + \varepsilon^2\int_{\Omega^\varepsilon}|\nabla u_1^\varepsilon(t)|^2dx\right). \tag{20}$$

Therefore, by integrating (19) over $(0,t)$ ($t \in (0,T]$) and taking into account (18) and (20), we are led to

$$\begin{aligned}
&\|u_1^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + d_1(2-\varepsilon^2C_1)\int_0^t\|\nabla u_1^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}ds \\
\le\ & C_1d_1\int_0^t\|u_1^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}ds + \varepsilon d_1C\|\psi\|_{L^2(0,T;\mathcal{C}(\overline{\Omega}\times\Gamma))}.
\end{aligned} \tag{21}$$

We therefore infer the boundedness of u_1^ε in $L^\infty(Q_\varepsilon)$ associated with (21) wherein there exists $\varepsilon_0 > 0$ such that (16) holds for $m = 1$ and $\|u_1^\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega^\varepsilon))} \le C\varepsilon^{\frac{1}{2}}$ for all $0 < \varepsilon \le \varepsilon_0$, where ε_0 is chosen such that $2 - \varepsilon^2C_1 \ge 1$, that is, $\varepsilon_0 \le C_1^{-\frac{1}{2}}$.

For $1 < m < M$, we proceed as for $m = 1$ and multiply $(2)_1$ by u_m^ε and integrate over Ω^ε; then, one obtains

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_m^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + d_m\|\nabla u_m^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \int_{\Omega^\varepsilon}\left(|u_m^\varepsilon|^2\sum_{j=1}^M a_{m,j}u_j^\varepsilon\right)dx \\
=\ & \int_{\Omega^\varepsilon} f_m^\varepsilon u_m^\varepsilon dx \le \|f_m^\varepsilon\|_{L^2(\Omega^\varepsilon)}\|u_m^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}.
\end{aligned}$$

By integrating over $(0,t)$ for $t \in (0,T]$, we obtain

$$\|u_m^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + 2d_m \int_0^t \|\nabla u_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds \le 2\|f_m^\varepsilon\|_{L^2(Q_\varepsilon)} \|u_m^\varepsilon\|^2_{L^2(Q_\varepsilon)}.$$

By using (15), we obtain at once

$$\|u_m^\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega^\varepsilon))} + \|\nabla u_m^\varepsilon\|^2_{L^2(Q_\varepsilon)} \le C\varepsilon^{\frac{1}{2}}.$$

Finally, the proof of (16) for $m = M$ is obtained exactly as the one for the case $1 < m < M$ mutatis mutandis (replace f_m^ε with g_ε).

Let us now prove (17). We proceed as above by distinguishing three cases.

For $m = 1$, we multiply $(1)_1$ by $\partial u_1^\varepsilon / \partial t$ and use $(1)_2$–$(1)_3$ to obtain

$$\begin{aligned} \int_{\Omega^\varepsilon} \left| \frac{\partial u_1^\varepsilon}{\partial t} \right|^2 dx + \frac{d_1}{2} \frac{\partial}{\partial t} \int_{\Omega^\varepsilon} |\nabla u_1^\varepsilon|^2 dx &= \varepsilon d_1 \int_{\Gamma^\varepsilon} \psi^\varepsilon \frac{\partial u_1^\varepsilon}{\partial t} d\sigma_\varepsilon(x) \\ &\quad - \int_{\Omega^\varepsilon} \left(u_1^\varepsilon \frac{\partial u_1^\varepsilon}{\partial t} \sum_{j=1}^M a_{1,j} u_j^\varepsilon \right) dx. \end{aligned}$$

But

$$\begin{aligned} \int_{\Omega^\varepsilon} \left(u_1^\varepsilon \frac{\partial u_1^\varepsilon}{\partial t} \sum_{j=1}^M a_{1,j} u_j^\varepsilon \right) dx &\le \left\| \frac{\partial u_1^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)} \left\| u_1^\varepsilon \sum_{j=1}^M a_{1,j} u_j^\varepsilon \right\|_{L^2(\Omega^\varepsilon)} \\ &\le \frac{1}{2} \left\| \frac{\partial u_1^\varepsilon}{\partial t} \right\|^2_{L^2(\Omega^\varepsilon)} + \frac{1}{2} \left\| u_1^\varepsilon \sum_{j=1}^M a_{1,j} u_j^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)}. \end{aligned}$$

Thus,

$$\begin{aligned} &\left\| \frac{\partial u_1^\varepsilon}{\partial t} \right\|^2_{L^2(\Omega^\varepsilon)} + d_1 \frac{\partial}{\partial t} \|\nabla u_1^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} \\ \le\; & 2\varepsilon d_1 \int_{\Gamma^\varepsilon} \psi^\varepsilon \frac{\partial u_1^\varepsilon}{\partial t} d\sigma_\varepsilon(x) + \left\| u_1^\varepsilon \sum_{j=1}^M a_{1,j} u_j^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)}. \end{aligned} \tag{22}$$

By integrating (22) over $(0,t)$ and using the boundedness property (15), we obtain after integration by parts

$$\begin{aligned} &\int_0^t \left\| \frac{\partial u_1^\varepsilon}{\partial s}(s) \right\|^2_{L^2(\Omega^\varepsilon)} ds + d_1 \left\| \nabla u_1^\varepsilon(t) \right\|^2_{L^2(\Omega^\varepsilon)} \le C\varepsilon \\ &+ 2\varepsilon d_1 \int_{\Gamma^\varepsilon} \psi^\varepsilon u_1^\varepsilon d\sigma_\varepsilon(x) - 2\varepsilon d_1 \int_0^t \int_{\Gamma^\varepsilon} \frac{\partial \psi^\varepsilon}{\partial s}(s) u_1^\varepsilon(s) d\sigma_\varepsilon(x) ds, \end{aligned} \tag{23}$$

where we use the fact that $\psi(0, \overline{x}, y) = 0$. Now, we use inequality (20); then, (23) becomes

$$\int_0^t \left\| \frac{\partial u_1^\varepsilon}{\partial s}(s) \right\|^2_{L^2(\Omega^\varepsilon)} ds + d_1 \|\nabla u_1^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}$$

$$\leq C\varepsilon + \varepsilon d_1 \left(\|\psi^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} + \|u_1^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} \right)$$

$$+ \varepsilon d_1 \int_0^t \left(\left\| \frac{\partial \psi^\varepsilon}{\partial s}(s) \right\|^2_{L^2(\Gamma^\varepsilon)} + \|u_1^\varepsilon(s)\|^2_{L^2(\Gamma^\varepsilon)} \right) ds$$

$$\leq C\varepsilon + C\varepsilon \left(\|\psi\|^2_{L^\infty(0,T;\mathcal{C}(\overline{\Omega}\times\Gamma))} + \left\| \frac{\partial \psi}{\partial t} \right\|^2_{L^2(0,T;\mathcal{C}(\overline{\Omega}\times\Gamma))} \right)$$

$$+ C\|u_1^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + C d_1 \varepsilon^2 \|\nabla u_1^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + C\|u_1^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + C\varepsilon^2 \|\nabla u_1^\varepsilon\|^2_{L^2(Q_\varepsilon)}.$$

It follows that

$$\int_0^t \left\| \frac{\partial u_1^\varepsilon}{\partial s}(s) \right\|^2_{L^2(\Omega^\varepsilon)} ds + d_1(1 - C\varepsilon^2) \|\nabla u_1^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} \leq C\varepsilon, \tag{24}$$

where, in (24), we took advantage of (15) and (16). Hence, by choosing $\varepsilon \leq \varepsilon_0$ to be sufficiently small so that $1 - C\varepsilon^2 \geq 0$, we obtain (17) for $m = 1$.

The proof of (17) in the case when $1 < m \leq M$ follows the same lines of reasoning as above, but it is much easier. It is therefore left to the reader. This completes the proof. □

The following result whose proof can be found in Theorem 3 in [17] will be useful in the sequel.

Lemma 1. *There exists a bounded linear operator $E_\varepsilon : H^1(\Omega^\varepsilon) \to H^1(\Omega_\varepsilon)$ such that, for all $v \in H^1(\Omega^\varepsilon)$, $E_\varepsilon v = v$ in Ω^ε and*

$$\|E_\varepsilon v\|_{L^2(\Omega_\varepsilon)} \leq C \left(\|v\|_{L^2(\Omega^\varepsilon)} + \varepsilon \|\nabla v\|_{L^2(\Omega^\varepsilon)} \right),$$

and

$$\|\nabla E_\varepsilon v\|_{L^2(\Omega_\varepsilon)} \leq C \|\nabla v\|_{L^2(\Omega^\varepsilon)}$$

for a positive constant independent of both ε and v.

By virtue of Lemma 1, we may define the extension operator from $L^2(0,T;H^1(\Omega^\varepsilon))$ into $L^2(0,T;H^1(\Omega_\varepsilon))$ via the following statement: for $v \in L^2(0,T;H^1(\Omega^\varepsilon))$, we have

$$(E_\varepsilon v)(t) = E_\varepsilon(v(t)) \text{ for a.e. } t \in (0,T).$$

Then, on account of Lemma 1 and Theorem 3, we have

$$\sup_{1 \leq m \leq M} \left(\|E_\varepsilon u_m^\varepsilon\|_{L^\infty(\Omega_\varepsilon^T)} + \|E_\varepsilon u_m^\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))} \right) \leq C\varepsilon^{\frac{1}{2}}, \tag{25}$$

where $C > 0$ is independent of ε and

$$\Omega_\varepsilon^T = (0,T) \times \Omega_\varepsilon. \tag{26}$$

We also need an estimate of $\partial u_m^\varepsilon / \partial t$ in $L^2(\Omega_\varepsilon^T)$. To this end, we proceed as in [18] and consider the restriction operator $R_\varepsilon : L^2(\Omega_\varepsilon) \to L^2(\Omega^\varepsilon)$, $R_\varepsilon v = v|_{\Omega^\varepsilon}$ (the restriction of v to Ω^ε). Then, it is a fact that R_ε is a bounded linear operator as

$$\|R_\varepsilon v\|_{L^2(\Omega^\varepsilon)} \leq \|v\|_{L^2(\Omega_\varepsilon)} \ \forall v \in L^2(\Omega_\varepsilon).$$

Now, if $R^* : L^2(\Omega^\varepsilon) \to L^2(\Omega_\varepsilon)$ denotes the adjoint operator of R_ε, then, for $v \in L^2(0,T;L^2(\Omega^\varepsilon)) = L^2(Q_\varepsilon)$, we define $R_\varepsilon^* v$ by

$$(R_\varepsilon^* v)(t) = R_\varepsilon^*(v(t)) \text{ for a.e. } t \in (0,T).$$

Then, one has

$$\langle R_\varepsilon^* u, v\rangle = \int_0^T \langle R_\varepsilon^*(u(t)), v(t)\rangle dt = \int_0^T \langle u(t), R_\varepsilon(v(t))\rangle dt$$

for all $u \in L^2(Q_\varepsilon)$ and $v \in L^2(\Omega_\varepsilon^T)$. It is therefore easy to see that $R_\varepsilon^* v = \chi_{\Omega^\varepsilon} v$ for all $v \in L^2(Q_\varepsilon)$, or equivalently

$$R_\varepsilon^* v = \chi_{\Omega^\varepsilon} E_\varepsilon v \text{ for all } v \in L^2(Q_\varepsilon), \tag{27}$$

where $\chi_{\Omega^\varepsilon}$ stands for the characteristic function of Ω^ε in Ω_ε.

Lemma 2. *Let the assumptions of Theorem 3 hold. It holds that*

$$\left\| \chi_{\Omega^\varepsilon} \frac{\partial E_\varepsilon u_m^\varepsilon}{\partial t} \right\|_{L^2(\Omega_\varepsilon^T)} \le C\varepsilon^{\frac{1}{2}} \text{ for all } 0 < \varepsilon \le \varepsilon_0, \tag{28}$$

where $C > 0$ is independent of ε, and ε_0 is defined in Theorem 3.

Proof. First, we have $R_\varepsilon^* \partial_t u_m^\varepsilon = \chi_{\Omega^\varepsilon} \partial_t E_\varepsilon u_m^\varepsilon$, where $\partial_t = \partial/\partial t$. Thus, it is sufficient to show that

$$\|R_\varepsilon^* \partial_t E_\varepsilon u_m^\varepsilon\|_{L^2(\Omega_\varepsilon^T)} \le C\varepsilon^{\frac{1}{2}}.$$

So, let $\varphi \in L^2(\Omega_\varepsilon^T)$; then,

$$\begin{aligned} |\langle R_\varepsilon^* \partial_t E_\varepsilon u_m^\varepsilon, \varphi\rangle| &= |\langle \partial_t E_\varepsilon u_m^\varepsilon, R_\varepsilon \varphi\rangle| \le \|\partial_t E_\varepsilon u_m^\varepsilon\|_{L^2(Q_\varepsilon)} \|R_\varepsilon \varphi\|_{L^2(Q_\varepsilon)} \\ &\le \|\partial_t u_m^\varepsilon\|_{L^2(Q_\varepsilon)} \|\varphi\|_{L^2(\Omega_\varepsilon^T)} \le C\varepsilon^{\frac{1}{2}} \|\varphi\|_{L^2(\Omega_\varepsilon^T)}. \end{aligned}$$

Whence the result. □

3. Two-Scale Convergence of Thin Heterogeneous Domains

The two-scale convergence of thin heterogeneous domains has been introduced in [19] and extended to thin porous surfaces in [12,17]. The notations used in this section are the same as in the previous ones. Specifically, the domain Ω_ε is defined as above, that is, $\Omega_\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$. When $\varepsilon \to 0$, Ω_ε shrinks to the "interface" $\Omega_0 = \Omega \times \{0\}$. We know that $Q_\varepsilon = (0,T) \times \Omega^\varepsilon$ and $\Omega_\varepsilon^T = (0,T) \times \Omega_\varepsilon$, and we set $Q = (0,T) \times \Omega_0$, $I = (-1,1)$, $Y = (0,1)^2$ and finally $Z = Y \times I$. Let $1 \le p < \infty$; by $L^p_{per}(Y; L^p(I))$, we denote the space of functions in $L^p_{loc}(\mathbb{R}^2; L^p(I))$ that are Y-periodic. Accordingly, we define $W^{1,p}_{per}(Y; W^{1,p}(I))$ as the subspace of $W^{1,p}_{loc}(Y; W^{1,p}(I))$ made of Y-periodic functions, and we set

$$W^{1,p}_{\#}(Y; W^{1,p}(I)) = \left\{ u \in W^{1,p}_{per}(Y; W^{1,p}(I)) : \int_Z u(\overline{y}, y_3) dy = 0 \right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{\#} = \left(\int_Z |\nabla u|^p dy \right)^{1/p}, \quad u \in W^{1,p}_{\#}(Y; W^{1,p}(I)).$$

Any x in $\mathbb{R}^3$ writes $(\overline{x}, x_3)$ or $(\overline{x}, \zeta)$, where $\overline{x} = (x_1, x_2)$. We identify Ω_0 with Ω so that the generic element in Ω_0 is also denoted by $\overline{x}$ instead of $(\overline{x}, 0)$.

We are now able to define the two-scale convergence of thin heterogeneous domains and thin boundaries.

Definition 1. *(a) The sequence* $(u_\varepsilon)_{\varepsilon>0} \subset L^p(\Omega_\varepsilon^T)$ $(1 \leq p < \infty)$ *is*

(i) *Weakly two-scale convergent in* $L^p(\Omega_\varepsilon^T)$ *to* $u_0 \in L^p(Q; L^p_{per}(Y; L^p(I)))$ *if whenever* $\varepsilon \to 0$, *one has*

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon^T} u_\varepsilon(t,x) f\left(t,\overline{x},\frac{x}{\varepsilon}\right) dxdt \to \iint_{Q\times Z} u_0(t,\overline{x},y) f(t,\overline{x},y)\, dyd\overline{x}dt$$

for any $f \in L^{p'}(Q; \mathcal{C}_{per}(Y; L^{p'}(I)))$ $(1/p' = 1 - 1/p)$*; we denote this by "*$u_\varepsilon \to u_0$ *in* $L^p(\Omega_\varepsilon^T)$*-weak 2s";*

(ii) *Strongly two-scale convergent in* $L^p(\Omega_\varepsilon^T)$ *towards* $u_0 \in L^p(Q; L^p_{per}(Y; L^p(I)))$ *if, as* $\varepsilon \to 0$, *one has* $u_\varepsilon \to u_0$ *in* $L^p(\Omega_\varepsilon^T)$*-weak 2s and*

$$\varepsilon^{-\frac{1}{p}}\|u_\varepsilon\|_{L^p(Q_\varepsilon)} \to \|u_0\|_{L^p(Q;L^p_{per}(Y;L^p(I)))}; \tag{29}$$

we denote this by "$u_\varepsilon \to u_0$ *in* $L^p(\Omega_\varepsilon^T)$*-strong 2s".*

(b) The sequence $(u_\varepsilon)_{\varepsilon>0}$ *in* $L^p((0,T)\times\Gamma^\varepsilon)$ *is weakly two-scale convergent in* $L^p((0,T)\times\Gamma^\varepsilon)$ *towards* $u_0 \in L^p(Q\times\Gamma)$ *if, whenever* $\varepsilon \to 0$, *one has*

$$\int_{(0,T)\times\Gamma^\varepsilon} u_\varepsilon(t,x) f\left(t,\overline{x},\frac{x}{\varepsilon}\right) d\sigma_\varepsilon(x)dt \to \iint_{Q\times\Gamma} u_0(t,\overline{x},y) f(t,\overline{x},y)\, d\sigma(y)d\overline{x}dt$$

for all $f \in L^{p'}(0,T; \mathcal{C}(\overline{\Omega}\times\Gamma))$ *that is* Y*-periodic in* $\overline{y}$*; we denote this by "*$u_\varepsilon \to u_0$ *in* $L^p((0,T)\times\Gamma^\varepsilon)$*-weak 2s".*

Remark 1. *It is easy to see that, if* $u_0 \in L^p(Q; \mathcal{C}_{per}(Y; L^p(I)))$*, then (29) is equivalent to*

$$\varepsilon^{-\frac{1}{p}}\|u_\varepsilon - u_0^\varepsilon\|_{L^p(\Omega_\varepsilon^T)} \to 0 \text{ as } \varepsilon \to 0, \tag{30}$$

where $u_0^\varepsilon(t,x) = u_0(t,\overline{x},x/\varepsilon)$ *for* $(t,x) \in \Omega_\varepsilon^T$.

We start with the following important result that should be used in the sequel; see Lemma 3.2.3 in [20] for the proof.

Lemma 3. *Let* $\psi \in L^p(0,T; \mathcal{C}(\overline{\Omega}\times\Gamma))$*, which is* Y*-periodic in* $\overline{y}$*. Then, by letting* $\psi^\varepsilon(t,x) = \psi(t,\overline{x},x/\varepsilon)$ *for* $(t,x) \in (0,T)\times\Gamma^\varepsilon$*, we have*

(i) $\|\psi^\varepsilon\|_{L^p((0,T)\times\Gamma^\varepsilon)} \leq \|\psi\|_{L^p(0,T;\mathcal{C}(\overline{\Omega}\times\Gamma))}$;

(ii) $\int_0^T \int_{\Gamma^\varepsilon} \psi(t,\overline{x},x/\varepsilon)\, d\sigma_\varepsilon(x)dt \to \iint_{Q\times\Gamma} \psi(t,\overline{x},y)\, d\overline{x}d\sigma(y)dt$.

Throughout this work, the letter E stands for any ordinary sequence $(\varepsilon_n)_{n\geq 1}$ with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ when $n \to \infty$. The generic term of E is merely denoted by ε, and $\varepsilon \to 0$ means $\varepsilon_n \to 0$ as $n \to \infty$. This being so, we have the following compactness results.

Theorem 4. *(i) Let* $(u_\varepsilon)_{\varepsilon\in E}$ *be a sequence in* $L^p(\Omega_\varepsilon^T)$ $(1 < p < \infty)$ *such that*

$$\sup_{\varepsilon\in E} \varepsilon^{-1/p}\|u_\varepsilon\|_{L^p(\Omega_\varepsilon^T)} \leq C$$

where C *is a positive constant independent of* ε*. Then, up to a subsequence* E' *of* E*, the sequence* $(u_\varepsilon)_{\varepsilon\in E'}$ *weakly two-scale converges in* $L^p(\Omega_\varepsilon^T)$ *to some* $u_0 \in L^p(Q; L^p_{per}(Y; L^p(I)))$.

(ii) Let $(u_\varepsilon)_{\varepsilon\in E}$ *be a sequence in* $L^p((0,T)\times\Gamma^\varepsilon)$ *such that*

$$\|u_\varepsilon\|_{L^p((0,T)\times\Gamma^\varepsilon)} \leq C,$$

with $C > 0$ being independent of ε. Then, we may find a subsequence E' of E such that the sequence $(u_\varepsilon)_{\varepsilon\in E'}$ weakly two-scale converges in $L^p((0,T)\times\Gamma^\varepsilon)$ towards some function $u_0 \in L^p(Q\times\Gamma)$.

In Theorem 4 above, the proof of part (i) can be found in [21], while the proof of part (ii) can be found in [20] (see also [12,17]).

Theorem 5. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^p(0,T;W^{1,p}(\Omega_\varepsilon))$ $(1 < p < \infty)$ such that*

$$\sup_{\varepsilon\in E}\left(\varepsilon^{-1/p}\|u_\varepsilon\|_{L^p(\Omega_\varepsilon^T)} + \varepsilon^{-1/p}\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon^T)}\right) \leq C$$

where $C > 0$ is independent of ε. Then, up to a subsequence E' extracted from E, we may find a vector function (u_0, u_1) with $u_0 \in L^p(0,T;W^{1,p}(\Omega))$ and $u_1 \in L^p(Q;W^{1,p}_\#(Y;W^{1,p}(I)))$ such that, when $E' \ni \varepsilon \to 0$, we have

$$u_\varepsilon \to u_0 \text{ in } L^p(\Omega_\varepsilon^T)\text{-weak } 2s,$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \text{ in } L^p(\Omega_\varepsilon^T)\text{-weak } 2s \text{ for } i = 1,2, \tag{31}$$

and

$$\frac{\partial u_\varepsilon}{\partial x_3} \to \frac{\partial u_1}{\partial y_3} \text{ in } L^p(\Omega_\varepsilon^T)\text{-weak } 2s. \tag{32}$$

For the proof of Theorem 5, we refer to [21].

Remark 2. If we set

$$\nabla_{\overline{x}} u_0 = \left(\frac{\partial u_0}{\partial x_1}, \frac{\partial u_0}{\partial x_2}, 0\right),$$

then (31) and (32) are equivalent to

$$\nabla u_\varepsilon \to \nabla_{\overline{x}} u_0 + \nabla_y u_1 \text{ in } L^p(\Omega_\varepsilon^T)^3\text{-weak } 2s.$$

The following result is sharper than its homologue in Theorem 5.

Theorem 6. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^2(0,T;H^1(\Omega_\varepsilon))$ such that*

$$\sup_{\varepsilon\in E}\varepsilon^{-\frac{1}{2}}\left(\|u_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))} + \|u_\varepsilon\|_{H^1(0,T;L^2(\Omega_\varepsilon))}\right) \leq C, \tag{33}$$

where C is a positive constant independent of ε. Finally, suppose that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Then, up to a subsequence E' of E, there is a vector function $(u, u^1) \in (L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))) \times L^2(Q;H^1_\#(Y;H^1(I)))$ such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u \text{ in } L^2(\Omega_\varepsilon^T)\text{-strong } 2s, \tag{34}$$

$$\nabla u_\varepsilon \to \nabla_{\overline{x}} u + \nabla_y u^1 \text{ in } L^2(\Omega_\varepsilon^T)^3\text{-weak } 2s, \tag{35}$$

and

$$\partial_t u_\varepsilon \to \partial_t u \text{ in } L^2(\Omega_\varepsilon^T)\text{-weak } 2s. \tag{36}$$

Proof. First, owing to Theorem 5, we derive the existence of a subsequence E' of E and of a vector function $(u, u^1) \in L^2(0,T;H^1(\Omega))) \times L^2(Q;H^1_\#(Y;H^1(I)))$ such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u \text{ in } L^2(\Omega_\varepsilon^T)\text{-weak } 2s,$$

$$\nabla u_\varepsilon \to \nabla_{\overline{x}} u + \nabla_y u^1 \text{ in } L^2(\Omega_\varepsilon^T)^3\text{-}weak\ 2s,$$

and

$$\partial_t u_\varepsilon \to \partial_t u \text{ in } L^2(\Omega_\varepsilon^T)\text{-}weak\ 2s.$$

Now, (34) remains to be proved. To this end, we set

$$M_\varepsilon u_\varepsilon(t, \overline{x}) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u_\varepsilon(t, \overline{x}, x_3) dx_3 \text{ for } (t, \overline{x}) \in Q. \tag{37}$$

Then, we easily see that $M_\varepsilon u_\varepsilon \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ with

$$\sup_{\varepsilon \in E} \Big(\|M_\varepsilon u_\varepsilon\|_{L^2(0,T;H^1(\Omega))} + \|M_\varepsilon u_\varepsilon\|_{H^1(0,T;L^2(\Omega))} \Big) \leq C. \tag{38}$$

Then, from (38), we derive the existence of a subsequence of E' still denoted by E' and of the function $u_0 \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ such that, as $E' \ni \varepsilon \to 0$,

$$M_\varepsilon u_\varepsilon \to u_0 \text{ in } L^2(0, T; L^2(\Omega))\text{-}strong. \tag{39}$$

We recall that (39) stems from the compactness of the continuous embedding $L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \hookrightarrow L^2(0, T; L^2(\Omega))$.

Now, from the Poincaré–Wirtinger inequality, it holds that

$$\varepsilon^{-\frac{1}{2}} \|u_\varepsilon - M_\varepsilon u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C\varepsilon \|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))},$$

so that

$$\varepsilon^{-\frac{1}{2}} \|u_\varepsilon - M_\varepsilon u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \to 0 \text{ as } E' \ni \varepsilon \to 0. \tag{40}$$

Thus, the inequality

$$\varepsilon^{-\frac{1}{2}} \|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon^T)} \leq \varepsilon^{-\frac{1}{2}} \|u_\varepsilon - M_\varepsilon u_\varepsilon\|_{L^2(\Omega_\varepsilon^T)} + \varepsilon^{-\frac{1}{2}} \|M_\varepsilon u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon^T)}$$

associated with the equality

$$\varepsilon^{-\frac{1}{2}} \|M_\varepsilon u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon^T)} = \sqrt{2} \|M_\varepsilon u_\varepsilon - u_0\|_{L^2(Q)}$$

yields (with the help of (39) and (40))

$$\varepsilon^{-\frac{1}{2}} \|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon^T)} \to 0 \text{ as } E' \ni \varepsilon \to 0.$$

This shows that $u_\varepsilon \to u_0$ in $L^2(\Omega_\varepsilon^T)$-*strong* $2s$, and so $u_0 = u$. The proof is complete. □

The next result and its corollary are proved exactly as their homologues in Theorem 6 and Corollary 5 in [22] (see also [23]).

Theorem 7. *Let $1 < p, q < \infty$ and $r \geq 1$ be such that $1/r = 1/p + 1/q \leq 1$. Suppose that $(u_\varepsilon)_{\varepsilon \in E} \subset L^q(\Omega_\varepsilon^T)$ weakly two-scale converges in $L^q(\Omega_\varepsilon^T)$ towards $u_0 \in L^q(Q; L^q_{per}(Y; L^q(I)))$ and $(v_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega_\varepsilon^T)$ strongly two-scale converges in $L^p(\Omega_\varepsilon^T)$ towards $v_0 \in L^p(Q; L^p_{per}(Y; L^p(I)))$. Then, $(u_\varepsilon v_\varepsilon)_{\varepsilon \in E}$ is weakly two-scale convergent in $L^r(\Omega_\varepsilon^T)$ to $u_0 v_0$.*

Corollary 1. *Assume the sequences $(u_\varepsilon)_{\varepsilon \in E}$ in $L^p(\Omega_\varepsilon^T)$ and $(v_\varepsilon)_{\varepsilon \in E}$ in $L^{p'}(\Omega_\varepsilon^T) \cap L^\infty(\Omega_\varepsilon^T)$ (with $1 < p < \infty$, $p' = p/(p-1)$) satisfy the following:*

(i) $u_\varepsilon \to u_0$ *in* $L^p(Q_\varepsilon)$*-weak* $2s$;
(ii) $v_\varepsilon \to v_0$ *in* $L^{p'}(Q_\varepsilon)$*-strong* $2s$;
(iii) $(v_\varepsilon)_{\varepsilon \in E}$ *is bounded in* $L^\infty(Q_\varepsilon)$.

Then, $u_\varepsilon v_\varepsilon \to u_0 v_0$ *in* $L^p(Q_\varepsilon)$*-weak* $2s$.

4. Derivation of the Homogenized Problem: Proofs of the Main Results

4.1. Preliminary Results

In this subsection, we aim to provide further important convergence results that will be very useful in the sequel. In that order, it is to be noted that Ω^ε can alternatively be defined as follows: $\Omega^\varepsilon = \cup_{k\in K_\varepsilon} Z_s^{\varepsilon,k}$, where $K_\varepsilon = \{k \in \mathbb{Z}^2 \times \{0\} : Z^{\varepsilon,k} \subset \Omega_\varepsilon\}$ with $\Omega_\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$ and $Z^{\varepsilon,k} = \{\varepsilon(k+y) : y \in Z\}$. We set $\Lambda_\varepsilon = \cup_{k\in K_\varepsilon} Z_s^{1,k}$, a periodic repetition of set Z_s. We denote by χ_ε the characteristic function of Λ_ε in Ω_ε: $\chi_\varepsilon \equiv \chi_{\Lambda_\varepsilon}$. Then, it holds that

$$\Omega^\varepsilon = \{x \in \Omega_\varepsilon : \chi_\varepsilon(\frac{x}{\varepsilon}) = 1\},$$

so that $\chi_{\Omega^\varepsilon}(x) = \chi_\varepsilon(\frac{x}{\varepsilon})$ for $x \in \Omega_\varepsilon$.

Lemma 4. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p(\Omega_\varepsilon^T)$ ($p > 1$ a real number), which is weakly two-scale convergent in $L^p(\Omega_\varepsilon^T)$ to $u_0 \in L^p(Q; L^p_{per}(Y; L^p(I)))$. Then, as $\varepsilon \to 0$,*

$$u_\varepsilon\chi_\varepsilon \to u_0\chi_{Z_s} \text{ in } L^p(\Omega_\varepsilon^T)\text{-weak } 2s. \tag{41}$$

If further two-scale convergence is strong, then (41) holds in the strong two-scale sense.

Proof. Set $v_\varepsilon(t,\overline{x},\zeta) = u_\varepsilon(t,\overline{x},\varepsilon\zeta)$ for $(t,\overline{x},\zeta) \in \Omega_1^T$. Then, since $u_\varepsilon \to u_0$ in $L^p(\Omega_\varepsilon^T)$-*weak* $2s$, it holds that $\|u_\varepsilon\|_{L^p(\Omega_\varepsilon^T)} \leq C\varepsilon^{1/2}$ (with $C > 0$ being independent of ε) so that $\|v_\varepsilon\|_{L^p(\Omega_1^T)} \leq C$. Hence, up to a subsequence, $v_\varepsilon \to v_0$ in $L^p(\Omega_1^T)$ in the usual classical two-scale weak sense, where $v_0 \in L^p(Q \times I; L^p_{per}(Y))$. Next, let $f \in \mathcal{C}(\overline{Q}; \mathcal{C}_{per}(Y; \mathcal{C}(\overline{I})))$. By passing to the limit (in the subsequence determined above) in the obvious equality

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon^T} u_\varepsilon(t,x) f\left(t,\overline{x},\frac{x}{\varepsilon}\right) dxdt = \int_{\Omega_1^T} v_\varepsilon(t,\overline{x},\zeta) f\left(t,\overline{x},\frac{\overline{x}}{\varepsilon},\zeta\right) d\overline{x}d\zeta dt,$$

we obtain at once $u_0 = v_0$.

This being so, by choosing f as above, one has

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon^T} u_\varepsilon(t,x)\chi_\varepsilon\left(\frac{x}{\varepsilon}\right) f\left(t,\overline{x},\frac{x}{\varepsilon}\right) dxdt &= \int_{\Omega_1^T} v_\varepsilon(t,\overline{x},\zeta)\chi_{\Lambda_1}\left(\frac{\overline{x}}{\varepsilon},\zeta\right) f\left(t,\overline{x},\frac{\overline{x}}{\varepsilon},\zeta\right) d\overline{x}d\zeta dt \\ &\equiv J_\varepsilon.\end{aligned}$$

Owing to the usual two-scale concept, we obtain, as $\varepsilon \to 0$,

$$J_\varepsilon \to \iint_{\Omega_1^T\times Y} u_0(t,\overline{x},\overline{y},\zeta)\chi_{Z_s}(\overline{y},\zeta) f(t,\overline{x},\overline{y},\zeta) d\overline{x}d\overline{y}d\zeta dt, \tag{42}$$

where, in (42), we used the fact that $u_0 = v_0$ as proven above. This concludes the proof. □

The following result will be crucial in the homogenization process. From now on, we set $\chi_s = \chi_{Z_s}$, the characteristic function of Z_s in Z.

Proposition 1. *Let $(u_m^\varepsilon)_{1\leq m\leq M}$ be the solution of (1)–(3). Given any ordinary sequence E, there exist a subsequence E' of E and functions $(u_m, u_m^1)_{1\leq m\leq M}$ with $u_m \in L^2(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$ and $u_m^1 \in L^2(Q; H^1_\#(Y; H^1(I)))$ such that, as $E' \ni \varepsilon \to 0$,*

$$\chi_\varepsilon u_m^\varepsilon \to \chi_s u_m \text{ in } L^2(\Omega_\varepsilon^T)\text{-strong } 2s, \tag{43}$$

$$\chi_\varepsilon \nabla u_m^\varepsilon \to \chi_s(\nabla_{\overline{x}} u_m + \nabla_y u_m^1) \text{ in } L^2(\Omega_\varepsilon^T)^3\text{-weak } 2s, \tag{44}$$

and

$$\chi_\varepsilon \partial_t u_m^\varepsilon \to \chi_s \partial_t u_m \textit{ in } L^2(\Omega_\varepsilon^T)\textit{-weak } 2s. \tag{45}$$

Proof. Since $E_\varepsilon u_m^\varepsilon = u_m^\varepsilon$ in Q_ε, we have

$$\chi_\varepsilon u_m^\varepsilon = \chi_\varepsilon E_\varepsilon u_m^\varepsilon. \tag{46}$$

Next, appealing to (25) and (28), we are in a condition to apply Theorem 6: given sequence E, we may find a subsequence E' of E together with a vector function $(u_m, u_m^1) \in (L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))) \times L^2(Q;H^1_\#(Y;H^1(I)))$ such that, as $E' \ni \varepsilon \to 0$,

$$E_\varepsilon u_m^\varepsilon \to u_m \text{ in } L^2(\Omega_\varepsilon^T)\textit{-strong}2s, \tag{47}$$

$$\nabla E_\varepsilon u_m^\varepsilon \to \nabla_{\overline{x}} u_m + \nabla_y u_m^1 \text{ in } L^2(\Omega_\varepsilon^T)^3\textit{-weak } 2s, \tag{48}$$

and

$$E_\varepsilon \partial_t u_m^\varepsilon \to \partial_t u_m \text{ in } L^2(\Omega_\varepsilon^T)\textit{-weak } 2s. \tag{49}$$

After applying Lemma 4 and accounting for (46), we are finished. □

4.2. Passage to the Limit

Assume that the functions u_m and u_m^1 are as in Proposition 1. Let $\varphi \in \mathcal{C}^1(\overline{Q})$ and $\varphi_1 \in \mathcal{C}^1(\overline{Q} \times \overline{I}; \mathcal{C}^1_{per}(Y))$, and define

$$\Phi_\varepsilon(t,x) = \varphi(t,\overline{x}) + \varepsilon\varphi_1(t,\overline{x},\frac{x}{\varepsilon}) \text{ for } (t,x) \in \Omega_\varepsilon^T.$$

We use Φ_ε as a test function in the variational form of (1)–(3):

$$\begin{cases} \frac{1}{\varepsilon}\int_{Q_\varepsilon} \frac{\partial u_1^\varepsilon}{\partial t}\Phi_\varepsilon dxdt + \frac{d_1}{\varepsilon}\int_{Q_\varepsilon} \nabla u_1^\varepsilon \cdot \nabla\Phi_\varepsilon dxdt + \frac{1}{\varepsilon}\int_{Q_\varepsilon} u_1^\varepsilon \sum_{j=1}^M a_{1,j} u_j^\varepsilon \Phi_\varepsilon dxdt \\ = \int_0^T \int_{\Gamma^\varepsilon} \psi(t,\overline{x},\frac{x}{\varepsilon})\Phi_\varepsilon(t,x) dt d\sigma_\varepsilon(x); \end{cases} \tag{50}$$

For $1 < m < M$,

$$\begin{cases} \frac{1}{\varepsilon}\int_{Q_\varepsilon} \frac{\partial u_m^\varepsilon}{\partial t}\Phi_\varepsilon dxdt + \frac{d_m}{\varepsilon}\int_{Q_\varepsilon} \nabla u_m^\varepsilon \cdot \nabla\Phi_\varepsilon dxdt + \frac{1}{\varepsilon}\int_{Q_\varepsilon} u_m^\varepsilon \sum_{j=1}^M a_{m,j} u_j^\varepsilon \Phi_\varepsilon dxdt \\ = \frac{1}{2\varepsilon}\int_{Q_\varepsilon} \sum_{j=1}^{m-1} a_{j,m-j} u_j^\varepsilon u_{m-j}^\varepsilon \Phi_\varepsilon dtdx; \end{cases} \tag{51}$$

and

$$\frac{1}{\varepsilon}\int_{Q_\varepsilon} \frac{\partial u_M^\varepsilon}{\partial t}\Phi_\varepsilon dxdt + \frac{d_M}{\varepsilon}\int_{Q_\varepsilon} \nabla u_M^\varepsilon \cdot \nabla\Phi_\varepsilon dxdt = \frac{1}{2}\sum_{j+k\geq M, j<M, k<M} \frac{1}{\varepsilon}\int_{Q_\varepsilon} a_{j,k} u_j^\varepsilon u_k^\varepsilon \Phi_\varepsilon dxdt. \tag{52}$$

Let us first deal with (50). We note that it is equivalent to

$$\begin{cases} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon^T} \chi_\varepsilon \frac{\partial u_1^\varepsilon}{\partial t}\Phi_\varepsilon dxdt + \frac{d_1}{\varepsilon}\int_{\Omega_\varepsilon^T} \chi_\varepsilon \nabla u_1^\varepsilon \cdot \nabla\Phi_\varepsilon dxdt + \frac{1}{\varepsilon}\int_{\Omega_\varepsilon^T} \chi_\varepsilon u_1^\varepsilon \sum_{j=1}^M a_{1,j} u_j^\varepsilon \Phi_\varepsilon dxdt \\ = \int_0^T \int_{\Gamma^\varepsilon} \psi(t,\overline{x},\frac{x}{\varepsilon})\Phi_\varepsilon(t,x) dt d\sigma_\varepsilon(x). \end{cases} \tag{53}$$

We have that

$$\nabla\Phi_\varepsilon(t,x) = \nabla_{\overline{x}}\varphi(t,\overline{x}) + \nabla_y \varphi_1((t,\overline{x},\frac{x}{\varepsilon}) + \varepsilon\nabla_{\overline{x}}\varphi_1((t,\overline{x},\frac{x}{\varepsilon}).$$

Thus, we may apply Proposition 1 to proceed to the passage to the limit in the first two terms of the left-hand side of (53), using Φ_ε as a test function in the two-scale concept. Concerning the right-hand side of (53), we use Lemma 3 to pass to the limit therein. We end up with the last term on the left-hand side, where the limit passage therein is more involved. Indeed, we use there the strong two-scale convergence of $\chi_\varepsilon u_1^\varepsilon$ towards $\chi_s u_1$ associated with the weak two-scale convergence of $\chi_\varepsilon u_j^\varepsilon$ $(1 \leq j \leq M)$ towards $\chi_s u_j$ to obtain from Corollary 1 that, for $1 \leq j \leq M$, we have, as $E' \ni \varepsilon \to 0$,

$$\chi_\varepsilon u_1^\varepsilon u_j^\varepsilon = (\chi_\varepsilon u_1^\varepsilon)(\chi_\varepsilon u_j^\varepsilon) \to \chi_s u_1 u_j \text{ in } L^2(\Omega_\varepsilon^T)\text{-}weak\ 2s. \tag{54}$$

Therefore, by using in that term the test function Φ_ε and taking into account all the processes described above after (53), we are led, as $E' \ni \varepsilon \to 0$ in (53), to

$$\begin{cases} \iint_{Q\times Z} \chi_s \frac{\partial u_1}{\partial t} \varphi d\overline{x}dydt + d_1 \iint_{Q\times Z} \chi_s (\nabla_{\overline{x}} u_1 + \nabla_y u_1^1) \cdot (\nabla_{\overline{x}} \varphi + \nabla_y \varphi_1) d\overline{x}dydt \\ + \iint_{Q\times Z} \chi_s u_1 \sum_{j=1}^{M} a_{1,j} u_j \varphi d\overline{x}dydt = \iint_{Q\times\Gamma} \psi\varphi d\overline{x}d\sigma(y)dt \\ \forall(\varphi,\varphi_1) \in \mathcal{C}^1(\overline{Q}) \times \mathcal{C}^1(\overline{Q}\times\overline{I}; \mathcal{C}^1_{per}(Y)). \end{cases} \tag{55}$$

We use the same process as for (53) to pass to the limit in (51) and in (52), and we obtain the following:
For $1 < m < M$,

$$\begin{cases} \iint_{Q\times Z} \chi_s \frac{\partial u_m}{\partial t} \varphi d\overline{x}dydt + d_m \iint_{Q\times Z} \chi_s (\nabla_{\overline{x}} u_m + \nabla_y u_m^1) \cdot (\nabla_{\overline{x}} \varphi + \nabla_y \varphi_1) d\overline{x}dydt \\ + \iint_{Q\times Z} \chi_s u_m \sum_{j=1}^{M} a_{m,j} u_j \varphi d\overline{x}dydt = \frac{1}{2} \iint_{Q\times Z} \chi_s \sum_{j=1}^{m-1} a_{j,m-j} u_j u_{m-j} \varphi d\overline{x}dydt \\ \text{for all } (\varphi,\varphi_1) \in \mathcal{C}^1(\overline{Q}) \times \mathcal{C}^1(\overline{Q}\times\overline{I}; \mathcal{C}^1_{per}(Y)); \end{cases} \tag{56}$$

and

$$\begin{cases} \iint_{Q\times Z} \chi_s \frac{\partial u_M}{\partial t} \varphi d\overline{x}dydt + d_M \iint_{Q\times Z} \chi_s (\nabla_{\overline{x}} u_M + \nabla_y u_M^1) \cdot (\nabla_{\overline{x}} \varphi + \nabla_y \varphi_1) d\overline{x}dydt \\ = \frac{1}{2} \sum_{j+k\geq M, j<M, k<M} \iint_{Q\times Z} \chi_s a_{j,k} u_j u_k \varphi d\overline{x}dydt \\ \text{for all } (\varphi,\varphi_1) \in \mathcal{C}^1(\overline{Q}) \times \mathcal{C}^1(\overline{Q}\times\overline{I}; \mathcal{C}^1_{per}(Y)). \end{cases} \tag{57}$$

We have proved the following result.

Theorem 8. *The functions $(u_m, u_m^1)_{1\leq m\leq M}$ determined by Proposition 1 solve variational problems (55)–(57).*

Our next goal is to derive the system whose solution is $(u_m)_{1\leq m\leq M}$. To this end, we start by uncoupling Equations (55)–(57). We first consider (55), and we see that it is equivalent to the following system consisting of (58) and (59) below:

$$\iint_{Q\times Z} \chi_s (\nabla_{\overline{x}} u_1 + \nabla_y u_1^1) \cdot \nabla_y \varphi_1 d\overline{x}dydt = 0 \ \ \forall \varphi_1 \in \mathcal{C}^1(\overline{Q}\times\overline{I}; \mathcal{C}^1_{per}(Y)), \tag{58}$$

$$\begin{cases} \iint_{Q\times Z}\chi_s\frac{\partial u_1}{\partial t}\varphi d\overline{x}dydt + d_1\iint_{Q\times Z}\chi_s(\nabla_{\overline{x}}u_1 + \nabla_y u_1^1)\cdot\nabla_{\overline{x}}\varphi d\overline{x}dydt \\ + \iint_{Q\times Z}\chi_s u_1\sum_{j=1}^{M} a_{1,j}u_j\varphi d\overline{x}dydt = \iint_{Q\times\Gamma}\psi\varphi d\overline{x}d\sigma(y)dt \;\; \forall\varphi\in\mathcal{C}^1(\overline{Q}). \end{cases} \tag{59}$$

Let us first consider Equation (58) and choose therein φ_1 under the form $\varphi_1(t,\overline{x},y) = \phi(t,\overline{x})\eta(y)$ with $\phi\in\mathcal{C}_0^\infty(Q)$ and $\eta\in\mathcal{C}_{per}^\infty(Y)\otimes\mathcal{C}^1(\overline{I})$; then, (58) becomes

$$\int_Z \chi_s(\nabla_{\overline{x}}u_1 + \nabla_y u_1^1)\cdot\nabla_y\eta dy = 0 \;\; \forall\eta\in\mathcal{C}_{per}^\infty(Y)\otimes\mathcal{C}^1(\overline{I}). \tag{60}$$

To solve (60), we instead consider the variation problem

$$\int_Z \chi_s(e_j + \nabla_y\omega_j)\cdot\nabla_y\eta dy = 0 \;\; \forall\eta\in\mathcal{C}_{per}^\infty(Y)\otimes\mathcal{C}^1(\overline{I}), \tag{61}$$

where e_j $(j = 1,2,3)$ denotes the jth vector of the canonical basis of $\mathbb{R}^3$. Then, (61) is equivalent to the cell problem

$$\begin{cases} -div_y(e_j + \nabla_y\omega_j) = 0 \text{ in } Z_s, \;\; (e_j + \nabla_y\omega_j)\cdot\nu = 0 \text{ on } \Gamma \\ \omega_j(.,y_3) \text{ is } Y\text{-periodic,} \end{cases} \tag{62}$$

where ν stands for the outward unit normal to Γ. It is an easy task to see that (62) possesses a solution in the space

$$H_\#^1(Y;H^1(I)) = \left\{u\in H_{per}^1(Y;H^1(I)) : \int_{Z_s} udy = 0\right\}$$

that is unique up to the addition of a function v_j such that $v_j = 0$ in Z_s. Now, by multiplying (61) by $\partial u_1/\partial x_j$ $(j = 1,2)$ and summing up the resulting equations, and then comparing the latter sum with (60), the following is yielded at once:

$$u_1^1(t,\overline{x},y) = \sum_{j=1}^{2}\omega_j(y)\frac{\partial u_1}{\partial x_j}(t,\overline{x}) \equiv \omega(y)\cdot\nabla_{\overline{x}}u_1(t,\overline{x}), \tag{63}$$

where $\omega = (\omega_1,\omega_2)$.

Next, by going back to (59) and replacing u_1^1 with the expression obtained in (63), we obtain

$$\begin{cases} \int_Q\left(\int_Z\chi_s dy\right)\frac{\partial u_1}{\partial t}\varphi d\overline{x}dt + d_1\int_Q\left(\int_Z\chi_s(I_2 + \nabla_{\overline{y}}\omega)dy\right)\nabla_{\overline{x}}u_1\cdot\nabla_{\overline{x}}\varphi d\overline{x}dt \\ + \int_Q\left(\int_Z\chi_s dy\right)u_1\sum_{j=1}^{M}a_{1,j}u_j\varphi d\overline{x}dt = \int_Q\left(\int_\Gamma\psi(.,.,y)d\sigma(y)\right)\varphi d\overline{x}dt \\ \text{for all } \varphi\in\mathcal{C}^1(\overline{Q}), \end{cases} \tag{64}$$

where I_2 is the identity 2×2 matrix.

This being so, we set

$$\theta = \int_Z\chi_s dy = |Z_s| > 0,\, A = I_2 + \nabla_{\overline{y}}\omega \text{ and } \widetilde{\psi}(t,\overline{x}) = \int_\Gamma\psi((t,\overline{x},y)d\sigma(y). \tag{65}$$

Then, A is a 2×2 symmetric positive definite matrix. Indeed, it is a fact that the entries of A have the form

$$A_{ij} = \int_{Z_s}(e_i + \nabla_y\omega_i)\cdot(e_j + \nabla_y\omega_j)dy,\;\; 1\le i,j\le 2;$$

this stems from (61), where we show that it is still valid for $\eta \in H^1_\#(Y; H^1(I))$ and then choose therein $\eta = \omega_i$. With the above notations in (65), we see that (64) is equivalent to the problem

$$\begin{cases} \theta\frac{\partial u_1}{\partial t} - div_{\overline{x}}(d_1 A\nabla_{\overline{x}} u_1) + \theta u_1 \sum\limits_{j=1}^{M} a_{1,j}u_j = d_1\widetilde{\psi} \text{ in } Q \\ A\nabla_{\overline{x}} u_1 \cdot n = 0 \text{ on } (0,T)\times\partial\Omega \\ u_1(0,\overline{x}) = 0 \text{ in } \Omega. \end{cases} \tag{66}$$

Proceeding as we did for (55), we easily show that (56) and (57) are equivalent to the variational formulations of the following PDEs:

For $1 < m < M$, (56) is equivalent to

$$\begin{cases} \theta\frac{\partial u_m}{\partial t} - div_{\overline{x}}(d_m A\nabla_{\overline{x}} u_m) + \theta u_m \sum\limits_{j=1}^{M} a_{m,j}u_j - \frac{\theta}{2}\sum\limits_{j=1}^{m-1} a_{j,m-j}u_j u_{m-j} = 0 \text{ in } Q \\ A\nabla_{\overline{x}} u_m \cdot n = 0 \text{ on } (0,T)\times\partial\Omega \\ u_m(0,\overline{x}) = 0 \text{ in } \Omega; \end{cases} \tag{67}$$

and for $m = M$, (57) is equivalent to

$$\begin{cases} \theta\frac{\partial u_M}{\partial t} - div_{\overline{x}}(d_M A\nabla_{\overline{x}} u_M) - \frac{\theta}{2}\sum\limits_{j+k\geq M, j<M, k<M} a_{j,k}u_j u_k = 0 \text{ in } Q \\ A\nabla_{\overline{x}} u_M \cdot n = 0 \text{ on } (0,T)\times\partial\Omega \\ u_M(0,\overline{x}) = 0 \text{ in } \Omega. \end{cases} \tag{68}$$

System (66)–(68) is the homogenized model arising from the microscale ε-problem (1)–(3). It is posed in a two-dimensional space, leading to a dimension reduction problem. We see in [2] that (66)–(68) possesses a unique solution. We are now in a position to prove Theorem 1.

4.3. Proof of Theorem 1

The proof of (5)–(7) follows easily from (47)–(49) associated with the properties of operator M_ε. The fact that $(u_m)_{1\leq m\leq M}$ solves (8)–(10) has been shown here above in Section 4.2. Now, if we proceed as in [1] (see also [2]), we obtain the well posedness of (8)–(10) in the space $(\mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}(Q))^M$, and, specifically, (11) holds true. Indeed, if we set $F = (F_1, \ldots, F_M)$, where

$$\begin{aligned} F_1(t,u) &= d_1\widetilde{\psi} - \theta u_1 \sum_{j=1}^{M} a_{1,j}u_j, \\ F_m(t,u) &= -\theta u_m \sum_{j=1}^{M} a_{m,j}u_j - \frac{\theta}{2}\sum_{j=1}^{m-1} a_{j,m-j}u_j u_{m-j} \text{ for } 1 < m < M, \\ F_M(t,u) &= \frac{\theta}{2}\sum_{\substack{j+k\geq M \\ j<M,\, k<M}} a_{j,k}u_j u_k. \end{aligned}$$

Then, F satisfies the assumptions of the appendix in [1]. Hence, Theorems 7.1 and 7.2 of [1] readily ensure the existence and uniqueness of the solution of (8)–(10) as claimed above. Finally, the fact that the whole sequence $[(u_m^\varepsilon)_{1\leq m\leq M}]_{\varepsilon>0}$ converges towards $(u_m)_{1\leq m\leq M}$ follows from the uniqueness of the solution of (8)–(10). This concludes the proof.

4.4. Proof of Theorem 2

First of all, we recall that $(u_m^1)^\varepsilon(t,x) = u_m^1(t,\overline{x},x/\varepsilon)$ for $(t,x) \in Q_\varepsilon$. This being so, for $1 \leq m \leq M$ to be freely fixed, let $r_m^\varepsilon = u_m^\varepsilon - u_m - \varepsilon(u_m^1)^\varepsilon$. Then, $\nabla r_m^\varepsilon = \nabla u_m^\varepsilon - \nabla_{\overline{x}} u_m - (\nabla_y u_m^1)^\varepsilon - \varepsilon(\nabla_{\overline{x}} u_m^1)^\varepsilon$. Assuming $u_m^1 \in L^2(0,T;H^1(\Omega)) \otimes \mathcal{C}^1_\#(Y;H^1(I))$, the functions u_m^1, $\nabla_y u_m^1$ and $\nabla_{\overline{x}} u_m^1$ belong to $L^2(Q;\mathcal{C}_{per}(Y;L^2(I)))$ so that they can be used as test functions in the definition of the two-scale convergence (see Definition 1).

This being so, let us first consider the case $m = 1$.

We have

$$d_1\|\nabla r_\varepsilon\|^2_{L^2(Q_\varepsilon)} = d_1\int_{Q_\varepsilon}\nabla r_\varepsilon\cdot\nabla r_\varepsilon dxdt.$$

Thus, by taking into account (43) (or (47)), proving Theorem 2 amounts to showing that $\varepsilon^{-1}\|\nabla r_\varepsilon\|^2_{L^2(Q_\varepsilon)} \to 0$ as $\varepsilon \to 0$. So, we have

$$\begin{aligned}\frac{d_1}{\varepsilon}\|\nabla r_\varepsilon\|^2_{L^2(Q_\varepsilon)} &= \frac{d_1}{\varepsilon}\int_{\Omega_\varepsilon^T}\chi_\varepsilon(\nabla u_1^\varepsilon - \nabla_{\overline{x}}u_1 - (\nabla_y u_1^1)^\varepsilon - \varepsilon(\nabla_{\overline{x}}u_1^1)^\varepsilon)\cdot(\nabla u_1^\varepsilon - \nabla_{\overline{x}}u_1 \\ &\quad - (\nabla_y u_1^1)^\varepsilon - \varepsilon(\nabla_{\overline{x}}u_1^1)^\varepsilon) \\ &= \frac{d_1}{\varepsilon}\int_{\Omega_\varepsilon^T}\chi_\varepsilon\nabla u_1^\varepsilon\cdot\nabla u_1^\varepsilon - \frac{d_1}{\varepsilon}\int_{\Omega_\varepsilon^T}\chi_\varepsilon\nabla u_1^\varepsilon\cdot(\nabla_{\overline{x}}u_1 + (\nabla_y u_1^1)^\varepsilon + \varepsilon(\nabla_{\overline{x}}u_1^1)^\varepsilon) \\ &\quad - \frac{d_1}{\varepsilon}\int_{\Omega_\varepsilon^T}\chi_\varepsilon(\nabla_{\overline{x}}u_1 + (\nabla_y u_1^1)^\varepsilon + \varepsilon(\nabla_{\overline{x}}u_1^1)^\varepsilon)\cdot\nabla u_m^\varepsilon \\ &\quad + \frac{d_1}{\varepsilon}\int_{\Omega_\varepsilon^T}\chi_\varepsilon(\nabla_{\overline{x}}u_1 + (\nabla_y u_1^1)^\varepsilon + \varepsilon(\nabla_{\overline{x}}u_1^1)^\varepsilon)\cdot(\nabla_{\overline{x}}u_1 + (\nabla_y u_1^1)^\varepsilon + \varepsilon(\nabla_{\overline{x}}u_1^1)^\varepsilon) \\ &= I_1 - I_2 - I_3 + I_4,\end{aligned}$$

where in the series of equalities above, we omitted $dxdt$ in the integrals just for the simplification of the presentation. We use $\nabla_y u_1^1$ and $\nabla_{\overline{x}}u_1^1$ as test functions to obtain at once

$$I_2 \to \iint_{Q\times Z}\chi_s(\nabla_{\overline{x}}u_1 + \nabla_y u_1^1)\cdot(\nabla_{\overline{x}}u_1 + \nabla_y u_1^1)d\overline{x}dydt, \tag{69}$$

$$I_4 \to \iint_{Q\times Z}\chi_s(\nabla_{\overline{x}}u_1 + \nabla_y u_1^1)\cdot(\nabla_{\overline{x}}u_1 + \nabla_y u_1^1)d\overline{x}dydt \tag{70}$$

and

$$I_3 \to \iint_{Q\times Z}\chi_s(\nabla_{\overline{x}}u_1 + \nabla_y u_1^1)\cdot(\nabla_{\overline{x}}u_1 + \nabla_y u_1^1)d\overline{x}dydt. \tag{71}$$

With regard to I_1, one has

$$I_1 = -\frac{1}{\varepsilon}\int_{\Omega_\varepsilon^T}\chi_\varepsilon\frac{\partial u_1^\varepsilon}{\partial t}u_1^\varepsilon - \frac{1}{\varepsilon}\int_{\Omega_\varepsilon^T}\chi_\varepsilon u_1^\varepsilon\sum_{j=1}^M a_{1,j}u_j^\varepsilon u_1^\varepsilon + \int_0^T\int_{\Gamma^\varepsilon}\psi(t,\overline{x},\frac{x}{\varepsilon})u_1^\varepsilon. \tag{72}$$

By appealing to (54) and using once more the strong two-scale convergence of $\chi_\varepsilon u_1^\varepsilon$ towards $\chi_s u_1$, we obtain

$$\chi_\varepsilon u_1^\varepsilon u_j^\varepsilon u_1^\varepsilon = (\chi_\varepsilon u_1^\varepsilon)(\chi_\varepsilon u_j^\varepsilon u_1^\varepsilon) \to \chi_s u_1 u_j u_1 \text{ in } L^2(\Omega_\varepsilon^T)\textit{weak } 2s. \tag{73}$$

Also, the strong two-scale convergence of $\chi_\varepsilon u_1^\varepsilon$ associated with the weak two-scale convergence of $\chi_\varepsilon\partial u_1^\varepsilon/\partial t$ gives, owing to Corollary 1,

$$\chi_\varepsilon\frac{\partial u_1^\varepsilon}{\partial t}u_1^\varepsilon \to \chi_s\frac{\partial u_1}{\partial t}u_1 \text{ in } L^1(\Omega_\varepsilon^T)\textit{weak } 2s. \tag{74}$$

Now, for the last term on the right-hand side of (72), we first notice that, from the well-known trace inequality

$$\varepsilon^{\frac{1}{2}}\|u_1^\varepsilon(t,.)\|_{L^2(\partial\Omega^\varepsilon)} \le C\Big(\|u_1^\varepsilon(t,.)\|_{L^2(\Omega^\varepsilon)} + \varepsilon\|\nabla u_1^\varepsilon(t,.)\|_{L^2(\Omega^\varepsilon)}\Big),$$

we have from (15) and (16)

$$\|u_1^\varepsilon\|_{L^2((0,T)\times\Gamma^\varepsilon)} \le C, \tag{75}$$

where $C > 0$ is independent of ε. It follows from part (ii) of Theorem 4 that (up to a subsequence) the trace of u_1^ε on $(0,T) \times \Gamma^\varepsilon$ two-scale converges in $L^2((0,T) \times \Gamma^\varepsilon)$, and its two-scale limit can be easily identified (by integration by parts) with the trace of u_1 on $Q \times \Gamma$, i.e.,

$$u_1^\varepsilon|_{(0,T)\times\Gamma^\varepsilon} \to u_1|_{Q\times\Gamma} \text{ in } L^2((0,T) \times \Gamma^\varepsilon)\text{-}weak\ 2s. \tag{76}$$

Thus, by using ψ as a test function, we obtain, up to a subsequence,

$$\int_0^T \int_{\Gamma^\varepsilon} \psi(t, \overline{x}, \frac{x}{\varepsilon}) u_1^\varepsilon(t,x) d\sigma_\varepsilon(x) dt \to \iint_{Q\times\Gamma} \psi u_1 d\overline{x} d\sigma(y) dt. \tag{77}$$

Now, in view of the uniqueness of u_1, the convergence result in (77) holds with the entire sequence $(u_1^\varepsilon)_{\varepsilon>0}$.

By collecting (73), (74) and (77), we obtain

$$I_1 \to \iint_{Q\times Z} \chi_s \frac{\partial u_1}{\partial t} u_1 - \iint_{Q\times Z} \chi_s \int_{\Omega_\varepsilon^T} \chi_s u_1 \sum_{j=1}^M a_{1,j} u_j u_1 + \iint_{Q\times\Gamma} \psi u_1 d\overline{x} d\sigma(y) dt. \tag{78}$$

Now, if we take u_1 as a test function in the variational form of (66) and account for (78), we see that

$$I_1 \to \iint_{Q\times Z} \chi_s (\nabla_{\overline{x}} u_1 + \nabla_y u_1^1) \cdot (\nabla_{\overline{x}} u_1 + \nabla_y u_1^1) d\overline{x} dy dt. \tag{79}$$

By putting together (69)–(71) and (79), we obtain the result of the case $m = 1$.

The proof in the case $1 < m \leq M$ is easier and follows the same steps as in the case $m = 1$. Theorem 2 is therefore proved.

5. Conclusions

In this work, we provided a qualitative multiscale analysis of a micro-model of Smoluchowski equations in thin heterogeneous domains. Starting from a three-dimensional problem, we proved that the upscaled equation is posed on a two-dimensional space, leading to a dimension reduction problem. We also addressed an approximation issue by proving a corrector-type result, showing that the solution u_m^ε can be approximated by the function $v_m^\varepsilon = u_m + \varepsilon(u_m^1)^\varepsilon$ *in* Q_ε where u_m and u_m^1 solve equations that are independent of ε. This is very useful in numerical computations and opens the door to the quantitative homogenization of (1), which aims to find the rate of convergence in the approximation of u_m^ε by v_m^ε.

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