mathematics

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Generalizations of Higher-Order Duality for Multiple Objective Nonlinear Programming under the Generalizations of Type-I Functions

Mohamed Abd El-Hady Kassem [1,*] and Huda M. Alshanbari [2]

[1] Department of Mathematics, Faculty of Science, Tanta University, Tanta 31111, Egypt
[2] Department of Mathematical Science, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
* Correspondence: mohamed.kassem@science.tanta.edu.eg

Abstract: In this study, we introduce new generalizations of higher-order type-I functions and higher-order pseudo-convexity type-I functions. The application of the notion of sublinear functionals to these generalizations of higher-order type-I and higher-order pseudo-convexity type-I functions is crucial to our main findings. Furthermore, under these generalizations of the higher-order type-I and higher-order pseudo-convexity type-I functions, we established and studied six new types of higher-order duality models and programs for multiple objective nonlinear programming problems. In addition, we use these generalizations of higher-order type-I functions and higher-order pseudo-convexity type-I functions, to formulate and prove the theorems of weak duality, strong duality, and strict converse duality for these new six types of higher-order model programs.

Keywords: multiple objective programming; nonlinear programming; higher-order duality; (F,ρ,σ)-type-I functions; pseudo-convexity functions

MSC: 90C30; 49M37; 65K05; 90C46

Citation: Kassem, M.A.E.-H.; Alshanbari, H.M. Generalizations of Higher-Order Duality for Multiple Objective Nonlinear Programming under the Generalizations of Type-I Functions. *Mathematics* **2023**, *11*, 889. https://doi.org/10.3390/math11040889

Academic Editors: Adrian Deaconu, Petru Adrian Cotfas and Daniel Tudor Cotfas

Received: 15 January 2023
Revised: 26 January 2023
Accepted: 27 January 2023
Published: 9 February 2023

1. Introduction

In [1], the author investigated higher-order duality for multi-objective programming problems. In [2–5], the authors studied second-order dual nonlinear programming problems. Reference [6] introduced the concept of invexity duality in programming problems. Reference [7] introduced invexity and nonconvex optimization and their applications to these programming problems. Reference [8] discussed v-invexity functions in vector optimization problems. These programming problems under ρ-convexity are presented [9,10]. [11], which was expanded to include (F,ρ)-convexity functions defined by [12,13]. The dual Mond-Weir type of these programming problems involving (F,ρ,σ)-type I functions was introduced by [14,15]. In [16–18], the authors discussed the higher-order duality of these programming problems. The second-order (F,ρ,σ)-type-I functions for nondifferentiable fractional programming problems were introduced by [19–23]. The higher-order vector optimization problems involving cone-invexity functions are given in [20]. In [24,25], they proposed a higher order for fractional programming problems.

In this work, we present new generalizations of higher-order type-I functions and higher-order pseudo-convexity type-I functions for multiple objective nonlinear programming (MONLP) problems. In addition, we establish and study of six new types of higher-order duality models and programs for multiple objective nonlinear programming problems. Furthermore, we formulate and prove the results of duality theorems under these generalizations of the higher-order type-I functions for these MONLP problems. Finally, we discuss the first four types of these higher-order duality models and programs with this condition $\alpha_i^1(x,u) = \alpha_j^2(x,u) = \alpha(x,u)$ and the other two types of higher-order duality models and programs without this condition.

2. Preliminaries and Definitions

Consider the MONLP problems that take the following form:

$$\begin{array}{l}\text{MONLP}: \ \min f(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \\ \text{subject to } x \in X = \{x \in R^n \,|\, g_j(x) \le 0,\ j = 1, 2, \ldots, k\}\end{array}$$

where the functions $f : X \to R^m$ *and* $g : X \to R^k$ have continuous differentiability.

Proposition 1 ([26]). *If the point $\bar{x}$ is weakly efficient for the MONLP problem, which satisfies the constraint qualification. Then $\exists \lambda \in R^m$, $y \in R^k$ satisfaction.*

$$\sum_{i=1}^{m} \nabla(\lambda_i f)(\bar{x}) + \sum_{j=1}^{k} \nabla(y_j g_j)(\bar{x}) = 0,$$

$$\sum_{j=1}^{k} y_j g_j(\bar{x}) = 0,$$

$$\lambda \ge 0,\ \sum_{i=1}^{m} \lambda_i = 1,\ y \ge 0.$$

Definition 1 ([10]). *A sublinear is a type of functional $F : X \times X \times R^n \to R$ that satisfies the following conditions:*

(i) $F(x, u; a_1 + a_2) \le F(x, u; a_1) + F(x, u; a_2)\ \forall a_1, a_2 \in R^n$,
(ii) $F(x, u; \alpha a) = \alpha F(x, u; a)\ \forall \alpha \in R, \alpha > 0, a \in R^n$

Let us define the functions $f_i : X \to R^m, g_j : X \to R^k, K_i : X \times R^m \to R$, $G_j : X \times R^k \to R$ that are differentiable and also define the following real-valued functions:

$$d : X \times X \to R,\ \alpha^1, \alpha^2 : X \times X \to R_+ \backslash \{0\},\ \rho,\ \sigma \in R.$$

The higher-order $(F, \alpha, \rho, \sigma, d)$—type-I, higher-order $(F, \alpha, \rho, \sigma, d)$—pseudo-convexity type-I, and higher-order strict $(F, \alpha, \rho, \sigma, d)$—pseudo-convexity type-I functions are defined in the new definitions that follow.

Definition 2. *The MONLP problem functions $f_i(x)$ and $g_j(x)$ are higher-order $(F, \alpha, \rho, \sigma, d)$—type-I at the point $u \in X$ with respect to (w. r. t.) the functions K_i, G_j $\forall i, j$ and $\rho_i, \sigma_j \in R$, $\alpha_i^1, \alpha_j^2 : X \times X \to R_+ \backslash \{0\}$ if $\forall x \in X$ we have*

$$f_i(x) - f_i(u) \ge F(x, u; \alpha_i^1(x, u)(\nabla f_i(u) + \nabla_p K_i(u, p))) + K_i(u, p) - p^T \nabla_p K_i(u, p) + \rho_i d^2(x, u),$$

and

$$-g_j(u) \ge F(x, u; \alpha_j^2(x, u)(\nabla g_j(u) + \nabla_q G_j(u, q))) + G_j(u, q) - q^T \nabla_q G_j(u, q) + \sigma_j d^2(x, u).$$

We note that, if $\alpha_i^1(x, u) = \alpha_j^2(x, u) = 1$, the higher-order $(F, \alpha, \rho, \sigma, d)$—type-I reduces to the higher-order (F, ρ, σ)—type-I defined in [15].

Definition 3. *The functions $f_i(x)$ and $g_j(x)$ of the MONLP problem are higher-order $(F, \alpha, \rho, \sigma, d)$—pseudo-convexity type-I at a given point $u \in X$ w. r. t. the functions K_i, G_j $\forall i, j$ and $\rho_i, \sigma_j \in R$, if $\forall\, x \in X$, we have*

$$F(x, u; \alpha_i^1(x, u)(\nabla f_i(u) + \nabla_p K_i(u, p))) + \rho_i d^2(x, u) \ge 0 \Rightarrow f_i(x) - f_i(u) \ge K_i(u, p) - p^T \nabla_p K_i(u, p),$$

and

$$F(x,u;\alpha_j^2(x,u)(\nabla g_j(u)+\nabla_q G_j(u,q)))+\sigma_j d^2(x,u)\geq 0 \Rightarrow$$
$$-g_j(u)\geq G_j(u,q)-q^T\nabla_q G_j(u,q).$$

Definition 4. *The functions $f_i(x)$ and $g_j(x)$ of the MONLP problem are higher-order strict (F,α,ρ,σ,d)—pseudo-convexity type-I at the point $u\in X$ where functions meet K_i, G_j if $\forall x\in X$, ρ_i, $\sigma_j\in R$, we have*

$$F(x,u;\alpha_i^1(x,u)(\nabla f_i(u)+\nabla_p K_i(u,p)))+\rho_i d^2(x,u)\geq 0 \Rightarrow$$
$$f_i(x)-f_i(u)>K_i(u,p)-p^T\nabla_p K_i(u,p),$$

and

$$F(x,u;\alpha_j^2(x,u)(\nabla g_j(u)+\nabla_q G_j(u,q)))+\sigma_j d^2(x,u)\geq 0 \Rightarrow$$
$$-g_j(u)>G_j(u,q)-q^T\nabla_q G_j(u,q).$$

Example 1. *Consider the problem.*

$$\min f(x)=\frac{x^2+x+2}{x+1}$$
$$\text{subject to } x\in X=\left\{x\in R\,|\,g(x)=1-x^2, x\geq 1\right\}$$

Let them $F(x,u;a)=|a|(x-u)^2$, $\alpha^1(x,u)=\alpha^2(x,u)=\frac{|x-u|}{5}$ and $\rho=-1$, $\sigma=-1$, $d(x,u)=0.2|x-u|, u=1$. And for each individual, as well as each family $x\in X$, $K(u,p)=\frac{p}{u+1}$, $G(u,q)=q(u^2+2)$, $f(x)-f(u)\geq F(x,u;\alpha^1(x,u)(\nabla f(u)+\nabla_p K(u,p)))-K(u,p)+p^T\nabla_p K(u,p)+\rho d^2(x,u)$, $-g(u)\geq F(x,u;\alpha^2(x,u)(\nabla g(u)+\nabla_q G(u,q)))-G(u,q)+q^T\nabla_q G(u,q)+\sigma d^2(x,u)$.

As a result, $f(x)$ and $g(x)$ they are higher-order (F,α,ρ,σ,d)—type I.

Example 2. *If $\alpha_1^1(x,u)=\alpha_1^2(x,u)=1$ in Example 1 we define the functions $K(u,p)=\frac{1}{2}p^T\nabla^2 f(u)p$ and $G(u,q)=\frac{1}{2}q^T\nabla^2 g(u)q$ then $f(x)$, $g(x)$ fail to be second-order (F,ρ,σ)—type-I functions (see [13]), because if $u=1, x=6, \rho=-1, \sigma=-0.5$ we have*

$$f(x)-f(u)-F(x,u;(\nabla f(u)+\nabla^2 f(u)p))+\frac{1}{2}p^T\nabla^2 f(u)p-\rho d^2(x,u)<0.$$

In addition,

$$-g(u)-F(x,u;(\nabla g(u)+\nabla^2 g(u)q))+\frac{1}{2}q^T\nabla^2 g(u)q-\sigma d^2(x,u)<0.$$

Remark 1. *If $\alpha_i^1(x,u)=\alpha_j^2(x,u)=1$ the higher-order (F,α,ρ,σ,d)—type-I is related to the following:*

1. *For example $K_i(u,p)=\frac{1}{2}p^T\nabla^2 f_i(u)p$, $G_j(u,q)=\frac{1}{2}q^T\nabla^2 g_j(u)q$ Definition 2 reduces to the second-order (F,ρ,σ)—type-I that is defined by [13].*
2. *If $F(x,u;a)=\eta(x,u)a, a\in R^n$ and $\eta: X\times X\to R^n$ so, then the higher-order invexity function becomes a special case of this higher-order (F,α,ρ,σ,d)—type-I.*
3. *For example $\rho_i=\sigma_j=0$, if we have the following functions,*

$$K_i(u,p)=\frac{1}{2}p^T\nabla^2 f_i(u)p,\ G_j(u,q)=\frac{1}{2}q^T\nabla^2 g_j(u)q,\ \eta: X\times X\to R^n,\ a\in R^n,\ F(x,u;a)=\eta(x,u)\,a.$$

The higher-order (F,α,ρ,σ,d)—type-I functions are related to the second-order type-I functions that are defined by [25].

3. The Six New Types of Higher-Order Duality Models for the MONLP Problems Are Described

In this section, we establish and study the new six types of higher-order duality model programs for the MONLP problems. We also define and show the theorems of weak duality, strong duality, and strict converse duality for these new six types of higher-order model programs using generalizations of the higher-order (F,α,ρ,σ,d)—type-I and higher-order (F,α,ρ,σ,d)—pseudo-convexity type-I functions.

3.1. The First Is in a Series of Six New Higher-Order Duality Models and Programs

Let us consider the first type of the new six types of higher-order duality model programs for the MONLP problems in the form

MONLD1:

max $\sum_{i=1}^{m}\lambda_i f_i(u)$

subject to

$$\sum_{i=1}^{m}\lambda_i\frac{\partial f_i(u)}{\partial u}+\sum_{j=1}^{k}y_j\frac{\partial g_j(u)}{\partial u}+\sum_{i=1}^{m}\lambda_i\frac{\partial K_i(u,p)}{\partial p}+\sum_{j=1}^{k}y_j\frac{\partial G_j(u,q)}{\partial q}=0, \tag{1}$$

$$\sum\nolimits_{j=1}^{k}y_j g_j(u)+\sum\nolimits_{j=1}^{k}y_j G_j(u,q)-q^T\sum\nolimits_{j=1}^{k}y_j\frac{\partial G_j(u,q)}{\partial q}\geq 0, \tag{2}$$

$$\sum\nolimits_{i=1}^{m}\lambda_i K_i(u,p)-p^T\sum\nolimits_{i=1}^{m}\lambda_i\frac{\partial K_i(u,p)}{\partial p}\geq 0, \tag{3}$$

$$y_j\geq 0,\ j=1,2,\ldots,k, \tag{4}$$

$$\lambda_i>0,\ \ i=1,2,\ldots,m;\ \sum_{i=1}^{m}\lambda_i=1. \tag{5}$$

We examine the weak duality, strong duality, and strict converse duality theorems for this first kind of duality model in this section.

Theorem 1 (Weak Duality)**.** *Assume that x is feasible for the MONLP problem and that (u,λ,y,p,q) is feasible for the MONLD1 problem, let the conditions be*

$$\alpha_i^1(x,u)=\alpha_j^2(x,u)=\alpha(x,u),(\sum_{i=1}^{m}\lambda_i\rho_i+\sum_{j=1}^{k}y_j\sigma_j)\geq 0$$

And choose one of the following:

(i) *The functions $f_i(x)$ and $g_j(x)$ are higher-order (F,α,ρ,σ,d)—type-I at point $u\in X$ w. r. t. K_i, G_j*

Or

(ii) *The functions $f_i(x)$ and $g_j(x)$ are higher-order (F,α,ρ,σ,d)—pseudo-convexity type-I at $u\in X$ w. r. t. K_i, G_j*

Then comes the

$$\sum\nolimits_{i=1}^{m}\lambda_i f_i(x)\geq\sum\nolimits_{i=1}^{m}\lambda_i f_i(u). \tag{6}$$

Proof. Using the assumption (i): Because the functions $f_i(x)$ and $g_j(x)$ are higher-order (F,α,ρ,σ,d)—type-I functions at $u\in X$ w. r. t. K_i, G_j we have

$$\begin{aligned}f_i(x)-f_i(u)\geq F(x,u;\alpha_i^1(x,u)(\nabla f_i(u)+\nabla_p K_i(u,p)))+K_i(u,p)-\\ p^T\nabla_p K_i(u,p)+\rho_i d^2(x,u),\end{aligned} \tag{7}$$

and

$$-g_j(u) \geq F(x,u;\alpha_j^2(x,u)(\nabla g_j(u) + \nabla_q G_j(u,q)))+ \\ G_j(u,q) - q^T \nabla_q G_j(u,q) + \sigma_j d^2(x,u). \tag{8}$$

since that time $y_j \geq 0,\ \lambda_i > 0, \alpha_i^1(x,u) = \alpha_j^2(x,u) = \alpha(x,u)$.

Multiply (7) by λ_i taking summation over i from $1 \to m$, and multiply (8) by y_j taking summation over j from $1 \to k$, and we get

$$\sum_{i=1}^{m} \lambda_i f_i(x) - \sum_{i=1}^{m} \lambda_i f_i(u) \geq F(x,u;\alpha(x,u)(\sum_{i=1}^{m} \lambda_i \tfrac{\partial f_i(u)}{\partial u} + \sum_{i=1}^{m} \lambda_i \tfrac{\partial K_i(u,p)}{\partial p}))+ \\ \sum_{i=1}^{m} \lambda_i K_i(u,p) \ - p^T \sum_{i=1}^{m} \lambda_i \tfrac{\partial K_i(u,p)}{\partial p} + d^2(x,u) \sum_{i=1}^{m} \lambda_i \rho_i, \tag{9}$$

and

$$-\sum_{j=1}^{k} y_j g_j(u) \geq F(x,u;\alpha(x,u)(\sum_{j=1}^{k} y_j \tfrac{\partial g_j(u)}{\partial u} + \sum_{j=1}^{k} y_j \tfrac{\partial G_j(u,q)}{\partial q}))+ \\ \sum_{j=1}^{k} y_j G_j(u,q) - \ q^T \sum_{j=1}^{k} y_j \tfrac{\partial G_j(u,q)}{\partial q} + d^2(x,u) \sum_{j=1}^{k} y_j \sigma_j. \tag{10}$$

By adding inequalities (9) and (10), we get

$$\sum_{i=1}^{m} \lambda_i f_i(x) - \sum_{i=1}^{m} \lambda_i f_i(u) - \sum_{j=1}^{k} y_j g_j(u) \geq F(x,u;\alpha(x,u)(\sum_{i=1}^{m} \lambda_i \tfrac{\partial f_i(u)}{\partial u} + \\ \sum_{i=1}^{m} \lambda_i \tfrac{\partial K_i(u,p)}{\partial p} + \sum_{j=1}^{k} y_j \tfrac{\partial g_j(u)}{\partial u} \ + \sum_{j=1}^{k} y_j \tfrac{\partial G_j(u,q)}{\partial q})) + \sum_{i=1}^{m} \lambda_i K_i(u,p) - \\ p^T \sum_{i=1}^{m} \lambda_i \tfrac{\partial K_i(u,p)}{\partial p} + \ \sum_{j=1}^{k} y_j G_j(u,q) - \ q^T \sum_{j=1}^{k} y_j \tfrac{\partial G_j(u,q)}{\partial q} + \\ d^2(x,u)\ (\sum_{i=1}^{m} \lambda_i \rho_i + \sum_{j=1}^{k} y_j \sigma_j) \tag{11}$$

We obtained the following: By applying the constraints (1)–(3) and applying the condition $(\sum_{i=1}^{m} \lambda_i \rho_i + \sum_{j=1}^{k} y_j \sigma_j) \geq 0$, $\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i=1}^{m} \lambda_i f_i(u)$.

Also, using the assumption (ii), the functions $f_i(x)$ and $g_j(x)$ are higher-order (F,α,ρ,σ,d)—pseudo-convexity type-I functions at $u \in X$ w. r. t. K_i and G_j we have

$$F(x,u;\alpha_i^1(x,u)(\nabla f_i(u) + \nabla_p K_i(u,p))) + \rho_i d^2(x,u) \geq 0 \Rightarrow \\ f_i(x) - f_i(u) \geq K_i(u,p) - p^T \nabla_p K_i(u,p), \tag{12}$$

and

$$F(x,u;\alpha_j^2(x,u)(\nabla g_j(u) + \nabla_q G_j(u,q))) + \sigma_j d^2(x,u) \geq 0 \Rightarrow \\ -g_j(u) \geq G_j(u,q) - q^T \nabla_q G_j(u,q). \tag{13}$$

Using constraints (4) and (5) and conditions $y_j \geq 0, \lambda_i > 0, \alpha_i^1(x,u) = \alpha_j^2(x,u) = \alpha(x,u)$.

Multiply (12) by λ_i taking summation over i from $1 \to m$, and multiply (13) by y_j taking summation over j from $1 \to k$. We got

$$F(x,u;\alpha(x,u)(\sum_{i=1}^{m} \lambda_i \tfrac{\partial f_i(u)}{\partial u} + \sum_{i=1}^{m} \lambda_i \tfrac{\partial K_i(u,p)}{\partial p})) + d^2(x,u) \sum_{i=1}^{m} \lambda_i \rho_i \geq 0 \Rightarrow \\ \sum_{i=1}^{m} \lambda_i f_i(x) - \sum_{i=1}^{m} \lambda_i f_i(u) \geq \sum_{i=1}^{m} \lambda_i K_i(u,p) - p^T \sum_{i=1}^{m} \lambda_i \tfrac{\partial K_i(u,p)}{\partial p}, \tag{14}$$

In addition,

$$F(x,u;\alpha(x,u)(\sum_{j=1}^{k} y_j \frac{\partial g_j(u)}{\partial u} + \sum_{j=1}^{k} y_j \frac{\partial G_j(u,q)}{\partial q})) + d^2(x,u)\sum_{j=1}^{k} y_j\sigma_j \geq 0 \Rightarrow$$
$$-\sum_{j=1}^{k} y_j g_j(u) \geq \sum_{j=1}^{k} y_j G_j(u,q) - q^T \sum_{j=1}^{k} y_j \frac{\partial G_j(u,q)}{\partial q}. \quad (15)$$

By adding inequalities (14) and (15), we get

$$F(x,u;\alpha(x,u)(\sum_{i=1}^{m} \lambda_i \frac{\partial f_i(u)}{\partial u} + \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p} + \sum_{j=1}^{k} y_j \frac{\partial g_j(u)}{\partial u} + \sum_{j=1}^{k} y_j \frac{\partial G_j(u,q)}{\partial q}))+$$
$$d^2(x,u)(\sum_{i=1}^{m} \lambda_i\rho_i + \sum_{j=1}^{k} y_j\sigma_j) \geq 0 \Rightarrow$$
$$\sum_{i=1}^{m} \lambda_i f_i(x) - \sum_{i=1}^{m} \lambda_i f_i(u) - \sum_{j=1}^{k} y_j g_j(u) \geq \sum_{i=1}^{m} \lambda_i K_i(u,p) -$$
$$p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p} + \sum_{j=1}^{k} y_j G_j(u,q) - q^T \sum_{j=1}^{k} y_j \frac{\partial G_j(u,q)}{\partial q}. \quad (16)$$

Use the constraints (1)–(3) that we have $\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i=1}^{m} \lambda_i f_i(u)$.

Then, the proof end. □

Theorem 2 (Strong Duality)**.** *Allow for the existence of $\overline{x}$ a weakly efficient solution to the MONLP problem that meets the constraint qualification, and the functions $K(u,0) = 0, G(u,0) = 0$* then *$\exists\overline{\lambda} \in R^m, \overline{y} \in R^k \ni \left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p} = 0, \overline{q} = 0\right)$ is feasible to solve the MONLD1 problem and the corresponding values of objective functions for the MONLP and MONLD1 problems are equal. If the hypotheses of Theorem 1 hold, then that point $\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p} = 0, \overline{q} = 0\right)$ is weakly efficient for the MONLD1 problem.*

Proof. We are $\overline{\lambda} \in R^m, \overline{y} \in R^k$ satisfied with the following: As the MONLP problem has $\overline{x}$ a weakly efficient solution that meets the constraint qualification,

$$\sum\nolimits_{i=1}^{m} \nabla\left(\overline{\lambda}_i f_i\right)(\overline{x}) + \sum\nolimits_{j=1}^{k} \nabla\left(\overline{y}_j g_j\right)(\overline{x}) = 0,$$

$$\sum\nolimits_{j=1}^{k} \overline{y}_j g_j(\overline{x}) = 0,$$
$$\overline{\lambda} \geq 0, \sum\nolimits_{i=1}^{m} \overline{\lambda}_i = 1, \overline{y} \geq 0.$$

Therefore, it $\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p} = 0, \overline{q} = 0\right)$ is feasible for the MONLD1 problem, and the corresponding values of the objective functions for the MONLP and MONLD1 problems are equal. If the hypotheses of Theorem 1 hold, then the point $\left(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p} = 0, \overline{q} = 0\right)$ is weakly efficient for the MONLD1 problem. □

Theorem 3 (Strict Converse Duality)**.** *Let's $\overline{x}$ be efficient for the MONLP problem and $\left(\overline{u}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right)$ optimal for the MONLD1 problem, respectively. Let the conditions be*

$$\alpha_i^1(\overline{x},\overline{u}) = \alpha_j^2(\overline{x},\overline{u}) = \alpha(\overline{x},\overline{u}),\ (\sum_{i=1}^{m} \overline{\lambda}_i\rho_i + \sum_{j=1}^{k} \overline{y}_j\ \sigma_j) \geq 0$$

And we assume that either

(i) At $\overline{u} \in X$ w. r. t. K_i, G_j, the functions $f_i(x)$ and $g_j(x)$ are of higher-order strict $(F,\alpha,\rho,\sigma,d)-$ type I.
Or

(ii) *At $\overline{u} \in X$ w. r. t. K_i,G_j the functions $f_i(x)$ and $g_j(x)$ are higher-order strict $(F, \alpha, \rho, \sigma, d)$—pseudo-convexity type-I.*

Then $\overline{u} = \overline{x}$.

That is $\overline{u}$ an efficient solution to the MONLP problem.

Proof. Consider the polar opposite, namely, $\overline{u} \neq \overline{x}$ since $\overline{x}$ is efficient for the MONLP problem and $(\overline{u}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q})$ optimal for the MONLD1 problem. Theorem 1 entails a

$$\sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{x}) \leq \sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{u}) \tag{17}$$

Because functions $f_i(x)$ and $g_j(x)$ are higher-order strict $(F, \alpha, \rho, \sigma, d)$—type-I at $\overline{u} \in X$ w. r. t. K_i, G_j assumption (i) $\Longrightarrow$

$$\begin{aligned} f_i(\overline{x}) - f_i(\overline{u}) > F(\overline{x}, \overline{u}; \alpha_i^1(\overline{x}, \overline{u})(\nabla f_i(\overline{u}) + \nabla_p K_i(\overline{u}, \overline{p}))) + K_i(\overline{u}, \overline{p}) - \\ \overline{p}^T \nabla_p K_i(\overline{u}, \overline{p}) + \rho_i d^2(\overline{x}, \overline{u}), \end{aligned} \tag{18}$$

and

$$\begin{aligned} -g_j(\overline{u}) > F(\overline{x}, \overline{u}; \alpha_j^2(\overline{x}, \overline{u})(\nabla g_j(\overline{u}) + \nabla_q G_j(\overline{u}, \overline{q}))) + G_j(\overline{u}, \overline{q}) - \\ \overline{q}^T \nabla_q G_j(\overline{u}, \overline{q}) + \sigma_j d^2(\overline{x}, \overline{u}). \end{aligned} \tag{19}$$

From constraints (4) and (5) and since $\overline{y}_j \geq 0$, $\overline{\lambda}_i > 0$, $\alpha_i^1(\overline{x}, \overline{u}) = \alpha_j^2(\overline{x}, \overline{u}) = \alpha(\overline{x}, \overline{u})$.

Multiply (18) by $\overline{\lambda}_i$ taking summation over i from 1 to m, and multiply (19) by $\overline{y}_j$ taking summation over j from 1 to k, and we get

$$\begin{aligned} \sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{x}) - \sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{u}) > F(\overline{x}, \overline{u}; \alpha(\overline{x}, \overline{u})(\sum_{i=1}^{m} \overline{\lambda}_i \tfrac{\partial f_i(\overline{u})}{\partial u} + \sum_{i=1}^{m} \overline{\lambda}_i \tfrac{\partial K_i(\overline{u}, \overline{p})}{\partial\ p})) + \\ \sum_{i=1}^{m} \overline{\lambda}_i K_i(\overline{u}, \overline{p}) - \overline{p}^T \sum_{i=1}^{m} \overline{\lambda}_i \tfrac{\partial K_i(\overline{u}, \overline{p})}{\partial p} + d^2(\overline{x}, \overline{u}) \sum_{i=1}^{m} \overline{\lambda}_i \rho_i, \end{aligned} \tag{20}$$

and

$$\begin{aligned} -\sum_{j=1}^{k} \overline{y}_j g_j(\overline{u}) > F(\overline{x}, \overline{u}; \alpha(\overline{x}, \overline{u})(\sum_{j=1}^{k} \overline{y}_j \tfrac{\partial g_j(\overline{u})}{\partial u} + \sum_{j=1}^{k} \overline{y}_j \tfrac{\partial G_j(\overline{u}, \overline{q})}{\partial q})) + \\ \sum_{j=1}^{k} \overline{y}_j G_j(\overline{u}, \overline{q}) - \overline{q}^T \sum_{j=1}^{k} \overline{y}_j \tfrac{\partial G_j(\overline{u}, \overline{q})}{\partial q} + d^2(\overline{x}, \overline{u}) \sum_{j=1}^{k} \overline{y}_j \sigma_j. \end{aligned} \tag{21}$$

By adding (20) and (21), we obtain

$$\begin{aligned} \sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{x}) - \sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{u}) - \sum_{j=1}^{k} \overline{y}_j g_j(\overline{u}) > F(\overline{x}, \overline{u}; \alpha(\overline{x}, \overline{u})(\sum_{i=1}^{m} \overline{\lambda}_i \tfrac{\partial f_i(\overline{u})}{\partial\ u} + \\ \sum_{i=1}^{m} \overline{\lambda}_i \tfrac{\partial K_i(\overline{u}, \overline{p})}{\partial p} + \sum_{j=1}^{k} \overline{y}_j \tfrac{\partial g_j(\overline{u})}{\partial\ u} + \sum_{j=1}^{k} \overline{y}_j \tfrac{\partial G_j(\overline{u}, \overline{q})}{\partial\ q})) + \sum_{i=1}^{m} \overline{\lambda}_i K_i(\overline{u}, \overline{p}) - \\ \overline{p}^T \sum_{i=1}^{m} \overline{\lambda}_i \tfrac{\partial K_i(\overline{u}, \overline{p})}{\partial p} + \sum_{j=1}^{k} \overline{y}_j G_j(\overline{u}, \overline{q}\ ) - \overline{q}^T \sum_{j=1}^{k} \overline{y}_j \tfrac{\partial G_j(\overline{u}, \overline{q})}{\partial\ q} + \\ d^2(\overline{x}, \overline{u})\ (\sum_{i=1}^{m} \overline{\lambda}_i \rho_i + \sum_{j=1}^{k} \overline{y}_j \sigma_j). \end{aligned} \tag{22}$$

By using constraints (1)–(3) as well as the condition, $(\sum_{i=1}^{m} \overline{\lambda}_i \rho_i + \sum_{j=1}^{k} \overline{y}_j \sigma_j) \geq 0$ we get

$$\sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{x}) > \sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{u})$$

That contradicts (17). Then $\overline{u} = \overline{x}$.

We can deduce from assumption (ii) that since the functions $f_i(x)$ and $g_j(x)$ are higher-order strict $(F, \alpha, \rho, \sigma, d)$—pseudo-convexity-type-I at $\overline{u} \in X$ w. r. t. K_i, G_j

$$F\big(\overline{x}, \overline{u}; \alpha_i^1(\overline{x}, \overline{u})\big(\nabla f_i(\overline{u}) + \nabla_p K_i(\overline{u}, \overline{p})\big)\big) + \rho_i d^2(\overline{x}, \overline{u}) \geq 0 \Rightarrow \\ f_i(\overline{x}) - f_i(\overline{u}) > K_i(\overline{u}, \overline{p}) - \overline{p}^T \nabla_p K_i(\overline{u}, \overline{p}), \tag{23}$$

and

$$F(\overline{x}, \overline{u}; \alpha_j^2(\overline{x}, \overline{u})(\nabla g_j(\overline{u}) + \nabla_q G_j(\overline{u}, \overline{q}))) + \sigma_j d^2(\overline{x}, \overline{u}) \geq 0 \Rightarrow \\ -g_j(\overline{u}) > G_j(\overline{u}, \overline{q}) - \overline{q}^T \nabla_q G_j(\overline{u}, \overline{q}). \tag{24}$$

Using constraints (4)–(5) and the conditions $\overline{y}_j \geq 0$, $\overline{\lambda}_i > 0$, $\alpha_i^1(x, u) = \alpha_j^2(x, u) = \alpha(x, u)$.

Furthermore, multiply (23) by $\overline{\lambda}_i$ taking summation over i from $1 \to m$, and multiply (24) by $\overline{y}_j$ taking summation over j from $1 \to k$ where we get

$$F(\overline{x}, \overline{u}; \alpha\,(\overline{x}, \overline{u})(\textstyle\sum_{i=1}^m \overline{\lambda}_i \frac{\partial f_i(\overline{u})}{\partial u} + \sum_{i=1}^m \overline{\lambda}_i \frac{\partial K_i(\overline{u}, \overline{p})}{\partial p})) + d^2(\overline{x}, \overline{u}) \sum_{i=1}^m \overline{\lambda}_i \rho_i \geq 0 \Rightarrow \\ \textstyle\sum_{i=1}^m \overline{\lambda}_i f_i(\overline{x}) - \sum_{i=1}^m \overline{\lambda}_i f_i(\overline{u}) > \sum_{i=1}^m \overline{\lambda}_i K_i(\overline{u}, \overline{p}) - \overline{p}^T \sum_{i=1}^m \overline{\lambda}_i \frac{\partial K_i(\overline{u}, \overline{p})}{\partial p} \tag{25}$$

In addition,

$$F(\overline{x}, \overline{u}; \alpha(\overline{x}, \overline{u})(\textstyle\sum_{j=1}^k \overline{y}_j \frac{\partial g_j(\overline{u})}{\partial u} + \sum_{j=1}^k \overline{y}_j \frac{\partial G_j(\overline{u}, \overline{q})}{\partial q})) + d^2(\overline{x}, \overline{u}) \sum_{j=1}^k \overline{y}_j \sigma_j \geq 0 \Rightarrow \\ -\textstyle\sum_{j=1}^k \overline{y}_j g_j(\overline{u}) > \sum_{j=1}^k \overline{y}_j G_j(\overline{u}, \overline{q}) - \overline{q}^T \sum_{j=1}^k \overline{y}_j \frac{\partial G_j(\overline{u}, \overline{q})}{\partial q}. \tag{26}$$

By adding the inequalities (25) and (26), we get

$$\begin{aligned} F(\overline{x}, \overline{u}; \alpha(\overline{x}, \overline{u})(&\sum_{i=1}^m \overline{\lambda}_i \frac{\partial f_i(\overline{u})}{\partial\, u} + \sum_{i=1}^m \overline{\lambda}_i \frac{\partial K_i(\overline{u}, \overline{p})}{\partial\, p} + \sum_{j=1}^k \overline{y}_j \frac{\partial g_j(\overline{u})}{\partial\, u} + \sum_{j=1}^k \overline{y}_j \frac{\partial G_j(\overline{u}, \overline{q})}{\partial\, q})) + \\ &d^2(\overline{x}, \overline{u})(\sum_{i=1}^m \overline{\lambda}_i \rho_i + \sum_{j=1}^k \overline{y}_j \sigma_j) \geq 0 \Rightarrow \\ &\sum_{i=1}^m \overline{\lambda}_i f_i(\overline{x}) - \sum_{i=1}^m \overline{\lambda}_i f_i(\overline{u}) - \sum_{j=1}^k \overline{y}_j g_j(\overline{u}) > \sum_{i=1}^m \overline{\lambda}_i K_i(\overline{u}, \overline{p}) - \\ &\overline{p}^T \sum_{i=1}^m \overline{\lambda}_i \frac{\partial K_i(\overline{u}, \overline{p})}{\partial p} + \sum_{j=1}^k \overline{y}_j G_j(\overline{u}, \overline{q}) - \overline{q}^T \sum_{j=1}^k \overline{y}_j \frac{\partial G_j(\overline{u}, \overline{q})}{\partial\, q}. \end{aligned} \tag{27}$$

Using the constraints (1)–(3) and the conditions $\left(\sum_{i=1}^m \overline{\lambda}_i \rho_i + \sum_{j=1}^k \overline{y}_j \sigma_j\right) \geq 0$, we obtain

$$\textstyle\sum_{i=1}^m \overline{\lambda}_i f_i(\overline{x}) > \sum_{i=1}^m \overline{\lambda}_i f_i(\overline{u})$$

That contradicts (17). After that $\overline{u} = \overline{x}$, the proof is complete. □

3.2. The Second of Six New Higher-Order Duality Models and Programs

Let us consider the second type of the new six types of higher-order duality model programs for the MONLP problems in the form:

MONLD2:
max $\sum_{i=1}^m \lambda_i f_i(u) + \sum_{j=1}^k y_j g_j(u)$
subject to

$$\sum_{i=1}^m \lambda_i \frac{\partial f_i(u)}{\partial u} + \sum_{j=1}^k y_j \frac{\partial g_j(u)}{\partial u} + \sum_{i=1}^m \lambda_i \frac{\partial K_i(u, p)}{\partial p} + \sum_{j=1}^k y_j \frac{\partial G_j(u, q)}{\partial q} = 0, \tag{28}$$

$$\textstyle\sum_{i=1}^m \lambda_i K_i(u, p) - p^T \sum_{i=1}^m \lambda_i \dfrac{\partial K_i(u, p)}{\partial p} \geq 0, \tag{29}$$

$$y_j \geq 0,\ j = 1, 2, \ldots, k, \tag{30}$$

$$\lambda_i > 0, \ \ i = 1, 2, \ldots, m; \ \sum_{i=1}^{m} \lambda_i = 1. \tag{31}$$

The duality theorems are covered for the second kind type in this section.

Theorem 4 (Weak Duality)**.** *Assume that x is feasible for the MONLP problem and that* $(\mathrm{u}, \lambda, \mathrm{y}, \mathrm{p}, \mathrm{q})$ *is feasible for the MONLD2 problem. Let the following conditions be met:*

$$\alpha_i^1(x, u) = \alpha_j^2(x, u) = \alpha(x, u), \ (\sum_{i=1}^{m} \lambda_i \rho_i + \sum_{j=1}^{k} y_j \sigma_j) \geq 0 \tag{32}$$

$$\sum_{j=1}^{k} y_j G_j(u, q) \geq q^T \sum_{j=1}^{k} y_j \frac{\partial G_j(u, q)}{\partial q}. \tag{33}$$

Additionally, we assume that either

(i) *The functions $f_i(x)$, $g_j(x)$ are higher-order $(F, \alpha, \rho, \sigma, d)-$ type-I at point $u \in X$ w. r. t. K_i, G_j*
Alternatively,

(ii) *At $u \in X$ w. r. t. K_i and G_j the functions $f_i(x)$, $g_j(x)$ are higher-order $(F, \alpha, \rho, \sigma, d)$—pseudo-convexity type-I functions.*

After that

$$\textstyle\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i=1}^{m} \lambda_i f_i(u) + \sum_{j=1}^{k} y_j g_j(u).$$

Proof. Using Theorem 1, we get:

By assuming (i) and the relations (28), (29), (32), and (33) in (16), we arrive at the following:

$$\textstyle\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i=1}^{m} \lambda_i f_i(u) + \sum_{j=1}^{k} y_j g_j(u)$$

Using the assumption (ii) and the relations (28), (29), (32) and (33) in (16), we get

$$\textstyle\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i=1}^{m} \lambda_i f_i(u) + \sum_{j=1}^{k} y_j g_j(u)$$

The proof is complete. □

Theorem 5 (Strong Duality)**.** *Let the point $\overline{x}$ satisfy the constraint qualification with the functions $K(u, 0) = 0, G(u, 0) = 0$ and be weakly efficient for the MONLP problem. Then $\exists\, \overline{\lambda} \in R^m, \overline{y} \in R^k$ a point that $(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p} = 0, \overline{q} = 0)$ is feasible for the MONLD2 problem, and the corresponding values of the objective functions for the MONLP and MONLD2 problems are equal. If the hypotheses of Theorem 4 are true, then the point $(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p} = 0, \overline{q} = 0)$ is weakly efficient for the MONLD2 problem.*

The proof is analogous to Theorem 2.

Theorem 6 (Strict Converse Duality)**.** *If $\overline{x}$ is efficient for the MONLP problem and $\left(\overline{u}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q}\right)$ is optimal for the MONLD2 problem, let the following conditions be met:*

$$\alpha_i^1(\overline{x}, \overline{u}) = \alpha_j^2(\overline{x}, \overline{u}) = \alpha(\overline{x}, \overline{u}), \ (\textstyle\sum_{i=1}^{m} \overline{\lambda}_i \rho_i + \sum_{j=1}^{k} \overline{y}_j \sigma_j) \geq 0 \tag{34}$$

$$\sum_{j=1}^{k} \overline{y}_j G_j(\overline{u}, \overline{q}) \geq \overline{q}^T \sum_{j=1}^{k} \overline{y}_j \frac{\partial G_j(\overline{u}, \overline{q})}{\partial q} \tag{35}$$

And we assume that either

(i) *At $\overline{u} \in X$ w. r. t. K_i, G_j, the functions $f_i(x)$, $g_j(x)$ are higher-order strict $(F, \alpha, \rho, \sigma, d)-$type-I.*
Or
(ii) *At $\overline{u} \in X$ w. r. t. K_i and G_j the functions $f_i(x)$, $g_j(x)$ are higher-order strict $(F, \alpha, \rho, \sigma, d)-$ pseudo-convexity type-I.*

Then $\overline{u} = \overline{x}$.
That is $\overline{u}$ an efficient solution to the MONLP problem.

Proof. Take a look at the polar opposite. That is $\overline{u} \neq \overline{x}$ since $\overline{x}$ is efficient for the MONLP problem and $(\overline{u}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q})$ optimal for the MONLD2 problem, we get the following inequality from Theorem 4:

$$\sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{x}) \leq \sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{u}) + \sum_{j=1}^{k} \overline{y}_j g_j(\overline{u}). \tag{36}$$

Assumptions (i) and the relations (28), (29), (34), and (35) are used in imports (22). $\sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{x}) > \sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{u}) + \sum_{j=1}^{k} \overline{y}_j g_j(\overline{u})$ That contradicts (36). Hence, we get $\overline{u} = \overline{x}$.
We get the following by substituting the assumptions (ii) and the inequalities (28), (29), (34), and (35) in the relation (27):
$\sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{x}) > \sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{u}) + \sum_{j=1}^{k} \overline{y}_j g_j(\overline{u})$ That contradicts (36).
Hence, the results follow. □

3.3. The Third Type of the New Six Types of Higher-Order Duality Model Programs

Let us consider the third type of the new six types of higher-order duality models programs for the MONLP problem in the form:
MONLD3:

$$\max \sum_{i=1}^{m} \lambda_i f_i(u) + \sum_{j=1}^{m} \lambda_i K_i(u, p) - p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u, p)}{\partial p}$$

subject to

$$\sum_{i=1}^{m} \lambda_i \frac{\partial f_i(u)}{\partial u} + \sum_{j=1}^{k} y_j \frac{\partial g_j(u)}{\partial u} + \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u, p)}{\partial p} + \sum_{j=1}^{k} y_j \frac{\partial G_j(u, q)}{\partial q} = 0, \tag{37}$$

$$\sum_{j=1}^{k} y_j g_j(u) + \sum_{j=1}^{k} y_j G_j(u, q) - q^T \sum_{j=1}^{k} y_j \frac{\partial G_j(u, q)}{\partial q} \geq 0, \tag{38}$$

$$y_j \geq 0,\ j = 1, 2, \ldots, k, \tag{39}$$

$$\lambda_i > 0,\ \ i = 1, 2, \ldots, m;\ \sum_{i=1}^{m} \lambda_i = 1. \tag{40}$$

The duality theorems for the third model type are covered in this section.

Theorem 7 (Weak Duality)**.** *Assume that x is feasible for the MONLP problem and that (u, λ, y, p, q) is feasible for the MONLD3 problem, let the following conditions be met:*

$$\alpha_i^1(x, u) = \alpha_j^2(x, u) = \alpha(x, u),\ (\sum_{i=1}^{m} \lambda_i \rho_i + \sum_{j=1}^{k} y_j \sigma_j) \geq 0 \tag{41}$$

And we assume that either

(i) *At $u \in X$ w. r. t. K_i, G_j, the functions $f_i(x)$, $g_j(x)$ are of higher-order $(F, \alpha, \rho, \sigma, d)-$type-I.*
Or
(ii) *At $u \in X$ w. r. t. K_i, G_j the functions $f_i(x)$, $g_j(x)$ are higher-order $(F, \alpha, \rho, \sigma, d)$—pseudo-convexity type-I.*

After that

$$\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i=1}^{m} \lambda_i f_i(u) + \sum_{j=1}^{m} \lambda_i K_i(u,p) - p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p}$$

Proof. Using Theorem 1, we have:

Using the assumption (i), substituting (37), (38), and (41) in (11), we get

$$\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i=1}^{m} \lambda_i f_i(u) + \sum_{j=1}^{m} \lambda_i K_i(u,p) - p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p}$$

Using assumption (ii), if we substitute (37), (38), and (41) into (16), we get

$$\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i=1}^{m} \lambda_i f_i(u) + \sum_{j=1}^{m} \lambda_i K_i(u,p) - p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p}$$

The proof is complete. □

Theorem 8 (Strong Duality)**.** *If there is $\overline{x}$ a weakly efficient solution to the MONLP problem that satisfies the constraint qualification with the functions, $K(u,0) = 0$, $G(u,0) = 0$, then we have $\overline{\lambda} \in R^m, \overline{y} \in R^k \Longrightarrow \exists(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p} = 0, \overline{q} = 0)$ a feasible solution to the MONLD3 problem as well, and the corresponding values of the objective functions for the MONLP and MONLD3 problems are equal. If the hypotheses of Theorem 7 are true, then that point $(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p} = 0, \overline{q} = 0)$ is weakly efficient for the MONLD3 problem.*

The proof is similar to Theorem 2.

Theorem 9 (Strict Converse Duality)**.** *If $\overline{x}$ is efficient for the MONLP problem and $(\overline{u}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q})$ is optimal for the MONLD3 problem, let the following conditions be met:*

$$\alpha_i^1(x,u) = \alpha_j^2(x,u) = \alpha(x,u),\ (\sum_{i=1}^{m} \overline{\lambda}_i \rho_i + \sum_{j=1}^{k} \overline{y}_j \sigma_j) \geq 0. \tag{42}$$

And we assume that either

(i) At $\overline{u} \in X$ w. r. t. K_i, G_j, the functions $f_i(x), g_j(x)$are of higher-order strict $(F, \alpha, \rho, \sigma, d)-$type-I. Or

(ii) At $\overline{u} \in X$ w. r. t. K_i, G_j the functions $f_i(x)$, $g_j(x)$ are higher-order strict $(F, \alpha, \rho, \sigma, d)$—pseudo-convexity type-I functions.

Then $\overline{u} = \overline{x}$.

That is $\overline{u}$ an efficient solution to the (MONLP) problem.

Proof. Consider the inverse; that is, $\overline{u} \neq \overline{x}$ we have from Theorem 7 that $\overline{x}$ and $(\overline{u}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q})$ are efficient and optimal for the MONLP and MONLD3 problems, respectively.

$$\sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{x}) \leq \sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{u}) + \sum_{j=1}^{m} \overline{\lambda}_i K_i(\overline{u},\overline{p}) - \overline{p}^T \sum_{i=1}^{m} \overline{\lambda}_i \frac{\partial K_i(\overline{u},\overline{p})}{\partial p} \tag{43}$$

We get the following from (22) by using the assumption (i) and the relations (37), (38), and (42).

(43) contradicts this. Hence, $\overline{u} = \overline{x}$.

Use (ii), (37), (38), and (42) in (27) and we get

$$\sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{x}) > \sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{u}) + \sum_{j=1}^{m} \overline{\lambda}_i K_i(\overline{u},\overline{p}) - \overline{p}^T \sum_{i=1}^{m} \overline{\lambda}_i \frac{\partial K_i(\overline{u},\overline{p})}{\partial p}$$

(43) contradicts this. Hence, the results follow. □

3.4. The Fourth of Six New Higher-Order Duality Model Programs

Let us consider the fourth type of the new six types of higher-order duality model programs for the MONLP problems in the form:

MONLD4:

max $\sum_{i=1}^{m} \lambda_i f_i(u) + \sum_{j=1}^{m} \lambda_i K_i(u,p) - p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p} +$

$\sum_{j=1}^{k} y_j g_j(u) + \sum_{j=1}^{k} y_j G_j(u,q) - q^T \sum_{j=1}^{k} y_j \frac{\partial G_j(u,q)}{\partial q}$

subject to

$$\sum_{i=1}^{m} \lambda_i \frac{\partial f_i(u)}{\partial u} + \sum_{j=1}^{k} y_j \frac{\partial g_j(u)}{\partial u} + \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p} + \sum_{j=1}^{k} y_j \frac{\partial G_j(u,q)}{\partial q} = 0, \tag{44}$$

$$y_j \geq 0,\ j = 1,2,\ldots,k, \tag{45}$$

$$\lambda_i > 0,\ \ i = 1,2,\ldots,m;\ \sum_{i=1}^{m} \lambda_i = 1. \tag{46}$$

These duality theorems for the fourth type are covered in this section.

Theorem 10 (Weak Duality)**.** *If x is feasible for the MONLP problem and (u,λ,y,p,q) is feasible for the MONLD4 problem, let the following conditions be met:*

$$\alpha_i^1(x,u) = \alpha_j^2(x,u) = \alpha(x,u),\ (\sum_{i=1}^{m} \lambda_i \rho_i + \sum_{j=1}^{k} y_j \sigma_j) \geq 0 \tag{47}$$

And we assume that either

(i) *At $u \in X$ w. r. t. K_i, G_j, the functions $f_i(x)$, $g_j(x)$ are of higher-order $(F,\alpha,\rho,\sigma,d)-$type I. Or*

(ii) *Atu $\in X$ w. r. t. K_i, G_j the functions $f_i(x), g_j(x)$ are higher-order $(F,\alpha,\rho,\sigma,d)-$pseudo-convexity type-I.*

Then

$$\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i=1}^{m} \lambda_i f_i(u) + \sum_{j=1}^{m} \lambda_i K_i(u,p) - p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p} + \sum_{j=1}^{k} y_j g_j(u) + \sum_{j=1}^{k} y_j G_j(u,q) - q^T \sum_{j=1}^{k} y_j \frac{\partial G_j(u,q)}{\partial q}.$$

Proof. From Theorem 1, we have:

Substitute (44) and (47) in (11) based on the assumption (i) to obtain

$$\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i=1}^{m} \lambda_i f_i(u) + \sum_{j=1}^{m} \lambda_i K_i(u,p) - p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p} + \sum_{j=1}^{k} y_j g_j(u) + \sum_{j=1}^{k} y_j G_j(u,q) - q^T \sum_{j=1}^{k} y_j \frac{\partial G_j(u,q)}{\partial q}$$

Using assumption (ii), substituting the relations (44) and (47) in (16), we get

$$\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i=1}^{m} \lambda_i f_i(u) + \sum_{j=1}^{m} \lambda_i K_i(u,p) - p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p} + \sum_{j=1}^{k} y_j g_j(u) + \sum_{j=1}^{k} y_j G_j(u,q) - q^T \sum_{j=1}^{k} y_j \frac{\partial G_j(u,q)}{\partial q}$$

The proof is complete. □

Theorem 11 (Strong Duality)**.** *If there is $\overline{x}$ a weakly efficient solution to the MONLP problem that satisfies the constraint qualification with the functions, $K(u,0) = 0$, $G(u,0) = 0$ then the point $(\overline{x},\overline{\lambda},\overline{y},\overline{p} = 0,\overline{q} = 0)$ is feasible for the MONLD4 problem as well, and the corresponding values of the objective functions for the two problems, MONLP and MONLD4, are equal. If the* *hypotheses of Theorem 10 are true, then the point* $(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p} = 0, \overline{q} = 0)$ *is weakly efficient for the MONLD4 problem.*

The proof is similar to Theorem 2.

Theorem 12 (Strict Converse Duality)**.** *If it* $\overline{x}$ *is efficient for the MONLP problem and* $(\overline{u}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q})$ *optimal for the MONLD4 problem, let the conditions be:*

$$\alpha_i^1(\overline{x}, \overline{u}) = \alpha_j^2(\overline{x}, \overline{u}) = \alpha(\overline{x}, \overline{u}),\ (\sum_{i=1}^{m} \overline{\lambda}_i \rho_i + \sum_{j=1}^{k} \overline{y}_j \sigma_j) \geq 0 \tag{48}$$

And we assume that either

(i) *At* $\overline{u} \in X$ *w. r. t.* K_i, G_j, *the functions* $f_i(x)$ *and* $g_j(x)$ *are of higher-order strict* $(F, \alpha, \rho, \sigma, d)$*—type I.*
Or
(ii) *At* $\overline{u} \in X$ *w. r. t.* K_i, G_j *the functions* $f_i(x)$ *and* $g_j(x)$ *are higher-order strict* $(F, \alpha, \rho, \sigma, d)$*—pseudo-convexity type-I.*

That is, there is $\overline{u}$ *an efficient solution to the MONLP problem.*

Proof. Assume the reverse, for example $\overline{u} \neq \overline{x}$. Since the point $\overline{x}$ is efficient for the MONLP problem and the point $(\overline{u}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q})$ is optimal for the MONLD4 problem, Theorem 10 deduces the following relationship:

$$\begin{gathered}\textstyle\sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{x}) \leq \sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{u}) + \sum_{j=1}^{m} \overline{\lambda}_i K_i(\overline{u}, \overline{p}) - \overline{p}^T \sum_{i=1}^{m} \overline{\lambda}_i \frac{\partial K_i(\overline{u},\overline{p})}{\partial p} \\ \textstyle\sum_{j=1}^{k} \overline{y}_j g_j(\overline{u}) + \sum_{j=1}^{k} \overline{y}_j G_j(\overline{u}, \overline{q}) - \overline{q}^T \sum_{j=1}^{k} \overline{y}_j \frac{\partial G_j(\overline{u},\overline{q})}{\partial q}\end{gathered} \tag{49}$$

From assumption (i), using (44) and (48) in (22) to obtain

$$\begin{gathered}\sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{x}) > \sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{u}) + \sum_{j=1}^{m} \overline{\lambda}_i K_i(\overline{u}, \overline{p}) - \overline{p}^T \sum_{i=1}^{m} \overline{\lambda}_i \frac{\partial K_i(\overline{u},\overline{p})}{\partial p} + \\ \textstyle\sum_{j=1}^{k} \overline{y}_j g_j(\overline{u}) + \sum_{j=1}^{k} \overline{y}_j G_j(\overline{u}, \overline{q}) - \overline{q}^T \sum_{j=1}^{k} \overline{y}_j \frac{\partial G_j(\overline{u},\overline{q})}{\partial q}\end{gathered}$$

This is contrary to (49). Hence, $\overline{u} = \overline{x}$ From assumption (ii), if we use (44) and (48) in (27), we get

$$\begin{gathered}\sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{x}) > \sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{u}) + \sum_{j=1}^{m} \overline{\lambda}_i K_i(\overline{u}, \overline{p}) - \overline{p}^T \sum_{i=1}^{m} \overline{\lambda}_i \frac{\partial K_i(\overline{u},\overline{p})}{\partial p} + \\ \textstyle\sum_{j=1}^{k} \overline{y}_j g_j(\overline{u}) + \sum_{j=1}^{k} \overline{y}_j G_j(\overline{u}, \overline{q}) - \overline{q}^T \sum_{j=1}^{k} \overline{y}_j \frac{\partial G_j(\overline{u},\overline{q})}{\partial q}\end{gathered}$$

This is contrary to (49). Hence, $\overline{u} = \overline{x}$ then the proof end. □

3.5. The Fifth of Six New Types of Higher-Order Duality Model Programs

Let us consider the fifth type of the new six types of higher-order duality model programs for the MONLP problems in the form:
MONLD5:
max $\sum_{i=1}^{m} \lambda_i f_i(u)$
subject to

$$\sum_{j=1}^{k} y_j g_j(u) + \sum_{j=1}^{k} y_j G_j(u, q) - q^T \sum_{j=1}^{k} y_j \frac{\partial G_j(u, q)}{\partial q} \geq 0, \tag{50}$$

$$\sum_{i=1}^{m} \lambda_i K_i(u, p) - p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u, p)}{\partial p} \geq 0, \tag{51}$$

$$y_j \geq 0,\ j = 1, 2, \ldots, k, \tag{52}$$

$$\lambda_i > 0,\ \ i = 1, 2, \ldots, m;\ \sum_{i=1}^{m} \lambda_i = 1. \tag{53}$$

This section deals with the fifth kind type category for duality theorems.

Theorem 13 (Weak Duality). *Assume that x is feasible for the MONLP problem and that (u, λ, y, p, q) is feasible for the MONLD5 problem, let the conditions be*

$$\begin{aligned}(\sum_{i=1}^{m} \lambda_i \rho_i + \sum_{j=1}^{k} y_j \sigma_j) \geq 0, \nabla_u f_i(u) = -\nabla_p K_i(u, p) \forall i,\\ \nabla_u g_j(u) = -\nabla_q G_j(u, q) \forall j\end{aligned} \tag{54}$$

Additionally, we assume that either

(i) At $\overline{u} \in X$ w. r. t. K_i, G_j, the functions $f_i(x)$, $g_j(x)$ are higher-order $(F, \alpha, \rho, \sigma, d)-$type-I. Or

(ii) At $\overline{u} \in X$ w. r. t. K_i, G_j the functions $f_i(x)$,$g_j(x)$ are higher-order $(F, \alpha, \rho, \sigma, d)$—pseudo-convexity type-I.

Then $\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i-1}^{m} \lambda_i f_i(u)$.

Proof. Using assumption (i), we can conclude that the functions $f_i(x)$ and $g_j(x)$ are higher-order $(F, \alpha, \rho, \sigma, d)$—type-I at $\overline{u} \in X$ w. r. t. K_i, G_j

$$\begin{aligned}f_i(x) - f_i(u) \geq F(x, u; \alpha_i^1(x, u)(\nabla f_i(u) + \nabla_p K_i(u, p))) + K_i(u, p) -\\ p^T \nabla_p K_i(u, p) + \rho_i d^2(x, u),\end{aligned} \tag{55}$$

and

$$\begin{aligned}-g_j(u) \geq F(x, u; \alpha_j^2(x, u)(\nabla g_j(u) + \nabla_q G_j(u, q))) +\\ G_j(u, q) - q^T \nabla_q G_j(u, q) + \sigma_j d^2(x, u).\end{aligned} \tag{56}$$

This is accomplished by combining (54) with the functional property F in relations (55) and (56).

$$f_i(x) - f_i(u) \geq K_i(u, p) - p^T \nabla_p K_i(u, p) + \rho_i d^2(x, u), \tag{57}$$

and

$$-g_j(u) \geq\ G_j(u, q) - q^T \nabla_q G_j(u, q) + \sigma_j d^2(x, u). \tag{58}$$

The restrictions (52) and (53), $\overline{y}_j \geq 0,\ \overline{\lambda}_i > 0$.

Multiplying (57) by $\overline{\lambda}_i$ taking summation over i from $1 \to m$ also multiplying (58) by $\overline{y}_j$ taking summation over j from $1 \to k$ then adding the results yields

$$\begin{aligned}\sum_{i=1}^{m} \lambda_i f_i(x) - \sum_{i=1}^{m} \lambda_i f_i(u) - \sum_{j=1}^{k} y_j g_j(u) \geq \sum_{i=1}^{m} \lambda_i K_i(u, p) - pi^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p} +\\ \sum_{j=1}^{k} y_j G_j(u, q) - q^T \sum_{j=1}^{k} y_j \frac{\partial G_j(u,q)}{\partial q} + d^2(x, u)(\sum_{i=1}^{m} \lambda_i \rho_i + \sum_{j=1}^{k} y_j \sigma_j)\end{aligned} \tag{59}$$

We get the following by using (50), (51), and the condition $(\sum_{i=1}^{m} \lambda_i \rho_i + \sum_{j=1}^{k} y_j \sigma_j) \geq 0$ in (59).

$$\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i-1}^{m} \lambda_i f_i(u).$$

We get the following because the functions $f_i(x)$ and $g_j(x)$ are higher-order $(F, \alpha, \rho, \sigma, d)$ —pseudo-convexity-type-I at the point $\overline{u} \in X$ w. r. t. K_i, G_j to assumption (ii):

$$\begin{aligned} F(x, u; \alpha_i^1(x, u)(\nabla f_i(u) + \nabla_p K_i(u, p))) + \rho_i d^2(x, u) \geq 0 \Rightarrow \\ f_i(x) - f_i(u) \geq K_i(u, p) - p^T \nabla_p K_i(u, p), \end{aligned} \tag{60}$$

and

$$\begin{aligned} F(x, u; \alpha_j^2(x, u)(\nabla g_j(u) + \nabla_q G_j(u, q))) + \sigma_j d^2(x, u) \geq 0 \Rightarrow \\ -g_j(u) \geq G_j(u, q) - q^T \nabla_q G_j(u, q). \end{aligned} \tag{61}$$

When we combine (54) with the functional property in (60) and (61), we get

$$\rho_i d^2(x, u) \geq 0 \ \Rightarrow f_i(x) - f_i(u) \geq K_i(u, p) - p^T \nabla_p K_i(u, p), \tag{62}$$

and

$$\sigma_j d^2(x, u) \geq 0 \ \Rightarrow -g_j(u) \geq G_j(u, q) - q^T \nabla_q G_j(u, q). \tag{63}$$

We get by multiplying (62) by $\overline{\lambda}_i$ taking summation over i from $1 \to m$ and also multiplying (63) by $\overline{y}_j$ taking summation over j from $1 \to k$ and adding the results with use constraints (52)–(53), $\overline{y}_j \geq 0$, $\overline{\lambda}_i > 0$.

$$\begin{aligned} d^2(x, u) \sum_{i=1}^{m} \lambda_i \rho_i \geq 0 \Rightarrow \\ \sum_{i=1}^{m} \lambda_i f_i(x) - \sum_{i=1}^{m} \lambda_i f_i(u) \geq \sum_{i=1}^{m} \lambda_i K_i(u, p) - p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p}, \end{aligned} \tag{64}$$

and

$$\begin{aligned} d^2(x, u) \sum_{j=1}^{k} y_j \sigma_j \geq 0 \Rightarrow \\ -\sum_{j=1}^{k} y_j g_j(u) \geq \sum_{j=1}^{k} y_j G_j(u, q) - q^T \sum_{j=1}^{k} y_j \frac{\partial G_j(u,q)}{\partial q}. \end{aligned} \tag{65}$$

The following is obtained by applying the condition $(\sum_{i=1}^{m} \lambda_i \rho_i + \sum_{j=1}^{k} y_j \sigma_j) \geq 0$ to relations (64) and (65):

$$\begin{aligned} & d^2(x, u)(\sum_{i=1}^{m} \lambda_i \rho_i + \sum_{j=1}^{k} y_j \sigma_j) \geq 0 \Rightarrow \\ & \sum_{i=1}^{m} \lambda_i f_i(x) - \sum_{i=1}^{m} \lambda_i f_i(u) - \sum_{j=1}^{k} y_j g_j(u) \geq \sum_{i=1}^{m} \lambda_i K_i(u, p) - p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p} + \\ & \qquad \sum_{j=1}^{k} y_j G_j(u, q) - q^T \sum_{j=1}^{k} y_j \frac{\partial G_j(u,q)}{\partial q}. \end{aligned} \tag{66}$$

We can use the constraints (50) and (51) as well as the conditions $(\sum_{i=1}^{m} \lambda_i \rho_i + \sum_{j=1}^{k} y_j \sigma_j) \geq 0$ in (66) to obtain

$$\textstyle\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i-1}^{m} \lambda_i f_i(u).$$

Hence, the proof is complete. □

Theorem 14 (Strong Duality)**.** *If $\overline{x}$ is weakly efficient for the MONLP problem and satisfies the constraint qualification with the functions, $K(u, 0) = 0$, $G(u, 0) = 0$ then the point $(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p} = 0, \overline{q} = 0)$ is feasible for the MONLD5 problem, and the corresponding values of objective functions for the MONLP and MONLD5 problems are equal. If the hypotheses of Theorem 13 are true, then the point $(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p} = 0, \overline{q} = 0)$ is weakly efficient for the MONLD5 problem.*

The proof is similar to Theorem 2.

Theorem 15 (Strict Converse Duality). *Let us proceed with that condition* $(\sum\limits_{i=1}^{m}\overline{\lambda}_i\rho_i+\sum\limits_{j=1}^{k}\overline{y}_j\sigma_j)\geq 0$ *and assume that it* $\overline{x}$ *is efficient for the MONLP problem and* $(\overline{u},\overline{\lambda},\overline{y},\overline{p},\overline{q})$ *optimal for the MONLD5 problem.*

In addition, we assume either one of the following:

(i) At $\overline{u}\in X$ *w. r. t.* K_i, G_j, *the functions* $f_i(x)$ *and* $g_j(x)$ *are of higher-order strict* $(F,\alpha,\rho,\sigma,d)-$*type I.*
Or

(ii) At $\overline{u}\in X$ *w. r. t.* K_i *and* G_j *the functions* $f_i(x)$ *and* $g_j(x)$ *are higher-order strict* $(F,\alpha,\rho,\sigma,d)-$ *pseudo-convexity type-I.*

Then $\overline{u}=\overline{x}$.

That is $\overline{u}$ *an efficient solution to the MONLP problem.*

Proof. Assume the inverse. For example, $\overline{u}\neq\overline{x}$, since $\overline{x}$ and $(\overline{u},\overline{\lambda},\overline{y},\overline{p},\overline{q})$ are efficient and optimal for the MONLP and MONLD4 problems, respectively, we get the following relation from Theorem 13:

$$\sum\nolimits_{i=1}^{m}\overline{\lambda}_i f_i(\overline{x})\leq\sum\nolimits_{i=1}^{m}\overline{\lambda}_i f_i(\overline{u}). \tag{67}$$

Because the functions $f_i(x)$ and $g_j(x)$ are higher-order strict (F,α,ρ,σ,d)—type-I at $\overline{u}\in X$ w. r. t. K_i, G_j, we get the following relations from assumption (i):

$$\begin{aligned} f_i(\overline{x})-f_i(\overline{u})>F(\overline{x},\overline{u};\alpha_i^1(\overline{x},\overline{u})(\nabla f_i(\overline{u})+\nabla_p K_i(\overline{u},\overline{p})))+K_i(\overline{u},\overline{p})-\\ \overline{p}^T\nabla_p K_i(\overline{u},\overline{p})+\overline{\rho}_i d^2(\overline{x},\overline{u}), \end{aligned} \tag{68}$$

and

$$\begin{aligned} -g_j(\overline{u})>F(\overline{x},\overline{u};\alpha_j^2(\overline{x},\overline{u})(\nabla g_j(\overline{u})+\nabla_q G_j(\overline{u},\hat{q})))+\\ G_j(\overline{u},\overline{q})-\overline{q}^T\nabla_q G_j(\overline{u},\overline{q})+\overline{\sigma}_j d^2(\overline{x},\overline{u}). \end{aligned} \tag{69}$$

When we combine (54) with the properties F in relations (68) and (69), we get

$$f_i(\overline{x})-f_i(\overline{u})>K_i(\overline{u},\overline{p})-\overline{p}^T\nabla_p K_i(\overline{u},\overline{p})+\overline{\rho}_i d^2(\overline{x},\overline{u}), \tag{70}$$

and

$$-g_j(\overline{u})>G_j(\overline{u},\overline{q})-\overline{q}^T\nabla_q G_j(\overline{u},\overline{q})+\overline{\sigma}_j d^2(\overline{x},\overline{u}). \tag{71}$$

The restrictions (52) and (53) apply as well $\overline{y}_j\geq 0,\overline{\lambda}_i>0$.

Multiply (70) by $\overline{\lambda}_i$ taking the summation over i from $1\to m$ also multiply (71) by $\overline{y}_j$ taking the summation over j from $1\to k$ and adding the results.

$$\begin{aligned} \sum_{i=1}^{m}\overline{\lambda}_i f_i(\overline{x})-\sum_{i=1}^{m}\overline{\lambda}_i f_i(\overline{u})-\sum_{j=1}^{k}\overline{y}_j g_j(\overline{u})>\sum_{i=1}^{m}\overline{\lambda}_i K_i(\overline{u},\overline{p})-\overline{p}^T\sum_{i=1}^{m}\overline{\lambda}_i\frac{\partial K_i(\overline{u},\overline{p})}{\partial p}+\\ \sum_{j=1}^{k}\overline{y}_j G_j(\overline{u},\overline{q})-\overline{q}^T\sum_{j=1}^{k}\overline{y}_j\frac{\partial G_j(\overline{u},\overline{q})}{\partial q}+d^2(\overline{x},\overline{u})(\sum_{i=1}^{m}\overline{\lambda}_i\rho_i+\sum_{j=1}^{k}\overline{y}_j\sigma_j) \end{aligned} \tag{72}$$

Combining (50) and (51) as well as the condition $(\sum\limits_{i=1}^{m}\overline{\lambda}_i\rho_i+\sum\limits_{j=1}^{k}\overline{y}_j\sigma_j)\geq 0$ in (72) yields $\sum_{i=1}^{m}\overline{\lambda}_i f_i(\overline{x})>\sum_{i=1}^{m}\overline{\lambda}_i f_i(\overline{u})$ that contradicts (67). Hence, $\overline{u}=\overline{x}$.

From assumption (ii), because the functions $f_i(x)$ and $g_j(x)$ are higher-order strict (F,α,ρ,σ,d)—pseudo-convexity type-I at $\overline{u}\in X$ w. r. t. K_i, G_j

$$\begin{aligned} F(\overline{x},\overline{u};\alpha_i^1(\overline{x},\overline{u})(\nabla f_i(\overline{u})+\nabla_p K_i(\overline{u},\overline{p})))+\rho_i d^2(\overline{x},\overline{u})\geq 0\Rightarrow\\ f_i(\overline{x})-f_i(\overline{u})>K_i(\overline{u},\overline{p})-\overline{p}^T\nabla_p K_i(\overline{u},\overline{p}), \end{aligned} \tag{73}$$

and

$$F(\overline{x},\overline{u};\alpha_j^2(\overline{x},\overline{u})(\nabla g_j(\overline{u})+\nabla_q G_j(\overline{u},\overline{q})))+\sigma_j d^2(\overline{x},\overline{u})\geq 0 \Rightarrow \\ -g_j(\overline{u}) > G_j(\overline{u},\overline{q})-\overline{q}^T\nabla_q G_j(\overline{u},\overline{q}). \tag{74}$$

Using the conditions (54) in the preceding relations (73) and (74) to get

$$\rho_i d^2(\overline{x},\overline{u})\geq 0 \ \Rightarrow f_i(\overline{x})-f_i(\overline{u}) > K_i(\overline{u},\overline{p})-\overline{p}^T\nabla_p K_i(\overline{u},\overline{p}) \tag{75}$$

and

$$\sigma_j d^2(\overline{x},\overline{u})\geq 0 \ \Rightarrow -g_j(\overline{u}) > G_j(\overline{u},\overline{q})-\overline{q}^T\nabla_q G_j(\overline{u},\overline{q}). \tag{76}$$

So we multiply (75) by $\overline{\lambda}_i$ summation over i from $1\rightarrow m$ and multiply (76) by $\overline{y}_j$ summation over j from $1\rightarrow k$, then add the results with constraints (52) and (53),$\overline{y}_j\geq 0$, $\overline{\lambda}_i>0$, $(\sum\limits_{i=1}^{m}\overline{\lambda}_i\rho_i+\sum\limits_{j=1}^{k}\overline{y}_j\sigma_j)\geq 0$ and we get

$$d^2(\overline{x},\overline{u})(\sum_{i=1}^{m}\overline{\lambda}_i\rho_i+\sum_{j=1}^{k}\overline{y}_j\sigma_j)\geq 0\Rightarrow \\ \sum_{i=1}^{m}\overline{\lambda}_i f_i(\overline{x})-\sum_{i=1}^{m}\overline{\lambda}_i f_i(\overline{u})-\sum_{j=1}^{k}\overline{y}_j g_j(\overline{u}) > \sum_{i=1}^{m}\overline{\lambda}_i K_i(\overline{u},\overline{p})-\overline{p}^T\sum_{i=1}^{m}\overline{\lambda}_i\frac{\partial K_i(\overline{u},\overline{p})}{\partial p}+ \\ \sum_{j=1}^{k}\overline{y}_j G_j(\overline{u},\overline{q})-\overline{q}^T\sum_{j=1}^{k}\overline{y}_j\frac{\partial G_j(\overline{u},\overline{q})}{\partial q}. \tag{77}$$

Using constraints (50) and (51) in (77), we get
$\sum_{i=1}^{m}\overline{\lambda}_i f_i(\overline{x}) > \sum_{i=1}^{m}\overline{\lambda}_i f_i(\overline{u})$ that contradicts (67).
Hence, $\overline{u}=\overline{x}$ the proof is complete. □

3.6. The Sixth of the Six New Types of Higher-Order Duality Model Programs

Let us consider the sixth type of the new six types of higher-order duality models programs for the MONLP problem in the form:
MONLD6:
max $\sum_{i=1}^{m}\lambda_i f_i(u)+\sum_{i=1}^{m}\lambda_i K_i(u,p)-p^T\sum_{i=1}^{m}\lambda_i\frac{\partial K_i(u,p)}{\partial p}$
subject to

$$\sum\nolimits_{j=1}^{k}y_j g_j(u)+\sum\nolimits_{j=1}^{k}y_j G_j(u,q)-q^T\sum\nolimits_{j=1}^{k}y_j\frac{\partial G_j(u,q)}{\partial q}\geq 0, \tag{78}$$

$$y_j\geq 0,\ j=1,2,\ldots,k, \tag{79}$$

$$\lambda_i>0,\ \ i=1,2,\ldots,m;\ \sum_{i=1}^{m}\lambda_i=1. \tag{80}$$

Finally, in this section, we look at the sixth kind for studying duality theorems.

Theorem 16 (Weak Duality)**.** *If x is feasible for the MONLP problem and (u,λ,y,p,q) is feasible for the MONLD6 problem, let the functions:*

$$\nabla_u f_i(u)=-\nabla_p K_i(u,p)\ \ \forall\ i,\ \nabla_u g_j(u)=-\nabla_q G_j(u,q)\forall j. \tag{81}$$

In addition, we assume

(i) The functions $f_i(x)$ and $g_j(x)$ are of higher-order $(F,\alpha,\rho,\sigma,d)-$type I at this point $\overline{u}\in X$ in terms of K_i, G_j.
Alternatively,
(ii) At $\overline{u}\in X$ w. r. t. K_i and G_j the functions $f_i(x)$ and $g_j(x)$ are higher-order (F,α,ρ,σ,d)—pseudo-convexity type-I

Then

$$\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i-1}^{m} \lambda_i f_i(u) + \sum_{i=1}^{m} \lambda_i K_i(u,p) - p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p}$$

Proof. According to Theorem 13, we have:

Using (78) and the condition $(\sum\limits_{i=1}^{m} \lambda_i \rho_i + \sum\limits_{j=1}^{k} y_j \sigma_j) \geq 0$ in (59) for assumption (i), we get

$$\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i-1}^{m} \lambda_i f_i(u) + \sum_{i=1}^{m} \lambda_i K_i(u,p) - p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p}.$$

Using constraints (78) and the condition $(\sum\limits_{i=1}^{m} \lambda_i \rho_i + \sum\limits_{j=1}^{k} y_j \sigma_j) \geq 0$ in (66) for assumption (ii), we get

$$\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i-1}^{m} \lambda_i f_i(u) + \sum_{i=1}^{m} \lambda_i K_i(u,p) - p^T \sum_{i=1}^{m} \lambda_i \frac{\partial K_i(u,p)}{\partial p}.$$

Hence, the proof is complete. □

Theorem 17 (Strong Duality)**.** *If $\overline{x}$ is weakly efficient for the MONLP problem and satisfies the constraint qualification with the functions, $K(u,0) = 0$, $G(u,0) = 0$ then we have $\overline{\lambda} \in R^m, \overline{y} \in R^k$ $\Longrightarrow (\overline{x}, \overline{\lambda}, \overline{y}, \overline{p} = 0, \overline{q} = 0)$ is feasible for the MONLD6 problem, and the corresponding values of objective functions for the MONLP and MONLD6 problems are equal. If the hypotheses of Theorem 16 are true, then that point $(\overline{x}, \overline{\lambda}, \overline{y}, \overline{p} = 0, \overline{q} = 0)$ is weakly efficient for the MONLD6 problem.*

The proof is analogous to Theorem 2.

Theorem 18 (Strict Converse Duality)**.** *If it $\overline{x}$ is efficient for the MONLP problem and $(\overline{u}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q})$ optimal for the MONLD6 problem, let the condition be:*

$$(\sum_{i=1}^{m} \overline{\lambda}_i \rho_i + \sum_{j=1}^{k} \overline{y}_j \sigma_j) \geq 0$$

We assume either the

(i) *At $\overline{u} \in X$ w. r. t. K_i and G_j the functions $f_i(x)$ and $g_j(x)$ are higher-order strict $(F, \alpha, \rho, \sigma, d)$−type-I.*
Alternatively,

(ii) *At $\overline{u} \in X$ w. r. t. K_i and G_j the functions $f_i(x)$ and $g_j(x)$ are higher-order strict $(F, \alpha, \rho, \sigma, d)$—pseudo-convexity type-I.*
Then $\overline{u} = \overline{x}$.
That is$\overline{u}$an efficient solution to the MONLP problem.

Proof. Assume the inverse. For the MONLP and MONLD4 problems, for example $\overline{u} \neq \overline{x}$, Theorem 16 implies the following: because it is $\overline{x}$ efficient and $(\overline{u}, \overline{\lambda}, \overline{y}, \overline{p}, \overline{q})$ optimal,

$$\sum_{i=1}^{m} \overline{\lambda}_i f_i(\overline{x}) \leq \sum_{i-1}^{m} \overline{\lambda}_i f_i(\overline{u}) + \sum_{i=1}^{m} \overline{\lambda}_i K_i(\overline{u}, \overline{p}) - \overline{p}^T \sum_{i=1}^{m} \overline{\lambda}_i \frac{\partial K_i(\overline{u}, \overline{p})}{\partial p} \quad (82)$$

We get from (78), and an assumption $(\sum\limits_{i=1}^{m}\overline{\lambda}_i\rho_i+\sum\limits_{j=1}^{k}\overline{y}_j\sigma_j)\geq 0$ is substituted into (72) for assumption (i).

$$\sum_{i=1}^{m}\overline{\lambda}_i f_i(\overline{x})>\sum_{i-1}^{m}\overline{\lambda}_i f_i(\overline{u})+\sum_{i=1}^{m}\overline{\lambda}_i K_i(\overline{u},\overline{p})-\overline{p}^T\sum_{i=1}^{m}\overline{\lambda}_i\frac{\partial K_i(\overline{u},\overline{p})}{\partial p}$$

That contradicts (82). Hence, $\overline{u}=\overline{x}$.

For assumption (ii), using constraint (78) and the condition $(\sum\limits_{i=1}^{m}\overline{\lambda}_i\rho_i+\sum\limits_{j=1}^{k}\overline{y}_j\sigma_j)\geq 0$ substitution in (77), we get an

$$\sum_{i=1}^{m}\overline{\lambda}_i f_i(\overline{x})>\sum_{i-1}^{m}\overline{\lambda}_i f_i(\overline{u})+\sum_{i=1}^{m}\overline{\lambda}_i K_i(\overline{u},\overline{p})-\overline{p}^T\sum_{i=1}^{m}\overline{\lambda}_i\frac{\partial K_i(\overline{u},\overline{p})}{\partial p}$$

That contradicts (82). Hence, $\overline{u}=\overline{x}$ the proof is complete. □

4. Conclusions

In this article, we established and studied six types of higher-order duality models and programs for MONLP problems under the generalizations of higher-order type-I functions. Furthermore, we formulated and proved the theorems of weak duality, strong duality, and strict converse duality of these six new types to higher-order models and programs for multiple objective nonlinear programming problems using these generalizations of higher-order type-I functions and higher-order pseudoconvex type-I functions.

Author Contributions: Conceptualization, M.A.E.-H.K.; Methodology, M.A.E.-H.K.; Formal analysis, M.A.E.-H.K.; Resources, H.M.A.; Writing—original draft, H.M.A.; Funding acquisition, H.M.A. All authors have read and agreed to the published version of the manuscript.

Funding: Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R 299), Princess Nourah Abdulrahman University, Riyadh, Saudi Arabia.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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