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Certain Quantum Operator Related to Generalized Mittag–Leffler Function

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Abstract: In this paper, we present a novel class of analytic functions in the form $h(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ in the unit disk. These functions establish a connection between the extended Mittag–Leffler function and the quantum operator presented in this paper, which is denoted by $\aleph_{q,p}^{n}(\mathcal{L}, a, b)$ and is also an extension of the Raina function that combines with the Jackson derivative. Through the application of differential subordination methods, essential properties like bounds of coefficients and the Fekete–Szegő problem for this class are derived. Additionally, some results of special cases to this study that were previously studied were also highlighted.

Keywords: Mittag–Leffler function; quantum calculus; Jackson differential operator; q-differentiation; q-integration; subordination relation; differential subordination; Fekete–Szegő function; operators in geometric function theory

MSC: 30C45; 30C80; 30C50; 05A30; 33E12; 33C05

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1. Introduction

We shall establish the definition of the class of analytic functions represented by $\mathbb{A}_p$ as follows

$$h(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \ (z \in \mathbb{D}, \ p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

where the set $\mathbb{D}$ encompasses all the values of z within the open unit disk $z \in \mathbb{C}$ satisfying $|z| < 1$.

Given two analytic functions, h_1 and h_2, within the domain $\mathbb{D}$, the relationship where h_1 is subordinated to h_2 is denoted as $h_1(z) \prec h_2(z)$. This implies the existence of a function ω, known as the Schwarz function, which is analytic in the open disk $\mathbb{D}$ and satisfies the criteria $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$; this function ω further satisfies the condition that $h_1(z) = h_2(\omega(z))$ for all $z \in D$. If the function $g \in S$ (S is the family of all functions that are univalent in the domain $\mathbb{D}$), then (cf., e.g., [1,2])

$$h_1(z) \prec h_2(z) \Leftrightarrow h_1(0) = h_2(0) \quad \text{and} \quad h_1(\mathbb{D}) \subset h_2(\mathbb{D}).$$

If we have two functions $h_1(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ and $h_2(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$ of $\mathbb{A}_p$, then the Hadamard product of these functions is defined by:

$$(h_1 \times h_2)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k, \ z \in \mathbb{D},$$

see for example [3].

Furthermore, let

$$\mathcal{P} := \left\{ \mathfrak{P} : \mathfrak{P}(z) = 1 + \mathfrak{b}_1 z + \mathfrak{b}_2 z^2 + \ldots,\ \mathrm{Re}\,\mathfrak{P}(z) > 0,\ z \in \mathbb{D} \right\}$$

denote all the Carathéodory functions (see [4,5]).

Quantum calculus holds substantial importance in many fields like hypergeometric series theory and quantum physics as well as other physical phenomena. The definition of both q differentiation and q integration was first defined by Jackson ([6,7]).

There are many authors who have studied the operators of quantum calculus through many diverse applications in geometric function theory; e.g., Attiya et al. [8] studied differential operators related to the q-Raina function, Ibrahim [9], Al-shbeil et al. [10] and Karthikeyan et al. [11] studied the q-convolution of a certain class of analytic functions related to the quantum differential operator in GFT, Ismail et al. [12] and Riaz et al. [13] studied starlike functions defined by q-fractional derivatives, Shaba et al. [14] studied coefficient inequalities of q-bi-univalent associated with q-hyperbolic tangent functions, Al-Shaikh et al. [15] studied a class of close-to-convex functions defined by a quantum difference operator, Sitthiwirattham et al. [16] studied Maclaurin's coefficients inequalities for convex functions in q-calculus, Al-Shaikhm [17] studied some classes of analytic functions associated with a Salagean quantum differential operator, and Tang et al. [18] studied the Hankel and Toeplitz determinant for certain subclasses of multivalent q-starlike functions.

We need the following definitions, lemmas and notations to obtain our results in the second and third sections in this paper.

Definition 1. *Raina's function ([19]; see also [8]) is defined by using gamma function* Γ *as follows:*

$$_{\mathcal{L}}\mathcal{H}_{a,b}(z) = \sum_{k=0}^{\infty} \frac{\mathcal{L}(k)}{\Gamma(ak+b)} z^k,\ z \in \mathbb{D},$$

where both a and b are complex values in the complex number field $\mathbb{C}$*, and provided that* $\mathrm{Re}(a) > 0$ *and* $\mathrm{Re}(b) > 0$*,* $\mathcal{L}(k)$ *is a member of the sequence* $\{\mathcal{L}(k)\}_{k\in\mathbb{N}_0}$ *which is a bounded sequence of arbitrary complex numbers.*

Raina's function is an extension of the Mittag–Leffler function:

The Mittag–Leffler function [20,21] is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$
$$(\alpha \in \mathbb{C};\ \mathrm{Re}(\alpha) > 0).$$

We denote the Pochhammer symbol by $(\delta)_n$, which is defined by:

$$(\delta)_n = \begin{cases} 1, & n = 0 \\ \delta(\delta+1)\ldots(\delta+n-1) & n \neq 0 \end{cases}.$$

Prabhakar [22] introduced the function $E^{\delta}_{\alpha,\beta}(z)$ $(z \in \mathbb{C})$ in the form

$$E^{\delta}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_k\, z^k}{\Gamma(\alpha k+\beta)\, k!}, \tag{2}$$
$$(\alpha, \beta, \delta \in \mathbb{C};\ \mathrm{Re}(\alpha) > 0;\ \mathrm{Re}(\beta) > 0;\ \mathrm{Re}(\delta) > 0),$$

noting that $E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$ $(z \in \mathbb{C})$ was introduced by Wiman ([23]).

For the Mittag–Leffler function and its generalizations, see for example [24–29].

Remark 1. *1. We obtain the Mittag–Leffler function [20,21], from Raina's function, if* $\mathcal{L}(k) = 1$ $(k \geq 0)$ *and* $b = 1$.

2. We obtain Wim's function $E_{\alpha,\beta}(z)$*, see [23], from Raina's function, if* $\mathcal{L}(k) = 1$ $(k \geq 0)$.

3. We obtain the function $E^{\delta}_{\alpha,\beta}(z)$ *given by (2) from Raina's function if* $\mathcal{L}(k) = \frac{(\delta)_k}{k!}$.

4. If $\mathcal{L}(k) = \frac{(a)_k (b)_k}{(c)_k}$*, in this case, Raina's function simplifies to the Gaussian hypergeometric function described below.*

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{\Gamma(k+1)}, \; z \in \mathbb{D}. $$

Definition 2 ([6])**.** *The Jackson derivative for the function* $h(z)$ *is provided as follows:*

$$ (\mathfrak{d}_q) h(z) := \frac{h(z) - h(qz)}{z(1-q)}, \; (0 < q < 1). $$

Then, we have

$$ \mathfrak{d}_q (z^k) = \frac{1-q^k}{1-q} z^{k-1}, \; k \in \mathbb{N} \cup \{0\}. $$

In the case where the function h takes the the form (1), then

$$ (\mathfrak{d}_q)\, h(z) = [p]_q z^p + \sum_{k=p+1}^{\infty} a_k \,[k]_q\, z^{k-1}, $$

where

$$ [k]_q := \frac{1-q^k}{1-q}. $$

Also, note that

$$ \mathfrak{d}_q \,\kappa = 0 \;\text{ and }\; \lim_{q \to 1^-} (\mathfrak{d}_q)\, h(z) = h'(z), $$

where κ represents a constant within the set of complex numbers.

In the case where $s \in \mathbb{C}$, the q-shifted factorial is established through the subsequent formula (see [6]):

$$ (s;q)_\tau := \prod_{j=0}^{\tau-1} \left(1 - q^j s\right), \; ((s;q)_0 = 1; \; \tau \in \mathbb{N} = \{1,2,\ldots\}). \tag{3} $$

If

$$ \Gamma_q(s+1) = \frac{\Gamma_q(s)(1-q)}{1-q}, \; 0 < q < 1 $$

and

$$ (s;q)_\infty = \prod_{j=0}^{\infty} \left(1 - q^j s\right), $$

by employing the expression (3), we are able to present the q-shifted gamma function as:

$$ \left(q^t;q\right)_\tau = \frac{(1-q)^\tau \Gamma(s+\tau)}{\Gamma_q(s)}, \Gamma_q(s) = \frac{(q;q)_\infty (1-q)^{1-s}}{(q;q)_\infty}. $$

For $\mathcal{L}(0) \neq 0$, the normalized function ${}_{\mathcal{L}}\aleph_{d,b}$ (see [8]) is defined by

$$ {}_{\mathcal{L}}\aleph_{d,b}(z) := z + \sum_{k=2}^{\infty} \frac{\mathcal{L}(k-1)\Gamma(b)}{\mathcal{L}(0)\Gamma(d(k-1)+b)} z^k, \; z \in \mathbb{D}. \tag{4} $$

If $\mathcal{L}(k) = (k+1)^{-s}$, $s \in \mathbb{R}$, $s > 0$, $a = 0$ and $b = 1$, the operator (4) can be described as the the Sălăgean integral operator with order s (see [30]).

Utilizing the q-gamma function, Attiya et al. [8] introduced the generalized normalized function ${}_{q,\mathcal{L}}\aleph_{d,b}(z)$, which is defined in the following manner:

$${}_{q,\mathcal{L}}\aleph_{d,b}(z) := z + \sum_{k=2}^{\infty} \Phi_k(d,b,\mathcal{L},q) z^k, \; z \in \mathbb{D},$$

where

$$\Phi_k(d,b,\mathcal{L},q) := \frac{\mathcal{L}(k-1)\Gamma_q(b)}{\mathcal{L}(0)\Gamma_q((k-1)d+b)}, \qquad (5)$$
$$(\operatorname{Re} d > 0; \operatorname{Re} b > 0; \mathcal{L}(0) \neq 0).$$

The q-Raina differential operator ${}_{\mathcal{L}}\aleph_q^n : \mathbb{A}_1 \to \mathbb{A}_1$ was introduced by Attiya et al. [8] as follows:

$$\begin{aligned}
{}_{\mathcal{L}}\aleph_q^0(d,b)h(z) &= h(z) *_{q,\mathcal{L}} \aleph_{d,b}(z), \\
{}_{\mathcal{L}}\aleph_q^1(d,b)h(z) &= z\mathfrak{d}_q\left({}_{\mathcal{L}}\aleph_q^0(d,b)h(z)\right), \\
{}_{\mathcal{L}}\aleph_q^2(d,b)h(z) &= {}_{\mathcal{L}}\aleph_q^1(d,b)\left({}_{\mathcal{L}}\aleph_q^1(d,b)h(z)\right), \\
&\dots \\
{}_{\mathcal{L}}\aleph_q^k(d,b)h(z) &= {}_{\mathcal{L}}\aleph_q^1(d,b)\left({}_{\mathcal{L}}\aleph_q^{k-1}(d,b)h(z)\right), \; \left(h \in \mathbb{A}_p; \; k \in \mathbb{N}; \; k \geq 2\right).
\end{aligned} \qquad (6)$$

Therefore, when h belongs to $\mathbb{A}_1$ in the form (1), we have:

$${}_{\mathcal{L}}\aleph_q^n(d,b)h(z) = z + \sum_{k=2}^{\infty} [k]_q^n \frac{\mathcal{L}(k-1)\Gamma_q(b)}{\mathcal{L}(0)\Gamma_q((k-1)d+b)} a_k z^k.$$

Now, analogously to ${}_{\mathcal{L}}\aleph_q^n$, we introduce a novel operator $\aleph_q^n(\mathcal{L},d,b)$ for functions in $\mathbb{A}_p$ in the form $h(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, $z \in \mathbb{D}$, as follows:

Definition 3. *Let the function $h(z) \in \mathbb{A}_p$ be in the form (1). The operator $\aleph_{q,p}^n(\mathcal{L},d,b)$ is defined as*

$$\aleph_{q,p}^n(\mathcal{L},d,b) : \mathbb{A}_p \to \mathbb{A}_p$$

$$\begin{aligned}
\aleph_{q,p}^n(\mathcal{L},d,b)h(z) &= z^p + \sum_{k=p+1}^{\infty} \left(\frac{[k]_q}{[p]_k}\right)^n \frac{\mathcal{L}(k-1)\Gamma_q(b)}{\mathcal{L}(0)\Gamma_q((k-1)d+b)} a_k z^k \\
&= z^p + \sum_{k=p+1}^{\infty} \left(\frac{[k]_q}{[p]_k}\right)^n \Phi_k(d,b,\mathcal{L},q) a_k z^k, \; z \in \mathbb{D},
\end{aligned} \qquad (7)$$

where $0 < q < 1$, $\operatorname{Re} d > 0, \operatorname{Re} b > 0, \mathcal{L}(0) \neq 0$ and $\Phi_k(d,b,\mathcal{L},q)$ is given by (5).

Remark 2. *(i) Setting $p = 1$ and $\mathcal{L}(k-1) = 1$ ($k \geq 1$), in (7), we have the q-differential operator of [31].*

(ii) Substituting $p = 1$, $\mathcal{L}(k-1) = 1$ ($k \geq 1$) and $d = 0$ in (7), we derive the Sălăgean q-differential operator defined in [32].

(iii) Putting $p = 1, q \to 1^-$ and $\mathcal{L}(k-1) = 1$ in (7), we have a class studied in [33] (see also [34]).

Remark 3. *Unless otherwise stated in this paper, we will use constraints on the parameters q, n, d, b, $\mathcal{L}(k)$ as follows:*

$0 < q < 1$, $n \in \mathbb{N}$, $\operatorname{Re} d > 0$, $\operatorname{Re} b > 0$ *and* $\mathcal{L}(k) \in \{\mathcal{L}(k)\}_{k \in \mathbb{N}_0}$ *which is a bounded sequence of arbitrary complex numbers with* $\mathcal{L}(0) \neq 0$.

Definition 4 ([35])**.** *Let us establish a definition for the convex analytic function* $\gamma_{j,\Im}$ *in domain* $\mathbb{D}$ *as follows*:

$$\gamma_{j,\Im}(z) := \begin{cases} \dfrac{1+z}{1-z}, & \text{if } j = 0, \\ \digamma_1(j,\Im), & \text{if } j = 1, \\ \digamma_2(j,\Im), & \text{if } 0 < j < 1, \\ \digamma_3(j,\Im), & \text{if } j > 1, \end{cases}$$

where $\Im \in \mathbb{C} \setminus \{0\}$, *and the subsequent functions are established by (see [35])*

$$\digamma_1(j,\Im)(z) = 1 + \frac{2\Im}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,$$
$$\digamma_2(j,\Im)(z) = 1 + \frac{2\Im}{1-j^2}\sinh^2\left(\frac{2}{\pi}\arccos(j)\operatorname{arctanh}(\sqrt{z})\right),$$
$$\digamma_3(j,\Im)(z) = 1 + \frac{\Im}{1-j^2} + \frac{\Im}{j^2-1}\sin\left(\frac{\pi}{2Y(t)}\int_0^{\ell(z)/\sqrt{t}} \frac{d\zeta}{\sqrt{1-\zeta^2}\sqrt{1-(\zeta t)^2}}\right).$$

Here, we select $\ell(z) = \dfrac{z-\sqrt{t}}{1-\sqrt{tz}}$ for $t \in (0,1)$ in such a way that $t = \cosh\left(\dfrac{\pi Y'(t)}{4Y(t)}\right)$, where $Y(t)$ represents Legendre's complete elliptic integral of the first kind, and $Y'(t)$ signifies the complementary integral of $Y(t)$, with $(Y'(t))^2 = 1 - (Y(t))^2$.

Now, we introduce the new class $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$ for functions belonging to $\mathbb{A}_p$.

Definition 5. *The function* $h \in \mathbb{A}_p$ *is in the class* $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$ *if we have the following subordination relation*

$$\frac{\left(\aleph^{n+1}_{q,p}(\mathcal{L},d,b)h(z)\right)}{[p]_q \aleph^{n}_{q,p}(\mathcal{L},d,b)h(z)} \prec \gamma_{j,\Im}(z), \tag{8}$$

where $\gamma_{j,\Im}$ *in the form (see also [8,35,36])*

$$\gamma_{j,\Im}(z) = 1 + \gamma_1 z + \gamma_2 z^2 + \ldots, \; z \in \mathbb{D} \tag{9}$$

is given by the Definition 4.

Definition 6. *The class* $\mathcal{S}^{n,j}_{\Im,p}(d,b)$ *is the class* $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$ *when* q *approaches* 1 *from the left.*

Lemma 1 ([37])**.** *Consider* $G(z) = \sum\limits_{k=0}^{\infty} g_k z^k$, *which represents a univalent convex function in the domain* $\mathbb{D}$, *and fulfills the relation:*

$$H(z) = \sum_{k=0}^{\infty} h_k z^k \prec G(z).$$

Then, $|h_k| \leq |g_1|$ *for all* $k \geq 1$.

Lemma 2 ([38])**.** *Assume* $P(z) = 1 + \sum\limits_{k=1}^{\infty} p_k z^k$ *is an analytic function in the domain* $\mathbb{D}$ *such that* $\operatorname{Re} P(z) > 0$ $(z \in \mathbb{D})$. *Then*

$$\left|p_2 - s\,p_1^2\right| \leq 2\max\{1; |2s-1|\}, \; s \in \mathbb{C}.$$

In our paper, we present the class $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$, which is related to the operator $\aleph^{n}_{q,p}(\mathcal{L},d,b)$. Important properties for the class $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$ are derived. Also, bounds of coefficients and the Fekete–Szegő problem for this class $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$ are obtained. Moreover, some results of special cases to this study that were previously studied were also highlighted.

2. Certain Properties for the Class $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$

The theorem presented below gives a new result of functions in the class $\mathcal{S}^{n,j}_{\Im}(d,b)$.

Theorem 1. *If h of the form (1) is in the class $\mathcal{S}^{n,j}_{\Im,p}(d,b)$, then*

$$\aleph^{n}_{q,p}(\mathcal{L},d,b)h(z) \prec z\exp\left(\int_0^z \frac{1}{\chi}\left(p\gamma_{j,\Im}(\omega(\chi))-1\right)d\chi\right),$$

where ω denotes a Schwarz function, $z\in\mathbb{D}$. Additionally, if $|z| := \varrho < 1$, we have

$$\exp\left(\int_0^1 \frac{1}{\varrho}\left(\gamma_{j,\Im}(-\varrho)-1\right)d\varrho\right) \le \left|\frac{1}{z}\left(\aleph^{n}_{q}(\mathcal{L},d,b)h(z)\right)\right| \le \exp\left(\int_0^1 \frac{1}{\varrho}\left(\gamma_{j,\Im}(\varrho)-1\right)d\varrho\right).$$

Proof. Since h belongs to $\mathcal{S}^{n,j}_{\Im,p}(d,b)$, then

$$\frac{\left(\aleph^{n}_{q,p}(\mathcal{L},d,b)h(z)\right)'}{p\aleph^{n}_{q,p}(\mathcal{L},d,b)h(z)} - \frac{1}{pz} = \frac{p\gamma_{j,\Im}(\omega(z))-1/p}{z},\ z\in\mathbb{D}. \tag{10}$$

By integrating both sides of the equation mentioned above, it can be deduced that

$$\aleph^{n}_{q,p}(\mathcal{L},d,b)h(z) \prec z\exp\left(\int_0^z \frac{1}{\chi}\left(p\gamma_{j,\Im}(\chi)-1\right)d\chi\right),$$

then, we have

$$\frac{\aleph^{n}_{q}(\mathcal{L},d,b)h(z)}{z} \prec \exp\left(\int_0^z \frac{1}{\chi}\left(p\gamma_{j,\Im}(\chi)-1\right)d\chi\right).$$

Since

$$\gamma_{j,\Im}(-\varrho|z|) \le \operatorname{Re}\left(\gamma_{j,\Im}(\omega(z\varrho))\right) \le \gamma_{j,\Im}(\varrho|z|),$$

therefore

$$\int_0^1 \frac{1}{\varrho}\left(p\gamma_{j,\Im}(-\varrho|z|)-1\right)d\varrho \le \int_0^1 \frac{1}{\varrho}\left(\operatorname{Re}\left(p\gamma_{j,\Im}(\omega(z\varrho))\right)-1\right)d\varrho \le \int_0^1 \frac{1}{\varrho}\left(p\gamma_{j,\Im}(\varrho|z|)-1\right)d\varrho.$$

then, we obtain

$$\int_0^1 \frac{1}{\varrho}\left(p\gamma_{j,\Im}(-\varrho|z|)-1\right)d\varrho \le \log\left|\frac{\aleph^{n}_{q,p}(\mathcal{L},d,b)h(z)}{z}\right| \le \int_0^1 \frac{1}{\varrho}\left(p\gamma_{j,\Im}(\varrho|z|)-1\right)d\varrho,$$

then

$$\exp\left(\int_0^1 \frac{1}{\varrho}\left(p\gamma_{j,\Im}(-\varrho)-1\right)d\varrho\right) \le \left|\frac{\aleph^{n}_{q,p}(\mathcal{L},d,b)h(z)}{z}\right| \le \exp\left(\int_0^1 \frac{1}{\varrho}\left(p\gamma_{j,\Im}(\varrho)-1\right)d\varrho\right).$$

□

Remark 4. *Theorem 1 extends the findings of various authors, including the following:*

1. *Setting $p=1$, in Theorem 1, we can attribute this result to Attiya et. al. [8] (Theorem 6).*

2. *Letting $p = 1$ and $\mathcal{L}(k) = 1$ $(k \geq 1)$, in Theorem 1, we can attribute this result to Noor and Razzaque [31] (Theorem 6).*
3. *Putting $p = 1, d = 0, \mathcal{L}(k) = 1$ $(k \geq 1)$ and $b = 1$, then in Theorem 1 we can attribute this result to Hussain et. al. [39] (Theorem 3.1).*

The following theorem and corollaries are related to the coefficient estimation for the class $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$.

Theorem 2. *If h of the form* (1) *is in the class $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$, then*

$$|a_{p+1}| \leq \frac{[p]_q|\gamma_1|}{q\left|\frac{q^p}{[p]_q}+1\right|^n \Phi_{p+1}(d,b,\mathcal{L},q)}, \text{ and}$$

$$|a_{p+k}| \leq \frac{[p]_q|\gamma_1|}{q^p[k]_q\left|1+\frac{q^p}{[p]_q}\frac{[k]_q}{}\right|^n \Phi_{p+k}(d,b,\mathcal{L},q)} \prod_{j=1}^{k-1}\left(1+\frac{[p]_q|\gamma_1|}{q^p[j]_q\left|1+\frac{q^p}{[p]_q}\frac{[j]_q}{}\right|^n}\right), \ (k \geq 2)$$

where γ_1 is defined by (9).

Proof. If we take:

$$P(z) = \frac{\left(\aleph^{n+1}_{q,p}(\mathcal{L},d,b)h(z)\right)}{\aleph^n_q(\mathcal{L},d,b)h(z)},$$

then,

$$\frac{z\mathfrak{d}_q\left(\aleph^n_q(\mathcal{L},d,b)h(z)\right)}{[p]_q\aleph^n_q(\mathcal{L},d,b)h(z)} =: P(z), \ z \in \mathbb{D},$$

putting $P(z) = 1 + \sum\limits_{k=1}^{\infty} p_k z^k$, in the above equation, we will obtain

$$z\mathfrak{d}_q\left(\aleph^n_q(\mathcal{L},d,b)h(z)\right) = [p]_q\left(\aleph^n_q(\mathcal{L},d,b)h(z)\right) P(z), \ z \in \mathbb{D}.$$

Then, we have

$$\begin{aligned} & z^p + \sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q}\left(\frac{[k]_q}{[p]_q}\right)^n \Phi_k(d,b,\mathcal{L},q)a_k z^k \\ &= \left(z^p + \sum_{k=p+1}^{\infty}\left(\frac{[k]_q}{[p]_q}\right)^n \Phi_k(d,b,\mathcal{L},q)a_k z^k\right)\left(1+\sum_{k=1}^{\infty} p_k z^k\right) \\ &= \sum_{k=0}^{\infty} p_k z^{k+p} + \sum_{k=0}^{\infty} p_k z^k \cdot \sum_{k=p+1}^{\infty}\left(\frac{[k]_q}{[p]_q}\right)^n \Phi_k(d,b,\mathcal{L},q)a_k z^k \quad (p_0 = 1) \\ &= z^p + \sum_{k=1}^{\infty}\left(p_k + \sum_{j=1}^{k}\left(\frac{[p]_q[j+p]_q}{[p]_q}\right)^n \Phi_{j+p}(d,b,\mathcal{L},q)\ a_{j+p}\ p_{k-j}\right) z^{p+k}. \end{aligned}$$

By equating the coefficients of z^k in the preceding equation, we obtain

$$\begin{aligned} \left(\frac{[p+k]_q}{[p]_q}\right)\left(\frac{[p+k]_q}{[p]_q}\right)^n \Phi_{p+k}(d,b,\mathcal{L},q)a_{p+k} &= p_k + \left(\frac{[p+k]_q}{[p]_q}\right)^n \Phi_k(d,b,\mathcal{L},q)a_{p+k} \\ &+ \sum_{j=1}^{k-1}\left(\frac{[p+j]_q}{[p]_q}\right)^n \Phi_{p+j}(d,b,\mathcal{L},q)\ a_{p+j}\ p_{k-j}, \end{aligned}$$

which gives

$$\left(\frac{[p+k]_q}{[p]_q}-1\right)\left(\frac{[p+k]_q}{[p]_q}\right)^n \Phi_{p+k}(d,b,\mathcal{L},q)a_{p+k} = p_k + \sum_{j=1}^{k-1}\left(\frac{[p+j]_q}{[p]_q}\right)^n \Phi_{p+j}(d,b,\mathcal{L},q)\; a_{p+j}\; p_{k-j}.$$

Consequently, we obtain

$$a_{p+k} = \frac{1}{\left(\frac{[p+k]_q[p]_q}{[p]_q}\right)^n\left(\frac{[p+k]_q}{[p]_q}-1\right)\Phi_{p+k}(d,b,\mathcal{L},q)}\left(\sum_{j=1}^{k}\left(\frac{[p+j-1]_q}{[p]_q}\right)^n \Phi_{p+j-1}(d,b,\mathcal{L},q)\; a_{p+j-1}\; p_{k-j-1}\right),$$

for some calculation implies that

$$a_{p+k} = \frac{1}{\left(\frac{[p+k]_q}{[p]_q}\right)^n\left(\frac{[p+k]_q}{[p]_q}-1\right)\Phi_{p+k}(d,b,\mathcal{L},q)}\sum_{j=1}^{k}\left(\frac{[p+j-1]_q}{[p]_q}\right)^n \frac{\mathcal{L}(j-1)\Gamma_q(b)}{\mathcal{L}(0)\Gamma_q(d(j-1)+b)} a_j\; p_{k-j}.$$

By Lemma 1, since $|p_k| \leq |\gamma_1|$, we obtain

$$|a_{p+k}| \leq \frac{|\gamma_1|}{\left|\frac{[p+k]_q[p]_q}{[p]_q}\right|^n\left(\frac{[p+k]_q}{[p]_q}-1\right)\Phi_{p+k}(d,b,\mathcal{L},q)}\sum_{j=1}^{k}\left|\frac{[p+j-1]_q}{[p]_q}\right|^n \frac{\mathcal{L}(j-1)\Gamma_q(b)}{\mathcal{L}(0)\Gamma_q(d(j-1)+b)}|a_{p+j-1}|.$$

For $k=1$, we obtain

$$\begin{aligned}|a_{p+1}| &\leq \frac{|\gamma_1|}{\frac{q^p}{[p]_q}\left|1+\frac{q^p}{[p]_q}\right|^n\Phi_{p+1}(d,b,\mathcal{L},q)}\sum_{j=1}^{1}\left|\frac{[p+j-1]_q}{[p]_q}\right|^n \frac{\mathcal{L}(j-1)\Gamma_q(b)}{\mathcal{L}(0)\Gamma_q(d(j-1)+b)}|a_j| \\ &= \frac{|\gamma_1|[p]_q}{q^p\left|1+\frac{q^p}{[p]_q}\right|^n\Phi_{p+1}(d,b,\mathcal{L},q)}.\end{aligned}$$

In the case where $k=2$, and employing the aforementioned inequality, then:

$$|a_{p+2}| \leq \frac{|\gamma_1|[p]_q}{q^p[2]_q\left|1+\frac{q^p[2]_q}{[p]_q}\right|^n\Phi_{p+2}(d,b,\mathcal{L},q)}\left(1+\frac{|\gamma_1|[p]_q}{q^p[2]_q\left|1+\frac{q^p[2]_q}{[p]_q}\right|^n}\right).$$

Assume that for a given value of $k \geq 3$, the following inequality holds true through mathematical induction:

$$|a_{p+k}| \leq \frac{|\gamma_1|[p]_q}{q^p[k]_q\left|1+\frac{q^p[2]_q}{[p]_q}\right|^n\Phi_{p+k}(d,b,\mathcal{L},q)}\prod_{j=1}^{k-1}\left(1+\frac{|\gamma_1|[p]_q}{q^p[j]_q\left|1+\frac{q^p[j]_q}{[p]_q}\right|^n}\right),\ (k\geq 2).$$

which completes the proof. □

Derived from Theorem 2 as special cases, we yield the following corollaries.

Corollary 1 ([40]). *If $h \in \mathcal{S}^{n,j}_{q,\Im,1}(d,b)$ of the form (1) with $p=1$, then*

$$\begin{aligned}|a_2| &\leq \frac{|\gamma_1|}{q|q+1|^n\Phi_2(d,b,\mathcal{L},q)}, \\ |a_k| &\leq \frac{|\gamma_1|}{q[k-1]_q|1+q\,[k-1]_q|^n\Phi_k(d,b,\mathcal{L},q)}\prod_{j=1}^{k-2}\left(1+\frac{|\gamma_1|}{q[j]_q|1+q\,[j]_q|^n}\right),\ (k\geq 3)\end{aligned}$$

where γ_1 is defined by (9).

Corollary 2 ([8] Theorem 2)**.** *If $h \in \mathcal{S}^{n,j}_{q,\Im,1}(1,d,b)$, of the form (1) with $p=1$, then*

$$|a_2| \le \frac{|\gamma_1|}{[2]_q^n\left([2]_q-1\right)\Phi_2(d,b,\mathcal{L},q)}, \text{ and}$$
$$|a_k| \le \frac{|\gamma_1|}{[k]_q^n\left([k]_q-1\right)\Phi_k(d,b,\mathcal{L},q)}\prod_{j=1}^{k-2}\left(1+\frac{|\rho_1|}{[j+1]_q-1}\right), \ (k\ge 3)$$

with γ_1 given by (9).

Corollary 3 ([31] Theorem 8)**.**
If $h \in \mathcal{S}^{n,j}_{q,\Im,1}(1,d,b)$ of the form (1) with $p=1$ and $\mathcal{L}(k)=1$ for all $k\ge 1$, then

$$|a_2| \le \frac{|\gamma_1|}{[2]_q^n\Phi_2(d,b,1,q)\left([2]_q-1\right)}, \text{ and}$$
$$|a_k| \le \frac{|\gamma_1|}{[k]_q^n\Phi_k(d,b,1,q)\left([k]_q-1\right)}\prod_{j=1}^{k-2}\left(1+\frac{|\gamma_1|}{[j+1]_q-1}\right), \ (k\ge 3),$$

with γ_1 given by (9).

Corollary 4 ([39] Theorem 3.2)**.** *If $h \in \mathcal{S}^{n,j}_{q,\Im,1}(1,0,1)$ of the form (1) with $p=1$ and $\mathcal{L}(k)=1$ for all $k\ge 1$, then*

$$|a_2| \le \frac{|\gamma_1|}{[2]_q^n\Phi_2(0,1,1,q)\left([2]_q-1\right)}, \text{and}$$
$$|a_k| \le \frac{|\gamma_1|}{[k]_q^n\Phi_k(0,1,1,q)\left([k]_q-1\right)}\prod_{j=1}^{k-2}\left(1+\frac{|\gamma_1|}{[j+1]_q-1}\right), \ (k\ge 3),$$

where γ_1 was given by (9).

3. Fekete–Szegő Problem Related to the Class $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$

In the upcoming theorem, we will provide an estimate for the Fekete–Szegő problem for the class $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$.

Theorem 3. *If $h \in \mathcal{S}^{n,j}_{q,\Im,p}(d,b)$ of the form (1), then*

$$|a_{p+2}-\psi a_{p+1}^2| \le \frac{|\gamma_1|}{2T\,\Phi_{p+2}(d,b,\mathcal{L},q)}\max\{1;|2\Psi-1|\},$$

where $\psi\in\mathbb{C}$, and

$$\Psi := \Psi(d,b,\mathcal{L},q) =$$
$$\frac{2T\Phi_{p+2}(d,b,\mathcal{L},q)}{\gamma_1}\left(\frac{U}{T\Phi_{p+2}(d,b,\mathcal{L},q)}-\frac{\psi\gamma_1^2}{\left(\frac{[p+1]_q}{[p]_q}-1\right)^2\left(\frac{[p+1]_q}{[p]_q}\right)^{2n}\Phi_2^2(d,b,\mathcal{L},q)}\right), \quad (11)$$

with

$$T=\left(1-\frac{[p+2]_q}{[p]_q}\right)\left(\frac{[p+2]_q}{[p]_q}\right)^n$$

and

$$U = \frac{\gamma_1^2}{\left(1 - \frac{[p+1]_q}{[p]_q}\right)} + \frac{1}{4}\left(\gamma_1 - \gamma_2^2\right),$$

where γ_1 *and* γ_2 *are defined by* (9).

Proof. Since $h \in \mathcal{S}_{q,\Im,p}^{n,j}(d,b)$, we have

$$\frac{\left(\aleph_{q,p}^{n+1}(\mathcal{L},d,b)h(z)\right)}{\aleph_{q,p}^{n}(\mathcal{L},d,b)h(z)} = \gamma_{j,\Im}(\omega(z)),$$

where ω is a Schwarz function.

If $v \in \mathcal{P}$ is defined by:

$$v(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + v_1 z + v_2 z^2 + \ldots,\ z \in \mathbb{D},$$

then

$$\omega(z) = \frac{v_1}{2}z + \frac{1}{2}\left(v_2 - \frac{v_1^2}{2}\right)z^2 + \ldots,\ z \in \mathbb{D}$$

and

$$\gamma_{j,\Im}(\omega(z)) = 1 + \frac{\gamma_1 v_1}{2}z + \left(\frac{\gamma_2 v_1^2}{4} + \frac{1}{2}\left(v_2 - \frac{v_1^2}{2}\right)\gamma_1\right)z^2 + \ldots, z \in \mathbb{D}.$$

Therefore,

$$\frac{\left(\aleph_{q,p}^{n+1}(\mathcal{L},d,b)h(z)\right)}{\aleph_{q,p}^{n}(\mathcal{L},d,b)h(z)} = 1 + \left(\frac{[p+1]_q}{[p]_q}\right)^n\left(\frac{[p+1]_q}{[p]_q} - 1\right)\Phi_{p+1}(d,b,\mathcal{L},q)a_{p+1}z$$

$$+\left(\left(1 - \frac{[p+1]_q}{[p]_q}\right)\left(\frac{[p+1]_q}{[p]_q}\right)^{2n}\Phi_{p+1}^2 a_{p+1}^2 - \left(1 - \frac{[p+2]_q}{[p]_q}\right)\left(\frac{[p+2]_q[p]_q}{[p]_q}\right)^n\Phi_{p+2}a_{p+2}\right)z^2 + \ldots,\ z \in \mathbb{D},$$

hence, the subsequent coefficients can be established in the following manner:

$$\begin{aligned}
a_{p+1} &= \frac{\gamma_1 v_1}{\left(\frac{[p+1]_q}{[p]_q} - 1\right)\left(\frac{[p+1]_q}{[p]_q}\right)^n \Phi_{p+1}(d,b,\mathcal{L},q)}, \\
a_{p+2} &= \frac{1}{T\Phi_{p+2}(d,b,\mathcal{L},q)}\left(-\frac{\gamma_1 v_2}{2} + v_1^2\left(-\frac{\gamma_2^2}{4} + \frac{\gamma_1}{4} - \frac{\gamma_1^2}{\left(\frac{[p+1]_q}{[p]_q} - 1\right)}\right)\right) \\
a_{p+2} - \psi a_{p+1}^2 &= \frac{1}{T\Phi_{p+2}(d,b,\mathcal{L},q)}\left(-\frac{\gamma_1 v_2}{2} + U\ p_1^2\right) \\
&\quad - \psi\left(\frac{\gamma_1 v_1}{\left(\frac{[p+1]_q}{[p]_q} - 1\right)\left(\frac{[p+1]_q}{[p]_q}\right)^n \Phi_{p+1}(d,b,\mathcal{L},q)}\right)^2.
\end{aligned}$$

By using some computation, we have

$$a_{p+2} - \psi a_{p+1}^2 = \frac{-\gamma_1}{2T\ \Phi_{p+2}(d,b,\mathcal{L},q)}(v_2 - \Psi v_1^2),$$

where Ψ is defined by (11) and $\psi \in \mathbb{C}$. So, by using Lemma 2, we achieve the desired result. $\square$

Theorem 3 generalizes some of the previous findings, including the following:

Corollary 5 ([8] Theorem 3)**.** *If $h \in \mathcal{S}^{n,j}_{q,\Im,1}(1,d,b)$ of the form (1) with $p=1$, then*

$$|a_3 - \psi a_2^2| \leq \frac{|\rho_1|}{2[3]_q^n \Phi_3(d,b,\mathcal{M},q)\big([3]_q - 1\big)} \max\{1; |2\Psi - 1|\},$$

where $\psi \in \mathbb{C}$, and

$$\Psi := \Psi(d,b,\mathcal{L},q) = \frac{1}{2}\left(1 - \frac{\rho_2}{\rho_1} - \rho_1\left(\frac{1}{[2]_q - 1} - \psi \frac{\big([3]_q - 1\big)[3]_q^n}{2\Big([2]_q^n\big([2]_q - 1\big)\Big)^2 \Phi_2(d,b,\mathcal{L},q)}\right)\right), \tag{12}$$

with ρ_1 and ρ_2 defined by (9).

Corollary 6 ([31] Theorem 10)**.** *If $h \in \mathcal{S}^{n,j}_{q,\Im,1}(1,d,b)$ of the form (1) with $p=1$ and $\mathcal{L}(k)=1$ for all $k \geq 1$, then*

$$|a_3 - \psi a_2^2| \leq \frac{|\gamma_1|}{2[3]_q^n \Phi_3(d,b,1,q)\big([3]_q - 1\big)} \max\{1; |2\widehat{\Psi} - 1|\},$$

where $\psi \in \mathbb{C}$, and

$$\widehat{\Psi} := \Psi(d,b,1,q) = \frac{1}{2}\left(1 - \frac{\gamma_2}{\gamma_1} - \gamma_1\left(\frac{1}{[2]_q - 1} - \psi \frac{\big([3]_q - 1\big)[3]_q^n}{2\Big([2]_q^n\big([2]_q - 1\big)\Big)^2 \Phi_2(d,b,1,q)}\right)\right),$$

with γ_1 and γ_2 given by (9).

Corollary 7 ([39] Theorem 3.3)**.** *If $h \in \mathcal{S}^{n,j}_{q,\Im,1}(1,0,1)$ of the form (1) with $p=1$ and $\mathcal{L}(k)=1$ for all $k \geq 1$, then*

$$|a_3 - \psi a_2^2| \leq \frac{|\gamma_1|}{2[3]_q^n \Phi_3(0,1,1,q)\big([3]_q - 1\big)} \max\{1; |2\widetilde{\Psi} - 1|\},$$

where $\psi \in \mathbb{C}$, and

$$\widetilde{\Psi} := \Psi(0,1,1,q) = \frac{1}{2}\left(1 - \frac{\gamma_2}{\gamma_1} - \gamma_1\left(\frac{1}{[2]_q - 1} - \psi \frac{\big([3]_q - 1\big)[3]_q^n}{2\Big([2]_q^n\big([2]_q - 1\big)\Big)^2 \Phi_2(0,1,1,q)}\right)\right),$$

with γ_1 and γ_2 defined by (9).

The following result is related to the sufficient condition of functions in the class $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$.

Theorem 4. *Assume $h \in \mathbb{A}_p$ in the form (1). If*

$$\sum_{k=p+1}^{\infty} \left((j+1)\left(\frac{[k]_q}{[p]_q} - 1\right) + |\Im|\right) |\Phi_k(d,b,\mathcal{L},q)| \left(\frac{[k]_q}{[p]_q}\right)^n |a_k| \leq |\Im|,$$

then $h \in \mathcal{S}^{n,j}_{q,\Im,p}(d,b)$.

Proof. Since

$$\left|\frac{z\eth_q\left({}_m\aleph^n_{q,p}(\mathcal{L},d,b)h(z)\right)}{[p]_m\aleph^n_{q,p}(\mathcal{L},d,b)h(z)}-1\right| = \left|\frac{z\eth_q\left({}_m\aleph^n_{q,p}(d,b)h(z)\right)-\aleph^n_{q,p}(\mathcal{L},d,b)h(z)}{[p]_m\aleph^n_{q,p}(\mathcal{L},d,b)h(z)}\right|$$

$$= \left|\frac{\sum_{k=p+1}^{\infty}\left(\frac{[k]_q}{[p]_q}-1\right)\left(\frac{[k]_q}{[p]_q}\right)^n\Phi_k(d,b,\mathcal{L},q)a_kz^k}{z^p+\sum_{k=p+1}^{\infty}\left(\frac{[k]_q}{[p]_q}\right)^n\Phi_k(d,b,\mathcal{L},q)a_kz^k}\right| \leq \frac{\sum_{k=p+1}^{\infty}\left|\left(\frac{[k]_q}{[p]_q}-1\right)\left(\frac{[k]_q}{[p]_q}\right)^n\Phi_k(d,b,\mathcal{L},q)\right||a_k|}{1-\sum_{k=2}^{\infty}\left|\left(\frac{[k]_q}{[p]_q}\right)^n\Phi_k(d,b,\mathcal{L},q)\right||a_k|},\ z\in\mathbb{D},$$

based on the theorem's assumption,

$$1-\sum_{k=p+1}^{\infty}\left|\left(\frac{[k]_q}{[p]_q}\right)^n\Phi_k(d,b,\mathcal{L},q)\right||a_k| > 0.$$

Since

$$\left|\frac{j}{\Im}\left(\frac{z\eth_{q,p}\left({}_m\aleph^n_{q,p}(\mathcal{L},d,b)h(z)\right)}{[p]\aleph^n_{q,p}(\mathcal{L},d,b)h(z)}-1\right)\right| - \mathrm{Re}\left(\frac{1}{\Im}\left(\frac{z\eth_{q,p}\left({}_m\aleph^n_{q,p}(\mathcal{L},d,b))h(z)\right)}{[p]_m\aleph^n_{q,p}(\mathcal{L},d,b)h(z)}-1\right)\right)$$

$$\leq \frac{j}{|\Im|}\left|\left(\frac{z\eth_{q,p}\left({}_m\aleph^n_{q,p}(\mathcal{L},d,b))h(z)\right)}{[p]_m\aleph^n_{q,p}(\mathcal{L},d,b)h(z)}-1\right)\right| + \left|\frac{1}{\Im}\right|\left|\frac{z\eth_{q,p}\left({}_m\aleph^n_{q,p}(\mathcal{L},d,b)h(z)\right)}{[p]\aleph^n_{q,p}(\mathcal{L},d,b)h(z)}-1\right|$$

$$= \frac{j+1}{|\Im|}\left|\left(\frac{z\eth_{q,p}\left({}_m\aleph^n_{q,p}(\mathcal{L},d,b))h(z)\right)}{[p]_m\aleph^n_{q,p}(\mathcal{L},d,b)h(z)}-1\right)\right| = \frac{j+1}{|\Im|}\left|\frac{z\eth_{q,p}\left({}_m\aleph^n_{q,p}(\mathcal{L},d,b))h(z)\right)-{}_m\aleph^n_{q,p}(\mathcal{L},d,b)h(z)}{[p]_m\aleph^n_{q,p}(\mathcal{L},d,b)h(z)}\right|$$

$$\leq \frac{j+1}{|\Im|}\left(\frac{\sum_{k=p+1}^{\infty}\left|\left(\frac{[k]_q}{[p]_q}-1\right)\left(\frac{[k]_q}{[p]_q}\right)^n\Phi_k(d,b,\mathcal{L},q)\right||a_k|}{1-\sum_{k=p+1}^{\infty}\left|\left(\frac{[k]_q}{[p]_q}\right)^n\Phi_k(d,b,\mathcal{L},q)\right||a_k|}\right) \leq 1,\ z\in\mathbb{D},$$

then, we obtain $h \in \mathcal{S}^{n,j}_{q,\Im,p}(d,b)$. □

It can be seen that Theorem 4 is a generalization of other previous results, for example:

Corollary 8 ([8] Theorem 4)**.** *Let* $h \in \mathbb{A}_1$ *be in the form* (1) *with* $p = 1$*. If*

$$\sum_{k=2}^{\infty}\left((j+1)\left([k]_q-1\right)+|\Im|\right)|\Phi_k(d,b,\mathcal{M},q)||[n]^k_q|a_k| \leq |\Im|,$$

then $h \in \mathcal{S}^{n,j}_{q,\Im,1}(1,d,b)$.

Corollary 9 ([31] Theorem 12)**.** *Let* $h \in \mathbb{A}_1$ *be of the form* (1) *with* $p = 1$*. If*

$$\sum_{k=2}^{\infty}\left((j+1)\left([k]_q-1\right)+|\Im|\right)|\Phi_k(d,b,1,q)||[k]^n_q|a_k| \leq |\Im|,$$

then

$$\frac{z\mathfrak{d}_q\left(\aleph_q^n(d,b)h(z)\right)}{\aleph_q^n(d,b)h(z)} \prec \gamma_{j,\Im}(z),$$

that is $h \in \mathcal{S}^{n,j}_{q,\Im,1}(d,b)$ *when* $\mathcal{L}(k) = 1$ *for all* $k \geq 1$.

Corollary 10 ([39] Theorem 3.4)**.** *Let* $h \in \mathbb{A}_1$ *be of the form* (1) *with* $p = 1$. *If*

$$\sum_{k=2}^{\infty}\left((j+1)\left([k]_q - 1\right) + |\Im|\right)|\Phi_k(0,1,1,q)||[k]_q^n|a_k| \leq |\Im|,$$

then

$$\frac{z\mathfrak{d}_q\left(\aleph_q^n(0,1)h(z)\right)}{\aleph_q^n(0,1)h(z)} \prec \gamma_{j,\Im}(z),$$

that is $h \in \mathcal{S}^{n,j}_{q,\Im,1}(0,1)$, *when* $\mathcal{L}(k) = 1$ *for all* $k \geq 1$.

4. Conclusions

Through the utilization of quantum calculus and the generalized Mittag–Leffler function, the operator $\aleph^n_{q,p}(\mathcal{L},d,b)$ introduced in Definition 3 is necessary to study the new class $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$ of analytic functions introduced and investigated in this paper, which is given in Definition 5. By employing the techniques of differential subordination in the geometric function theorem, we derived new and interesting results. In the second section, coefficients inequalities for the class $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$ and the subordination relation for a special case of this class are obtained. Also, in the third section, the Fekete–Szegő problem and the sufficient conditions for functions in the class $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$ are derived. Additionally, some results of special cases of the class $\mathcal{S}^{n,j}_{q,\Im,p}(d,b)$ that were previously studied were also highlighted.

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