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Estimates for the Coefficients of Subclasses Defined by the Bell Distribution of Bi-Univalent Functions Subordinate to Gegenbauer Polynomials

Ala Amourah [1,*], Omar Alnajar [2], Maslina Darus [2,*], Ala Shdouh [3] and Osama Ogilat [4]

[1] Department of Mathematics, Faculty of Science and Technology, Irbid National University, Irbid 21110, Jordan
[2] Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor, Malaysia
[3] Faculty General Education and Foundation Program, Rabdan Academy, Abu Dhabi 00971, United Arab Emirates
[4] Department of Basic Sciences, Faculty of Arts and Science, Al-Ahliyya Amman University, Amman 19328, Jordan
* Correspondence: dr.alm@inu.edu.jo (A.A.); maslina@ukm.edu.my (M.D.)

Abstract: In the real world there are many applications that find the Bell distribution to be a useful and relevant model. One of these is the normal distribution. In this paper, we develop a new subclass of analytic bi-univalent functions by making use of the Bell distribution as a building block. These functions involve the Gegenbauer polynomials, and we use them to establish our new subclass. In this study, we solve the Fekete–Szegö functional problem and analyse various different estimates of the Maclaurin coefficients $|D_2|$ and $|D_3|$ for functions that belong to the built class.

Keywords: Gegenbauer polynomials; bell distribution; bi-univalent functions; Fekete–Szegö problem; analytic functions

MSC: 30C45

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1. Definitions and Preliminaries

As soon as Legendre discovered orthogonal polynomials, they were thoroughly researched by Legendre (1784) [1]. Orthogonal polynomials frequently appear in the mathematical study of model issues to locate solutions to ordinary differential equations under specific model-imposed constraints. There is no question concerning the significance of orthogonal polynomials for modern mathematics or the variety of uses they have in physics and engineering. It is common knowledge that these polynomials are crucial in issues with approximation theory. Both mathematical statistics and the theory of differential equations contain them. They have also been used in the fields of signal analysis, automatic control, quantum physics, scattering theory, and axially symmetric potential theory [2].

The Gegenbauer polynomial is a great example of a polynomial that is orthogonal. Fekete–Szegö (1933) [3] discovered a sharp bound for the functional $|D_3 - \eta D_2^2|$, with real $\eta (0 \leq \eta \leq 1)$ for a univalent function f. Since then, the challenge of establishing sharp bounds for this function of any compact family of functions $f \in A$ with any complex n as defined by the Fekete–Szegö inequality has been one of the most well-known problems associated with the coefficient of univalent analytic functions. $|D_2| \leq 1.51$ was discovered by Lewin (1967) [4] while researching the bi-univalent function class Σ.

Assume that A represents the classification of all analytical functions, where f is defined on the open unit disc $\mathbb{F} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ where $f(0) = 0$ and $f\prime(0) - 1 = 0$ are the necessary conditions. This leads to an expansion in each $f \in A$ form according to the Taylor series:

$$f(\xi) = \xi + D_2\xi^2 + D_3\xi^3 + \cdots = \xi + \sum_{k=2}^{\infty} D_k\xi^k, \quad (\xi \in \mathbb{F}). \tag{1}$$

Furthermore, the letter S will stand for the group of all functions $f \in A$ that are univalent in $\mathbb{F}$.

Let us make the assumption that the functions f and g are analytical in $\mathbb{F}$. It is conceivable for one function, given by the notation $f \prec g$, to be subordinate to another function, g. This is possible if there is a Schwarz function ϖ that is analytical in $\mathbb{F}$ with respect to

$$\varpi(0) = 0 \text{ and } |\varpi(\xi)| < 1 \qquad (\xi \in \mathbb{F})$$

Similar to

$$f(\xi) = g(\varpi(\xi)). \quad (\xi \in \mathbb{F})$$

One other thing to keep in mind is that if the function g is univalent in $\mathbb{F}$, then the equivalence stated in the following sentence is:

$$f(\xi) \prec g(\xi) \quad \text{if and only if the condition is met} \quad f(0) = g(0)$$

and

$$f(\mathbb{F}) \subset g(\mathbb{F}).$$

It is well known that for every function $f \in \mathcal{S}$, there is an inverse or opposite, named f^{-1}. The following describes what f^{-1} is:

$$f^{-1}(f(\xi)) = \xi$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4})$$

where

$$f^{-1}(w) = w - D_2w^2 + (2D_2^2 - D_3)w^3 - (5D_2^3 - 5D_2D_3 + D_4)w^4 + \cdots. \tag{2}$$

When both $f(\xi)$ and $f^{-1}(\xi)$ are judged to be bi-univalent in $\mathbb{F}$, we refer to a function as being bi-univalent in $\mathbb{F}$.

Let us designate the class of bi-univalent functions by the symbol Σ in the unit space $\mathbb{F}$ given by (1). For Σ subclasses that include interesting functions, see [5–19].

Amourah et al. [20] conducted research to examine the following Gegenbauer polynomial generating function:

$$\partial_\alpha(\gamma, \xi) = \frac{1}{(1 - 2\gamma\xi + \xi^2)^\alpha}, \tag{3}$$

where $\gamma \in [-1, 1]$ and $\xi \in \mathbb{F}$. Because the function ∂_α is analytic in $\mathbb{F}$ when γ is held constant, it is possible to expand it using a Taylor series as follows:

$$\partial_\alpha(\gamma, \xi) = \sum_{k=0}^{\infty} C_k^\alpha(\gamma)\xi^k, \tag{4}$$

where $C_k^\alpha(\gamma)$ represents a polynomial with degree k belonging to the Gegenbauer family.

∂_α, obviously accomplishes nothing when $\alpha = 0$. The Gegenbauer polynomial's generating function is therefore set to be

$$\partial_0(\gamma, \xi) = 1 - \log\left(1 - 2\gamma\xi + \xi^2\right) = \sum_{k=0}^{\infty} C_k^0(\gamma)\xi^k \tag{5}$$

for $\alpha = 0$. In addition, it is important to highlight the fact that it is preferable for the normalization to be higher than -0.5, as stated in [21]. Recurrence relations, such as the ones shown below, can also be used to define Gegenbauer polynomials.

$$C_k^{\alpha}(\gamma) = \frac{1}{k}\left[2\gamma(k+\alpha-1)C_{k-1}^{\alpha}(\gamma) - (k+2\alpha-2)C_{k-2}^{\alpha}(\gamma)\right], \tag{6}$$

with the starting values in mind

$$C_0^{\alpha}(\gamma) = 1, C_1^{\alpha}(\gamma) = 2\alpha\gamma \text{ and } C_2^{\alpha}(\gamma) = 2\alpha(1+\alpha)\gamma^2 - \alpha. \tag{7}$$

The Chebyshev polynomials are obtained when $\alpha = 1$ is used, while the Legendre polynomials are obtained when $\alpha = 0.5$ is used. These are all special cases of the Gegenbauer polynomials $C_k^{\alpha}(\gamma)$.

The distributions of random variables, which represent the distribution of probabilities over the values of the random variable, serve a fundamental role in the statistics and probability and are widely used to describe and model a variety of real-world occurrences [22]. Geometric function theory has used some of the fundamental distributions, including the Poisson, Pascal, logarithmic, binomial and Borel distributions, see [23,24].

The Bell distribution was originally presented by Castellares et al. [25], in 2018, marking a significant improvment from the Bell numbers [26].

Using the Bell distribution, one can write X, a discrete random variable, as well as the probability density function associated with it by using the formula:

$$P(X = m) = \frac{\lambda^m e^{e^{\left(-\lambda^2\right)+1}} L_m}{m!}; \;\; m = 1, 2, 3, \ldots \tag{8}$$

where $L_m = \frac{1}{e}\sum_{b=0}^{\infty}\frac{b^m}{m!}$ are the Bell numbers, $m \geq 2$ and $\lambda > 0$.

The first few Bell numbers are $L_2 = 2$, $L_3 = 5$, $L_4 = 15$ and $L_5 = 52$.

Now, we are going to provide a new power series, and the coefficients of this series will be the probabilities of the Bell distribution

$$\mathbb{L}(\lambda, \xi) = \xi + \sum_{k=2}^{\infty}\frac{\lambda^{k-1} e^{e^{\left(-\lambda^2\right)+1}} L_k}{(k-1)!}\xi^k, \qquad \xi \in \mathbb{F}. \text{ where } \lambda > 0. \tag{9}$$

Let us now look at the Hadamard product or convolution, which defines the linear operator, represented by the symbol $\mathbb{P}_\lambda : \mathcal{A} \to \mathcal{A}$

$$\mathbb{P}_\lambda f(\xi) = \mathbb{L}(\lambda, \xi) * f(\xi) = \xi + \sum_{k=2}^{\infty}\frac{\lambda^{k-1} e^{e^{\left(-\lambda^2\right)+1}} L_k}{(k-1)!}D_k\xi^k, \qquad \xi \in \mathbb{F}. \tag{10}$$

The relationships between orthogonal polynomials and bi-univalent functions have been studied by a great deal of academics in recent years (see references [27–31]). Regarding the Gegenbauer polynomial, as far as we are aware, there is very little work in the literature that is linked with bi-univalent functions.

With the Gegenbauer polynomial and the Bell distribution, we create a new subclass of functions in this new class, primarily influenced by the research of Amourah et al. [32,33], given the upper bounds for the Fekete–Szegö functional and the Taylor–Maclaurin coefficients, $|D_2|$ and $|D_3|$.

2. Boundaries for the Class Coefficients $\mathfrak{G}_{\Sigma}^{\alpha}(\gamma, \lambda, \nu, \beta)$

This section begins by defining the new subclass $\mathfrak{G}_{\Sigma}^{\alpha}(\gamma, \lambda, \nu, \beta)$ associated with the Bell distribution.

Definition 1. *If the conditions in the subordinations that follow are met, the function $f \in \Sigma$ denoted in (1) is a member of the class $\mathfrak{G}_{\Sigma}^{\alpha}(\gamma, \lambda, \nu, \beta)$,*

$$(1-\nu)\frac{\mathbb{P}_{\lambda} f(\xi)}{\xi} + \nu(\mathbb{P}_{\lambda} f(\xi))' + \beta\xi(\mathbb{P}_{\lambda} f(\xi))'' \prec \partial_{\alpha}(\gamma, \xi) \tag{11}$$

and

$$(1-\nu)\frac{\mathbb{P}_{\lambda} f(w)}{w} + \nu(\mathbb{P}_{\lambda} f(w))' + \beta w(\mathbb{P}_{\lambda} f(w))'' \prec \partial_{\alpha}(\gamma, w), \tag{12}$$

when $\alpha > 0$, $\nu, \beta \geq 0$, $\gamma \in (\frac{1}{2}, 1]$ and the function $g = f^{-1}$ are both supplied by Equation (2), and the function ∂_{α}, that generates the Gegenbauer polynomial, is given by Equation (3).

By specialising the parameter ν, one can obtain multiple new Σ subclasses, as the next example will demonstrate.

Remark 1. *We have $\mathfrak{G}_{\Sigma}^{\alpha}(\gamma, \lambda, 1, 0)$, for $\nu = 1$, and $\beta = 0$ where $\mathfrak{G}_{\Sigma}^{\alpha}(\gamma, \lambda, 1, 0)$ is the collection of functions $f \in \Sigma$ provided by (1) and meet the following criteria*

$$(\mathbb{P}_{\lambda} f(\xi))' \prec \partial_{\alpha}(\gamma, \xi) \tag{13}$$

and

$$(\mathbb{P}_{\lambda} f(w))' \prec \partial_{\alpha}(\gamma, w), \tag{14}$$

when $\alpha > 0$, $\gamma \in (0.5, 1]$, and the function $g = f^{-1}$ are both supplied by Equation (2), and the function ∂_{α}, the generates the Gegenbauer polynomial, is given by Equation (3).

Remark 2. *We have $\mathfrak{G}_{\Sigma}^{\alpha}(\gamma, \lambda, \nu, 0)$, for $\beta = 0$, where $\mathfrak{G}_{\Sigma}^{\alpha}(\gamma, \lambda, \nu, 0)$ is the collection of functions $f \in \Sigma$ provided by (1) and meet the following criteria*

$$(1-\nu)\frac{\mathbb{P}_{\lambda} f(\xi)}{\xi} + \nu(\mathbb{P}_{\lambda} f(\xi))' \prec \partial_{\alpha}(\gamma, \xi) \tag{15}$$

and

$$(1-\nu)\frac{\mathbb{P}_{\lambda} f(w)}{w} + \nu(\mathbb{P}_{\lambda} f(w))' \prec \partial_{\alpha}(\gamma, w), \tag{16}$$

when $\alpha > 0$, $\gamma \in (0.5, 1]$ and the function $g = f^{-1}$ are both supplied by Equation (2), and the function ∂_{α}, that generates the Gegenbauer polynomial, is given by Equation (3).

In this paper, we will assume that $\alpha > 0$, $\theta \geq 0$ and $\gamma \in (0.5, 1]$.

To begin, we provide some estimates for the coefficients that belong to the class $\mathfrak{G}_{\Sigma}^{\alpha}(\gamma, \lambda, \nu, \beta)$, as described in Definition 1.

Theorem 1. *Assume that the function $f \in \Sigma$, in Definition(1), is a member of the class $\mathfrak{G}_{\Sigma}^{\alpha}(\gamma, \lambda, \nu, \beta)$. Then*

$$|D_2| \leq \frac{4\alpha\gamma\sqrt{\alpha\gamma}}{\sqrt{\left|5\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}(2\alpha\gamma)^2 - 8\lambda^2(1+\nu+2\beta)^2 e^{e^{(1-\lambda^2)}}(2\alpha(1+\alpha))\gamma^2 - \alpha\right|}},$$

and

$$|D_3| \leq \frac{\alpha^2\gamma^2}{\lambda^2(1+\nu+2\beta)^2 e^{2e^{(1-\lambda^2)}}} + \frac{4|\alpha|\gamma}{5\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}}.$$

Proof. Assume $f \in \mathfrak{G}_{\Sigma}^{\alpha}(\gamma,\lambda,\nu,\beta)$. For certain analytical tasks w and τ, we can write $\xi, w \in \mathbb{F}$ such that $w(0) = \tau(0) = 0$ and $|w(\xi)| < 1$, and $|\tau(w)| < 1$ for all functions from the Definition 1.

$$(1-\nu)\frac{\mathbb{P}_{\lambda}f(\xi)}{\xi} + \nu(\mathbb{P}_{\lambda}f(\xi))' + \beta\xi(\mathbb{P}_{\lambda}f(\xi))'' = \partial_{\alpha}(\gamma, w(\xi)) \tag{17}$$

and

$$(1-\nu)\frac{\mathbb{P}_{\lambda}f(w)}{w} + \nu(\mathbb{P}_{\lambda}f(w))' + \beta w(\mathbb{P}_{\lambda}f(w))'' = \partial_{\alpha}(\gamma, \tau(w)). \tag{18}$$

This is what we obtain as a result of the equalities shown in (17) and (18).

$$(1-\nu)\frac{\mathbb{P}_{\lambda}f(\xi)}{\xi} + \nu(\mathbb{P}_{\lambda}f(\xi))' + \beta\xi(\mathbb{P}_{\lambda}f(\xi))'' = 1 + C_1^{\alpha}(\gamma)c_1\xi + \left[C_1^{\alpha}(\gamma)c_2 + C_2^{\alpha}(\gamma)c_1^2\right]\xi^2 + \cdots \tag{19}$$

and

$$(1-\nu)\frac{\mathbb{P}_{\lambda}f(w)}{w} + \nu(\mathbb{P}_{\lambda}f(w))' + \beta w(\mathbb{P}_{\lambda}f(w))'' = 1 + C_1^{\alpha}(\gamma)d_1 w + \left[C_1^{\alpha}(\gamma)d_2 + C_2^{\alpha}(\gamma)d_1^2\right])w^2 + \cdots. \tag{20}$$

It is common knowledge that if

$$|w(\xi)| = \left|c_1\xi + c_2\xi^2 + c_3\xi^3 + \cdots\right| < 1, \ \ (\xi \in \mathbb{F})$$

and

$$|\tau(w)| = \left|d_1 w + d_2 w^2 + d_3 w^3 + \cdots\right| < 1, \ \ (w \in \mathbb{F}),$$

then

$$|c_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{21}$$

Therefore, after comparing the relevant coefficients in (19) and (20), we come to the conclusion that

$$2\lambda(1+\nu+2\beta)e^{e^{(1-\lambda^2)}}D_2 = C_1^{\alpha}(\gamma)c_1, \tag{22}$$

$$\frac{5}{2}\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}D_3 = C_1^{\alpha}(\gamma)c_2 + C_2^{\alpha}(\gamma)c_1^2, \tag{23}$$

$$-2\lambda(1+\nu+2\beta)e^{e^{(1-\lambda^2)}}D_2 = C_1^{\alpha}(\gamma)d_1, \tag{24}$$

and

$$\frac{5}{2}\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}\left[2D_2^2 - D_3\right] = C_1^{\alpha}(\gamma)d_2 + C_2^{\alpha}(\gamma)d_1^2. \tag{25}$$

It follows from (22) and (24) that

$$c_1 = -d_1 \tag{26}$$

and

$$8\lambda^2(1+\nu+2\beta)^2 e^{2e^{(1-\lambda^2)}}D_2^2 = [C_1^{\alpha}(\gamma)]^2\left(c_1^2 + d_1^2\right). \tag{27}$$

If we add (23) and (25), we obtain

$$5\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}D_2^2 = C_1^{\alpha}(\gamma)(c_2+d_2) + C_2^{\alpha}(\gamma)\left(c_1^2 + d_1^2\right). \tag{28}$$

The value of $\left(c_1^2+d_1^2\right)$ from (27) has been swapped into the right side of (28), implying that

$$\left[5(1+2\nu+6\beta)-8(1+\nu+2\beta)^2e^{e^{(1-\lambda^2)}}\frac{C_2^\alpha(\gamma)}{\left[C_1^\alpha(\gamma)\right]^2}\right]\lambda^2e^{e^{(1-\lambda^2)}}D_2^2$$
$$=C_1^\alpha(\gamma)(c_2+d_2). \tag{29}$$

Moreover, using computations with (7), (21) and (29), we find that

$$|D_2|\leq \frac{4\alpha\gamma\sqrt{\alpha\gamma}}{\sqrt{\left|5\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}(2\alpha\gamma)^2-8\lambda^2(1+\nu+2\beta)^2e^{e^{(1-\lambda^2)}}(2\alpha(1+\alpha))\gamma^2-\alpha\right|}}.$$

Moreover, if we subtract (25) from (23), we obtain

$$5\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}\left(D_3-D_2^2\right)=C_1^\alpha(\gamma)(c_2-d_2)+C_2^\alpha(\gamma)\left(c_1^2-d_1^2\right). \tag{30}$$

Then, in view of (27), Equation (30) becomes

$$D_3=\frac{\left[C_1^\alpha(\gamma)\right]^2}{8\lambda^2(1+\nu+2\beta)^2e^{2e^{(1-\lambda^2)}}}\left(c_1^2+d_1^2\right)$$
$$+\frac{C_1^\alpha(\gamma)}{5\lambda^2(1+2\nu++6\beta)e^{e^{(1-\lambda^2)}}}(c_2-d_2).$$

Thus, applying (7), we conclude that

$$|D_3|\leq\frac{\alpha^2\gamma^2}{\lambda^2(1+\nu+2\beta)^2e^{2e^{(1-\lambda^2)}}}+\frac{4|\alpha|\gamma}{5\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}}.$$

The proof of Theorem 1 is now complete. □

We can use the values of D_2^2 and D_3. to derive what comes next in the Fekete–Szegö inequality for the class $\mathfrak{G}_\Sigma^\alpha(\gamma,\lambda,\nu,\beta)$ functions.

Theorem 2. *Assume that the function $f\in\Sigma$, in Definition (1), is a member of the class $\mathfrak{G}_\Sigma^\alpha(\gamma,\lambda,\nu,\beta)$. Then*

$$\left|D_3-\eta D_2^2\right|\leq\begin{cases}\dfrac{4|\alpha\gamma|}{5\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}}, & |\eta-1|\leq\varrho\\[2ex] \dfrac{2(2\alpha\gamma)^3(1-\eta)}{\lambda^2e^{e^{(1-\lambda^2)}}\left[5(1+2\nu+6\beta)(2\alpha\gamma)^2-8(1+\nu+2\beta)^2e^{e^{(1-\lambda^2)}}(2\alpha(1+\alpha)\gamma^2-\alpha)\right]}, & |\eta-1|\geq\varrho,\end{cases}$$

where

$$\varrho=\left|1-\frac{8(1+\nu+2\beta)^2e^{e^{(1-\lambda^2)}}(2\alpha(1+\alpha)\gamma^2-\alpha)}{5(1+2\nu+6\beta)(2\alpha\gamma)^2}\right|.$$

Proof. From (29) and (30)

$$
\begin{aligned}
& D_3 - \eta D_2^2 \\
&= (1-\eta)\frac{\left[C_1^{\alpha}(\gamma)\right]^3 (c_2 + d_2)}{\lambda^2 e^{e^{(1-\lambda^2)}}\left[5(1+2\nu+6\beta)\left[C_1^{\alpha}(x)\right]^2 - 8(1+\nu+2\beta)^2 e^{e^{(1-\lambda^2)}} C_2^{\alpha}(\gamma)\right]} \\
&+ \frac{C_1^{\alpha}(\gamma)}{5\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}}(c_2 - d_2) \\
&= C_1^{\alpha}(\gamma)\left[h(\eta) + \frac{1}{5\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}}\right]c_2 \\
&+ C_1^{\alpha}(\gamma)\left[h(\eta) - \frac{1}{5\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}}\right]d_2,
\end{aligned}
$$

where

$$
h(\eta) = \frac{\left[C_1^{\alpha}(\gamma)\right]^2 (1-\eta)}{\lambda^2 e^{e^{(1-\lambda^2)}}\left[5(1+2\nu+6\beta)\left[C_1^{\alpha}(\gamma)\right]^2 - 8(1+\nu+2\beta)^2 e^{e^{(1-\lambda^2)}} C_2^{\alpha}(\gamma)\right]},
$$

Given (7), we must therefore conclude that

$$
\left|D_3 - \eta D_2^2\right| \leq \begin{cases} \frac{2|C_1^{\alpha}(\gamma)|}{5\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}} & 0 \leq |h(\eta)| \leq \frac{1}{5\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}}, \\ 2\left|C_1^{\alpha}(\gamma)\right|\left|h(\eta)\right| & |h(\eta)| \geq \frac{1}{5\lambda^2(1+2\nu+6\beta)e^{e^{(1-\lambda^2)}}}. \end{cases}
$$

The proof of Theorem 2 is now complete. □

3. Corollaries and Consequences

The following is a list of corollaries that can be deduced from Theorems 1 and 2, which correlate with Remarks 1 and 2.

Corollary 1. *Assume that the function $f \in \Sigma$, in Definition (1), is a member of the class $\mathfrak{G}_{\Sigma}^{\alpha}(\gamma, \lambda, 1, 0)$. Then*

$$
|D_2| \leq \frac{4\alpha\gamma\sqrt{\alpha\gamma}}{\sqrt{\left|15\lambda^2 e^{e^{(1-\lambda^2)}}(2\alpha\gamma)^2 - 32\lambda^2 e^{e^{(1-\lambda^2)}}(2\alpha(1+\alpha))\gamma^2 - \alpha\right|}},
$$

$$
|D_3| \leq \frac{\alpha^2\gamma^2}{4\lambda^2 e^{2e^{(1-\lambda^2)}}} + \frac{4|\alpha|\gamma}{15\lambda^2 e^{e^{(1-\lambda^2)}}}.
$$

and

$$
\left|D_3 - \eta D_2^2\right| \leq \begin{cases} \frac{4|\alpha\gamma|}{15\lambda^2 e^{e^{(1-\lambda^2)}}}, & |\eta - 1| \leq M \\ \frac{2(2\alpha\gamma)^3(1-\eta)}{\lambda^2 e^{e^{(1-\lambda^2)}}\left[15(2\alpha\gamma)^2 - 32 e^{e^{(1-\lambda^2)}}(2\alpha(1+\alpha)\gamma^2 - \alpha)\right]}, & |\eta - 1| \geq M. \end{cases}
$$

where

$$
M = \left|1 - \frac{32 e^{e^{(1-\lambda^2)}}(2\alpha(1+\alpha)\gamma^2 - \alpha)}{15(2\alpha\gamma)^2}\right|
$$

Corollary 2. *Assume that the function $f \in \Sigma$, in Definition (1), is a member of the class $\mathfrak{G}_{\Sigma}^{\alpha}(\gamma, \lambda, v, 0)$. Then*

$$|D_2| \leq \frac{4\alpha\gamma\sqrt{\alpha\gamma}}{\sqrt{\left|5\lambda^2(1+2\nu)e^{e^{(1-\lambda^2)}}(2\alpha x)^2 - 8\lambda^2(1+\nu)^2 e^{e^{(1-\lambda^2)}}(2\alpha(1+\alpha))x^2 - \alpha\right|}},$$

$$|D_3| \leq \frac{\alpha^2\gamma^2}{4\lambda^2(1+\nu)^2 e^{2e^{(1-\lambda^2)}}} + \frac{4|\alpha|\gamma}{5\lambda^2(1+2\nu)e^{e^{(1-\lambda^2)}}}.$$

and

$$\left|D_3 - \eta D_2^2\right| \leq \begin{cases} \frac{4|\alpha\gamma|}{5\lambda^2(1+2\nu)e^{e^{(1-\lambda^2)}}}, & |\eta - 1| \leq q \\ \frac{2(2\alpha\gamma)^3(1-\eta)}{\lambda^2 e^{e^{(1-\lambda^2)}}\left[5(1+2\nu)(2\alpha\gamma)^2 - 8(1+\nu)^2 e^{e^{(1-\lambda^2)}}(2\alpha(1+\alpha)\gamma^2 - \alpha)\right]}, & |\eta - 1| \geq q. \end{cases}$$

where

$$q = \left|1 - \frac{8(1+\nu)^2 e^{e^{(1-\lambda^2)}}(2\alpha(1+\alpha)\gamma^2 - \alpha)}{5(1+2\nu)(2\alpha\gamma)^2}\right|$$

Remark 3. *More research was conducted on the conclusions from this study could result in a wide range of other novel findings for the classes $\mathfrak{G}_{\Sigma}^{1}(x, \lambda, \nu, \beta)$ of the Chebyshev polynomials and $\mathfrak{G}_{\Sigma}^{0.5}(x, \lambda, \nu, \beta)$ of the Legendre polynomials.*

4. Conclusions

In this study, we created a new class $\mathfrak{G}_{\Sigma}^{\alpha}(\gamma, \lambda, \nu, \beta)$ of normalized analytic and bi-univalent functions connected to the Bell distribution. We found estimates for the Taylor–Maclaurin coefficients, $|D_2|$ and $|D_3|$, and the Fekete–Szegö functional problem for functions that belong to this class. Furthermore, by correctly specializing the parameter, one can find the results for the subclass $\mathfrak{G}_{\Sigma}^{\alpha}(\gamma, \lambda, 1, 0)$, defined in Remarks 1 and 2 and linked to the Bell distribution. Using the Bell distribution series in (10), researchers could estimate the Taylor–Maclaurin coefficients, $|D_2|$ and $|D_3|$, and the Fekete–Szegö functional problem for functions in new bi-univalent function subclasses defined by the associated Gegenbauer polynomials.

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