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Weighted Hardy–Rellich Inequality for Dunkl Operators

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Abstract: In this paper, we proved a weighted Hardy–Rellich inequality for Dunkl operators based on the spherical h-harmonic decomposition theory of Dunkl operators. Moreover, we obtained the explicit constant of the inequalities, which is optimal in some cases. Our results extend some known inequalities.

Keywords: Hardy inequalities; Hardy–Rellich inequalities; Dunkl operators

MSC: 26D10; 20F55; 42B37

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Citation: Lyu, J.; Jin, Y.; Shen, S.; Tang, L. Weighted Hardy–Rellich Inequality for Dunkl Operators. *Mathematics* **2023**, *11*, 1487. https://doi.org/10.3390/math11061487

Academic Editors: Calogero Vetro and Omar Bazighifan

Received: 24 February 2023
Revised: 14 March 2023
Accepted: 16 March 2023
Published: 18 March 2023

1. Introduction

The classical Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \geq \left|\frac{N-p}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx, \tag{1}$$

holds for $u \in C_0^\infty(\mathbb{R}^N)$ when $1 < p < N$ and for $u \in C_0^\infty(\mathbb{R}^N \backslash \{0\})$ when $N < p < \infty$. Hardy's inequality plays an important role in analysis and has extensive applications in partial differential equations and physics. Since Hardy in [1] firstly proved this inequality in the case of one dimension, many researchers devoted themselves to it and made great progress, not only in Euclidean spaces, there are many counterparts in Carnot groups and Riemannian manifolds, see [2–12] and the references therein.

Davis and Hinz obtained in [13] the following Rellich inequality

$$\int_{\mathbb{R}^N} |\Delta u|^p dx \geq \left(\frac{N(N-2p)(p-1)}{p^2}\right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}} dx, u \in C_0^\infty(\mathbb{R}^N), \tag{2}$$

where $1 < p < \frac{N}{2}$, and the constant $\left(\frac{N(N-2p)(p-1)}{p^2}\right)^p$ is sharp. It is a generalization to the second-order derivative of Hardy inequality. In [14], Tertikas and Zographopoulos obtained a Hardy–Rellich type inequality

$$\int_{\mathbb{R}^N} |x|^m |\Delta u|^2 dx \geq C_{m,N} \int_{\mathbb{R}^N} |x|^m \frac{|\nabla u|^2}{|x|^2} dx, u \in C_0^\infty(\mathbb{R}^N) \tag{3}$$

where $N \geq 5, 4-N < m \leq 0$, and

$$C_{m,N} := \min_{n=0,1,2,\cdots} \frac{\left(\frac{(N-2+m)(N-m)}{4} + n(n+N-2)\right)^2}{\frac{(N-2+m)^2}{4} + n(n+N-2)}.$$

The constant $C_{m,N}$ was proven to be sharp. Particularly, when $\frac{N+4-2\sqrt{N^2-N+1}}{3} \leq m \leq 0$, $C_{m,N} = \frac{(N-m)^2}{4}$, whereas when $4-N \leq m < \frac{N+4-2\sqrt{N^2-N+1}}{3}$, $0 < C_{m,N} < \frac{(N-m)^2}{4}$. Furthermore, a new proof of inequality (3) for $m = 0$ was given in [15]. Particularly, the author proved that, for $N = 3, 4$, the sharp constants $C_{0,N}$ are $\frac{25}{36}$ and 3, respectively. Moreover, the authors in [16] obtained an improved Hardy–Rellich inequality associated with operators

$$\mathcal{L} = -\frac{\partial^2}{\partial r^2} - \frac{N_k - 1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda_\omega,$$

where Λ_ω is a non-negative, self-adjoint operator on $\mathbb{S}^{N-1}$.

In recent years, there has been considerable interest in studying the Hardy-type inequality for Dunkl operators. It is well known that there exists a constant such that the following Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla_k u|^p d\mu_k \geq C \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} d\mu_k, \tag{4}$$

holds for all $u \in C_0^\infty(\mathbb{R}^N)$ (see [17–19]). When $p > N_k = N + 2\gamma$, γ is defined in Section 2, the best constant $C = \left(\frac{p-N_k}{p}\right)^p$ was obtained by different method in [17,18]. When $p = 2$, Velicu proved in [19] for any $u \in C_0^\infty(\mathbb{R}^N)$ and $N_k > 2$, the following sharp inequality

$$\int_{\mathbb{R}^N} |\nabla_k u|^2 d\mu_k \geq \frac{(N_k - 2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} d\mu_k, \tag{5}$$

holds. This results is based on a L^2 norm comparison of $|\nabla_k u|$ and $|\nabla u|$ for any $u \in C_0^1(\mathbb{R}^N)$ which is obtained in [20] by investigating the carré-du-champ operator. For any $1 < p < N_k$, the authors in [17,19] get explicit constants $C = \left(\frac{N_k - 2\gamma p - p}{p}\right)^p$ and $\left(\frac{N_k - p}{p}\right)^{p-1} \left(\frac{N_k - p}{p} - 2\gamma(p-1)\right)$, respectively. However, the best constant of L^p Dunkl–Hardy inequality for any $1 < p < N_k$ is still an open question.

The author in [19] also obtained a Rellich inequality for Dunkl–Laplacian

$$\int_{\mathbb{R}^N} |\Delta_k u|^2 d\mu_k \geq \frac{N_k^2 (N_k - 4)^2}{16} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^4} d\mu_k, \tag{6}$$

and the constant $\frac{N_k^2(N_k-4)^2}{16}$ is sharp.

In this paper, we proved the following weighted Hardy–Rellich inequality for Dunkl operators with an explicit constant

$$\int_{\mathbb{R}^N} |x|^a |\Delta_k u|^2 d\mu_k \geq C_{a,N_k} \int_{\mathbb{R}^N} |x|^a \frac{|\nabla_k u|^2}{|x|^2} d\mu_k. \tag{7}$$

It is an extension of inequality (3) in the case of Dunkl operators. In [18], the authors proved that for $a = 0$ and $a = 2$ the best constants of inequality (7) are, respectively, $\frac{N_k^2}{4}$ ($N_k \geq 5 + 4\gamma$) and $\frac{(N_k-2)^2}{4}$ ($N_k \neq 2$). When $a = 0$, we have $C_{a,N_k} = \frac{N_k^2}{4}$ for any $N_k \geq H_{0,\gamma}$, where $H_{a,\gamma}$ is defined as the largest real zero points of cubic function

$$f_{a,\gamma}(x) := x^3 + [a(5 + 2\gamma) - (5 + 4\gamma)]x^2 + 4a(a-1)x + 4(a-1)^2.$$

Note that $H_{0,\gamma} \leq 5 + 4\gamma$, so our result improved the inequality in [18]. When $\gamma = 0$ and $a = m = 0$, Δ_k and ∇_k degenerate, respectively, to Δ and ∇, and the inequality (7) return to the inequality (3).

The plan of this paper is as follows. In Section 2, we introduce some definitions and basic conceptions of Dunkl operators. In Section 3, we obtained weighted Hardy–Rellich type inequalities for Dunkl operators by using the spherical h-harmonic decomposition.

2. Dunkl Operators

In this section, we will introduce some fundamental concepts and notations of Dunkl operators, see also [21,22] for more details.

We call R a root system, if $R \subset \mathbb{R}^N \setminus \{0\}$ is a finite set such that $R \cap \alpha\mathbb{R} = \{-\alpha, \alpha\}$ and $\sigma_\alpha(R) = R$ for any $\alpha \in R$, then denote σ_α as a reflection on the hyperplane which is orthogonal to the root α, written as

$$\sigma_\alpha x = x - 2\frac{\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

We write G as the group generated by all the reflections σ_α for $\alpha \in R$, it is a finite group. Let $k : R \longrightarrow [0, \infty)$ be a G-invariant function, i.e., $k(\alpha) = k(v\alpha)$ for all $v \in G$ and all $\alpha \in R$, simply written $k_\alpha = k(\alpha)$. R can be decomposed as $R = R_+ \cup (-R_+)$, when $\alpha \in R_+$, then$-\alpha \in -R_+$, and R_+ is called a positive subsystem. Fix a positive subsystem R_+ in a root system R. Without loss of generality, we assume that for all $\alpha \in R$, $|\alpha|^2 = 2$. For $i = 1, ..., N$, the Dunkl operators on $C^1(\mathbb{R}^N)$ is defined as

$$T_i u(x) = \partial_i u(x) + \sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{u(x) - u(\sigma_\alpha x)}{\langle \alpha, x\rangle}.$$

By this definition, we can see that even if the decomposition of R is not unique, the different choices of positive subsystems make no difference in the definitions due to the G-invariance of k. Denote by $\nabla_k = (T_1, \ldots, T_N)$ the Dunkl gradient, $\Delta_k = \sum\limits_{i=1}^{N} T_i^2$ the Dunkl–Laplacian. Especially, for $k = 0$ we have $\nabla_0 = \nabla$ and $\Delta_0 = \Delta$. The Dunkl–Laplacian can be written in terms of the usual gradient and Laplacian as follows,

$$\Delta_k u(x) = \Delta u(x) + 2\sum_{\alpha \in R_+} k_\alpha \left[\frac{\langle \nabla u(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{u(x) - u(\sigma_\alpha x)}{\langle \alpha, x\rangle^2}\right].$$

The weight function naturally associated with Dunkl operators is

$$\omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x\rangle|^{2k_\alpha}.$$

This is a homogeneous function of degree 2γ, where

$$\gamma := \sum_{\alpha \in R_+} k_\alpha.$$

We will work in spaces $L^p(\mathbb{R}^N, |x|^a \mu_k)$, where $d\mu_k = \omega_k(x)dx$ is the weighted measure. For this weighted measure, we have a formula of integration by parts

$$\int_{\mathbb{R}^N} T_i(u) v d\mu_k = -\int_{\mathbb{R}^N} u T_i(v) d\mu_k.$$

If at least one of the functions u, v is G-invariant, the following Leibniz rule holds.

$$T_i(uv) = uT_i v + vT_i u$$

Spherical h-harmonics. We introduce some concepts and basic facts for spherical h-harmonic theory, see [21] for more details. then we called homogeneous polynomial p of degree n an h-harmonic polynomial of degree n if it satisfies

$$\Delta_k p = 0.$$

Spherical h-harmonics (or shortly h-harmonics) of degree n are the restrictions of h-harmonic polynomials of degree n to the unit sphere $\mathbb{S}^{N-1}$. We denote the space of

h-harmonics of degree n as $\mathcal{P}_n$. Denote the dimension of $\mathcal{P}_n$ as $d(n)$, which is finite and given by following equation:

$$d(n) = \binom{n+N-1}{N-1} - \binom{n+N-3}{N-1}.$$

Furthermore, one can decompose the space $L^2(\mathbb{S}^{N-1}, \omega_k(\xi)d\xi)$ as the orthogonal direct sum of the spaces $\mathcal{P}_n$, for $n = 0, 1, 2, \ldots$.

Let $\{Y_i^n\}$, $i = 1, \ldots, d(n)$ be a set of orthogonal basis of $\mathcal{P}_n$, In the spherical polar coordinates $x = r\xi$, for $r \in [0, \infty)$ and $\xi \in \mathbb{S}^{N-1}$, we can write the Dunkl–Laplacian as

$$\Delta_k = \frac{\partial^2}{\partial r^2} + \frac{N_k - 1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{k,0},$$

where $\Delta_{k,0}$ is a generalization of the classical Laplace-=Beltrami operator on the sphere, which only acts on the ξ variable. The spherical h-harmonics Y_i^n are all eigenfunctions of $\Delta_{k,0}$, and the corresponding eigenvalues are given by

$$\Delta_{k,0} Y_i^n = -n(n + N_k - 2)Y_i^n =: \lambda_n Y_i^n.$$

The h-harmonic expansion of a function $u \in L^2(\mu_k)$ can be expressed as

$$u(r\xi) = \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} u_{n,i}(r) Y_i^n(\xi),$$

where

$$u_{n,i}(r) = \int_{\mathbb{S}^{N-1}} u(r\xi) Y_i^n(\xi) \omega_k(\xi) d\nu(\xi),$$

and ν is the surface measure on the sphere $\mathbb{S}^{N-1}$.

3. Hardy–Rellich Type Inequalities for Dunkl Operators

In this section, we prove the weighted L^2 Hardy–Rellich inequalities for Dunkl operators.

Theorem 1. *Assume $a \in \mathbb{R}$, $N_k \neq 2 - a$ and $N_k \neq a$. If one of the following conditions is satisfied:*
(1) $a \geq 2$, and $N_k \geq \max\{1, 2 - a + 2\sqrt{(a-2)(a-1)}\}$;
(2) $0 \leq a < 2$, and $N_k \geq a + H_{a,\gamma}$;
(3) $a < 0$, $N_k \geq a + H_{a,\gamma}$, and

$$\begin{cases} 2 - \sqrt{3N_k - 1} \leq a < 0, & if \;\; \max\{\frac{5}{3}, \frac{(2-a)^2}{4}\} \leq N_k \leq 1 + \frac{(2-a)^2}{2}; \\ 2 - \sqrt{2(N_k - 1)} \leq a < 0, & if \;\; N_k \geq 1 + \frac{(2-a)^2}{2}, \end{cases}$$

then, for any $u \in C_0^\infty(\mathbb{R}^N)$, the following inequality holds

$$\int_{\mathbb{R}^N} |x|^a |\Delta_k u|^2 d\mu_k \geq \frac{(N_k - a)^2}{4} \int_{\mathbb{R}^N} |x|^a \frac{|\nabla_k u|^2}{|x|^2} d\mu_k. \tag{8}$$

Remark 1. *Note that when $a = 0$, the function*

$$f_{0,\gamma}(x) = x^3 - (5 + 4\gamma)x^2 + 4,$$

it follows that

$$4 + 4\gamma < H_{0,\gamma} < 5 + 4\gamma.$$

Thus, Theorem 1 improves the results given in [18].

Remark 2. $f_{0,0}(4) = -12$, $f_{0,0}(5) = 4$, *so* $4 < H_{0,0} < 5$. *This means that the condition $N \geq 5$ is reasonable when $a = \gamma = 0$ in inequality (8).*

Remark 3. *Especially, we have $H_{1,\gamma} = 2\gamma$. Therefore, inequality (8) holds for any $N_k \geq 1 + 2\gamma$.*

Proposition 1. *Assume $a < 1$, then $a + H_{a,\gamma} > 1 + 2\gamma$.*

Proof. We only need to prove $f_{a,\gamma}(1 + 2\gamma - a) < 0$. By direct computation we have

$$\begin{aligned} f_{a,\gamma}(1+2\gamma-a) =&(1+2\gamma-a)^3 + [a(5+2\gamma) - (5+4\gamma)](1+2\gamma-a)^2 \\ &+ 4a(a-1)(1+2\gamma-a) + 4(a-1)^2 \\ =&(a-1)((1+2\gamma)(4+2\gamma) - 2a\gamma)2\gamma \\ &- (1-a)^2((1+2\gamma)(4+2\gamma) - (4+2a\gamma)). \end{aligned}$$

Since $a < 1$, it is clear that $f_{a,\gamma}(1 + 2\gamma - a) < 0$. □

We prove firstly an estimate of the right-hand side of inequality (8) which is different from the result for Euclidean gradient. In fact, in the case of the Euclidean gradient, the following inequality (9) is exactly an equality for any $a \in \mathbb{R}$.

Lemma 1. *For any $u \in C_0^\infty(\mathbb{R}^N)$, we have the inequalities:*
(1) When $a \geq 2$,

$$\int_{\mathbb{R}^N} \frac{|\nabla_k u|^2}{|x|^{2-a}} d\mu_k \leq \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_0^{+\infty} \Big[|u_{n,i}'|^2 r^{N_k+a-3} - \lambda_n u_{n,i}^2 r^{N_k+a-5}\Big] dr. \tag{9}$$

(2) When $a < 2$,

$$\int_{\mathbb{R}^N} \frac{|\nabla_k u|^2}{|x|^{2-a}} d\mu_k \geq \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_0^{+\infty} \Big[|u_{n,i}'|^2 r^{N_k+a-3} - \lambda_n u_{n,i}^2 r^{N_k+a-5}\Big] dr, \tag{10}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|\nabla_k u|^2}{|x|^{2-a}} d\mu_k \leq& \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_0^{+\infty} \Big[|u_{n,i}'|^2 r^{N_k+a-3} - \lambda_n u_{n,i}^2 r^{N_k+a-5}\Big] dr \\ &+ 2(2-a)\gamma \sum_{n=1}^{\infty} \sum_{i=1}^{d(n)} \int_0^{+\infty} u_{n,i}^2 r^{N_k+a-5} dr. \end{aligned} \tag{11}$$

Proof. By integration by parts,

$$\int_{\mathbb{R}^N} \frac{|\nabla_k u|^2}{|x|^{2-a}} d\mu_k = -\int_{\mathbb{R}^N} \frac{\Delta_k u \cdot u}{|x|^{2-a}} d\mu_k + (2-a) \int_{\mathbb{R}^N} u \frac{x \cdot \nabla_k u}{|x|^{4-a}} d\mu_k, \tag{12}$$

where

$$\begin{aligned} \int_{\mathbb{R}^N} u \frac{x \cdot \nabla_k u}{|x|^{4-a}} d\mu_k =& -\int_{\mathbb{R}^N} u \cdot \nabla_k \Big(\frac{xu}{|x|^{4-a}}\Big) d\mu_k \\ =& -\int_{\mathbb{R}^N} u \left(\frac{N_k + a - 4}{|x|^{4-a}} u + \frac{x}{|x|^{4-a}} \nabla_k u - \frac{2}{|x|^{4-a}} \sum_{\alpha \in R_+} k_\alpha (u - u(\sigma_\alpha x)) \right) d\mu_k. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N} u \frac{x \cdot \nabla_k u}{|x|^{4-a}} d\mu_k = -\frac{N_k + a - 4}{2} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{4-a}} d\mu_k + \sum_{\alpha \in R_+} k_\alpha \int_{\mathbb{R}^N} \frac{(u - u(\sigma_\alpha x))u}{|x|^{4-a}} d\mu_k. \tag{13}$$

Inserting (13) into (12),

$$\int_{\mathbb{R}^N} \frac{|\nabla_k u|^2}{|x|^{2-a}} d\mu_k = -\int_{\mathbb{R}^N} \frac{\Delta_k u \cdot u}{|x|^{2-a}} d\mu_k - \frac{(2-a)(N_k+a-4)}{2} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{4-a}} d\mu_k + (2-a) \sum_{\alpha\in R_+} k_\alpha \int_{\mathbb{R}^N} \frac{(u-u(\sigma_\alpha x))u}{|x|^{4-a}} d\mu_k. \tag{14}$$

When $a \geq 2$, since $|x|^{a-4}$ and $d\mu_k$ are G-invariant, by Hölder's inequality we have

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^{a-4} u(\sigma_\alpha x) \cdot u d\mu_k &\leq \left(\int_{\mathbb{R}^N} |x|^{a-4} |u(\sigma_\alpha x)|^2 d\mu_k\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |x|^{a-4} |u|^2 d\mu_k\right)^{\frac{1}{2}} \\ &= \int_{\mathbb{R}^N} |x|^{a-4} |u|^2 d\mu_k.\end{aligned} \tag{15}$$

It follows from (14) and (15),

$$\int_{\mathbb{R}^N} \frac{|\nabla_k u|^2}{|x|^{2-a}} d\mu_k \leq -\int_{\mathbb{R}^N} \frac{\Delta_k u \cdot u}{|x|^{2-a}} d\mu_k - \frac{(2-a)(N_k+a-4)}{2} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{4-a}} d\mu_k.$$

By using the spherical decomposition for Dunkl operators,

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|\nabla_k u|^2}{|x|^{2-a}} d\mu_k &\leq -\int_{\mathbb{R}^N} \frac{u \cdot \Delta_k u}{|x|^{2-a}} d\mu_k - \frac{(2-a)(N_k+a-4)}{2} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{4-a}} d\mu_k \\ &= -\sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_0^{+\infty} \left[u_{n,i} (u''_{n,i} + \frac{N_k-1}{r} u'_{n,i} + \frac{\lambda_n}{r^2} u_{n,i}) r^{N_k+a-3} \right] dr \\ &\quad - \frac{(2-a)(N_k+a-4)}{2} \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_0^{+\infty} u_{n,i}^2 r^{N_k+a-5} dr \\ &= \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_0^{+\infty} \left[|u'_{n,i}|^2 r^{N_k+a-3} - \lambda_n u_{n,i}^2 r^{N_k+a-5} \right] dr.\end{aligned}$$

When $a < 2$, inequality (10) can be obtained similarly. On the other hand, by spherical decomposition we have

$$u = \sum_{n=0}^{+\infty} \sum_{i=1}^{d(n)} u_{n,i}(r) Y_i^n(\xi),$$

$$u(\sigma_\alpha x) = \sum_{n=0}^{+\infty} \sum_{i=1}^{d(n)} \widetilde{u}_{n,i}(r) Y_i^n(\xi),$$

where

$$u_{0,1}(r) = \frac{1}{\omega_d^k} \int_{\mathbb{S}^{N-1}} u(r\xi) \omega_k(\xi) d\nu(\xi),$$

$$\widetilde{u}_{0,1}(r) = \frac{1}{\omega_d^k} \int_{\mathbb{S}^{N-1}} u(r \cdot \sigma_\alpha(\xi)) \omega_k(\xi) d\nu(\xi),$$

where $\omega_d^k := \int_{\mathbb{S}^{N-1}} \omega_k(\xi) d\nu(\xi)$ is the spherical measure. Note that $\omega_k(\xi) d\nu(\xi)$ is G-invariant, by a change of variables $\sigma_\alpha \xi \to \xi$, we obtain

$$\widetilde{u}_{0,1}(r) = u_{0,1}(r),$$

$$u - u(\sigma_\alpha x) = \sum_{n=1}^{+\infty} \sum_{i=1}^{d(n)} (u_{n,i}(r) - \widetilde{u}_{n,i}(r)) Y_i^n(\xi).$$

From Parseval's identity, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{1}{|x|^{4-a}}(u-u(\sigma_\alpha x))u d\mu_k &= \sum_{n=1}^{+\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}(u_{n,i}(r)-\widetilde{u}_{n,i}(r))\cdot u_{n,i} r^{\overline{N}+a-5}dr \\ &= \int_{\mathbb{R}^N}\frac{1}{|x|^{4-a}}[(u-u_{0,1})-(u(\sigma_\alpha x)-\widetilde{u}_{0,1})](u-u_{0,1})d\mu_k.\end{aligned}$$

Moreover

$$\begin{aligned}&-\int_{\mathbb{R}^N}\frac{1}{|x|^{4-a}}(u(\sigma_\alpha x)-\widetilde{u}_{0,1})(u-u_{0,1})d\mu_k \\ &\leq \left(\int_{\mathbb{R}^N}\frac{1}{|x|^{4-a}}(u(\sigma_\alpha x)-\widetilde{u}_{0,1})^2 d\mu_k\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\frac{1}{|x|^{4-a}}(u-u_{0,1})^2 d\mu_k\right)^{\frac{1}{2}} \\ &= \int_{\mathbb{R}^N}\frac{1}{|x|^{4-a}}(u-u_{0,1})^2 d\mu_k.\end{aligned}$$

Then we have

$$\int_{\mathbb{R}^N}\frac{1}{|x|^{4-a}}(u-u(\sigma_\alpha x))u d\mu_k \leq 2\int_{\mathbb{R}^N}\frac{1}{|x|^{4-a}}(u-u_{0,1})^2 d\mu_k. \tag{16}$$

By (16) and spherical decomposition of Dunkl operators, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|\nabla_k u|^2}{|x|^{2-a}}d\mu_k &\leq -\int_{\mathbb{R}^N}\frac{u\cdot\Delta_k u}{|x|^{2-a}}d\mu_k - \frac{(2-a)(N_k+a-4)}{2}\int_{\mathbb{R}^N}\frac{u^2}{|x|^{4-a}}d\mu_k \\ &\quad + 2(2-a)\gamma\int_{\mathbb{R}^N}\frac{1}{|x|^{4-a}}(u-u_{0,1})^2 d\mu_k \\ &= -\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}\left[u_{n,i}(u''_{n,i}+\frac{N_k-1}{r}u'_{n,i}+\frac{\lambda_n}{r^2}u_{n,i})r^{N_k+a-3}\right]dr \\ &\quad - \frac{(2-a)(N_k+a-4)}{2}\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}u_{n,i}^2 r^{N_k+a-5}dr \\ &\quad + 2(2-a)\gamma\sum_{n=1}^{\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}u_{n,i}^2 r^{N_k+a-5}dr \\ &= \sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}\left[|u'_{n,i}|^2 r^{N_k+a-3}-\lambda_n u_{n,i}^2 r^{N_k+a-5}\right]dr \\ &\quad + 2(2-a)\gamma\sum_{n=1}^{\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}u_{n,i}^2 r^{N_k+a-5}dr.\end{aligned}$$

□

Now it's time to prove Theorem 1.

Proof of Theorem 1. Since

$$\int_{\mathbb{R}^N}|x|^a|\Delta_k u|^2 d\mu_k = \sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}\left(u''_{n,i}+\frac{N_k-1}{r}u'_{n,i}+\frac{\lambda_n}{r^2}u_{n,i}\right)^2 r^{N_k+a-1}dr,$$

when $a \geq 2$, by Lemma 1 we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |x|^a |\Delta_k u|^2 d\mu_k - C\int_{\mathbb{R}^N} \frac{|\nabla_k u|^2}{|x|^{2-a}} d\mu_k \\
\geq &\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)} \int_0^{+\infty} \left[\left(u''_{n,i} + \frac{N_k - 1}{r} u'_{n,i} + \frac{\lambda_n}{r^2} u_{n,i} \right)^2 r^{N_k+a-1} - C|u'_{n,i}|^2 r^{N_k+a-3} + \lambda_n C u_{n,i}^2 r^{N_k+a-5} \right] dr \\
= &\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)} \int_0^{+\infty} \left[|u''_{n,i}|^2 r^{N_k+a-1} + ((N_k-1)(1-a) - 2\lambda_n - C)|u'_{n,i}|^2 r^{N_k+a-3} \right] dr \\
&+ \lambda_n(\lambda_n - (2-a)(N_k+a-4) + C) \sum_{n=0}^{\infty}\sum_{i=1}^{d(n)} \int_0^{+\infty} u_{n,i}^2 r^{N_k+a-5} dr.
\end{aligned}
$$

Denote

$$
\begin{aligned}
I_{n,i} = &\int_0^{+\infty} \left[|u''_{n,i}|^2 r^{N_k+a-1} + ((N_k-1)(1-a) - 2\lambda_n - C)|u'_{n,i}|^2 r^{N_k+a-3} \right] dr \\
&+ \lambda_n(\lambda_n - (2-a)(N_k+a-4) + C) \int_0^{+\infty} u_{n,i}^2 r^{N_k+a-5} dr.
\end{aligned}
$$

Using the following weighted Hardy inequality

$$
\int_0^{+\infty} |u'|^2 r^{N_k+a-1} dr \geq \frac{(N_k+a-2)^2}{4} \int_0^{+\infty} u^2 r^{N_k+a-3} dr, \tag{17}
$$

$$
\int_0^{+\infty} |u'|^2 r^{N_k+a-3} dr \geq \frac{(N_k+a-4)^2}{4} \int_0^{+\infty} u^2 r^{N_k+a-5} dr, \tag{18}
$$

we have

$$
\begin{aligned}
I_{n,i} \geq &\left(\frac{(N_k-a)^2}{4} - 2\lambda_n - C \right) \int_0^{+\infty} |u'_{n,i}|^2 r^{N_k+a-3} dr \\
&+ \lambda_n(\lambda_n - (2-a)(N_k+a-4) + C) \int_0^{+\infty} u_{n,i}^2 r^{N_k+a-5} dr.
\end{aligned}
$$

Let $C \leq \frac{(N_k-a)^2}{4} - 2\lambda_n$, then $C_{max} = \frac{(N_k-a)^2}{4}$. Taking $C = \frac{(N_k-a)^2}{4}$, we have

$$
I_{n,i} \geq \lambda_n \left(\lambda_n - \frac{(N_k-a)(N_k+3a-8)}{4} \right) \int_0^{+\infty} u_{n,i}^2 r^{N_k+a-5} dr.
$$

If $N_k \geq a$, $\frac{(N_k-a)(N_k+3a-8)}{4} \geq 0$, then $I_{n,i} \geq 0$. When $1 \leq N_k \leq a$, $\frac{(N_k-a)(N_k+3a-8)}{4} \leq 0$, then $I_{n,i} \geq 0$ if $\lambda_1 \leq \frac{(N_k-a)(N_k+3a-8)}{4}$, thus

$$
\max\{1, 2-a+2\sqrt{(a-2)(a-1)}\} \leq N_k \leq a.
$$

When $a < 2$,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |x|^a |\Delta_k u|^2 d\mu_k - C \int_{\mathbb{R}^N} \frac{|\nabla_k u|^2}{|x|^{2-a}} d\mu_k \\
\geq &\sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_0^{+\infty} \left[\left(u''_{n,i} + \frac{N_k - 1}{r} u'_{n,i} + \frac{\lambda_n}{r^2} u_{n,i} \right)^2 r^{N_k+a-1} - C|u'_{n,i}|^2 r^{N_k+a-3} + \lambda_n C u_{n,i}^2 r^{N_k+a-5} \right] dr \\
&- 2(2-a)C\gamma \sum_{n=1}^{\infty} \sum_{i=1}^{d(n)} \int_0^{+\infty} u_{n,i}^2 r^{N_k+a-5} dr \\
= &\sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} \int_0^{+\infty} \left[|u''_{n,i}|^2 r^{N_k+a-1} + A_n |u'_{n,i}|^2 r^{N_k+a-3} + B_n u_{n,i}^2 r^{N_k+a-5} \right] dr.
\end{aligned}
$$

By integration by parts, we obtain

$$
A_n = (N_k - 1)(1-a) - 2\lambda_n - C,
$$

$$
B_n = \begin{cases} \lambda_0(\lambda_0 - (2-a)(N_k + a - 4) + C), & n = 0; \\ \lambda_n(\lambda_n - (2-a)(N_k + a - 4) + C) - 2(2-a)C\gamma, & n \geq 1, \end{cases}
$$

since $\lambda_0 = 0$, then $B_0 = 0$.

Denote

$$
J_{n,i} = \int_0^{+\infty} \left[|u''_{n,i}|^2 r^{N_k+a-1} + A_n |u'_{n,i}|^2 r^{N_k+a-3} + B_n u_{n,i}^2 r^{N_k+a-5} \right] dr,
$$

then from inequality (17) we have

$$
J_{n,i} \geq \left[A_n + \frac{(N_k + a - 2)^2}{4} \right] \int_0^{+\infty} |u'_{n,i}|^2 r^{N_k+a-3} dr + B_n \int_0^{+\infty} u_{n,i}^2 r^{N_k+a-5} dr. \tag{19}
$$

For $n = 0$,

$$
J_{0,1} \geq \left(\frac{(N_k - a)^2}{4} - C \right) \int_0^{+\infty} |u'_{0,1}|^2 r^{N_k+a-3} dr,
$$

so we get $C \leq \frac{(N_k-a)^2}{4}$.

For $n \geq 1$, take $C = \frac{(N_k-a)^2}{4}$, from inequality (18) we get

$$
\begin{aligned}
J_{n,i} &\geq -2\lambda_n \int_0^{+\infty} |u'_{n,i}|^2 r^{N_k+a-3} dr + B_n \int_0^{+\infty} u_{n,i}^2 r^{N_k+a-5} dr \\
&\geq D_n \int_0^{+\infty} u_{n,i}^2 r^{N_k+a-5} dr,
\end{aligned}
$$

where

$$
D_n = \lambda_n \left(\lambda_n - \frac{(N_k - a)(N_k + 3a - 8)}{4} \right) - \frac{(2-a)}{2} (N_k - a)^2 \gamma,
$$

1. If $0 \leq a < 2$, we can rewrite D_1 as

$$
\begin{aligned}
D_1 =& \frac{(N_k - a)^3 + [a(5 + 2\gamma) - (5 + 4\gamma)](N_k - a)^2 + 4a(a-1)(N_k - a) + 4(a-1)^2}{4} \\
=& \frac{1}{4} f_{a,\gamma}(N_k - a).
\end{aligned}
$$

Thus, $D_1 \geq 0$ if $N_k - a \geq H_{a,\gamma}$. Moreover, D_n can be seen as a quadratic function of λ_n. Since

$$
\lambda_n \leq \lambda_2 = -2N_k \leq -2 \leq -\frac{(2-a)^2}{2} \leq \frac{(N_k - a)(N_k + 3a - 8)}{8},
$$

for $0 \le a < 2$ and $n \ge 3$, we have

$$D_n \ge D_2 = \frac{N + a\gamma}{4} {N_k}^2 + \frac{a(3a - 8 - 2N)}{4} \lambda_n + \frac{a}{2}(2 - a)(2N_k - a)\gamma \ge 0.$$

So $D_n \ge 0$ for any $n \ge 1$, i.e., $J_{n,i} \ge 0$ for any $n \ge 0$. Thus, the inequality (8) holds.

2. If $a < 0$, we can also have $D_1 \ge 0$ for

$$N_k \ge a + H_{a,\gamma}.$$

Moreover, $D_n \ge D_2 \ge D_1 \ge 0$ if $\lambda_2 \le \lambda_1 \le -\frac{(2-a)^2}{2}$ or $\lambda_2 \le -\frac{(2-a)^2}{2} \le \lambda_1$ and $-\frac{(2-a)^2}{2} - \lambda_2 \ge \lambda_1 + \frac{(2-a)^2}{2}$. Computing directly we have

$$\begin{cases} 2 - \sqrt{3N_k - 1} \le a < 0, & if \ \ \max\{\frac{5}{3}, \frac{(2-a)^2}{4}\} \le N_k \le 1 + \frac{(2-a)^2}{2}; \\ 2 - \sqrt{2(N_k - 1)} \le a < 0, & if \ \ N_k \ge 1 + \frac{(2-a)^2}{2}. \end{cases}$$

Combining all the arguments above, we obtain the inequality (8).

Next we prove the optimality of the constant $\frac{(N_k - a)^2}{4}$. For any $\epsilon > 0$, take

$$u_\epsilon(r) = \begin{cases} r & , \quad 0 < r < 1; \\ r^{-\frac{N_k + a - 4 + \epsilon}{2}}, & \quad r \ge 1. \end{cases}$$

Recall that $N_k \ne 2 - a$, calculating directly we have

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{\int_{\mathbb{R}^N} |\Delta_k u_\epsilon|^2 d\mu_k}{\int_{\mathbb{R}^N} \frac{|\nabla_k u_\epsilon|^2}{|x|^2} d\mu_k} &= \lim_{\epsilon \to 0} \frac{\frac{(N_k + a - 4 + \epsilon)^2 (N_k - a - \epsilon)^2}{16} + \frac{(N_k - 1)^2}{N_k + a - 2}\epsilon}{\frac{(N_k + a - 4 + \epsilon)^2}{4} + \frac{1}{N_k + a - 2}\epsilon} \\ &= \frac{(N_k - a)^2}{4}. \end{aligned}$$

□

Theorem 2. *Assume $0 \le a < 1$, $N_k \ne 2$, and $1 + 2\gamma \le N_k < a + H_{a,\gamma}$. Then, for any $u \in C_0^\infty(\mathbb{R}^N)$, we have the inequality*

$$\int_{\mathbb{R}^N} |\Delta_k u|^2 d\mu_k \ge C_{a,N,\gamma} \int_{\mathbb{R}^N} \frac{|\nabla_k u|^2}{|x|^2} d\mu_k, \tag{20}$$

where

$$C_{a,N,\gamma} := \frac{\left(N_k^2 + a^2 - 4a - 4\right)^2}{4(N_k + a - 2)^2 + 16(2 - a)(1 + 2\gamma)}.$$

Proof. When $n \ge 1$, choosing $0 < C < \frac{(N_k - a)^2}{4}$, then we have from the inequalities (18) and (19)

$$J_{n,i} \ge E_n \int_0^{+\infty} u_{n,i}^2 r^{N_k + a - 5} dr,$$

where

$$E_n = \left(\lambda_n - \frac{(N_k - a)(N_k + a - 4)}{4}\right)^2 - \left(\frac{(N_k + a - 4)^2}{4} + (2 - a)2\gamma - \lambda_n\right) C.$$

Firstly we can rewrite E_1 as

$$E_1 = \left(\frac{N_k^2 + a^2 - 4a - 4}{4}\right)^2 - \frac{(N_k + a - 2)^2 + 4(2 - a)(1 + 2\gamma)}{4} C.$$

From $E_1 \geq 0$ we have

$$C_{max} = \frac{\left(N_k^2 + a^2 - 4a - 4\right)^2}{4(N_k + a - 2)^2 + 16(2-a)(1+2\gamma)}.$$

$C_{max} < \frac{(N_k - a)^2}{4}$ if and only if $1 + 2\gamma \leq N_k < a + H_{a,\gamma}$. Taking $0 < C \leq C_{max}$, then,

$$\begin{aligned} E_2 - E_1 =& \lambda_2 \left(\lambda_2 - \frac{(N_k - a)(N_k + a - 4)}{2} \right) - \lambda_1 \left(\lambda_1 - \frac{(N_k - a)(N_k + a - 4)}{2} \right) \\ &+ (\lambda_2 - \lambda_1) \left(C - \frac{N_k^4}{4} \right) \\ =& N_k \left(N_k^2 + a(4-a) \right) - \frac{(N_k - 1)\left((N_k - 1)^2 - 2(N_k - 1) + a(4-a)\right)}{2} \\ &+ (N_k + 1) \left(\frac{(N_k - a)^2}{4} - C \right) \\ &\geq 0 \end{aligned}$$

for any $0 \leq a < 1$.

On the other hand, for any $n \geq 2$,

$$\begin{aligned} \lambda_n - \frac{(N_k - a)(N_k + a - 4)}{2} + C \leq& \lambda_2 - \frac{(N_k - a)(N_k + a - 4)}{2} + C \\ =& -\frac{N_k^2 + a(4-a)}{2} + C \leq 0, \end{aligned}$$

we have $E_n \geq E_2 \geq E_1 \geq 0$. Thus, $J_{n,i} \geq 0$, which implies that inequality (20) holds. □

Remark 4. *If $a = 0$, $\gamma = 0$, then*

$$C_{0,N,0} = \begin{cases} \frac{25}{36}, & N = 3; \\ 3, & N = 4, \end{cases}$$

which recovers the results in [15].

4. Conclusions

In this paper, by using the spherical h-harmonic decomposition theory, we obtained some weighted Hardy–Rellich inequalities associated with Dunkl operators. Particularly, we obtained the explicit constants of these inequalities and proved the sharpness of the constant in some cases.

Author Contributions: All authors contribute equally to the paper. All authors have read and agreed to the published version of the manuscript.

Funding: This work is partially supported by National Natural Science Foundation of China (Grant No. 12071431).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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