# Hopf Bifurcation and Control for the Bioeconomic PredatorPrey Model with Square Root Functional Response and Nonlinear Prey Harvesting 

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#### Abstract

In this essay, we introduce a bioeconomic predator-prey model which incorporates the square root functional response and nonlinear prey harvesting. Due to the introduction of nonlinear prey harvesting, the model demonstrates intricate dynamic behaviors in the predator-prey plane. Economic profit serves as a bifurcation parameter for the system. The stability and Hopf bifurcation of the model are discussed through normal forms and bifurcation theory. These results reveal richer dynamic features of the bioeconomic predator-prey model which incorporates the square root functional response and nonlinear prey harvesting, and provides guidance for realistic harvesting. A feedback controller is introduced in this paper to move the system from instability to stability. Moreover, we discuss the biological implications and interpretations of the findings. Finally, the results are validated by numerical simulations.


Keywords: bioeconomic systems; predator-prey models; non-linear prey harvesting; Hopf bifurcation
MSC: 34D20; 92D25

## 1. Introduction

Biomathematics is a branch of applied mathematics. Biomathematics focuses on the application of mathematical methods and techniques to a variety of problems in biology. This field encompasses a wide range of disciplines, including ecology, epidemiology, bioinformatics, pharmacokinetics, genetics, and so on. The main goal of biomathematics is to use mathematical tools to solve complex problems in biology and to develop mathematical models to better understand and predict the behavior of biological systems. Biomathematics describes biological processes and systems by building mathematical models. These models can involve mathematical tools such as differential equations, difference equations, and stochastic processes and are used to simulate and analyze the dynamics of biological systems. The development of biomathematics has promoted a deep integration between biology and mathematics, providing a powerful tool for solving complex biological problems. Research in this interdisciplinary field has not only expanded our understanding of the life sciences but has also had a profound impact on practical applications such as medicine, ecology, and agriculture. Among them, many researchers [1-15] focus on the study of predator-prey systems in biomathematics. The study of predator-prey systems not only contributes to the development of scientific theories, but is also directly related to the sustainable development of human society and the health of ecosystems.

Currently, researchers are particularly interested in the following research. Zhang et al. [16] explored the stability analysis and Hopf bifurcation of differential-algebraic models of bioeconomic systems. The model uses linear prey harvesting and uniquely
considers economic factors. Through the application of the Hopf bifurcation theorem and the normal form, this study explores the system's behavior as a bifurcation parameter increases. The analysis reveals the transition from stable stationary states to periodic solutions as the bifurcation parameter approaches a specific limit. Liu et al. [17] proposed a biological economic system which includes the Holling type II functional response and prey harvesting. Utilizing both differential-algebraic system theory and Hopf bifurcation theory, the researchers examine the potential of Hopf bifurcation within the specified system. The analysis incorporates economic factors into the model. Economic profit is selected as a bifurcation parameter, and the investigation unveils the occurrence of Hopf bifurcation when the economic profit exceeds a defined threshold. Kar et al. [18] explored the use of harvesting efforts as a strategic control measure for managing a predator-prey system with the Holling type III functional response. Li et al. [19] introduced a model that integrates Holling Type II functional responses and nonlinear prey harvesting. The model is expressed as a bioeconomic differential-algebraic equation, examining how economic profit serves as a bifurcation parameter and influences the system. Applying the normal form of the differential-algebraic model and bifurcation theory, the research explores stability and various bifurcation scenarios. These results offer detailed insights into the intricate dynamics of bioeconomic predator-prey models and provide practical recommendations for implementing harvesting strategies. According to the methodologies of Zhang et al. [16], Liu et al. [17], Karl et al. [18], and Li et al. [19], we depict the system through differentialalgebraic equations, taking into account economic benefits. The analysis includes an examination of the stability and Hopf bifurcation in a predator-prey system featuring nonlinear prey harvesting and the square root functional response. Furthermore, we discuss reasonable economic profits that contribute to sustaining the stability of predatorprey systems.

In brief, this paper presents a bioeconomic differential-algebraic model that describes the dynamics of predator-prey interactions. The model integrates the square root functional response and nonlinear prey harvesting. The dynamic behavior of the system is extensively examined, with particular emphasis on Hopf bifurcation, where economic profit is used as a bifurcation parameter.

The primary objective of this research is to unveil the dynamic features inherent in a bioeconomic differential-algebraic model portraying predator-prey dynamics. The model includes elements of the square root functional response and nonlinear prey harvesting. The ultimate goal is to identify a reasonable economic profit range that can provide valuable guidance for the effective management of bioeconomic systems.

This research is centered around the system derived from the predator-prey model proposed by Braza [20] and Bera [21].

$$
\left\{\begin{array}{l}
\frac{d x}{d t}=r x\left(1-\frac{x}{k}\right)-\frac{\alpha \sqrt{x} y}{1+t_{h} \alpha \sqrt{x}}  \tag{1}\\
\frac{d y}{d t}=-\beta y+\frac{e \alpha \sqrt{x} y}{1+t_{h} \alpha \sqrt{x}}
\end{array}\right.
$$

In this model, $x$ represents the density of the prey population, $y$ represents the density of the predator population, $r$ expresses the prey population growth rate, and $k$ is the environmental carrying capacity. Additional parameters include $t_{h}$ for the average processing time after the prey has been preyed upon, $e$ for the depletion rate, $\alpha$ for the efficiency of prey searching by the predator, and $\beta$ for the natural predator mortality rate in the absence of prey.

Nonlinear harvesting plays a crucial role in regulating the interactions between predators and prey and maintaining the stability of ecosystems. By introducing nonlinear prey harvesting, predator-prey systems display more complex dynamics.

Subsequently, Mortuja [14] extended the model by introducing nonlinear harvesting in prey populations.

$$
\left\{\begin{array}{l}
\frac{d x(t)}{d t}=r x\left(1-\frac{x}{k}\right)-\frac{\alpha \sqrt{x} y}{1+t_{h} \alpha \sqrt{x}}-\frac{q E x}{m_{1} E+m_{2} x}  \tag{2}\\
\frac{d y(t)}{d t}=-\beta y+\frac{e \alpha \sqrt{x} y}{1+t_{h} \alpha \sqrt{x}}
\end{array}\right.
$$

where $q$ is a coefficient representing the harvesting capacity, $E$ is the harvesting effort, and $m_{1}$ and $m_{2}$ are intrinsic constants.

In addition, taking economic profit into account in modeling predator-prey systems can provide the researcher with a more comprehensive perspective, making the analysis more realistic and relevant. Consequently, we incorporate an algebraic equation to account for the economic aspects of harvesting. Following Gordon's economic theory [22], the Net Economic Revenue (NER) is defined as the difference between Total Revenue (TR) and Total Cost (TC).

In system (2), the Total Revenue (TR) and Total Cost (TC) are denoted as

$$
\begin{aligned}
T R & =\frac{q E x}{m_{1} E+m_{2} x} p \\
T C & =\frac{q E}{m_{1} E+m_{2} x} c
\end{aligned}
$$

where $p$ represents the cost of harvesting per unit of biomass, $c$ denotes the cost per unit of harvest, and the economic profit $\rho$ is synonymous with the Net Economic Revenue. It is determined by the following equation:

$$
N E R=T R-T C=(p x-c) \frac{q E}{m_{1} E+m_{2} x}=\rho
$$

Combined with the bioeconomic algebraic equations, system (2) can be characterized through differential algebraic equations:

$$
\left\{\begin{array}{l}
\frac{d x(t)}{d t}=r x\left(1-\frac{x}{k}\right)-\frac{\alpha \sqrt{x} y}{1+t_{h} \alpha \sqrt{x}}-\frac{q E x}{m_{1} E+m_{2} x}  \tag{3}\\
\frac{d y(t)}{d t}=-\beta y+\frac{e \alpha \sqrt{x} y}{1+t_{h} \alpha \sqrt{x}} \\
0=\frac{q E}{m_{1} E+m_{2} x}(p x-c)-\rho
\end{array}\right.
$$

For the sake of simplicity, we assume

$$
\begin{gathered}
f(\rho, \bar{X})=\left[\begin{array}{l}
f_{1}(\rho, \bar{X}) \\
f_{2}(\rho, \bar{X})
\end{array}\right]=\left[\begin{array}{c}
r x\left(1-\frac{x}{k}\right)-\frac{\alpha \sqrt{x} y}{1+t_{h} \alpha \sqrt{x}}-\frac{q E x}{m_{1} E+m_{2} x} \\
y\left(-\beta+\frac{e \alpha \sqrt{x}}{1+t_{h} \alpha \sqrt{x}}\right)^{2}
\end{array}\right] \\
g(\rho, \bar{X})=\frac{q E}{m_{1} E+m_{2} x}(p x-c)-\rho, \bar{X}=[x, y, E]^{T}
\end{gathered}
$$

Moving forward, specifically in system (3), we conduct a detailed analysis of how economic profit influences dynamics within the region $R_{+}^{3}=\left\{[x, y, E]^{T} \mid x>0, y>0, E>0\right\}$.

The following sections are arranged in the subsequent manner: In Section 2, we discuss the local stability of non-negative equilibrium points through the analysis of corresponding characteristic equations. In Section 3, we investigate Hopf bifurcation for positive equilibrium points, utilizing economic profit $\rho$ as the bifurcation parameter in system (3). In Section 4, we propose a feedback controller and discuss the effect of the feedback controller. In Section 5, we employ numerical simulations to validate the mathematical conclusions. In Section 6, we discuss the biological implications and interpretations of the findings. Finally, in Section 7, we encapsulate discussions and conclusions.

Remark 1. The innovations of this paper are as follows: unlike the works of Kar et al. ([18]), this paper adds differential algebraic equations; unlike Zhang ([16]) and Liu et al. ([17]), this paper introduces nonlinear harvesting. In contrast to the study of Li et al. ([19]), this paper uses a different response function and incorporates a feedback controller.

## 2. Equilibrium Point and Local Stability Analysis

In the context of the system, the positive equilibrium point $\bar{X}_{0}=\left[x_{0}, y_{0}, E_{0}\right]^{T}$ in system (3) is determined by the following set of equations:

$$
\left\{\begin{array}{l}
0=r x\left(1-\frac{x}{k}\right)-\frac{\alpha \sqrt{x} y}{1+t_{h} \alpha \sqrt{x}}-\frac{q E x}{m_{1} E+m_{2} x}  \tag{4}\\
0=-\beta y+\frac{e \alpha \sqrt{x} y}{1+t_{h} \alpha \sqrt{x}} \\
0=\frac{q E}{m_{1} E+m_{2} x}(p x-c)-\rho
\end{array}\right.
$$

Upon computation, it becomes evident that Equation (4) possesses a sole real solution $\bar{X}_{0}=\left[x_{0}, y_{0}, E_{0}\right]^{T}$, where

$$
\begin{aligned}
& x_{0}=\left(\frac{\beta}{\alpha\left(e-t_{h} \beta\right)}\right)^{2}, \\
& y_{0}=\frac{\sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)}{\alpha}\left(r-\frac{r}{k} x_{0}-\frac{q E_{0}}{m_{1} E_{0}+m_{2} x_{0}}\right), \\
& E_{0}=\frac{\rho m_{2} x_{0}}{q p x_{0}-q c-\rho m_{1}} .
\end{aligned}
$$

It is important to emphasise that we focus only on the internal equilibrium of system (3). The reason for the focus on internal balance is that it biologically represents the coexistence of prey, predator, and harvesting effort. Therefore, this paper assumes that

$$
r-\frac{r}{k} x_{0}-\frac{q E_{0}}{m_{1} E_{0}+m_{2} x_{0}}>0, \quad q p x_{0}-q c-\rho m_{1}>0 .
$$

By employing the local parameterization for system (3), we derive

$$
\bar{X}=\bar{\psi}(\rho, \aleph)=\bar{X}_{0}(\rho)+\bar{U}_{0} \aleph+\bar{V}_{0} h(\rho, \aleph), \aleph=\left[\aleph_{1}, \aleph_{2}\right]^{T}, g(\rho, \bar{\psi}(\rho, \aleph))=0
$$

where

$$
\bar{U}_{0}=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right)^{T}
$$

and

$$
\bar{V}_{0}=\left(\begin{array}{lll}
0 & 0 & 1
\end{array}\right)^{T}
$$

For additional details regarding the local parameterization, refer to [23]. The parametric representation of system (3) can be expressed as

$$
\left\{\begin{array}{l}
\grave{\aleph}_{1}=f_{1}(\rho, \bar{\psi}(\rho, \aleph))  \tag{5}\\
\dot{\aleph_{2}}=f_{2}(\rho, \bar{\psi}(\rho, \aleph))
\end{array}\right.
$$

With the provided definition and system (5), the Jacobian matrix $D$ at $\aleph=0$ in system (5) demonstrates

$$
\begin{align*}
D & =\left(\begin{array}{cc}
D_{\aleph_{1}} f_{1}(\rho, \bar{\psi}(\rho, \aleph)) & D_{\aleph_{2}} f_{1}(\rho, \bar{\psi}(\rho, \aleph)) \\
D_{\aleph_{1}} f_{2}(\rho, \bar{\psi}(\rho, \aleph)) & D_{\aleph_{2}} f_{2}(\rho, \bar{\psi}(\rho, \aleph))
\end{array}\right)_{\aleph=0} \\
& =\binom{D_{\bar{X}} g\left(\rho, \bar{X}_{0}(\rho)\right)}{\bar{U}_{0}^{T}}\binom{0}{I_{2}}  \tag{6}\\
& =\left(\begin{array}{cc}
\frac{q p x_{0} E_{0}}{\left(p x_{0}-c\right)\left(m_{1} E_{0}+m_{2} x_{0}\right)}-\frac{r}{k} x_{0}+\frac{\alpha y_{0}\left(1+2 t_{h} \alpha \sqrt{x_{0}}\right)}{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}} & -\frac{\alpha \sqrt{x_{0}}}{1+t_{h} \alpha \sqrt{x_{0}}} \\
\frac{e \alpha y_{0}}{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}} & 0
\end{array}\right)
\end{align*}
$$

Theorem 1. Regarding the point of positive equilibrium denoted as $\bar{X}_{0}$ within the system (3):
(i) If $a_{1}^{2}(\rho)>4 a_{2}(\rho)$, the positive equilibrium point $\bar{X}_{0}$ is asymptotically stable when $a_{1}(\rho)>0$, and it is unstable when $a_{1}(\rho)<0$.
(ii) If $a_{1}^{2}(\rho)<4 a_{2}(\rho)$, then the positive equilibrium point $\bar{X}_{0}$ is a sink when $a_{1}(\rho)>0$, and it is a source when $a_{1}(\rho)<0$.

Proof. The characteristic equation of matrix $D$ can be expressed as

$$
\begin{equation*}
\lambda^{2}+a_{1}(\rho) \lambda+a_{2}(\rho)=0 \tag{7}
\end{equation*}
$$

where

$$
a_{1}(\rho)=\frac{r}{k} x_{0}-\frac{q p x_{0} E_{0}}{\left(p x_{0}-c\right)\left(m_{1} E_{0}+m_{2} x_{0}\right)}-\frac{\alpha y_{0}\left(1+2 t_{h} \alpha \sqrt{x_{0}}\right)}{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}
$$

and

$$
a_{2}(\rho)=\frac{e \alpha^{2} y_{0}}{2\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{3}}
$$

We denote $\Delta$ by

$$
\begin{aligned}
\Delta & =\left(\frac{r}{k} x_{0}-\frac{q p x_{0} E_{0}}{\left(p x_{0}-c\right)\left(m_{1} E_{0}+m_{2} x_{0}\right)}-\frac{\alpha y_{0}\left(1+2 t_{h} \alpha \sqrt{x_{0}}\right)}{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}\right)^{2}-\frac{2 e \alpha^{2} y_{0}}{\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{3}} \\
& =a_{1}^{2}(\rho)-4 a_{2}(\rho)
\end{aligned}
$$

Clearly, when $a_{1}^{2}(\rho)>4 a_{2}(\rho)$ and $a_{1}(\rho)>0$, the roots of Equation (7) all have negative real parts. Conversely, when $a_{1}^{2}(\rho)>4 a_{2}(\rho)$ and $a_{1}(\rho)<0$, the roots of Equation (7) all have positive real parts. Therefore, the first part has been proven, and the second part can likewise be proven.

## 3. Hopf Bifurcation Analysis of Positive Equilibrium

Hopf bifurcations are important in the study of predator-prey systems, providing key insights into the understanding of system dynamics and stability. Hopf bifurcations provide a way to understand the stability of predator-prey systems. By studying the Hopf bifurcation phenomenon in depth, we are able to better understand the dynamic behavior in ecosystems, contributing to sustainable resource management and ecological balance.

In this section, we investigate the Hopf bifurcation of system (3), utilizing economic profit $\rho$ as a bifurcation parameter.

Assuming $a_{1}(\rho)=0$, we determine the bifurcation value as

$$
\rho_{0}=\frac{r\left(3 x_{0}+4 t_{h} \alpha x_{0}^{\frac{3}{2}}-2 k t_{h} \alpha \sqrt{x_{0}}-k\right)\left(p x_{0}-c\right)^{2}}{k\left(c+2 c t_{h} \alpha \sqrt{x_{0}}+p x_{0}\right)} .
$$

If $a_{1}^{2}(\rho)<4 a_{2}(\rho)$, then Equation (7) possesses a pair of conjugate complex roots:

$$
\lambda_{1,2}=-\frac{1}{2} a_{1}(\rho) \pm i \sqrt{a_{2}(\rho)-\frac{\left(a_{1}(\rho)\right)^{2}}{4}}:=\alpha(\rho) \pm i \omega(\rho)
$$

Moreover,

$$
\begin{gathered}
\alpha\left(\rho_{0}\right)=0 \\
\alpha_{\rho_{0}}^{\prime}=\frac{p x_{0}}{2\left(p x_{0}-c\right)^{2}}-\frac{1+2 \alpha t_{h} \sqrt{x_{0}}}{4\left(p x_{0}-c\right)\left(1+\alpha t_{h} \sqrt{x_{0}}\right)} \\
\omega_{0}=\sqrt{\frac{e \alpha^{2} y_{0}}{2\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{3}}}>0
\end{gathered}
$$

Subsequently, Hopf bifurcation occurs at the bifurcation value $\rho_{0}$.

To understand the properties of the Hopf bifurcation in system (3), we adopt the methodology outlined in [23-26]. This involves transforming system (3) into the following form:

$$
\left\{\begin{align*}
\aleph_{1}= & \omega_{0} \aleph_{2}+\frac{1}{2} a_{11}^{1} \aleph_{1}^{2}+a_{12}^{1} \aleph_{1} \aleph_{2}+\frac{1}{2} a_{22}^{1} \aleph_{2}^{2}+\frac{1}{6} a_{111}^{1} \aleph_{1}^{3}  \tag{8}\\
& +\frac{1}{2} a_{112}^{1} \aleph_{1}^{2} \aleph_{2}+\frac{1}{2} a_{122}^{1} \aleph_{1} \aleph_{2}^{2}+\frac{1}{6} a_{222}^{1} \aleph_{2}^{3}+o\left(|\aleph|^{4}\right), \\
\dot{\aleph_{2}}= & -\omega_{0} \aleph_{1}+a_{11}^{2} \aleph_{1}^{2}+a_{12}^{2} \aleph_{1} \aleph_{2}+\frac{1}{2} a_{22}^{2} \aleph_{2}^{2}+\frac{1}{6} \aleph_{111}^{2} \aleph_{1}^{3} \\
& +\frac{1}{2} a_{112}^{2} \aleph_{1}^{2} \aleph_{2}+\frac{1}{2} a_{122}^{2} \aleph_{1} \aleph_{2}^{2}+\frac{1}{6} a_{222}^{2} \aleph_{2}^{3}+o\left(|\aleph|^{4}\right) .
\end{align*}\right.
$$

where $\omega_{0}=\omega\left(\rho_{0}\right), \bar{X}=\bar{X}_{0}$, and $\rho=\rho_{0}$.
Subsequently, we delve into the analysis of the parametric system (5) and obtain

$$
\begin{align*}
& +\frac{1}{2} f_{\left.1 \aleph_{1} \aleph_{1} \aleph_{2}(\rho, \bar{X}) \aleph_{1}^{2} \aleph_{2}+\frac{1}{2} f_{1 \aleph_{1} \aleph_{2} \aleph_{2}}(\rho, \bar{X}) \aleph_{1} \aleph_{2}^{2},{ }^{2}\right)} \\
& +\frac{1}{6} f_{1 \aleph_{2} \aleph_{2} \aleph_{2}}(\rho, \bar{X}) \aleph_{2}^{3}+o\left(|\aleph|^{4}\right) \text {, } \\
& \aleph_{2}=f_{2 \aleph_{1}}(\rho, \bar{X}) \aleph_{1}+f_{2 \aleph_{2}}(\rho, \bar{X}) \aleph_{2}+\frac{1}{2} f_{2 \aleph_{1} \aleph_{1}}(\rho, \bar{X}) \aleph_{1}^{2}  \tag{9}\\
& +f_{2 \aleph_{1} \aleph_{2}}(\rho, \bar{X}) \aleph_{1} \aleph_{2}+\frac{1}{2} f_{2 \aleph_{2} \aleph_{2}}(\rho, \bar{X}) \aleph_{2}^{2}+\frac{1}{6} f_{2 \aleph_{1} \aleph_{1} \aleph_{1}}(\rho, \bar{X}) \aleph_{1}^{3} \\
& +\frac{1}{2} f_{2 \aleph_{1}} \aleph_{1} \aleph_{2}(\rho, \bar{X}) \aleph_{1}^{2} \aleph_{2}+\frac{1}{2} f_{2 \aleph_{1} \aleph_{2} \aleph_{2}}(\rho, \bar{X}) \aleph_{1} \aleph_{2}^{2} \\
& +\frac{1}{6} f_{2 \aleph_{2} \aleph_{2} \aleph_{2}}(\rho, \bar{X}) \aleph_{2}^{3}+o\left(|\aleph|^{4}\right) \text {. }
\end{align*}
$$

Utilizing the local parameterization method [23-26], we reach the parametric representation of system (9) (refer to Appendix A for detailed calculations):

Performing the subsequent nonsingular linear transformation on system (10):

$$
\binom{\aleph_{1}}{\aleph_{2}}=H\binom{u_{1}}{u_{2}}, \quad H=\left(\begin{array}{cc}
\frac{\alpha \sqrt{x_{0}}}{1+t_{h} \alpha \sqrt{x_{0}}} & 0 \\
0 & -\omega_{0}
\end{array}\right)
$$

While still utilizing $\aleph=\left[\aleph_{1}, \aleph_{2}\right]^{T}$ to express $u=\left[u_{1}, u_{2}\right]^{T}$, the normal form of system (3) is ultimately presented as

$$
\left\{\begin{align*}
\aleph_{1}= & \omega_{0} \aleph_{2}+\frac{1}{2}\left(\frac{\alpha \sqrt{x_{0}}}{1+t_{h} \alpha \sqrt{x_{0}}}\right)\left(\frac{\alpha y_{0}\left(1+3 t_{h} \alpha \sqrt{x_{0}}\right)}{4 x_{0}^{\frac{3}{2}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{3}}-\frac{2 r}{k}+\frac{2 E_{0}^{2} m_{1} m_{2} q}{\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}}\right.  \tag{11}\\
& \left.-\frac{2 E_{0} q\left(m_{1} p E_{0}+m_{2} c\right)\left(c m_{1} E_{0}+m_{2} p^{2} x_{0}\right)}{\left(p x_{0}-c\right)^{2}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}}\right) \aleph_{1}^{2}+\frac{\alpha \omega_{0}}{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}} \aleph_{1} \aleph_{2} \\
& +\frac{1}{6}\left(\frac{\alpha \sqrt{x_{0}}}{1+t_{h} \alpha \sqrt{x_{0}}}\right)^{2}\left(\frac{3 \alpha y_{0}\left(1+4 t_{n} \alpha \sqrt{x_{0}}+5 t_{h}^{2} \alpha^{2} x_{0}\right)}{8 x_{0}^{\frac{5}{2}}\left(1+t_{n} \alpha \sqrt{x_{0}}\right)^{4}}+\frac{4 E_{0}^{2} m_{1}+m_{2}^{2} q}{\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}}\right. \\
& +\frac{6 E_{0} m_{2} q\left(c m_{2}+p m_{1} E_{0}\right)\left(m_{2} x_{0} p^{2}+c m_{1} E_{0}\right)}{\left(p x_{0}-c\right)^{2}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{4}}-\frac{2 E_{0} m_{2} p^{2} q\left(c m_{2}+m_{1} p E_{0}\right)}{\left(p x_{0}-c\right)^{2}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}} \\
& \left.+\frac{4 E_{0} p q\left(c m_{2}+p m_{1} E_{0}\right)\left(m_{2} x_{0} p^{2}+c m_{1} E_{0}\right)}{\left(p x_{0}-c\right)^{3}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}}\right) \aleph_{1}^{3} \\
& +\frac{2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right)\left(2 E_{0}^{2} c^{2} m_{1}^{2} m_{2} x-E^{2} c^{2} m_{1}^{2}+2 E c^{2} m_{1} m_{2}^{2} x^{2}+2 E c^{2} m_{1} m_{2} x+c m_{2}^{2} p^{2} x_{0}^{2}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{4}} \\
& +\frac{2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right)\left(-4 E_{0} c m_{1} m_{2}^{2} p x_{0}^{3}-2 E_{0} c m_{1} m_{2} p^{3} x_{0}+2 E_{0}^{2} m_{1}^{2} m_{2} p^{2} x_{0}^{3}-E_{0}^{2} m_{1} p^{3} x_{0}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{2} E_{0}+m_{2} x_{0}\right)^{4}} \\
& +\frac{2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right)\left(2 E_{0} m_{1} m_{2}^{2} p^{2} x_{0}^{4}+2 E_{0} m_{1} m_{2} p^{3} x_{0}^{4}-4 E^{2} c m_{1}^{2} m_{2} p x^{3}+3 p x_{0} E^{2} c m_{1}^{2}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{4}} \\
& +\left(\frac{\alpha^{2} \omega_{0}\left(3 \alpha t_{h} \sqrt{x_{0}}+1\right)}{4 x_{0}\left(1+\alpha t_{h} \sqrt{x_{0}}\right)^{4}}\right) \aleph_{1}^{2} \aleph_{2}+o\left(|\aleph|^{4}\right), \\
\aleph_{2}= & -\omega_{0} \aleph_{1}+\frac{1}{2}\left(\frac{\alpha \omega_{0}\left(1+3 t_{h} \alpha \sqrt{x_{0}}\right)}{4 x_{0}^{\frac{1}{2}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}\right) \aleph_{1}^{2}+\frac{e \alpha^{2}}{2\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{3} \aleph_{1} \aleph_{2}} \\
& +\frac{1}{8}\left(\frac{3 \alpha^{2} \omega_{0}\left(4 \alpha t_{h} \sqrt{x_{0}}+5 \alpha^{2} t_{h}^{2} x_{0}+1\right)}{x_{0}\left(1+\alpha t_{h} \sqrt{x_{0}}\right)^{4}}\right) \aleph_{1}^{3}+\frac{1}{2}\left(\frac{\alpha^{3} e\left(3 \alpha t_{h} \sqrt{x_{0}}+1\right)}{4 x_{0}^{\frac{1}{2}}\left(1+\alpha t_{h} \sqrt{x_{0}}\right)^{5}}\right) \aleph_{1}^{2} \aleph_{2}+o\left(|\aleph|^{4}\right)
\end{align*}\right.
$$

Through the transformation and the application of the Hopf bifurcation theorem [23-26], the following theorem is derived.

Theorem 2. In system (3), a positive constant 1 and two neighborhoods ( $\Phi$ and $\Psi$ ) around the equilibrium point $\bar{X}_{0}(\rho)$ exist, satisfying $0<\imath \ll 1$ and $\Phi \subset \Psi$.
(i) If $\frac{\alpha a_{11}^{1}\left(3 t_{h} \alpha \sqrt{x_{0}}-1\right)}{4 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}+a_{111}^{1}>\frac{\alpha^{3} e\left(1+3 t_{h} \alpha \sqrt{x_{0}}\right)}{8 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{5}}$, then:
(1) For $\rho_{0}<\rho<\rho_{0}+\imath, \bar{X}_{0}(\rho)$ is unstable, excluding all points in $\Psi$.
(2) For $\rho_{0}-\imath<\rho<\rho_{0}$, at least one periodic solution exists in $\bar{\Phi}$, with one of them excluding all points in $\bar{X}_{0}(\rho)$. Simultaneously, another periodic solution (possibly the same one) excludes all points in $\Psi$, and $\bar{X}_{0}(\rho)$ is locally asymptotically stable.
(ii) If $\frac{\alpha a_{11}^{1}\left(3 t_{h} \alpha \sqrt{x_{0}}-1\right)}{4 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}+a_{111}^{1}<\frac{\alpha^{3} e\left(1+3 t_{h} \alpha \sqrt{x_{0}}\right)}{8 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{5}}$, then:
(1) For $\rho_{0}-\delta<\rho<\rho_{0}, \bar{X}_{0}(\rho)$ is locally asymptotically stable, attracting all points in $\Psi$.
(2) For $\rho_{0}<\rho<\rho_{0}+1$, there is at least one periodic solution in $\bar{\Phi}$, with one of them attracting all points in $\bar{X}_{0}(\rho)$. Simultaneously, there also exists another periodic solution (possibly the same one) attracting all points in $\Psi \backslash \bar{\Phi}$, and $\bar{X}_{0}(\rho)$ is asymptotically unstable, where

$$
\begin{aligned}
a_{11}^{1}= & \frac{\alpha \sqrt{x_{0}}}{1+t_{h} \alpha \sqrt{x_{0}}}\left(\frac{\alpha y_{0}\left(1+3 t_{h} \alpha \sqrt{x_{0}}\right)}{4 x_{0}^{\frac{3}{2}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{3}}-\frac{2 r}{k}+\frac{2 E_{0}^{2} m_{1} m_{2} q}{\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}}\right. \\
& \left.-\frac{2 E_{0} q\left(m_{1} p E_{0}+m_{2} c\right)\left(c m_{1} E_{0}+m_{2} p^{2} x_{0}\right)}{\left(p x_{0}-c\right)^{2}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}}\right),
\end{aligned}
$$

$$
\begin{aligned}
a_{111}^{1}= & \frac{\alpha^{2} x_{0}}{\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}\left(\frac{3 \alpha y_{0}\left(1+4 t_{n} \alpha \sqrt{x_{0}}+5 t_{h}^{2} \alpha^{2} x_{0}\right)}{8 x_{0}^{\frac{5}{2}}\left(1+t_{n} \alpha \sqrt{x_{0}}\right)^{4}}+\frac{4 E_{0}^{2} m_{1}+m_{2}^{2} q}{\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}}\right. \\
& +\frac{6 E_{0} m_{2} q\left(c m_{2}+p m_{1} E_{0}\right)\left(m_{2} x_{0} p^{2}+c m_{1} E_{0}\right)}{\left(p x_{0}-c\right)^{2}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{4}} \\
& +\frac{4 E_{0} p q\left(c m_{2}+p m_{1} E_{0}\right)\left(m_{2} x_{0} p^{2}+c m_{1} E_{0}\right)}{\left(p x_{0}-c\right)^{3}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}}-\frac{2 E_{0} m_{2} p^{2} q\left(c m_{2}+m_{1} p E_{0}\right)}{\left(p x_{0}-c\right)^{2}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}} \\
& +2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right) \frac{\left(2 E_{0}^{2} c^{2} m_{1}^{2} m_{2} x-E^{2} c^{2} m_{1}^{2}+2 E c^{2} m_{1} m_{2}^{2} x^{2}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{4}} \\
& +2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right) \frac{\left(2 E c^{2} m_{1} m_{2} x+c m_{2}^{2} p^{2} x_{0}^{2}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{4}} \\
& -2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right) \frac{\left(4 E_{0} c m_{1} m_{2}^{2} p x_{0}^{3}+2 E_{0} c m_{1} m_{2} p^{3} x_{0}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{2} E_{0}+m_{2} x_{0}\right)^{4}} \\
& +2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right) \frac{\left(2 E_{0}^{2} m_{1}^{2} m_{2} p^{2} x_{0}^{3}-E_{0}^{2} m_{1} p^{3} x_{0}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{2} E_{0}+m_{2} x_{0}\right)^{4}} \\
& +2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right) \frac{\left(2 E_{0} m_{1} m_{2}^{2} p^{2} x_{0}^{4}+2 E_{0} m_{1} m_{2} p^{3} x_{0}^{4}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{4}} \\
& \left.+2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right) \frac{\left(-4 E^{2} c m_{1}^{2} m_{2} p x^{3}+3 p x_{0} E^{2} c m_{1}^{2}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{4}}\right)
\end{aligned}
$$

Proof. Based on parameter systems (8) and (11), we can derive that

$$
\begin{aligned}
& a_{12}^{1}=\frac{\alpha \omega_{0}}{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}, a_{12}^{2}=\frac{e \alpha^{2}}{2\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{3}}, a_{22}^{1}=a_{22}^{2}=0 \\
& a_{11}^{2}=\frac{\alpha \omega_{0}\left(1+3 t_{h} \alpha \sqrt{x_{0}}\right)}{4 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}} \\
& a_{122}^{1}=a_{222}^{2}=0, a_{112}^{2}=-\frac{e \alpha^{3}\left(1+3 t_{h} \alpha \sqrt{x_{0}}\right)}{4 x_{0}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{4}}
\end{aligned}
$$

Applying the Hopf bifurcation theorem [23-26], we evaluate the value of $\omega$ :

$$
\begin{aligned}
16 \omega= & {\left[a_{11}^{1}\left(a_{11}^{2}-a_{12}^{1}\right)+a_{22}^{2}\left(a_{12}^{2}-a_{22}^{1}\right)+\left(a_{11}^{2} a_{12}^{2}-a_{12}^{1} a_{22}^{1}\right)\right] / \omega_{0} } \\
& +\left(a_{111}^{1}+a_{122}^{1}+a_{112}^{2}+a_{222}^{2}\right) \\
= & \frac{\alpha a_{11}^{1}\left(3 t_{h} \alpha \sqrt{x_{0}}-1\right)}{4 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}-\frac{\alpha^{3} e\left(1+3 t_{h} \alpha \sqrt{x_{0}}\right)}{8 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{5}}+a_{111}^{1}
\end{aligned}
$$

We proceed to examine two cases: $\omega>0$ and $\omega<0$. The subsequent steps closely parallel the proof of [23-26], and are therefore omitted.

Remark 2. In summary, according to Theorem 2, system (3) undergoes a Hopf bifurcation at the internal equilibrium point $\bar{X}_{0}(\rho)$ when $\rho=\rho_{0}$. Typically, the internal equilibrium point $\bar{X}_{0}(\rho)$ remains stable for $\rho<\rho_{0}$ and becomes unstable for $\rho>\rho_{0}$.

Consequently, the economically viable range for stabilizing bio-economic differential-algebraic predator-prey systems is $\rho<\rho_{0}$. Considering the practical expectation of positive economic profit, it is advisable to ensure the economic profit remains within the reasonable range of $0<\rho<\rho_{0}$.

## 4. Feedback Controller for System (3)

Compared with alternative controllers, the state feedback controller offers advantages such as simplicity, efficiency, and maneuverability. It finds extensive applications in the dynamic control of various systems [27]. In this section, we devise a feedback controller.

$$
\begin{equation*}
u_{1}=k_{1}\left(x-x_{0}\right) \tag{12}
\end{equation*}
$$

where $k_{1}$ represents the feedback gain. By manipulating it, Hopf bifurcations can be controlled separately to achieve the desired behavior.

Firstly, let us examine controller $u_{1}$. By incorporating it into system (3), we obtain

$$
\left\{\begin{array}{l}
\frac{d x(t)}{d t}=r x\left(1-\frac{x}{k}\right)-\frac{\alpha \sqrt{x} y}{1+t_{h} \alpha \sqrt{x}}-\frac{q E x}{m_{1} E+m_{2} x}-k_{1}\left(x-x_{0}\right)  \tag{13}\\
\frac{d y(t)}{d t}=-\beta y+\frac{e \alpha \sqrt{x} y}{1+t_{h} \alpha \sqrt{x}} \\
0=\frac{q E}{m_{1} E+m_{2} x}(p x-c)-\rho
\end{array}\right.
$$

Theorem 3. Considering the positive equilibrium point $\bar{X}_{0}(\rho)$ in system (13), if
$k_{1}>2 \sqrt{\frac{e \alpha^{2} y_{0}}{2\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{3}}}-\frac{r}{k} x_{0}+\frac{q p x_{0} E_{0}}{\left(p x_{0}-c\right)\left(m_{1} E_{0}+m_{2} x_{0}\right)}+\frac{\alpha y_{0}\left(1+2 t_{h} \alpha \sqrt{x_{0}}\right)}{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}$, the positive equilibrium point $\bar{X}_{0}(\rho)$ is asymptotically stable.

Proof. The Jacobian matrix $D_{1}$ for system (13) at $\bar{X}_{0}(\rho)$ is characterized by

$$
D_{1}=\left(\begin{array}{cc}
\frac{q p x_{0} E_{0}}{\left(p x_{0}-c\right)\left(m_{1} E_{0}+m_{2} x_{0}\right)}-\frac{r}{k} x_{0}+\frac{\alpha y_{0}\left(1+2 t_{h} \alpha \sqrt{x_{0}}\right)}{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}-k_{1} & -\frac{\alpha \sqrt{x_{0}}}{1+t_{h} \alpha \sqrt{x_{0}}}  \tag{14}\\
\frac{c \alpha y_{0}}{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}} & 0
\end{array}\right)
$$

The characteristic equation of matrix $D_{1}$ can be expressed as

$$
\begin{equation*}
\lambda^{2}+b_{1}(\rho) \lambda+b_{2}(\rho)=0 \tag{15}
\end{equation*}
$$

where

$$
b_{1}(\rho)=k_{1}+\frac{r}{k} x_{0}-\frac{q p x_{0} E_{0}}{\left(p x_{0}-c\right)\left(m_{1} E_{0}+m_{2} x_{0}\right)}-\frac{\alpha y_{0}\left(1+2 t_{h} \alpha \sqrt{x_{0}}\right)}{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}
$$

and

$$
b_{2}(\rho)=\frac{e \alpha^{2} y_{0}}{2\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{3}}
$$

We denote $\Delta_{1}$ by

$$
\begin{aligned}
\Delta_{1} & =\left(k_{1}+\frac{r}{k} x_{0}-\frac{q p x_{0} E_{0}}{\left(p x_{0}-c\right)\left(m_{1} E_{0}+m_{2} x_{0}\right)}-\frac{\alpha y_{0}\left(1+2 t_{h} \alpha \sqrt{x_{0}}\right)}{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}\right)^{2}-\frac{2 e \alpha^{2} y_{0}}{\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{3}} \\
& =b_{1}^{2}(\rho)-4 b_{2}(\rho)
\end{aligned}
$$

Clearly, when $k_{1}>2 \sqrt{\frac{e \alpha^{2} y_{0}}{2\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{3}}}-\frac{r}{k} x_{0}+\frac{q p x_{0} E_{0}}{\left(p x_{0}-c\right)\left(m_{1} E_{0}+m_{2} x_{0}\right)}+\frac{\alpha y_{0}\left(1+2 t_{h} \alpha \sqrt{x_{0}}\right)}{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}$, the roots of Equation (15) all have negative real parts. The positive equilibrium point $\bar{X}_{0}(\rho)$ is asymptotically stable. The proof of the Theorem 3 is complete.

## 5. Numerical Simulation

In this section, we demonstrate the validity of the conclusions through simulation. We provide the following parameters:

$$
\begin{equation*}
r=1, k=2, t_{h}=1, \beta=1, \alpha=1, q=1, e=2, c=1, m_{1}=1, m_{2}=1, p=3, k_{1}=0.75 \tag{16}
\end{equation*}
$$

This indicates that system (3) possesses an equilibrium point $\bar{X}_{0}(\rho)=\left[1, \frac{2}{3}, \frac{1}{5}\right]^{T}$ and a bifurcation value $\rho_{0}=\frac{1}{3}$.

The simulation results can be summarized as follows:
(i) For $\rho=0.28<\rho_{0}$, the positive equilibrium point $\bar{X}_{0}(\rho)$ is locally asymptotically stable (see Figure 1).
(ii) For $\rho=0.332<\rho_{0}$, periodic solutions emerge at positive equilibrium points $\bar{X}_{0}(\rho)$ (see Figure 2).
(iii) For $\rho=0.343>\rho_{0}$, the positive equilibrium point $\bar{X}_{0}(\rho)$ is locally unstable (see Figure 3).
(iv) For $k_{1}=0.75>\frac{4 \sqrt{6}-3}{24}$, the positive equilibrium point $\bar{X}_{0}(\rho)$ under the action of the controller is locally asymptotically stable according to Theorem 3 (see Figure 4).
Numerical simulation analyses show that by keeping the economic profit $(\rho)$ in the range of $0<\rho<\rho_{0}=\frac{1}{3}$, system (3) is stable. This range of economic profit can be used as a practical guide for sustainable harvesting in bioeconomic systems. Operating within the suggested range of economic profit ensures a delicate balance between economic profit and system stability, promoting long-term sustainability.

Remark 3. Based on the simulation results, it is evident that maintaining economic profit ( $\rho$ ) within the range of $0<\rho<\rho_{0}=\frac{1}{3}$ is crucial for ensuring the stability of system (3). This reasonable range serves as a guideline for harvesting in bioeconomic systems.


Figure 1. When $\rho=0.28<\rho_{0}$, and with initial conditions $x_{0}=0.99, y_{0}=0.7, E_{0}=0.18$, the positive equilibrium point $\bar{X}_{0}(\rho)$ is locally asymptotically stable.


Figure 2. For $\rho=0.332<\rho_{0}$, periodic solutions emerge from the positive equilibrium point $\bar{X}_{0}(\rho)$ in system (3), given the initial conditions $x_{0}=0.99, y_{0}=0.7, E_{0}=0.18$.


Figure 3. For $\rho=0.343>\rho_{0}$, the positive equilibrium point $\bar{X}_{0}(\rho)$ exhibits instability with the given initial conditions $x_{0}=0.99, y_{0}=0.7, E_{0}=0.18$.


Figure 4. For $k_{1}=0.75>\frac{4 \sqrt{6}-3}{24}$, the positive equilibrium point $\bar{X}_{0}(\rho)$, subject to control, is stable given the initial conditions $x_{0}=0.99, y_{0}=0.7, E_{0}=0.18$.

## 6. Biological Commentary

In this paper, we focus on stability and Hopf bifurcation in the predator-prey system with nonlinear prey harvesting and the square root functional response. Stability and Hopf bifurcation in the predator-prey system have important implications for ecology and biology, providing insights into understanding about the dynamics of interacting populations.

Biological significance of stability: It is well known that economic profit has a strong influence on the dynamic evolution of populations. Most of the harvesting of biological resources is carried out to realize economic benefits, which motivates the introduction of harvesting in predator-prey models. In the model of this paper, according to Theorem 1 and simulation results, when the economic profit is maintained at $0<\rho<\rho_{0}=\frac{1}{3}$, the system is stable, and the population densities of predator and prey, the prey harvest, and the economic profit will be maintained at a relatively balanced level, which allows them to coexist. Moreover, a stable predator-prey system helps to maintain ecological balance and prevent drastic fluctuations in population size, thus protecting the ecosystem structure and function.

Biological significance of Hopf bifurcation: We further understand the effect of economic profit on the stability of the modeled system. Theorem 2 and numerical simulations show that economic profit is responsible for the stability switching of the model system, and the Hopf bifurcation phenomenon occurs when the economic profit delay increases to a certain critical value $\left(\rho_{0}=\frac{1}{3}\right)$. The Hopf bifurcation accompanies the transition of the system from equilibrium to oscillation. Furthermore, the system is unstable when $\rho_{0}>\frac{1}{3}$. Both Hopf bifurcation and instability are undesirable states when $\rho_{0} \geq \frac{1}{3}$.

Subsequently, we introduced the feedback controller. According to Theorem 3 and numerical simulation results, the feedback controller can move the system from instability to stability. The feedback controller maintains the relatively stable state of the ecosystem by dynamically adjusting the prey population.

Overall, the study of stability and Hopf bifurcation in predator-prey systems provides profound ecological insights into understanding how biological systems adapt and respond to environmental change. The understanding of these concepts has broad applications for fields such as ecology, conservation biology, and sustainable resource management.

## 7. Discussion and Conclusions

This study examines differential-algebraic predator-prey systems with nonlinear prey harvesting, emphasizing the impact of predation and, in particular, the reality of nonlinear interactions. The analysis introduces economic profit as a bifurcation parameter and proves its effect on the stability of the system. Taking Theorem 2 as an example, the results show that the internal equilibrium point $\rho_{0}$ transitions from stable to unstable due to the Hopf bifurcation.

Notably, the results have bioeconomic implications, including the coexistence of predator, prey, and harvesting activities. The results identify a reasonable economic range $\left(0<\rho<\rho_{0}\right)$ that ensures population equilibrium and promotes sustainable harvesting, resulting in positive economic profits.

In addition, significant advances in research have included the integration of nonlinear prey harvesting, the study of Hopf bifurcation effects, and the introduction of feedback controllers. Potential future research directions may include integrating nonlinear predator harvesting, enhancing the utility of the model, and delving into more advanced control strategies as well as more complex bifurcations. In the future, we will explore the codimension2 bifurcations associated with this model using methods from the literature [28]. And we will extend to plotting the Hopf bifurcation curve and delving into a comprehensive explanation of the codimension-2 bifurcation that takes place along this curve.

Author Contributions: Writing-original draft, model formulation, analysis, H.G.; investigation, validation, J.H.; conceptualization, methodology, supervision, G.Z. All authors contributed equally to the interpretation and discussion of results. All authors have read and agreed to the published version of the manuscript.

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## Abbreviations

The following abbreviations are used in this manuscript:

| NER | Net Economic Revenue |
| :--- | :--- |
| TR | Total Revenue |
| TC | Total Cost |

## Appendix A

The coefficients in the differential equation of Equation (10) are computed as follows. Through the method in [23,26], we can determine that

$$
\left.\begin{array}{rl}
D_{\bar{X}} f_{1}(\rho, \bar{X})= & \left(r-\frac{2 r x}{k}-\frac{\alpha y}{2 \sqrt{x}\left(1+t_{h} \alpha \sqrt{x}\right)^{2}}-\frac{q m_{1} E^{2}}{\left(m_{1} E+m_{2} x\right)^{2}},\right. \\
& \left.-\frac{\alpha \sqrt{x}}{1+\alpha t_{h} \sqrt{x}},-\frac{q m_{2} x^{2}}{\left(m_{1} E+m_{2} x\right)^{2}}\right), \\
D_{\bar{X}} f_{2}(\rho, \bar{X})= & \left(\frac{e \alpha y}{2 \sqrt{x}\left(1+t_{h} \alpha \sqrt{x}\right)^{2}},-\beta+\frac{e \alpha \sqrt{x}}{1+t_{h} \alpha \sqrt{x}}, 0\right), \\
D_{\bar{X}} g(\rho, \bar{X})= & \left(\frac{q E\left(p m_{1} E+m_{2} c\right)}{\left(m_{1} E+m_{2} x\right)^{2}}, 0, \frac{q(p x-c) m_{2} x}{\left(m_{1} E+m_{2} x\right)^{2}}\right), \\
D_{\bar{X} \bar{\psi}}(\rho, \aleph)= & \binom{D_{\bar{X}} g(\rho, \bar{X}(\rho))}{\bar{U}_{0}^{T}}^{-1}\binom{0}{I_{2}}=\left(\begin{array}{ll}
1 & 0 \\
0 & E \frac{E\left(p m_{1} E+m_{2} c\right)}{\left(m_{2} x\right)(p x-c)}
\end{array}\right)
\end{array}\right) .
$$

Thus, we obtain

$$
\begin{align*}
f_{1 \aleph_{1}}(\rho, \bar{X})= & D_{\bar{X}} f_{1}(\rho, \bar{X}) D_{\aleph_{1}} \bar{\psi}(\rho, \aleph)=r-\frac{2 r x}{k}-\frac{\alpha y}{2 \sqrt{x}\left(1+t_{h} \alpha \sqrt{x}\right)^{2}} \\
& -\frac{q m_{1} E^{2}}{\left(m_{1} E+m_{2} x\right)^{2}}+\frac{q E x\left(m_{1} p E+m_{2} c\right)}{(p x-c)\left(m_{1} E+m_{2} x\right)^{2}}, \\
f_{1 \aleph_{2}}(\rho, \bar{X})= & D_{\bar{X}} f_{1}(\rho, \bar{X}) D_{\aleph_{2}} \bar{\psi}(\rho, \aleph)=-\frac{\alpha \sqrt{x}}{1+t_{h} \alpha \sqrt{x}},  \tag{A1}\\
f_{2 \aleph_{1}}(\rho, \bar{X})= & D_{\bar{X}} f_{2}(\rho, \bar{X}) D_{\aleph_{1}} \bar{\psi}(\rho, \aleph)=\frac{e \alpha y}{2 \sqrt{x}\left(1+t_{h} \alpha \sqrt{x}\right)^{2}}, \\
f_{2 \aleph_{2}}(\rho, \bar{X})= & D_{\bar{X}} f_{2}(\rho, \bar{X}) D_{\aleph_{2}} \bar{\psi}(\rho, \aleph)=-\beta+\frac{e \alpha \sqrt{x}}{1+t_{h} \alpha \sqrt{x}} .
\end{align*}
$$

By (A1), we have

$$
\begin{align*}
D_{\bar{X}} f_{1 \aleph_{1}}(\rho, \bar{X})= & \left(-\frac{2 r}{k}+\frac{\alpha y\left(1+3 t_{h} \alpha \sqrt{x}\right)}{4 x^{\frac{3}{2}}\left(1+t_{h} \alpha \sqrt{x}\right)^{3}}+2 \frac{q m_{1} m_{2} E^{2}}{\left(m_{1} E+m_{2} x\right)^{3}}\right. \\
& +\frac{q E\left(p m_{1} E+m_{2} c\right)\left(m_{2} x c-m_{1} E c-2 p m_{2} x^{2}\right)}{(p x-c)^{2}\left(m_{1} E+m_{2} x\right)^{3}}, \\
& -\frac{\alpha}{2 \sqrt{x}\left(1+t_{h} \alpha \sqrt{x}\right)^{3}}, \frac{q m_{2} x\left(2 m_{1} p E x+m_{2} x c-m_{1} E c\right)}{(p x-c)\left(m_{1} E+m_{2} x\right)^{3}} \\
& \left.-\frac{2 E m_{1} m_{2} q x}{\left(m_{1} E+m_{2} x\right)^{3}}\right),  \tag{A2}\\
D_{\bar{X}} f_{1 \aleph_{2}}(\rho, \bar{X})= & \left(-\frac{\alpha}{2 \sqrt{x}\left(1+t_{h} \alpha \sqrt{x}\right)^{2}}, 0,0\right), \\
D_{\bar{X}} f_{2 \aleph_{1}}(\rho, \bar{X})= & \left(-\frac{\alpha e y\left(1+3 t_{h} \alpha \sqrt{x}\right)}{4 x^{\frac{3}{2}}\left(1+t_{h} \alpha \sqrt{x}\right)^{3}}, \frac{e \alpha}{2 \sqrt{x}\left(1+t_{h} \alpha \sqrt{x}\right)^{2}}, 0\right), \\
D_{\bar{X}} f_{2 \aleph_{2}}(\rho, \bar{X})= & \left(\frac{e \alpha}{2 \sqrt{x}\left(1+t_{h} \alpha \sqrt{x}\right)^{2}}, 0,0\right) .
\end{align*}
$$

Thus, we obtain

$$
\begin{align*}
f_{1 \aleph_{1} \aleph_{1}}(\rho, \bar{X})= & \frac{\alpha y\left(1+3 t_{h} \alpha \sqrt{x}\right)}{4 x^{\frac{3}{2}}\left(1+t_{h} \alpha \sqrt{x}\right)^{3}}-\frac{2 r}{k}+\frac{2 E^{2} m_{1} m_{2} q}{\left(m_{1} E+m_{2} x\right)^{3}} \\
& -\frac{2 E q\left(m_{1} p E+m_{2} c\right)\left(c m_{1} E+m_{2} p^{2} x\right)}{(p x-c)^{2}\left(m_{1} E+m_{2} x\right)^{3}}, \\
f_{1 \aleph_{1} \aleph_{2}}(\rho, \bar{X})= & -\frac{\alpha}{2 \sqrt{x}\left(1+t_{h} \alpha \sqrt{x}\right)^{2}}, \\
f_{1 \aleph_{2} \aleph_{2}}(\rho, \bar{X})= & 0,  \tag{A3}\\
f_{2 \aleph_{1} \aleph_{1}}(\rho, \bar{X})= & -\frac{\alpha e y\left(1+3 t_{h} \alpha \sqrt{x}\right)}{4 x^{\frac{3}{2}}\left(1+t_{h} \alpha \sqrt{x}\right)^{3}}, \\
f_{2 \aleph_{1} \aleph_{2}}(\rho, \bar{X})= & \frac{e \alpha}{2 \sqrt{x}\left(1+t_{h} \alpha \sqrt{x}\right)^{2}}, \\
f_{2 \aleph_{2} \aleph_{2}}(\rho, \bar{X})= & 0 .
\end{align*}
$$

Substituting $\rho_{0}$ and $\bar{X}_{0}$ into (A1) and (A3), we obtain

$$
\begin{aligned}
f_{1 \aleph_{1}}\left(\rho_{0}, \bar{X}_{0}\right) & =0, \\
f_{2 \aleph_{2}}\left(\rho_{0}, \bar{X}_{0}\right) & =0, \\
f_{1 \aleph_{2}}\left(\rho_{0}, \bar{X}_{0}\right) & =-\frac{\alpha \sqrt{x_{0}}}{1+\alpha t_{h} \sqrt{x_{0}}}, \\
f_{2 \aleph_{1}}\left(\rho_{0}, \bar{X}_{0}\right) & =\frac{e \alpha y_{0}}{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}, \\
f_{1 \aleph_{1} \aleph_{1}}\left(\rho_{0}, \bar{X}_{0}\right) & =\frac{\alpha y_{0}\left(1+3 t_{h} \alpha \sqrt{x_{0}}\right)}{4 x_{0}^{\frac{3}{2}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{3}}-\frac{2 r}{k}+\frac{2 E_{0}^{2} m_{1} m_{2} q}{\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}} \\
& -\frac{2 E_{0} q\left(m_{1} p E_{0}+m_{2} c\right)\left(c m_{1} E_{0}+m_{2} p^{2} x_{0}\right)}{\left(p x_{0}-c\right)^{2}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}}, \\
f_{1 \aleph_{1} \aleph_{2}}\left(\rho_{0}, \bar{X}_{0}\right) & =-\frac{\alpha}{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}}, \\
f_{1 \aleph_{2} \aleph_{2}}\left(\rho_{0}, \bar{X}_{0}\right) & =0, \\
f_{2 \aleph_{2} \aleph_{2}}\left(\rho_{0}, \bar{X}_{0}\right) & =0, \\
f_{2 \aleph_{1} \aleph_{1}}\left(\rho_{0}, \bar{X}_{0}\right) & =-\frac{\alpha e y_{0}\left(1+3 t_{h} \alpha \sqrt{x_{0}}\right)}{4 x_{0}^{\frac{3}{2}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{3}}, \\
f_{2 \aleph_{1} \aleph_{2}}\left(\rho_{0}, \bar{X}_{0}\right) & =\frac{e \alpha \sqrt{2 \sqrt{x_{0}}\left(1+t_{h} \alpha \sqrt{x_{0}}\right)^{2}} .}{}
\end{aligned}
$$

Now, by (A3), we have

$$
\begin{aligned}
D_{\bar{X}} \aleph_{1} \aleph_{1} \aleph_{2}(\rho, \bar{X})= & \left(\frac{\alpha\left(3 \alpha t_{h} \sqrt{x}+1\right)}{4 x^{\frac{3}{2}}\left(1+\alpha t_{h} \sqrt{x}\right)^{3}}, 0,0\right), \\
D_{\bar{X}} \aleph_{2} \aleph_{1} \aleph_{2}(\rho, \bar{X})= & \left(-\frac{e \alpha\left(3 \alpha t_{h} \sqrt{x}+1\right)}{4 x^{\frac{3}{2}}\left(1+\alpha t_{h} \sqrt{x}\right)^{2}}, 0,0\right), \\
D_{\bar{X}} \aleph_{2} \aleph_{1} \aleph_{1}(\rho, \bar{X})= & \left(-\frac{3 \alpha e y\left(4 \alpha t_{h} \sqrt{x}+5 \alpha^{2} t_{h}^{2} x+1\right)}{8 x^{\frac{5}{2}}\left(1+\alpha t_{h} \sqrt{x}\right)^{4}},-\frac{\alpha e\left(3 \alpha t_{h} \sqrt{x}+1\right)}{4 x^{\frac{3}{2}}\left(1+\alpha t_{h} \sqrt{x}\right)^{3}}, 0\right), \\
D_{\bar{X}} \aleph_{1} \aleph_{1} \aleph_{1}(\rho, \bar{X})= & \left(\frac{3 \alpha y\left(4 \alpha t_{h} \sqrt{x}+5 \alpha^{2} t_{h}^{2} x+1\right)}{8 x^{\frac{5}{2}}\left(1+\alpha t_{h} \sqrt{x}\right)^{4}}+\frac{4 E^{2} m_{1} m_{2}^{2} q}{\left(m_{1} E+m_{2} x\right)^{4}}\right. \\
& -\frac{2 E m_{2} p^{2} q\left(m_{1} p E+c m_{2}\right)}{(p x-c)^{2}\left(m_{1} E+m_{2} x\right)^{4}} \\
& +\frac{6 E m_{2} q\left(m_{1} p E+c m_{2}\right)\left(m_{2} x p^{2}+E c m_{1}\right)}{(p x-c)^{2}\left(m_{1} E+m_{2} x\right)^{4}} \\
& +\frac{4 E p q\left(c m_{2}+m_{1} E p\right)\left(m_{2} x p+c m_{1} E\right)}{(p x-c)^{3}\left(m_{1} E+m_{2} x\right)^{3}},-\frac{\alpha\left(1+3 t_{h} \alpha \sqrt{x}\right)}{4 x^{\frac{3}{2}}\left(1+t_{h} \alpha \sqrt{x}\right)^{3}}, \\
& \frac{4 E^{2} m_{1}^{2} m_{2} q}{\left(m_{1} E+m_{2} x\right)^{3}}-\frac{2 q\left(m_{1} p E+m_{2} c\right)\left(m_{2} x p^{2}+E c m_{1}\right)}{(p x-c)^{3}\left(m_{1} E+m_{2} x\right)^{3}} \\
& -\frac{4 E m_{1} m_{2} q}{\left(m_{1} E+m_{2} x\right)^{2}}-\frac{2 E m_{1} p q\left(m_{2} x p^{2}+E c m_{1}\right)}{(p x-c)^{2}\left(m_{1} E+m_{2} x\right)^{3}} \\
& \left.+\frac{6 E m_{1} q\left(m_{2} c+m_{1} p E\right)\left(m_{2} x p^{2}+E m_{1} c\right)}{(p x-c)^{2}\left(m_{1} E+m_{2} x\right)^{4}}\right) .
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
& f_{1} \aleph_{1} \aleph_{2} \aleph_{2}\left(\rho_{0}, \bar{X}_{0}\right)=0, \\
& f_{1} \aleph_{2} \aleph_{2} \aleph_{2}\left(\rho_{0}, \bar{X}_{0}\right)=0 \text {, } \\
& f_{2} \aleph_{1} \aleph_{1} \aleph_{1}\left(\rho_{0}, \bar{X}_{0}\right)=-\frac{3 \alpha e y_{0}\left(4 \alpha t_{h} \sqrt{x_{0}}+5 \alpha^{2} t_{h}^{2} x_{0}+1\right)}{8 x_{0}^{\frac{5}{2}}\left(1+\alpha t_{h} \sqrt{x_{0}}\right)^{4}}, \\
& f_{2} \aleph_{1} \aleph_{1} \aleph_{2}\left(\rho_{0}, \bar{X}_{0}\right)=-\frac{\alpha e\left(3 \alpha t_{h} \sqrt{x_{0}}+1\right)}{4 x_{0}^{\frac{3}{2}}\left(1+\alpha t_{h} \sqrt{x_{0}}\right)^{3}}, \\
& f_{2} \aleph_{2} \aleph_{2} \aleph_{2}\left(\rho_{0}, \bar{X}_{0}\right)=0, \\
& f_{2} \aleph_{1} \aleph_{2} \aleph_{2}\left(\rho_{0}, \bar{X}_{0}\right)=0, \\
& f_{1} \aleph_{1} \aleph_{1} \aleph_{2}\left(\rho_{0}, \bar{X}_{0}\right)=-\frac{\alpha\left(3 \alpha t_{h} \sqrt{x_{0}}+1\right)}{4 x_{0}^{\frac{3}{2}}\left(1+\alpha t_{h} \sqrt{x_{0}}\right)^{3}}, \\
& f_{1} \aleph_{1} \aleph_{1} \aleph_{1}\left(\rho_{0}, \bar{X}_{0}\right)=\frac{3 \alpha y_{0}\left(1+4 t_{n} \alpha \sqrt{x_{0}}+5 t_{h}^{2} \alpha^{2} x_{0}\right)}{8 x_{0}^{\frac{5}{2}}\left(1+t_{n} \alpha \sqrt{x_{0}}\right)^{4}}+\frac{4 E_{0}^{2} m_{1}+m_{2}^{2} q}{\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}} \\
& +\frac{6 E_{0} m_{2} q\left(c m_{2}+p m_{1} E_{0}\right)\left(m_{2} x_{0} p^{2}+c m_{1} E_{0}\right)}{\left(p x_{0}-c\right)^{2}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{4}} \\
& +\frac{4 E_{0} p q\left(c m_{2}+p m_{1} E_{0}\right)\left(m_{2} x_{0} p^{2}+c m_{1} E_{0}\right)}{\left(p x_{0}-c\right)^{3}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}} \\
& -\frac{2 E_{0} m_{2} p^{2} q\left(c m_{2}+m_{1} p E_{0}\right)}{\left(p x_{0}-c\right)^{2}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{3}} \\
& +2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right) \frac{\left(2 E_{0}^{2} c^{2} m_{1}^{2} m_{2} x-E^{2} c^{2} m_{1}^{2}+2 E c^{2} m_{1} m_{2}^{2} x^{2}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{4}} \\
& +2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right) \frac{\left(2 E c^{2} m_{1} m_{2} x+c m_{2}^{2} p^{2} x_{0}^{2}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{4}} \\
& -2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right) \frac{\left(4 E_{0} c m_{1} m_{2}^{2} p x_{0}^{3}+2 E_{0} c m_{1} m_{2} p^{3} x_{0}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{2} E_{0}+m_{2} x_{0}\right)^{4}} \\
& +2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right) \frac{\left(2 E_{0}^{2} m_{1}^{2} m_{2} p^{2} x_{0}^{3}-E_{0}^{2} m_{1} p^{3} x_{0}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{2} E_{0}+m_{2} x_{0}\right)^{4}} \\
& +2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right) \frac{\left(2 E_{0} m_{1} m_{2}^{2} p^{2} x_{0}^{4}+2 E_{0} m_{1} m_{2} p^{3} x_{0}^{4}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{4}} \\
& +2 E_{0} q\left(c m_{2}+E_{0} m_{1} p\right) \frac{\left(-4 E^{2} c m_{1}^{2} m_{2} p x^{3}+3 p x_{0} E^{2} c m_{1}^{2}\right)}{x_{0}\left(p x_{0}-c\right)^{3}\left(m_{1} E_{0}+m_{2} x_{0}\right)^{4}},
\end{aligned}
$$

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