mathematics

MDPI

Article

Distributed Observers for State Omniscience with Stochastic Communication Noises

Kairui Chen [1,2], Zhangmou Zhu [1], Xianxian Zeng [3,*] and Junwei Wang [4]

1 School of Mechanical and Electrical Engineering, Guangzhou University, Guangzhou 510006, China; kray@gzhu.edu.cn (K.C.); 2112107125@e.gzhu.edu.cn (Z.Z.)
2 School of Computer & Information, Qiannan Normal University for Nationalities, Duyun 558000, China
3 School of Computer Science, Guangdong Polytechnic Normal University, Guangzhou 510000, China
4 School of Mathematics and Statistics, Guangdong University of Foreign Studies, Guangzhou 510006, China; jwwang@gdufs.edu.cn
* Correspondence: zengxianxian@gpnu.edu.cn

Abstract: The focus of this paper is on solving the state estimation problem for general continuous-time linear systems through the use of distributed networked observers. To better reflect the communication environment, stochastic noises are considered when observers exchange information. In the networked observers, each local observer measures only part of the system output, and the state estimation can not be accomplished within a single observer. Then, all observers communicate through a pre-specified graph to make up information in the remaining system output. By solving a parametric algebraic Riccati equation (ARE), a simple method to calculate parameters in the observers is proposed. Furthermore, using the stability theory of stochastic differential equations, state omniscience is discussed in almost sure sense and in the mean square sense for the cases of state-dependent noises and non-state-dependent noises, respectively. It is shown that, for observable linear systems, the resulting observers work in a coordinated mode to reach state omniscience under the connected graph. Illustrative examples are provided to show the effectiveness of the distributed observers.

Keywords: state estimation; distributed observer; communication noises; algebraic Riccati equation

MSC: 37N35

check for updates

Citation: Chen, K.; Zhu, Z.; Zeng, X.; Wang, J. Distributed Observers for State Omniscience with Stochastic Communication Noises. *Mathematics* **2023**, *11*, 1997. https://doi.org/10.3390/math11091997

Academic Editors: Yurii Vladimirovich Mitrishkin and Nikolay M. Kartsev

Received: 8 April 2023
Revised: 20 April 2023
Accepted: 21 April 2023
Published: 23 April 2023

1. Introduction

Many traditional control methods are designed based on full state variables, which may not be measured directly due to the economic or technological reasons of existing sensor equipment. In modern control theory, one of the most critical challenges is incorporating state estimation or reconstruction techniques to effectively utilize traditional control methods [1,2]. Apart from estimating the states of a system centrally, the mass production of sensors provides a cooperative way for networked sensors to deal with the problem of real-time state estimation of large-scale systems. The cooperative control of networked systems enjoys many advantages, such as robustness, scalability, and reliability. Thus, estimating states via a cooperative network, which is called cooperative distributed estimation, is an interesting topic with great potential for application [3,4].

In cooperative distributed estimation, a notable characteristic is that individual local observers can only obtain partial measurement outputs and communicate with neighboring observers to collaboratively estimate the full state of the system [5,6]. The basis for distributed estimation was established by Saber in 2005 with the introduction of a distributed Kalman-filtering algorithm [7]. Inspired by [7], a kind of Kalman-consensus filtering was developed in [8], which consists of a state update stage that utilizes a Kalman filter and a data fusion stage that employs a consensus strategy. In 2018, Liu et al. further the classical Kalman-consensus filtering algorithm by considering the link failure [9]. Another kind of distributed estimation is studied under the assumption of joint observability for linear time-invariant (LTI) systems. In [10,11], states of a discrete LTI system were estimated by a network of observers with an augmented state. In [12], a kind of distributed observer was designed for continuous-time LTI systems, which could estimate singular with a pre-assigned convergence rate. While [12] sought to construct distributed observers with the least dimension, a kind of networked Luenberger observers with a dimension equal to the state of the system was proposed for LTI systems in [13–15]. Particularly, under the joint observability assumption, the distributed observers achieve asymptotic omniscience if the connectivity of the communication network is satisfied and the parameters are chosen properly. In [16], the authors borrowed the idea of observability decomposition from [17], where the observable part was estimated locally, and the unobservable part was obtained by reaching a consensus with its neighboring observers. The idea of observability decomposition was employed by [18] for finite time distributed observer and was extended to agent-wise detectability decomposition for a kind of completely decentralized design method for the distributed observer. Recently, Yang et al. designed a kind of distributed observer with the consideration of unknown input in [19] and switching communication topology in [20], respectively. In [21,22], the idea of the distributed observer was applied to solve the output regulation problem. In [23], the scheme of the distributed observer was employed in the formation trajectory tracking problem of a leader-following multi-AUV system.

Although distributed estimation provides us with unprecedented opportunities, challenges along with the introduction of communication network call for our further studies. While communication network is often imperfect in distributed systems, issues, such as limited bandwidth, time delay, and noises, are inevitable to degrade the performance of the networked systems. The primary focus of this paper is to examine the impact of communication noise on distributed observers, a topic that has not yet been thoroughly investigated in the existing literature. In networked systems, we can categorize noises into two classes, namely non-state-dependent (additive) noises and state-dependent (multiplicative) noises [24,25]. The description of state-dependent noises captures the fact that communication noises' intensities are time-varying and dependent on the relative states of the communicating nodes. The state-dependent noises are particularly suitable for representing communication via analog-fading channels, where the uncertainties in the measured states were affected by the states of the system [26]. Considering these two kinds of noises appears to be important in network systems, for example, the networked observers in this paper, to ensure their performance. However, little work has been devoted to distributed estimation with either kind of noise.

In this paper, the state estimation problem for general continuous-time linear systems by distributed networked observers is studied. The stability of observer error systems is analyzed by a low-gain feedback method. The gain matrix of the observer is determined by solving a parametric ARE. Moreover, based on the stability theory of stochastic systems, the distributed observers for state omniscience subjecting to non-state-dependent noises and state-dependent are analyzed in almost sure sense and mean square sense, respectively. The contributions of this paper are summarized as follows:

1. Distributed observers for state omniscience are considered with the existence of communication noises, i.e., state-dependent noises and non-state-dependent noises.
2. A framework of distributed observers is proposed, and sufficient conditions to confront noises are derived based on parametric ARE.

The structure of this paper is outlined as follows: Section 2 will provide an overview of essential background information and the problem formulation. Section 3 will delve into the intricate design of the distributed observers for linear systems with two different types of noises. Following that, Section 4 will report the outcomes of the simulations conducted. Finally, Section 5 will provide concluding remarks.

2. Preliminaries and Problem Formulation

2.1. Graph Theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph with the node set $\mathcal{V} = \{1, \ldots, N\}$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge in $\mathcal{E}$ is denoted by an unordered pair of distinct nodes (i, j), and $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$. The neighbor set of node i is denoted by $N_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. $G = [g_{ij}] \in \mathbb{R}^{N \times N}$ represents the adjacency matrix associated with $\mathcal{G}$, where $g_{ij} = g_{ji} > 0$ if $(i, j) \in \mathcal{E}$ and $g_{ij} = g_{ji} = 0$ otherwise. Correspondingly, the Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ is defined as $l_{ii} = \sum_{k \in N_i} g_{ik}$ and $l_{ij} = -g_{ij}$ for $j \neq i$.

Lemma 1 ([27])**.** *For an undirected graph $\mathcal{G}$ with N nodes, its eigenvalues of the Laplacian matrix L are real and are arranged in an ascending order as $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$. Moreover, if $\mathcal{G}$ is connected, $\lambda_1 = 0$ is a simple eigenvalue of L with corresponding eigenvector 1_N.*

2.2. Stability of Stochastic Differential Equations

Consider the following stochastic differential equation of Itô form

$$dx = f(t, x)dt + g(t, x)dB, \ x(0) = x_0 \tag{1}$$

where $x \in \Omega \subseteq \mathbb{R}^m$ is the state of the system, $f : \mathbb{R}^+ \cup \{0\} \times \Omega \mapsto \mathbb{R}^m$, $g : \Omega \times \mathbb{R}^+ \cup \{0\} \mapsto \mathbb{R}^{m \times p}$ are nonlinear functions and B is a Brownian motion. Assume that Ω is an invariant set for system (1), for example, for $x_0 \in \Omega$, $x(t)$ of (1) exists and $x(t) \in \Omega$. In addition, $f(0, t) = g(0, t) = 0$ and $x = 0$ is the trivial solution of (1).

Let $V(t, x) \in \mathcal{C}^{2 \times 1} : \Omega \times \mathbb{R}^+ \cup \{0\} \mapsto \mathbb{R}^+$ be a nonnegative function. Then, the stochastic differential of Itô process $V(t, x)$ is given by

$$dV(t, x) = \mathcal{L}V(t, x)dt + V_x(t, x)g(t, x)dB \tag{2}$$

where $\mathcal{L}V(t, x) = V_t(t, x) + V_x(t, x)f(t, x) + \frac{1}{2}tr(g^T(t, x)V_{xx}(t, x)g(t, x))$.

Lemma 2. *For a nonnegative function $V(t, x) \in \mathcal{C}^{2 \times 1}$ and constants $p > 0$, $c_1 > 0$, $c_2 \in \mathbb{R}$, $c_3 \geq 0$, such that $\forall x \neq 0$ and $t \in \mathbb{R}^+$: (a) $c_1|x|^p \leq V(t, x)^p$; (b) $\mathcal{L}V(t, x) \leq c_2 V(t, x)$; $|V_x(t, x)g(t, x)|^2 \geq c_3 V(t, x)^2$. Then: $\lim_{t \to \infty} \sup \frac{1}{t} \log(|x(t)|) \leq -\frac{c_3 - 2c_2}{p}$, a.s. In particular, if $c_3 > 2c_2$, the trivial solution of (1) is almost surely exponentially stable.*

2.3. Problem Formulation

The estimation problem is introduced as follows. For a linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix} x = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}, \end{aligned} \tag{3}$$

where $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ is the state to be estimated, $y \in \mathbb{R}^q$ denotes the measurement output, and $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{q \times n}$ are system matrix and output matrix of the system, respectively. Denote y as $y = [y_1^T, \ldots, y_N^T]^T$ by letting $C = [C_1^T, \ldots, C_N^T]^T$, where $C_i \in \mathbb{R}^{q_i \times n}$ and $y_i \in \mathbb{R}^{q_i}$.

This paper will design N networked observers so that each one of them estimates the state of (3) by communicating with its neighboring observers. The primary difficulty in tackling this distributed estimation problem arises from the constraints outlined below: (i) The ith local observer only measures part of the output y_i, which is insufficient to estimate full state x individually and (ii) the exchange of information between observers is susceptible to noise, potentially rendering the exchanged information invalid. To achieve the aim of the paper, a framework of distributed observers is proposed as follows:

$$\dot{\hat{x}}_i = A\hat{x}_i + H_i(y_i - C_i\hat{x}_i) + k\sum_{j\in N_i} g_{ij}(\check{x}_{ij}(t) - \hat{x}_i(t)) \tag{4}$$

where $\hat{x}_i = [\hat{x}_{i1}, \dots, \hat{x}_{in}]^T \in \mathbb{R}^n$ is the state of the ith observer aiming to estimate x in (3). $H_i \in \mathbb{R}^{n\times q_i}$ is the gain matrix and $k > 0$ is the coupling strength to be designed. Note that $\check{x}_{ij}$ is the perturbed version of $\hat{x}_j$, $j \in N_i$, which takes the following two forms.

$$\check{x}_{ij}(t) = \hat{x}_j(t) + \alpha\delta_{ij}(t, \hat{x}_i(t) - \hat{x}_j(t))\omega_{ij}(t) \tag{5a}$$

$$\check{x}_{ij}(t) = \hat{x}_j(t) + \alpha\omega_j(t) \tag{5b}$$

where $\alpha \neq 0$; ω_{ij} and $\omega_j \in \mathbb{R}^n$ are independent m-dimensional white noise processes. $\delta_{ij} \in \mathbb{R}^{n\times m}$ depends on the state of the network and features how noises diffuse in the network. We assume that $\delta_{ij} = 0$ for $j \notin N_i$.

In (5a), the magnitudes of noises are linked to the relative states between the communicating nodes, and these are referred to as state-dependent noises. In (5b), the noises are induced by nodes and uncorrelated with the states of the nodes, which are called non-state-dependent noises. Both cases will be analyzed in the following sections.

Definition 1. *The distributed observers (4) are said to fulfill state omniscience for system (3) in the almost sure sense if*

$$\lim_{t\to+\infty}\sup\frac{1}{t}\log(|e_i(t)|) < 0 \;\; a.s. \;\; \forall i = 1, \dots, N.$$

where $e_i = \hat{x}_i - x$ is the observation error.

Definition 2. *The distributed observers (4) are said to fulfill state omniscience for system (3) in the mean square sense if there are $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$ so that*

$$\sum_{i=1}^{N}\mathbb{E}\{\|e_i(t)\|^2\} \le c_1 e^{-c_2 t}\sum_{i=1}^{N}\mathbb{E}\{\|e_i(0)\|^2\} + c_3.$$

Assumption 1. *There is a constant K_δ such that $\|\delta_{ij}(t,x)\| \le K_\delta|x|$ for all $x \in \mathbb{R}^n$.*

Assumption 2. *The pair (A, C) is observable.*

Remark 1. *Assumption 1 imposes limitations on the diffusion function δ_{ij}, as it is reasonable to expect that noise would spread in the communication network with limited intensity. In this paper, none of the observers is required to fulfill the estimation task individually, i.e., (A, C_i) is not necessarily observable. Under Assumption 2, each observer estimates part of the states of the system (3) locally and communicates with other observers through the connected graph for the remaining part of the states. Therefore, observers need to work in a coordinated mode to achieve state omniscience.*

Before presenting the method for designing distributed observers, we introduce an important technical lemma for the parameterized ARE:

$$PA^T + AP - PC^TCP = -\varepsilon P \tag{6}$$

where $\varepsilon > 0$ is a positive constant.

Lemma 3 ([28])**.** *Under Assumption 2, there is a unique positive definite matrix $P(\epsilon) = W^{-1}(\epsilon)$ solving ARE (6), and $W(\epsilon) > 0$ is the unique solution to*

$$W\left(A + \frac{\varepsilon}{2} I_n\right) + \left(A + \frac{\varepsilon}{2} I_n\right)^T W = C^T C. \tag{7}$$

Moreover, the solution matrix $P(\varepsilon)$ has the following properties:

- $tr(CP(\varepsilon)C^T) = 2tr(A) + n\varepsilon$;
- $P(\varepsilon)C^T CP(\varepsilon) \leq (2tr(A) + n\varepsilon)P(\varepsilon)$.

To simplify the derivation of this paper, we only consider the case that $tr(A) = 0$ in the following analysis. The case of $tr(A) \neq 0$ can be derived by extending the results in this paper.

Lemma 4. *(Schur Complement): For a given symmetric matrix $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$ with $M_{11} = M_{11}^T$ and $M_{22} = M_{22}^T$, $M < 0$ is equivalent to $M_{11} < 0$, $M_{22} - M_{12}^T M_{11}^{-1} M_{12} < 0$.*

3. Distributed Observer Design under State-Dependent Noises

In this section, a distributed observer with state-dependent noises will be analyzed, and sufficient conditions for stochastic state omniscience in an almost sure sense are given.

Combining (4) and (5a) yields the closed-loop system of Itô type

$$d\hat{x}_i = \left[A\hat{x}_i + H_i(y_i - C_i\hat{x}_i) + k\sum_{j\in N_i} g_{ij}(\hat{x}_j - \hat{x}_i)\right]dt + k\alpha \sum_{j\in N_i} \delta_{ij}(t, \hat{x}_i - \hat{x}_j)db_{ij} \tag{8}$$

where b_{ij} represents the Brownian motion that is associated with the white noise process ω_{ij}.

Let $\hat{X} = [\hat{x}_1^T, \ldots, \hat{x}_N^T]^T$, $X = 1_{N\times 1} \otimes x$, $\bar{C} = \text{Diag}\{C_1, C_2, \ldots, C_N\}$, $\bar{H} = \text{Diag}\{H_1, H_2, \ldots, H_N\}$, $B = [b_1^T, \ldots, b_N^T]^T$ is a mN^2-dimensional vector, $b_i = [b_{i1}^T, \ldots, b_{iN}^T]^T$, and $\Delta(t, \hat{X}) = \text{Diag}(\Delta_i(t, \hat{X}))$, with $\Delta_i = [\delta_{i1}(t, \hat{x}_i - \hat{x}_1), \ldots, \delta_{iN}(t, \hat{x}_i - \hat{x}_N)]$. Therefore, we can get the compact form of the systems:

$$d\hat{X} = \left[(I_N \otimes A)\hat{X} + \bar{H}\bar{C}(X - \hat{X}) - k(L \otimes I_n)\hat{X}\right]dt + k\alpha\Delta(t, \hat{X})dB \tag{9}$$

Denote the whole observation error by $e = [e_1^T, \ldots, e_N^T]^T$, then we have

$$de = F(e, t)dt + \tilde{\Delta}(t, \hat{X})dB \tag{10}$$

where $F(e) = [(I_N \otimes A) - \bar{H}\bar{C} - k(L \otimes I_n)]e$ and $\tilde{\Delta}(t, \hat{X}) = k\alpha\Delta(t, \hat{X})$.

Theorem 1. *Under an undirected connected graph and Assumptions 1 and 2, consider the distributed observers (4) where state-dependent noises (5a) exist. Then, the state omniscience for system (3) can be achieved in almost sure sense if H_i and k are chosen as*

$$H_i = \frac{1}{2}NP(\varepsilon)C_i^T,\ k > \frac{\rho(\varepsilon)}{2\lambda_2} \tag{11}$$

where $\rho(\varepsilon) = \eta + (n-1)\varepsilon + \frac{1}{\varepsilon - \eta}[(nN+1)^2 + n^2 - 2n + 2Nn^2]\varepsilon^2$ and $\frac{\bar{\lambda}(P)}{\underline{\lambda}(P)}\lambda_N(k\alpha K_\delta)^2 < \eta < \varepsilon$.

Proof. According to Lemma 2, to prove the distributed observers for state omniscience in an almost sure sense, we need to prove the almost sure exponential stability of the observation error e. To this aim, construct the Lyapunov candidate function as

$$V(t) = \frac{1}{2}e^T(I_N \otimes P^{-1})e \tag{12}$$

where P is the unique solution to ARE (6) and the ARE is solvable under Assumption 2. □

In light of Lemma 2, we need to calculate $\mathcal{L}V(e)$ and show $\mathcal{L}V(e) \leq -cV(e)$ for some $c > 0$. By the definition of infinitesimal operator $\mathcal{L}V(\cdot)$, we have

$$\mathcal{L}V(e) = V_t(e) + V_e(e)F(e) + \frac{1}{2}tr(\tilde{\Delta}(t,e)^T V_{ee}\tilde{\Delta}(t,e)) \tag{13}$$

Note that $V_t(e) = 0$, then for the second term, we have

$$\begin{aligned} V_e(e)F(e) =& e^T(I_N \otimes P^{-1})[(I_N \otimes A) - \bar{H}\bar{C} - k(L \otimes I_n)]e \\ =& \frac{1}{2}e^T\Pi(k)e \end{aligned} \tag{14}$$

where

$$\begin{aligned} \Pi(k) =& I_N \otimes (A^TP^{-1} + P^{-1}A) - N\bar{C}^T\bar{C} - 2kL \otimes P^{-1} \\ =& \bar{\Upsilon} - 2kL \otimes P^{-1} \end{aligned}$$

with $\bar{\Upsilon} = \text{Diag}\{\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_N\}$ and $\Upsilon_i = A^TP^{-1} + P^{-1}A - NC_i^TC_i$.

The forthcoming explanation will demonstrate the existence of a positive value k such that

$$\Pi(k) < -\eta(I_N \otimes P^{-1}) \tag{15}$$

For an undirected connected graph, we have $U_0 = \frac{1}{\sqrt{N}}1_N$ being the left eigenvector of the Laplacian matrix L that is associated with the zero eigenvalues. Denote the orthogonal matrix $U = [U_0\ U_1]$, then

$$U^TLU = \text{Diag}\{0, \hat{L}\}$$

with $\hat{L} = \text{Diag}\{\lambda_2, \ldots, \lambda_N\}$. Therefore, (15) can be written as

$$(U^T \otimes I_n)\Pi(k)(U \otimes I_n) < -\eta(I_N \otimes P^{-1}) \tag{16}$$

By using Lemma 1 and Assumption 2, we have

$$\begin{aligned} (U_0^T \otimes I_n)\bar{\Upsilon}(U_0 \otimes I_n) =& \frac{1}{N}\sum_{i=1}^{N}\Upsilon_i \\ =& \frac{1}{N}(N(A^TP^{-1} + P^{-1}A) - NC^TC) = -\varepsilon P^{-1} \end{aligned} \tag{17}$$

Note that $U = [U_0\ U_1]$, we can rewrite (16) in block matrices as follows

$$\begin{pmatrix} (\eta - \varepsilon)P^{-1} & \frac{1}{\sqrt{N}}(1_N^T \otimes I_n)\bar{\Upsilon}(U_1 \otimes I_n) \\ * & (U_1^T \otimes I_n)\bar{\Upsilon}(U_1 \otimes I_n) + (\eta I_{N-1} - k\hat{L}) \otimes P^{-1} \end{pmatrix} < 0. \tag{18}$$

According to Lemma 4 , with $\eta < \varepsilon$, the inequality can be analyzed by

$$\begin{aligned} \Psi =& (U_1^T \otimes I_n)\bar{\Upsilon}(U_1 \otimes I_n) + (\eta I_{N-1} - k\hat{L}) \otimes P^{-1} \\ &+ \frac{1}{(\varepsilon - \eta)N}(U_1^T \otimes I_n)\bar{\Upsilon}(1_N1_N^T \otimes P)\bar{\Upsilon}(U_1 \otimes I_n) < 0. \end{aligned} \tag{19}$$

According to Lemma 3, we have

$$\Upsilon_i = -\varepsilon P^{-1} + C^T C - N C_i^T C_i \leq -\varepsilon P^{-1} + C^T C \leq (n-1)\varepsilon P^{-1},$$

and, therefore, $\Upsilon_i \leq (n-1)\varepsilon(I_N \otimes P^{-1})$.

Since $U_1^T U_1 = I_{N-1}$, we have

$$(U_1^T \otimes I_n)\bar{\Upsilon}(U_1 \otimes I_n) \leq (n\varepsilon - \varepsilon)\left(I_{N-1} \otimes P^{-1}\right)$$

By Lemma 3, we have $\mathrm{tr}(C_i P C_i^T) \leq \mathrm{tr}(C P C^T) = n\varepsilon$ and $C_i^T C_i \leq C^T C \leq n\varepsilon P^{-1}$, and then $C_i^T C_i P C_i^T C_i \leq C_i^T \mathrm{tr}(C_i P C_i^T) C_i \leq (n\varepsilon) C_i^T C_i \leq (n\varepsilon)^2 P^{-1}$.

Hence, we have

$$\begin{aligned}
&\Upsilon_i P \Upsilon_i \\
=&\varepsilon^2 P^{-1} - 2\varepsilon C^T C + 2\varepsilon N C_i^T C_i + C^T C P C^T C \\
&- N C^T C P C_i^T C_i - N C_i^T C_i P C^T C + N^2 C_i^T C_i P C_i^T C_i \\
\leq&\varepsilon^2 P^{-1} + 2\varepsilon(N-1) C^T C + (N+1) C^T C P C^T C \\
&+ (N^2+N) C_i^T C_i P C_i^T C_i \\
\leq&((nN+1)^2 + n^2 - 2n + 2Nn^2)\varepsilon^2 P^{-1}.
\end{aligned} \tag{20}$$

Thus, it comes to

$$\begin{aligned}
&(U_1^T \otimes I_n)\bar{\Upsilon}(1_N 1_N^T \otimes P)\bar{\Upsilon}(U_1 \otimes I_n) \\
\leq&N(U_1^T \otimes I_n)\bar{\Upsilon}(I_N \otimes P)\bar{\Upsilon}(U_1 \otimes I_n) \\
\leq&N((nN+1)^2 + n^2 - 2n + 2Nn^2)\varepsilon^2\left(I_{N-1} \otimes P^{-1}\right)
\end{aligned}$$

In sum, we have

$$\Psi \leq -(2k\lambda_2 - \rho(\varepsilon))(I_{N-1} \otimes P^{-1})$$

and, therefore,

$$V_e(e)F(e) \leq -2\eta V(e). \tag{21}$$

Therefore, by (11), we have $V_e F(e) \leq -\eta e^T(I_n \otimes P^{-1})e$.

For the third term in $\mathcal{L}V(e)$, we have $V_{ee} = I_N \otimes P^{-1}$, then

$$\begin{aligned}
&\frac{1}{2} tr(\tilde{\Delta}(t,e)^T V_{ee} \tilde{\Delta}(t,e)) \\
=&\frac{1}{2} tr(\tilde{\Delta}(t,e)^T (I_N \otimes P^{-1})\tilde{\Delta}(t,e)) \\
\leq&\frac{1}{2}(k\alpha)^2 \sum_{i,j=1}^{N} tr(\delta_{ij}(t, e_i - e_j)^T P^{-1} \delta_{ij}(t, e_i - e_j))
\end{aligned}$$

Recalling Assumption 1 and $\delta_{ij} = 0$ for $j \notin N_i$, we have

$$\begin{aligned}
&\frac{1}{2} tr(\tilde{\Delta}(t,e)^T V_{ee} \tilde{\Delta}(t,e)) \\
\leq&\frac{(k\alpha)^2}{2\underline{\lambda}(P)} K_\delta^2 \sum_{i=1}^{N} \sum_{j \in N_i} (e_i - e_j)^T (e_i - e_j) \\
=&\frac{(k\alpha)^2}{\underline{\lambda}(P)} K_\delta^2 e^T (L \otimes I_n) e \\
\leq&\frac{2(k\alpha)^2 \bar{\lambda}(P)}{\underline{\lambda}(P)} K_\delta^2 \lambda_N V(e)
\end{aligned} \tag{22}$$

Upon combining Equations (21) and (22), we arrive at

$$\mathcal{L}V(e) \leq -2\left(\eta - \frac{(k\alpha)^2\bar{\lambda}(P)}{\underline{\lambda}(P)}K_\delta^2\lambda_N\right)V(e) \leq -cV(e) \tag{23}$$

where $c > 0$. In accordance with Lemma 2, if $c_3 = 0$, the state omniscience for system (3) can be achieved in an almost sure sense.

Remark 2. *The sufficient conditions in Theorem 1 imply* $\frac{\bar{\lambda}(P)}{\underline{\lambda}(P)}\lambda_N(k\alpha K_\delta)^2 < \varepsilon$, *where* ε *is a parameter in ARE (6). In other words, the low bound of* ε *is jointly determined by the strength of noises* α, *the topology structure* L, *and the coupling strength* k.

Remark 3. *To implement the distributed observers in Theorem 1, the following steps can be followed.*

1. *Decide the number of observers according to the output matrix in (3).*
2. *By the knowledge of noises in (5a),* α *and* K_δ *are obtained.*
3. *Decide* ε *by the condition in Theorem 1, and solve the ARE (6) for* P.
4. *Calculate* H_i *and* k *by (11).*
5. *Construct the distributed observer in the form of (4).*

4. Distributed Observer Design under Non-State-Dependent Noises

In this section, distributed observers with node-induced noises will be analyzed, and sufficient conditions for stochastic state omniscience in a mean square sense are given.

Combining (4) and (5b) yields the closed-loop system of Itô type

$$d\hat{x}_i = \left[A\hat{x}_i + H_i(y_i - C_i\hat{x}_i) + k\sum_{j\in N_i} g_{ij}(\hat{x}_j - \hat{x}_i)\right]dt + k\alpha\sum_{j\in N_i} g_{ij}db_j \tag{24}$$

where b_j represents the Brownian motion that is linked to the white noise process ω_j.

The dynamics of the observation error could be written, with $F(t,e)$ given in (10), as follows

$$de = F(e,t)dt + \alpha k(G \otimes I_n)dB \tag{25}$$

where $B = [b_1^T, \ldots, b_N^T]$ is a mN-dimensional vector.

Now we can present another of our main results.

Theorem 2. *Under an undirected connected graph and Assumptions 1 and 2, consider the distributed observers (4) where non-state-dependent noises (5b) exist. Then, the state omniscience for system (3) can be achieved in mean square sense if* H_i *and* k *satisfy*

$$H_i = NP(\varepsilon)C_i^T,\ k > \frac{\rho(\varepsilon)}{\lambda_2} \tag{26}$$

where $\rho(\varepsilon) = \eta + (n-1)\varepsilon - \frac{1}{\eta-\varepsilon}[(nN+1)^2 + n^2 - 2n + 2Nn^2]\varepsilon^2$ *and* $\eta < \epsilon$.

Proof. By selecting the same Lyapunov function as Theorem 1, we can obtain that

$$\mathcal{L}V(e) \leq -2\eta V(e) + \frac{\alpha^2k^2}{2}tr\left((G\otimes I_n)^T V_{ee}(G\otimes I_n)\right) \tag{27}$$

Recall that $V_{ee} = I_N \otimes P^{-1}$, we have

$$
\begin{aligned}
\mathcal{L}V(e) \leq & -2\eta V(e) + \frac{\alpha^2 k^2}{2} tr(G^T G \otimes P^{-1}) \\
\leq & -2\eta V(e) + \frac{\alpha^2 k^2}{2} tr\big((G \otimes P^{-\frac{1}{2}})^T (G \otimes P^{-\frac{1}{2}})\big) \\
\leq & -2\eta V(e) + \frac{\alpha^2 k^2}{2} \|(G \otimes P^{-\frac{1}{2}})\|_F^2 \\
\leq & -2\eta V(e) + \varpi
\end{aligned} \tag{28}
$$

where $\varpi = \frac{\alpha^2 k^2}{2} \|(G \otimes P^{-\frac{1}{2}})\|_F^2$ is bounded.

Applying the Itô differential formula, the stochastic differential of $V(t)$ is

$$
dV(e) = \mathcal{L}V(t)dt + \alpha k e^T (G \otimes P^{-1}) dB \tag{29}
$$

Taking expectation of (29), we have

$$
\frac{\mathbb{E}\{dV(t)\}}{dt} = \mathbb{E}\{\mathcal{L}V(t)\} \leq -2\eta \mathbb{E}\{V(t)\} + \varpi \tag{30}
$$

According to the Comparison principle [29], we have

$$
\begin{aligned}
\mathbb{E}\{V(t)\} \leq & e^{-2\eta t} \mathbb{E}\{V(0)\} + \frac{1 - e^{-2\eta}}{2\eta} \varpi \\
\leq & e^{-2\eta t} \mathbb{E}\{V(0)\} + \frac{\varpi}{2\eta}
\end{aligned} \tag{31}
$$

In light of Definition 2, the distributed observers fulfill state omniscience in the mean square sense. □

Remark 4. *Theorems 1 and 2 examine two types of noise in the communication network of distributed observers. While state omniscience is in an almost sure sense for the case of state-dependent noises, it can only achieve state omniscience in a mean square sense for the case of non-state-dependent noises. This is consistent with our understanding of non-state-dependent noise, as it will not disappear due to changes in the states of observers. Based on the findings of these theorems, it is possible to design distributed observers using equation (4) irrespective of the type of noise involved. Specifically, we can choose H_i according to (11) and select a large k to meet conditions in both theorems.*

In both Theorems 1 and 2, we assume that noises only occur when the two nodes are connected. However, since noises appear randomly in the networked systems, it could be interesting to explore the situation that the noisy topology differs from the communication topology. We can build a two-layer graph to describe the heterogeneous topologies, where the second layer regarding to the noisy topology is described as follows.

Let a_{ij}^n be the elements of the adjacency matrix $\mathcal{A}^n = [a_{ij}^n] \in \mathbb{R}^{N \times N}$ representing the noisy coupling topology $\mathcal{G}^n = (\mathcal{V}, \mathcal{E}^n)$. In such a noisy topology $\mathcal{G}^n = (\mathcal{V}, \mathcal{E}^n)$, $N_i^n = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}^n\}$ is the neighbor set of node i, and L^n is the Laplacian matrix associated with the noisy topology.

Corollary 1. *Under an undirected connected graph and Assumption 1, consider the distributed observers (4) where state-dependent noises (5a) exist. Then, the distributed observers achieve stochastic state omniscience if H_i and k are designed as (11) and $\alpha \leq \sqrt{\frac{\eta \underline{\lambda}(P)}{k^2 K_\delta^2 \lambda_N(L^n) \bar{\lambda}(P)}}$, where $\lambda_N(L^n)$ is the largest eigenvalue of L^n.*

In this case, the assumption that $\delta_{ij} = 0$ for $j \in N_i$ does not hold. The proof for Corollary 1 follows that of Theorem 1, but replacing L in (22) with L^n defined in the corollary statement.

5. Numerical Example

In this section, we consider the model of spacecraft formation flying in the low Earth orbit [30]. The simulation will be conducted using the Euler–Maruyama method. The sample period is 3×10^{-13} s. The system matrix is given as

$$A = \begin{bmatrix} 0_3 & I_3 \\ A_1 & A_2 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3\omega^2 & 0 \\ 0 & 0 & -\omega^2 \end{bmatrix} A_2 = \begin{bmatrix} 0 & 2\omega & 0 \\ -2\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{32}$$

where the angular rate of the virtual satellite ω is set to 0.001, the state variable x_1, x_2, and x_3 are the position components and x_4, x_5, and x_6 are the corresponding velocities. Due to the velocities are often unmeasurable, the measurement matrices are set to $C_1 = [1, 0, 0, 0, 0]$, $C_2 = [0, 1, 0, 0, 0, 0]$, and $C_3 = [0, 0, 1, 0, 0, 0]$. Apparently, (A, C) is observable and none of (A, C_i) is observable. The Laplacian matrix is

$$L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

The solution to the ARE (6) with $\varepsilon = 0.05$ is

$$P(\varepsilon) = 10^{-2} \begin{bmatrix} 15.14 & 4.78 & 0 & 0.88 & -0.19 & 0 \\ 04.78 & 4.86 & 0 & 0.91 & 0.11 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0.25 \\ 0.88 & 0.91 & 0 & 0.17 & 0.02 & 0 \\ -0.19 & 0.11 & 0 & 0.02 & 0.03 & 0 \\ 0 & 0 & 0.25 & 0 & 0 & 0.11 \end{bmatrix}.$$

For the case of state-dependent noises, we let the noise diffusing function $\delta_{ij}(t, x) = x$ and $\alpha = 0.05$. Then, we have $\rho(\varepsilon) = 150.54$ in (18) with $\eta = 0.04$, which gives the the bound for the coupling strength $k > 50.18$. Let $k = 50.28$ and calculate H_i by (18); the trajectories of the distributed observers are depicted in Figure 1, and observer errors are shown in Figure 2. According to these results, the distributed observers fulfill state omniscience in almost sure sense as defined in Definition 2. For the case of non-state-dependent noises, calculate k and H_i by Theorem 2, the trajectories of the distributed observers are shown in Figure 3 and the observer errors in the form of 2-norm are plotted in Figure 4, which imply the distributed observers fulfill state omniscience in a mean square sense as defined in Definition 2.

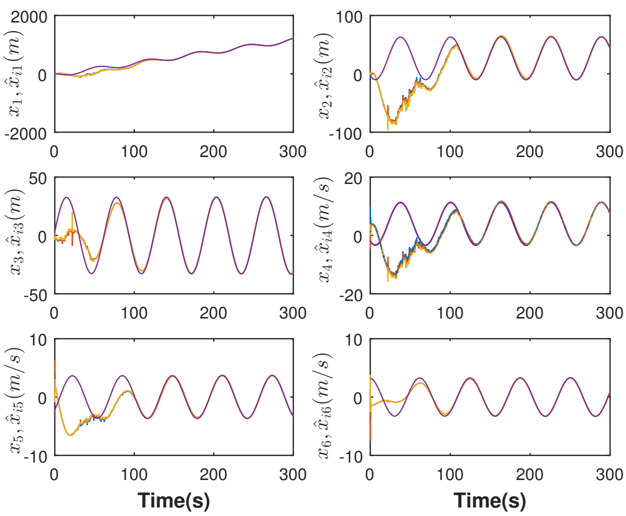

Figure 1. Trajectories of the plant and the distributed observers with state-dependent noises.

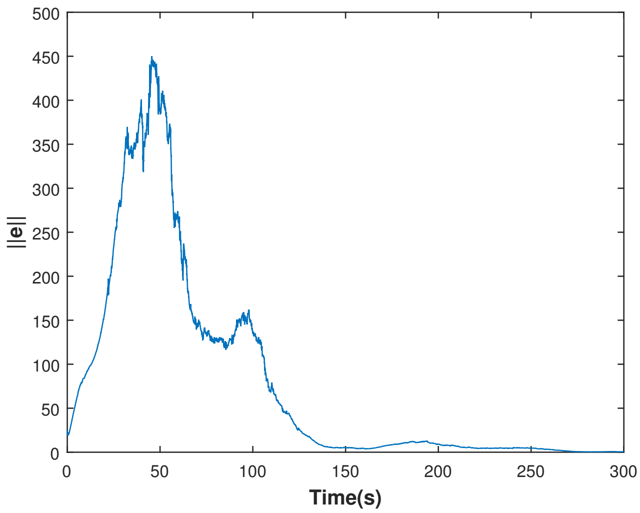

Figure 2. Observer errors with state-dependent noises.

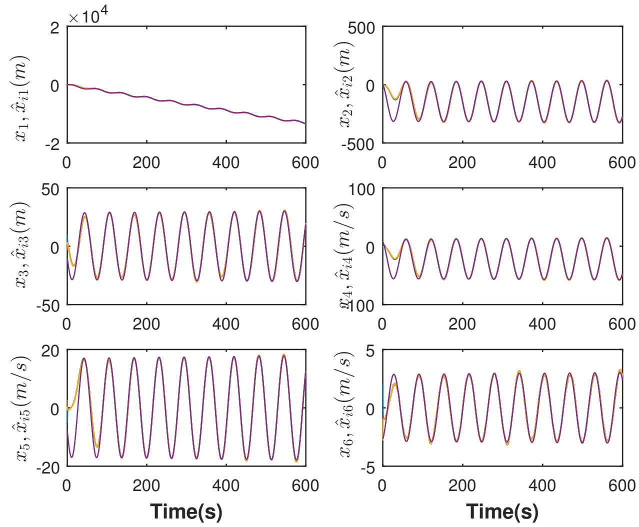

Figure 3. Trajectories of the plant and the distributed observers with non-state-dependent noises.

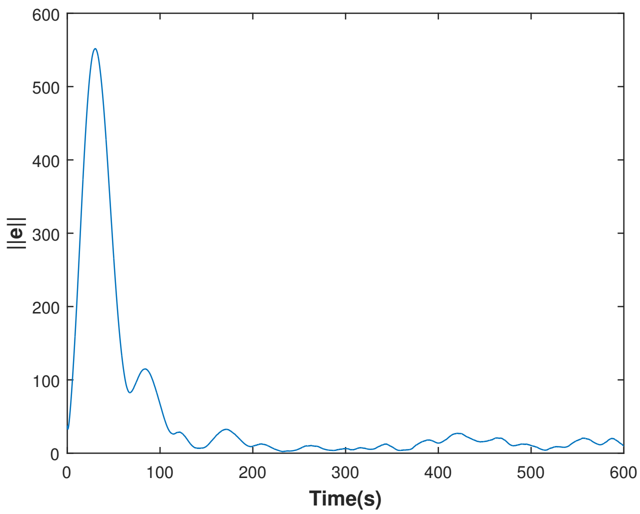

Figure 4. Observer errors with non-state-dependent noises.

6. Conclusions

This paper addresses the issue of distributed state estimation for linear time-invariant plants with continuous-time dynamics, considering the presence of communication noise. The distributed observers proposed in this paper are composed of networked local observers. Each local observer generates a local state estimation by utilizing its own output measurement and the estimations of its neighbors, as determined by a connected communication graph. Notably, we have considered two types of communication noises: state-dependent noises and non-state-dependent noises. By solving a parametric ARE, the coupling strength and the gain matrices are designed properly. It is demonstrated that, assuming joint observability and connectivity of the communication graph, the resulting observer operates in a coordinated manner to attain state omniscience even in the presence of communication noises.

It is important to acknowledge that only white noise is considered, and the assumption of the diffuse function is quite strong, which may limit the application of the obtained results. Additionally, the simplified model used in the paper may not capture important practical considerations such as nonlinear plant dynamics, transmission errors, communication delays, and directed communication graphs. We believe that addressing these practical considerations would be valuable for advancing the field of distributed observer design and improving the performance of distributed estimation in complex systems.

Author Contributions: Methodology, J.W.; Software, Z.Z.; Validation, X.Z.; Investigation, X.Z.; Writing—original draft, K.C. and Z.Z.; Writing—review & editing, K.C., Z.Z. and X.Z.; Supervision, J.W.; Project administration, J.W.; Funding acquisition, K.C. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported in part by the National Natural Science Foundation of China under Grants 62103115 and Grants 11771102, in part by the Guangdong Basic and Applied Basic Research Foundation under Grants 2021A1515011636, 2023A1515011025, in part by the basic and applied basic research projects jointly funded by Guangzhou and schools (colleges) under Grant 202201020233, and in part by the Project of Guangzhou Science and Technology Plan under Grant 202102020313.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

x	State to be estimated.
x_i	State of the ith observers.
y_i	Partial measurement output of the ith observers.
n	Dimensions of x.
N	Number of observers.
A, C	System matrix, output matrix.
$\mathcal{G}, \mathcal{V}, \mathcal{E}$	Undirected graph, node set, edge set.
g_{ij}	Adjacency weight between node i and node j.
N_i	Neighbor set of node i.
L	Laplacian matrix.
λ_i	The ith eigenvalue of L.
$dV(\cdot)$	Stochastic differential.
$\mathcal{L}V(\cdot)$	Infinitesimal operator.
$\check{x}_{ij}$	Noise perturbed state of the jth observer, $j \in N_i$.
H_i	Gain matrix of the ith observer.
k	Coupling strength of observers.
ω_i, ω_{ij}	White noise processes.
b_i, b_{ij}	Brownian motions associate with ω_i, ω_{ij} .
α	Noise intensity.
$\delta_{ij}(\cdot)$	Noise diffuse function.

e_i	Observation error of the ith observer.
$\mathbb{E}(\cdot)$	Mathematical expectation.
$\mathrm{tr}(\cdot)$	Trace of a matrix.
$\rho(\varepsilon)$	Function of ε for brief expression.
$\bar{\lambda}(\cdot), \underline{\lambda}(\cdot)$	Maximum and minimum eigenvalue of a matrix.
η	An intermediate constant.

References

1. Roman, R.C.; Precup, R.E.; Petriu, E.M. Hybrid data-driven fuzzy active disturbance rejection control for tower crane systems. *Eur. J. Control* **2021**, *58*, 373–387. [CrossRef]
2. Chi, R.; Li, H.; Shen, D.; Hou, Z.; Huang, B. Enhanced P-type control: Indirect adaptive learning from set-point updates. *IEEE Trans. Autom. Control* **2023**, *68*, 1600–1613. [CrossRef]
3. Xiao, L.; Boyd, S.; Lall, S. A scheme for robust distributed sensor fusion based on average consensus. In Proceedings of the IEEE Fourth International Symposium on Information Processing in Sensor Networks, Boise, ID, USA, 15 April 2005; pp. 63–70.
4. Khan, U.A.; Moura, J.M. Distributing the Kalman filter for large-scale systems. *IEEE Trans. Signal Process.* **2008**, *56*, 4919–4935. [CrossRef]
5. Subbotin, M.V.; Smith, R.S. Design of distributed decentralized estimators for formations with fixed and stochastic communication topologies. *Automatica* **2009**, *45*, 2491–2501. [CrossRef]
6. Viegas, D.; Batista, P.; Oliveira, P.; Silvestre, C. Decentralized H_2 observers for position and velocity estimation in vehicle formations with fixed topologies. *Syst. Control Lett.* **2012**, *61*, 443–453. [CrossRef]
7. Olfati-Saber, R. Distributed Kalman filter with embedded consensus filters. In Proceedings of the IEEE 44th IEEE Conference on Decision and Control, Seville, Spain, 15 December 2005; pp. 8179–8184.
8. Khan, U.A.; Jadbabaie, A. On the stability and optimality of distributed Kalman filters with finite-time data fusion. In Proceedings of the IEEE 2011 American Control Conference, San Francisco, CA, USA, 29 June–1 July 2011; pp. 3405–3410.
9. Liu, Q.; Wang, Z.; He, X.; Zhou, D. On Kalman-consensus filtering with random link failures over sensor networks. *IEEE Trans. Autom. Control* **2018**, *63*, 2701–2708. [CrossRef]
10. Park, S.; Martins, N.C. An augmented observer for the distributed estimation problem for LTI systems. In Proceedings of the IEEE 2012 American Control Conference (ACC), Montreal, QC, Canada, 27–29 June 2012; pp. 6775–6780.
11. Park, S.; Martins, N.C. Design of distributed LTI observers for state omniscience. *IEEE Trans. Autom. Control* **2017**, *62*, 561–576. [CrossRef]
12. Wang, L.; Morse, A.S. A distributed observer for a time-invariant linear system. *IEEE Trans. Autom. Control* **2018**, *63*, 2123–2130. [CrossRef]
13. Zhu, H.; Liu, K.; Lü, J.; Lin, Z.; Chen, Y. On the cooperative observability of a continuous-time linear system on an undirected network. In Proceedings of the IEEE 2014 International Joint Conference on Neural Networks (IJCNN), Beijing, China, 6–11 July 2014; pp. 2940–2944.
14. Liu, K.; Chen, Y.; Duan, Z.; Lü, J. Cooperative output regulation of LTI plant via distributed observers with local measurement. *IEEE Trans. Cybern.* **2018**, *48*, 2181–2191. [CrossRef]
15. Liu, K.; Lü, J.; Lin, Z. Design of distributed observers in the presence of arbitrarily large communication delays. *IEEE Trans. Neural Netw. Learn. Syst.* **2018**, *29*, 4447–4461. [CrossRef]
16. Han, W.; Trentelman, H.L.; Wang, Z.; Shen, Y. A simple approach to distributed observer design for linear systems. *IEEE Trans. Autom. Control* **2019**, *64*, 329–336. [CrossRef]
17. Kim, T.; Shim, H.; Cho, D.D. Distributed Luenberger observer design. In Proceedings of the 2016 IEEE 55th Conference on Decision and Control (CDC), Las Vegas, NV, USA, 12–14 December 2016; pp. 6928–6933.
18. Silm, H.; Efimov, D.; Michiels, W.; Ushirobira, R.; Richard, J.P. A simple finite-time distributed observer design for linear time-invariant systems. *Syst. Control Lett.* **2020**, *141*, 104707. [CrossRef]
19. Yang, G.; Barboni, A.; Rezaee, H.; Parisini, T. State estimation using a network of distributed observers with unknown inputs. *Automatica* **2022**, *146*, 110631. [CrossRef]
20. Yang, G.; Rezaee, H.; Alessandri, A.; Parisini, T. State estimation using a network of distributed observers with switching communication topology. *Automatica* **2023**, *147*, 110690. [CrossRef]
21. Su, Y.; Cai, H.; Huang, J. The cooperative output regulation by the distributed observer approach. *Int. J. Netw. Dyn. Intell.* **2022**, 20–35. [CrossRef]
22. Chen, K.; Wang, J.; Zhao, Z.; Lai, G.; Lyu, Y. Output consensus of heterogeneous multiagent systems: A distributed observer-based approach. *IEEE Trans. Syst. Man, Cybern. Syst.* **2022**, *52*, 370–376. [CrossRef]
23. Yan, Z.; Zhang, C.; Tian, W.; Cai, S.; Zhao, L. Distributed observer-based formation trajectory tracking method of leader-following multi-AUV system. *Ocean Eng.* **2022**, *260*, 112019. [CrossRef]
24. Zhang, K.; Zhang, Y.; Zong, X.; Li, R. Mean square and almost sure consensus of heterogeneous multi-agent systems with multiplicative noise. *Int. J. Robust Nonlinear Control* **2022**, *32*, 2399–2419. [CrossRef]
25. Du, Y.; Wang, Y.; Zuo, Z.; Zhang, W. Event-triggered bipartite consensus for multi-agent systems subject to multiplicative and additive noises. *Appl. Math. Comput.* **2022**, *429*, 127235. [CrossRef]

26. Liu, Q.; Wang, Z.; He, X.; Zhou, D.H. Event-based distributed filtering with stochastic measurement fading. *IEEE Trans. Ind. Inform.* **2015**, *11*, 1643–1652. [CrossRef]
27. Godsil, C.; Royle, G.F. *Algebraic Graph Theory*; Springer Science & Business Media: Berlin, Germany, 2001.
28. Zhou, B.; Lin, Z.; Duan, G.R. Properties of the parametric Lyapunov equation-based low-gain design with applications in stabilization of time-delay systems. *IEEE Trans. Autom. Control* **2009**, *54*, 1698–1704. [CrossRef]
29. Kim, B.H.; Baldick, R. A comparison of distributed optimal power flow algorithms. *IEEE Trans. Power Syst.* **2000**, *15*, 599–604. [CrossRef] [PubMed]
30. Ren, W. Formation keeping and attitude alignment for multiple spacecraft through local interactions. *J. Guid. Control Dyn.* **2007**, *30*, 633–638. [CrossRef]