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Ansatz and Averaging Methods for Modeling the (Un)Conserved Complex Duffing Oscillators

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Abstract: In this study, both the ansatz and averaging methods are carried out for analyzing the complex Duffing oscillators including the undamped/conserved complex Duffing oscillator (CDO) and the damped/unconserved CDO to obtain some approximate analytical solutions. To analyze the conserved CDO, it is reduced to two decoupled conserved Duffing oscillators. After that, the exact solution of the conserved Duffing oscillator is employed to derive an approximation of the conserved CDO in terms of the Jacobi elliptic function. To analyze the damped CDO, two methodologies are considered. For the first methodology, the damped CDO is reduced to two decoupled damped Duffing oscillators, and the ansatz method is devoted to analyzing the damped Duffing oscillator. Accordingly, an approximation of the damped CDO in terms of trigonometric functions is obtained. In the second methodology, the averaging method is applied directly to the damped CDO to derive an approximation in terms of trigonometric functions. All the obtained solutions are compared with the fourth-order Runge–Kutta (RK4) numerical approximations. This study may help many researchers interested in the field of plasma physics to interpret their laboratory and observations results.

Keywords: complex Duffing oscillators; damped complex oscillator; trigonometric functions; Jacobi elliptic functions

MSC: 34C15

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1. Introduction

Numerous scholars have effectively used the theory of linear oscillations to analyze and model oscillatory devices. However, the nonlinear behavior can be found in a wide range of real applications [1–6]. Consequently, nonlinear oscillation is one of the most popular and widely researched fields due to its diverse applications in automobiles, sensing, micro- and nanoscale, fluid and solid interaction, nonlinear oscillations in plasma physics, bioengineering, and nonlinear oscillations in optics. Many different and various equations of motion have been used to model several nonlinear oscillations in different physical and engineering systems [7,8]. For instance, many nonlinear conserved and unconserved oscillators/equations were used to model the nonlinear oscillations in different plasma models such as the Duffing-type oscillators [9], the damped (non-conserved) and undamped (conserved) Helmholtz–Duffing oscillators [10], the damped (non-conserved) Helmholtz oscillator and its applications in electronic circuits and plasma physics [11], and the Helmholtz–Fangzhu oscillator [12]) and its applications in different physical and engineering systems. The Duffing-type equation is one of the most famous and important equations that succeeded in explaining many different oscillations in different engineering, physical systems, and statistical mechanics [9–12]. A complex Duffing oscillator

(CDO) is an example of the nonlinear differential equations with complex variables [13–17]. Cveticanin [14] analyzed and studied the following complex Duffing-type oscillator

$$\ddot{z} + c_1 z + 3c_3 z|z|^2 - c_3\bar{z}^3 = \epsilon G(z, \dot{z}, cc), \tag{1}$$

where $z \equiv z(t) = x + iy$ is a complex function, and $\bar{z}$ is the conjugate of z, whereas $x \equiv x(t)$ and $y \equiv y(t)$ indicate the real and imaginary parts, respectively, $z\bar{z} = |z|^2$, $G(z, \dot{z}, cc)$ represents a complex defection, "cc" indicates the complex conjugate function, and $i = \sqrt{-1}$. Cveticanin [13,14] used the elliptic Krylov–Bogolubov method (eKBM) and compared his analytical results with the numerical one. Moreover, there are many analytical methods such as the Bogolubov–Mitropolski procedure [15] and the averaging method [18,19] that have succeeded in analyzing and studying a weakly nonlinear coupled system of differential equations and deriving some approximate solutions. In Ref. [1], the authors investigated the cubic Duffing equation and presented the desired solutions in the different forms of functions such as the Jacobi and Weierstrass elliptic functions. In Ref. [2], the CDO has been studied via the Wiener–Hermite expansion. Moreover, both the multiple scales method (MSM) and the Krylov–Bogoliubov–Mitropolsky (KBM) method were applied to find some approximations to a generalized damped forced CDO [16]. In the current study, we investigate both the conserved/undamped CDO and the unconserved/damped CDO, respectively,

$$\begin{cases} \ddot{z} + \alpha z + \beta z|z|^2 + \gamma\bar{z}^3 = 0, \\ \ddot{z} + 2\varepsilon\dot{z} + \alpha z + \beta z|z|^2 + \gamma\bar{z}^3 = 0, \end{cases} \tag{2}$$

using two different approaches, known as the ansatz method [20–22] and the averaging method [18,19]. The ansatz method has succeeded in finding the exact and approximate solutions to many strong nonlinear problems in different scientific fields, especially plasma physics and the physical and engineering problems related to several pendulum oscillators. For instance, some approximations to different models for the damped forced complex Duffing oscillator have been derived and analyzed via the KBM approach and the ansatz method [17]. Therefore, the ansatz method is be applied in our study to analyze and find some approximations for the CDO family. Furthermore, the averaging method is a classic and crucial tool for solving and analyzing different types of nonlinear oscillations. The success of this method has been proven in analyzing many complex issues, which indicates its efficiency. Accordingly, this method is implemented for analyzing both conserved and unconserved complex Duffing oscillators (2). For applying the ansatz method, both the conserved and unconserved complex Duffing oscillators (2) are reduced to two decoupled conserved and unconserved standard Duffing oscillators. However, the averaging method is applied directly to analyze the unconserved CDO without decoupling.

2. The Ansatz Method for Analyzing the (Un)Conserved CDO

Let us rewrite the unconserved CDO in the form of the initial value problem (i.v.p.)

$$\begin{cases} \mathbb{R}_2 \equiv \ddot{z} + 2\varepsilon\dot{z} + \alpha z + \beta z|z|^2 + \gamma\bar{z}^3 = 0, \\ z(0) = z_0 \text{ and } \dot{z}(0) = \dot{z}_0, \end{cases} \tag{3}$$

which is equivalent to

$$\begin{cases} \ddot{x} + 2\varepsilon\dot{x} + x\left[\alpha + (\beta - 3\gamma)y^2\right] + (\beta + \gamma)x^3 = 0, \\ \ddot{y} + 2\varepsilon\dot{y} + y\left[\alpha + (\beta - 3\gamma)x^2\right] + (\beta + \gamma)y^3 = 0, \\ x(0) = x_0 \text{ and } \dot{x}(0) = \dot{x}_0, \\ y(0) = y_0 \text{ and } \dot{y}(0) = \dot{y}_0. \end{cases} \tag{4}$$

Suppose that both (x, y) obey the following damped Duffing oscillators

$$\begin{cases} \ddot{x} = -\left(2\varepsilon\dot{x} + px + qx^3\right), \\ \ddot{y} = -\left(2\varepsilon\dot{y} + ry + sy^3\right). \end{cases} \tag{5}$$

Here, the coefficients (p, q, r, s) need to be determined.

Now, by inserting the values of $(\ddot{x}, \ddot{y})$ given in system (5) into system (4), we obtain

$$\begin{cases} (\alpha - p)x + (\beta - 3\gamma)xy^2 + (\beta + \gamma - q)x^3 = 0, \\ (\alpha - r)y + (\beta - 3\gamma)x^2y + (\beta + \gamma - s)y^3 = 0. \end{cases} \tag{6}$$

Equating the coefficients of (x^i, y^i) in system (6) to zero and solving the obtained system, we finally obtain the values of (p, q, r, s) as follows

$$\begin{cases} p = r = \alpha, \\ q = s = \frac{4\beta}{3}, \\ \gamma = \frac{\beta}{3}. \end{cases} \tag{7}$$

Thus, the damped complex Duffing i.v.p. (3) is reduced to two decoupled damped Duffing oscillators, as given in system (5). In general, to find the solution to the original problem (3), we only need to solve the following damped Duffing equation

$$\ddot{u} + 2\varepsilon\dot{u} + pu + qu^3 = 0. \tag{8}$$

It is known that the Duffing oscillator (8) is not integrable except when the following condition is met

$$\alpha = \frac{8}{9}\varepsilon^2. \tag{9}$$

For this reason, we must find an approximation for Equation (8).

In the absence of the damping term ($\varepsilon = 0$), the i.v.p. (8) reduces to the conserved CDO, and in this case, we need only to solve the following conserved Duffing oscillator

$$\begin{cases} \ddot{u} + pu + qu^3 = 0, \\ u(0) = u_0 \ \& \ \dot{u}(0) = \dot{u}_0. \end{cases} \tag{10}$$

The solution to Equation (10) in terms of Jacobi elliptic functions is expressed by

$$u(t) = \frac{u_0 \mathrm{cn}(\sqrt{\omega}t, m) + \frac{\dot{u}_0}{\sqrt{\omega}} \mathrm{sn}(\sqrt{\omega}t, m) \mathrm{dn}(\sqrt{\omega}t, m)}{1 + b\, \mathrm{sn}^2(\sqrt{\omega}t, m)}, \tag{11}$$

with

$$\begin{cases} b = \frac{q(1-2m)u_0^2}{2p} - m, \ \omega = \frac{p}{1-2m} \\ m = \frac{1}{2}\left(1 \pm \frac{p}{\sqrt{(p + qu_0^2)^2 + 2q\dot{u}_0^2}}\right). \end{cases} \tag{12}$$

To analyze the damped Duffing oscillator (8), we rewrite this problem in the following i.v.p.

$$\begin{cases} \mathbb{R}_3 \equiv \ddot{u} + 2\varepsilon\dot{u} + pu + qu^3 = 0, \\ u(0) = u_0 \text{ and } \dot{u}(0) = \dot{u}_0, \end{cases} \tag{13}$$

and we suppose that

$$\lim_{t \to +\infty} u(t) = 0. \tag{14}$$

Now, the ansatz method can be applied to analyze the i.v.p. (13). To start, we assume that the solution to the i.v.p. (13) is defined by the following ansatz form

$$u = e^{-\varepsilon t}(u_0 \cos(w) + C \sin(w)), \tag{15}$$

where the function $w \equiv w(t)$ and the constant C are to be determined later.

Putting ansatz (15) into $\mathbb{R}_3 = 0$, we have

$$\mathbb{R}_3 = S_1 \cos(w) + S_2 \sin(w) + \text{higher order term} \tag{16}$$

with

$$\begin{cases} S_1 = \frac{1}{4}e^{-3\varepsilon t}\left[\begin{array}{c} u_0\left(3C^2q + 4pe^{2\varepsilon t} + 3qu_0^2 - 4\varepsilon^2 e^{2\varepsilon t}\right) \\ +4Ce^{2\varepsilon t}\ddot{w} - 4u_0 e^{2\varepsilon t}\dot{w}^2 \end{array}\right], \\ S_2 = \frac{1}{4}e^{-3\varepsilon t}\left[\begin{array}{c} C\left(3C^2q + 4pe^{2\varepsilon t} + 3qu_0^2 - 4\varepsilon^2 e^{2\varepsilon t}\right) \\ -4Ce^{2\varepsilon t}\dot{w}^2 - 4u_0 e^{2\varepsilon t}\ddot{w} \end{array}\right]. \end{cases} \tag{17}$$

By solving $S_1 = 0$ and $S_2 = 0$ and by eliminating $\ddot{w}$ from the obtained results, we obtain

$$\dot{w}^2 = \left(p - \varepsilon^2\right) + \frac{3}{4}qe^{-2t\varepsilon}\left(C^2 + u_0^2\right). \tag{18}$$

Solving Equation (18) yields

$$\begin{aligned} w &= \int_0^t \sqrt{(p - \varepsilon^2) + \frac{3}{4}q(C^2 + u_0^2)e^{-2\varepsilon\tau}}d\tau \\ &= W(t) - W(0), \end{aligned} \tag{19}$$

with

$$W(t) = \frac{\sqrt{-p_\varepsilon + \frac{3}{4}C_q e^{-2\varepsilon t}}}{\varepsilon\sqrt{-C_q}\sqrt{3 - \frac{4p_\varepsilon e^{2\varepsilon t}}{C_q}}}\left(2\sqrt{p_\varepsilon}e^{\varepsilon t}\sinh^{-1}\left(\frac{e^{\varepsilon t}\sqrt{4p_\varepsilon}}{\sqrt{-3C_q}}\right) - \sqrt{-C_q}\sqrt{3 - \frac{4p_\varepsilon e^{2\varepsilon t}}{C_q}}\right), \tag{20}$$

where $C_q = q\left(C^2 + u_0^2\right)$, and $p_\varepsilon = \left(\varepsilon^2 - p\right)$.

The value of the constant C can be obtained from the initial condition $\dot{u}(0) = \dot{u}_0$. Accordingly, the number C is a solution to the following algebraic equation

$$(\varepsilon u_0 + \dot{u}_0)^2 - \left[\left(\left(p - \varepsilon^2\right) + \frac{3}{4}qu_0^2\right)C^2 + \frac{3}{4}qC^4\right] = 0. \tag{21}$$

Solving Equation (21) yields

$$C = \pm\frac{1}{\sqrt{6}}\sqrt{\frac{-4p - 3qu_0^2 + 4\varepsilon^2 \pm \sqrt{(-4\varepsilon^2 + 4p + 3qu_0^2)^2 + 48q(\varepsilon u_0 + \dot{u}_0)^2}}{q}}. \tag{22}$$

By applying solution (11), we can obtain the solutions to both the real and imaginary components (x, y) of the Duffing system (5) for $\varepsilon = 0$. In Figure 1, both the analytical absolute solution $|z| = \sqrt{x^2 + y^2}$ and the fourth-order Runge–Kutta (RK4) numerical approximation to the i.v.p. (5) for $\varepsilon = 0$ are considered for $(\alpha, \beta, \gamma) = (1, 1, \beta/3)$ and $(x, \dot{x}, y, \dot{y}) = (1, 2, 0, 0)$. Moreover, the global maximum distance error (GMDE L_d) for the whole study domain is estimated to be $L_d = 8.65719 \times 10^{-7}$. Further, by using the obtained solution (15) of the damped Duffing oscillator in system (5), we finally obtain the solutions to both the real and imaginary components (x, y) and, accordingly, the solution to the complex function $|z| = \sqrt{x^2 + y^2}$. The analytical approximation based on solution (15) and the RK4 numerical approximation to the i.v.p. (3) are presented in Figure 2 for $(x, \dot{x}, y, \dot{y}) = (0, 0.2, 0.1, 0.1)$. The numerical value of the trigonometric solution $z = x + iy$ based on solution (15) and for $(\varepsilon, \alpha, \beta, \gamma) = (0.1, 1, 1, \beta/3)$ reads

$$z = e^{-0.1t}\left[\begin{array}{c} -0.197173\sin\left(h_1(t) + 12.955 - \frac{35.6824e^{0.1t}\sqrt{1.98 + 0.0777547e^{-0.2t}}\sinh^{-1}\left(5.04626e^{0.1t}\right)}{\sqrt{25.4647e^{0.2t} + 1}}\right) \\ -0.109348i\sin\left(h_1(t) + 15.8395 - \frac{47.4806e^{0.1t}\sqrt{1.98 + 0.0439141e^{-0.2t}}\sinh^{-1}\left(6.71476e^{0.1t}\right)}{\sqrt{45.0881e^{0.2t} + 1}}\right) \\ +0.1i\cos\left(h_1(t) + 15.8395 - \frac{47.4806e^{0.1t}\sqrt{1.98 + 0.0439141e^{-0.2t}}\sinh^{-1}\left(6.71476e^{0.1t}\right)}{\sqrt{45.0881e^{0.2t} + 1}}\right) \end{array}\right], \tag{23}$$

where $h_1(t) = 7.07107\sqrt{1.98 + 0.0777547e^{-0.2t}}$. In addition, the GMDE L_d for the whole study domain is estimated to be $L_d = 0.00143931$.

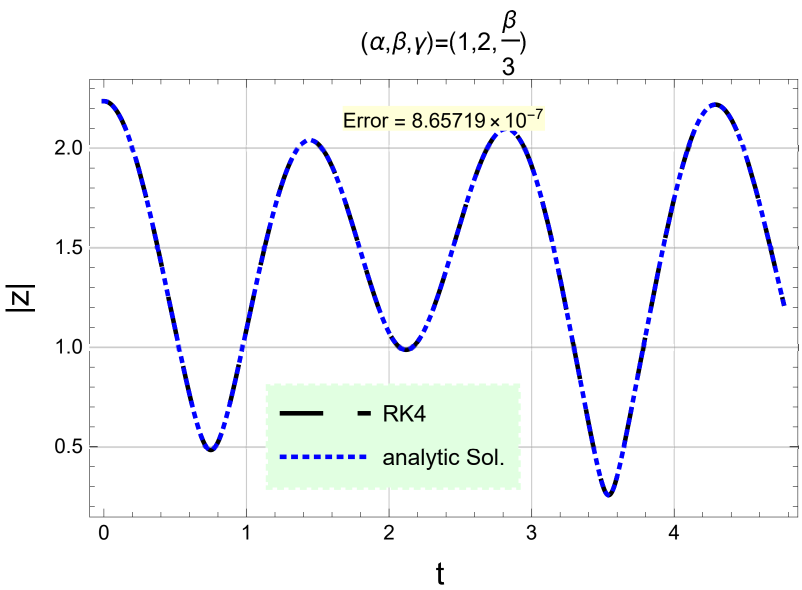

Figure 1. The analytical absolute solution $|z|$ and the RK4 numerical approximation of the i.v.p. (5) for $\varepsilon = 0$ are considered for $(x, \dot{x}, y, \dot{y}) = (1, 2, 0, 0)$.

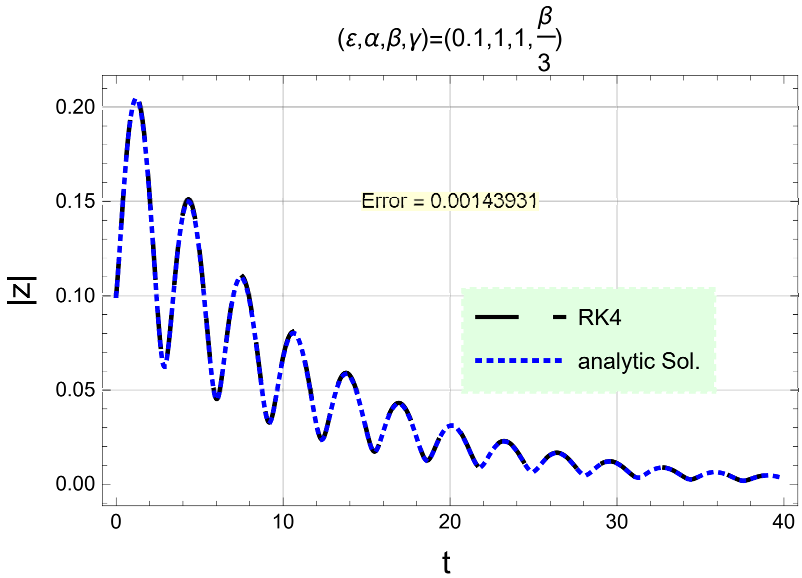

Figure 2. The analytical approximation based on solution (15) and the RK4 numerical approximation of the i.v.p. (3) are considered for $(x, \dot{x}, y, \dot{y}) = (0, 0.2, 0.1, 0.1)$.

3. The Averaging Method for Analyzing the Damped CDO

Now, let us summarize the algorithm of the averaging method for analyzing the i.v.p. (3) or (4) in the following brief points:

Step (I) We assume the approximations to the system (4) are defined by

$$\begin{cases} x(t) = a\cos(\Theta_1), \\ y(t) = b\cos(\Theta_2), \end{cases} \tag{24}$$

with

$$\begin{cases} \dot{x}(t) \approx -\omega_0 a\sin(\Theta_1), \\ \dot{y}(t) \approx -\omega_0 b\sin(\Theta_2), \end{cases} \tag{25}$$

where $\Theta_1 = \omega_0 t + \phi(t)$, $\Theta_2 = \omega_0 t + \psi(t)$, $(a(t), b(t)) \equiv (a, b)$, and $(\phi(t), \psi(t)) \equiv (\phi, \psi)$. Note here that we can take $\omega_0^2 = \alpha$, and superscript dot "$\cdot$" represents the derivative with time.

Step (II) By differentiating system (24) with the holding system (25), we have

$$\begin{cases} \dot{x}(t) = \dot{a}\cos(\Theta_1) - a(\dot{\phi} + \omega_0)\sin(\Theta_1), \\ \dot{y}(t) = \dot{b}\cos(\Theta_2) - b(\dot{\psi} + \omega_0)\sin(\Theta_2). \end{cases} \tag{26}$$

Now, by equating the respective expressions in systems (25) and (26) and solving the obtained results in $\dot{a}$ and $\dot{b}$, we obtain

$$\begin{cases} \dot{a} = a\dot{\phi}\tan(\Theta_1), \\ \dot{b} = b\dot{\psi}\tan(\Theta_2). \end{cases} \tag{27}$$

Step (III) By differentiating system (25), we have

$$\begin{cases} \quad \ddot{x}(t) \approx -\omega_0 \frac{d}{dt}[a(t)\sin(\Theta_1)] \\ = -\omega_0 \dot{a}(t)\sin(\Theta_1) - \omega_0 a(t)(\dot{\phi}(t) + \omega_0)\cos(\Theta_1), \\ \quad \ddot{y}(t) \approx -\omega_0 \frac{d}{dt}[b(t)\sin(\Theta_2)] \\ = -\omega_0 \dot{b}(t)\sin(\Theta_2) - \omega_0 b(t)(\dot{\psi}(t) + \omega_0)\cos(\Theta_2). \end{cases} \tag{28}$$

Step (IV) Now, plugging the expressions $\ddot{x}(t)$ and $\ddot{y}(t)$ into system (4) and solving the obtained results for the first derivatives of the unknown functions gives us the following system of ODE

$$\begin{cases} \dot{a} = \frac{1}{\omega_0}\left[a\sin(\Theta_1)\left(a^2(\beta+\gamma)\cos^3(\Theta_1) + b^2(\beta-3\gamma)\cos^2(\Theta_2)\cos(\Theta_1) - 2\varepsilon\omega_0\sin(\Theta_1)\right)\right], \\ \dot{b} = \frac{1}{\omega_0}\left[b\sin(\Theta_2)\left(a^2(\beta-3\gamma)\cos(\Theta_2)\cos^2(\Theta_1) + b^2(\beta+\gamma)\cos^3(\Theta_2) - 2\varepsilon\omega_0\sin(\Theta_2)\right)\right], \\ \dot{\psi} = \frac{1}{\omega_0}\left[\cos(\Theta_2)\left(a^2(\beta-3\gamma)\cos(\Theta_2)\cos^2(\Theta_1) + b^2(\beta+\gamma)\cos^3(\Theta_2) - 2\varepsilon\omega_0\sin(\Theta_2)\right)\right], \\ \dot{\phi} = \frac{1}{\omega_0}\left[\cos(\Theta_1)\left(a^2(\beta+\gamma)\cos^3(\Theta_1) + b^2(\beta-3\gamma)\cos^2(\Theta_2)\cos(\Theta_1) - 2\varepsilon\omega_0\sin(\Theta_1)\right)\right]. \end{cases} \tag{29}$$

Now, we integrate the system (29) over $0 \le t \le 2\pi/\omega_0$ and then multiply the obtained integrals by $\omega_0/(2\pi)$ to obtain the following averaged equations

$$\begin{cases} \dot{a} = \frac{a}{8\omega_0}\left(b^2(\beta-3\gamma)\sin(2(\phi-\psi)) - 8\varepsilon\omega_0\right), \\ \dot{b} = -\frac{b}{8\omega_0}\left(a^2(\beta-3\gamma)\sin(2(\phi-\psi)) + 8\varepsilon\omega_0\right), \\ \dot{\phi} = \frac{1}{8\omega_0}\left[3a^2(\beta+\gamma) + b^2(\beta-3\gamma)\cos(2(\phi-\psi)) + 2b^2(\beta-3\gamma)\right], \\ \dot{\psi} = \frac{1}{8\omega_0}\left[a^2(\beta-3\gamma)\cos(2(\phi-\psi)) + 2a^2(\beta-3\gamma) + 3b^2(\beta+\gamma)\right]. \end{cases} \tag{30}$$

Step (V) System (30) cannot be integrated in closed form. Thus, the approximate analytical solution to the proposed problem is obtained by taking the following hypotheses into account: $a^2 \ll 1$ and $b^2 \ll 1$, which lead to

$$\begin{cases} b^2(\beta-3\gamma)\sin(2(\phi-\psi)) \approx 0, \\ b^2(\beta-3\gamma)\cos(2(\phi-\psi)) \approx 0\,, \\ a^2(\beta-3\gamma)\sin(2(\phi-\psi)) \approx 0, \\ a^2(\beta-3\gamma)\cos(2(\phi-\psi)) \approx 0. \end{cases} \tag{31}$$

Thus, the averaged system (30) simplifies to

$$\begin{cases} \dot{a} = -\varepsilon a, \\ \dot{b} = -\varepsilon b, \\ \dot{\phi} = \frac{1}{8\omega_0}\left[3a^2(\beta+\gamma) + 2b^2(\beta-3\gamma)\right], \\ \dot{\psi} = \frac{1}{8\omega_0}\left[2a^2(\beta-3\gamma) + 3b^2(\beta+\gamma)\right]. \end{cases} \tag{32}$$

Step (VI) The approximate solutions to system (32) are given by

$$\begin{cases} a = c_0 e^{-\varepsilon t}, b = d_0 e^{-\varepsilon t}, \\ \phi = \frac{3c_0^2(\beta+\gamma)+d_0^2(\beta-3\gamma)}{8\varepsilon\omega_0} e^{-\varepsilon t}\sinh(\varepsilon t) + c_1, \\ \psi = \frac{c_0^2(\beta-3\gamma)+3d_0^2(\beta+\gamma)}{8\varepsilon\omega_0} e^{-\varepsilon t}\sinh(\varepsilon t) + d_1. \end{cases} \tag{33}$$

This solution is exact when $\gamma = \beta/3$. The constants (c_0, d_0, c_1, d_1) are obtained from the initial conditions (ICs).
The explicit form to the approximate solutions (24) reads

$$\begin{cases} x(t) = c_0 e^{-\varepsilon t}\cos\left(\omega_0 t + c_1 + \frac{3c_0^2(\beta+\gamma)+d_0^2(\beta-3\gamma)}{8\varepsilon\omega_0} e^{-\varepsilon t}\sinh(\varepsilon t)\right), \\ y(t) = d_0 e^{-\varepsilon t}\cos\left(\omega_0 t + d_1 + \frac{c_0^2(\beta-3\gamma)+3d_0^2(\beta+\gamma)}{8\varepsilon\omega_0} e^{-\varepsilon t}\sinh(\varepsilon t)\right). \end{cases} \tag{34}$$

Note that solutions (11) are valid for any value of γ. Both analytical approximations based on solution (34) and the RK4 numerical approximation of the i.v.p. (3) are compared with each other as shown in Figure 3 for $(x, \dot{x}, y, \dot{y}) = (0, 0.2, 0.1, 0.1)$. The numerical value of the solution (34) $z = x + iy$ according to the values of parameters $(\varepsilon, \alpha, \beta, \gamma) = (0.1, 1, 1, \beta/3)$ reads

$$z = e^{-0.1t}\left(\begin{array}{c} 0.196222\cos\left(-t + 0.0962581e^{-0.2t} + 7.75772\right) \\ +(0.147784i\cos\left(-t + 0.0546e^{-0.2t} + 0.772973\right) \end{array}\right). \tag{35}$$

Moreover, the GMDE L_d for this method is estimated to be $L_d = 0.00376382$. It is clear that the approximations of the i.v.p. (3) or (4) using the ansatz method are better than the averaging method.

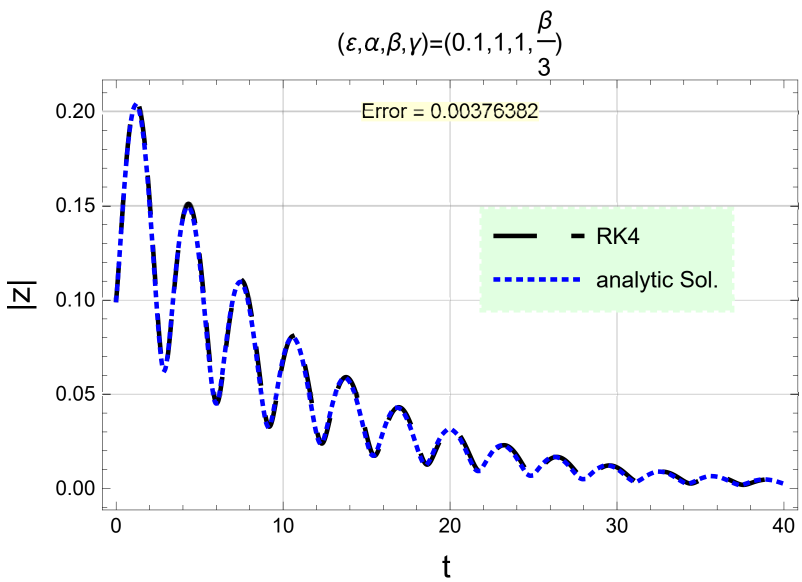

Figure 3. Both the analytical approximation based on solution (34) and the RK4 numerical approximation of the i.v.p. (3) are compared with each other for $(x, \dot{x}, y, \dot{y}) = (0, 0.2, 0.1, 0.1)$.

Formulas for the Residual Error Estimation

Some formulas for the residual error of the approximation (34) are estimated using the Chebyshev approximation, as follows.

Formula (i) For $(\alpha, \gamma) = (1, 0)$, $0 \le \varepsilon \le 0.2$, and $0 \le \beta \le 1$, the following residual error formula of the approximation (34) is obtained

$$\begin{aligned}
&\mathcal{E}(\varepsilon,\beta)|_{\gamma=0} = 474385\beta^5\varepsilon^5 - 291424\beta^5\varepsilon^4 + 65894.1\beta^5\varepsilon^3 - 6610\beta^5\varepsilon^2 + 280.531\beta^5\varepsilon \\
&-3.98336\beta^5 - 1.11208\times 10^6\beta^4\varepsilon^5 + 683694\beta^4\varepsilon^4 - 154353\beta^4\varepsilon^3 + 15382.2\beta^4\varepsilon^2 \\
&-643.877\beta^4\varepsilon + 9.18581\beta^4 + 873777\beta^3\varepsilon^5 - 537175\beta^3\varepsilon^4 + 120786\beta^3\varepsilon^3 \\
&-11878.6\beta^3\varepsilon^2 + 483.213\beta^3\varepsilon - 6.94163\beta^3 - 198831\beta^2\varepsilon^5 + 123020\beta^2\varepsilon^4 \\
&-27379.3\beta^2\varepsilon^3 + 2559.53\beta^2\varepsilon^2 - 90.5068\beta^2\varepsilon + 1.28073\beta^2 - 92610.4\beta\varepsilon^5 \\
&+55768.3\beta\varepsilon^4 - 12770.6\beta\varepsilon^3 + 1390.73\beta\varepsilon^2 - 72.0946\beta\varepsilon + 1.34397\beta \\
&+36.3947\varepsilon^5 - 13.6746\varepsilon^4 + 1.58342\varepsilon^3 - 0.0165635\varepsilon^2 + 0.0896355\varepsilon.
\end{aligned} \tag{36}$$

The error $\mathcal{E}$ according to Formula (36) is numerically estimated, as illustrated in Table 1.

Table 1. The error $\mathcal{E}$ according to Formula (36) for $\gamma = 0$.

ε	β	$\mathcal{E}(\varepsilon,\beta)$	ε	β	$\mathcal{E}(\varepsilon,\beta)$	ε	β	$\mathcal{E}(\varepsilon,\beta)$
0.05	0.1	0.00869611	0.12	0.3	0.00992028	0.16	0.4	0.0111127
0.06	0.1	0.00614833	0.13	0.1	0.00812582	0.17	0.2	0.00905933
0.07	0.1	0.00442296	0.13	0.2	0.0053519	0.17	0.3	0.00739511
0.08	0.1	0.00298519	0.13	0.3	0.00923724	0.17	0.4	0.0102543
0.08	0.2	0.0106368	0.14	0.1	0.00903857	0.18	0.2	0.0107507
0.09	0.1	0.00372497	0.14	0.2	0.00635553	0.18	0.3	0.00823168
0.09	0.2	0.00855087	0.14	0.3	0.00850884	0.18	0.4	0.00936358
0.1	0.1	0.00479024	0.15	0.1	0.0103468	0.19	0.2	0.011775
0.1	0.2	0.00668055	0.15	0.2	0.00746561	0.19	0.3	0.00950246
0.11	0.1	0.00591684	0.15	0.3	0.00774254	0.19	0.4	0.00752799
0.11	0.2	0.00609337	0.15	0.4	0.0119317	0.2	0.3	0.0106144
0.11	0.3	0.0107847	0.16	0.1	0.0114729	0.2	0.4	0.00864199
0.12	0.1	0.00681545	0.16	0.2	0.00835907	0.2	0.5	0.0113734
0.12	0.2	0.00546545	0.16	0.3	0.00643335			

Formula (ii) For $(\alpha,\gamma) = (1, 0.1)$, $0 \le \varepsilon \le 0.2$, and $0 \le \beta \le 1$, the following residual error formula of the approximation (34) is obtained

$$\begin{aligned}
&\mathcal{E}(\varepsilon,\beta)|_{\gamma=0.1} = -1.42608\times 10^6\beta^5\varepsilon^5 + 879247\beta^5\varepsilon^4 - 202528\beta^5\varepsilon^3 + 21410.2\beta^5\varepsilon^2 - 1028.82\beta^5\varepsilon + \\
&18.8501\beta^5 + 3.81256\times 10^6\beta^4\varepsilon^5 - 2.34166\times 10^6\beta^4\varepsilon^4 + 537306\beta^4\varepsilon^3 - 56581.2\beta^4\varepsilon^2 + 2707.29\beta^4\varepsilon - \\
&49.4232\beta^4 - 3.35444\times 10^6\beta^3\varepsilon^5 + 2.05351\times 10^6\beta^3\varepsilon^4 - 469530\beta^3\varepsilon^3 + 49253.9\beta^3\varepsilon^2 - 2345.77\beta^3\varepsilon + \\
&42.6662\beta^3 + 940121\beta^2\varepsilon^5 - 574266\beta^2\varepsilon^4 + 130807\beta^2\varepsilon^3 - 13632.4\beta^2\varepsilon^2 + 642.115\beta^2\varepsilon - 11.4954\beta^2 + \\
&22012.2\beta\varepsilon^5 - 13070.5\beta\varepsilon^4 + 3009.27\beta\varepsilon^3 - 336.912\beta\varepsilon^2 + 18.1092\beta\varepsilon - 0.39929\beta - 32781.5\varepsilon^5 + \\
&19850.8\varepsilon^4 - 4514.37\varepsilon^3 + 475.271\varepsilon^2 - 22.8054\varepsilon + 0.423489.
\end{aligned} \tag{37}$$

The error $\mathcal{E}$ according to Formula (37) is numerically estimated, as illustrated in Table 2.

Table 2. The error $\mathcal{E}$ according to Formula (37) for $\gamma = 0.1$.

ε	β	$\mathcal{E}(\varepsilon,\beta)$	ε	β	$\mathcal{E}(\varepsilon,\beta)$	ε	β	$\mathcal{E}(\varepsilon,\beta)$
0.01	0.3	0.00351056	0.09	0.4	0.00484636	0.14	0.5	0.00624417
0.02	0.3	0.00329974	0.09	0.5	0.00908224	0.15	0.4	0.0093985
0.03	0.3	0.00350766	0.1	0.3	0.00866673	0.15	0.5	0.00726692
0.04	0.3	0.00399564	0.1	0.4	0.00471321	0.15	0.6	0.00960439
0.05	0.3	0.00456364	0.1	0.5	0.00827798	0.16	0.5	0.00813551
0.05	0.4	0.00857651	0.11	0.3	0.00946396	0.16	0.6	0.0089799

Table 2. *Cont.*

ε	β	$\mathcal{E}(\varepsilon,\beta)$	ε	β	$\mathcal{E}(\varepsilon,\beta)$	ε	β	$\mathcal{E}(\varepsilon,\beta)$
0.06	0.3	0.00540216	0.11	0.4	0.00574467	0.17	0.5	0.00909218
0.06	0.4	0.00666374	0.11	0.5	0.00791905	0.17	0.6	0.00730655
0.07	0.3	0.00611684	0.12	0.4	0.00668723	0.18	0.6	0.00804375
0.07	0.4	0.00530534	0.12	0.5	0.0075032	0.19	0.6	0.00921011
0.08	0.3	0.00683354	0.13	0.4	0.00774797	0.2	0.7	0.00856223
0.08	0.4	0.00510303	0.13	0.5	0.00703465			
0.09	0.3	0.00778361	0.14	0.4	0.00865272			

Formula (iii) For $\alpha = 1$, $0 \leq \varepsilon \leq 0.2$, and$0 \leq (\beta,\gamma) \leq 1$, the following residual error formula of the approximation (34) is obtained

$$
\begin{aligned}
\mathcal{E}(\varepsilon,\beta,\gamma) = {} & -910.221\gamma^5\beta^5 + 2203.12\gamma^4\beta^5 - 1743.14\gamma^3\beta^5 + 410972\gamma^5\varepsilon^3\beta^5 - 982851\gamma^4\varepsilon^3\beta^5 + \\
& 757391\gamma^3\varepsilon^3\beta^5 - 174264.\gamma^2\varepsilon^3\beta^5 - 15572.3\gamma\varepsilon^3\beta^5 + 2389.97\varepsilon^3\beta^5 + 446.149\gamma^2\beta^5 - 184710\gamma^5\varepsilon^2\beta^5 + \\
& 445951\gamma^4\varepsilon^2\beta^5 - 351359\gamma^3\varepsilon^2\beta^5 + 88153.6\gamma^2\varepsilon^2\beta^5 + 3627.47\gamma\varepsilon^2\beta^5 - 888.094\varepsilon^2\beta^5 + 11.4512\gamma\beta^5 + \\
& 24494.\gamma^5\varepsilon\beta^5 - 59399.2\gamma^4\varepsilon\beta^5 + 47278.8\gamma^3\varepsilon\beta^5 - 12330.3\gamma^2\varepsilon\beta^5 - 241.874\gamma\varepsilon\beta^5 + 103.941\varepsilon\beta^5 - \\
& 3.98336\beta^5 + 2250.73\gamma^5\beta^4 - 5260.09\gamma^4\beta^4 + 3921.76\gamma^3\beta^4 - 1.02347\times 10^6\gamma^5\varepsilon^3\beta^4 + \\
& 2.36671\times 10^6\gamma^4\varepsilon^3\beta^4 - 1.71875\times 10^6\gamma^3\varepsilon^3\beta^4 + 327463.\gamma^2\varepsilon^3\beta^4 + 58426.\gamma\varepsilon^3\beta^4 - 5398.96\varepsilon^3\beta^4 - \\
& 854.827\gamma^2\beta^4 + 461311\gamma^5\varepsilon^2\beta^4 - 1.08027\times 10^6\gamma^4\varepsilon^2\beta^4 + 808533\gamma^3\varepsilon^2\beta^4 - 176600\gamma^2\varepsilon^2\beta^4 - \\
& 16962.6\gamma\varepsilon^2\beta^4 + 1980.87\varepsilon^2\beta^4 - 75.5725\gamma\beta^4 - 61206.9\gamma^5\varepsilon\beta^4 + 144016\gamma^4\varepsilon\beta^4 - 109065\gamma^3\varepsilon\beta^4 + \\
& 25092.7\gamma^2\varepsilon\beta^4 + 1642.06\gamma\varepsilon\beta^4 - 231.567\varepsilon\beta^4 + 9.18583\beta^4 - 2080.95\gamma^5\beta^3 + 4676.47\gamma^4\beta^3 - \\
& 3267.1\gamma^3\beta^3 + 956720\gamma^5\varepsilon^3\beta^3 - 2.13947\times 10^6\gamma^4\varepsilon^3\beta^3 + 1.47074\times 10^6\gamma^3\varepsilon^3\beta^3 - 235069\gamma^2\varepsilon^3\beta^3 - \\
& 61143.1\gamma\varepsilon^3\beta^3 + 3896.9\varepsilon^3\beta^3 + 589.257\gamma^2\beta^3 - 429731\gamma^5\varepsilon^2\beta^3 + 974946\gamma^4\varepsilon^2\beta^3 - \\
& 694313\gamma^3\varepsilon^2\beta^3 + 133301\gamma^2\varepsilon^2\beta^3 + 18965.9\gamma\varepsilon^2\beta^3 - 1393.68\varepsilon^2\beta^3 + 97.2138\gamma\beta^3 + \\
& 56977.4\gamma^5\varepsilon\beta^3 - 129815\gamma^4\varepsilon\beta^3 + 93503.7\gamma^3\varepsilon\beta^3 - 19040.9\gamma^2\varepsilon\beta^3 - 2010.22\gamma\varepsilon\beta^3 + \\
& 162.445\varepsilon\beta^3 - 6.94165\beta^3 + 1173.26\gamma^5\beta^2 - 2724.9\gamma^4\beta^2 + 2084.36\gamma^3\beta^2 - 561523\gamma^5\varepsilon^3\beta^2 + \\
& 1.31585\times 10^6\gamma^4\varepsilon^3\beta^2 - 1.02028\times 10^6\gamma^3\varepsilon^3\beta^2 + 274089\gamma^2\varepsilon^3\beta^2 - 6220.02\gamma\varepsilon^3\beta^2 - 585.496\varepsilon^3\beta^2 - \\
& 550.393\gamma^2\beta^2 + 243286\gamma^5\varepsilon^2\beta^2 - 572674\gamma^4\varepsilon^2\beta^2 + 448332\gamma^3\varepsilon^2\beta^2 - 124220\gamma^2\varepsilon^2\beta^2 + \\
& 4529.49\gamma\varepsilon^2\beta^2 + 170.54\varepsilon^2\beta^2 + 13.6017\gamma\beta^2 - 31741.7\gamma^5\varepsilon\beta^2 + 74655.1\gamma^4\varepsilon\beta^2 - 58432.8\gamma^3\varepsilon\beta^2 + \\
& 16285.3\gamma^2\varepsilon\beta^2 - 671.3\gamma\varepsilon\beta^2 - 18.4299\varepsilon\beta^2 + 1.28074\beta^2 - 385.697\gamma^5\beta + 990.263\gamma^4\beta - 911.029\gamma^3\beta + \\
& 192437\gamma^5\varepsilon^3\beta - 500608\gamma^4\varepsilon^3\beta + 468582\gamma^3\varepsilon^3\beta - 188005\gamma^2\varepsilon^3\beta + 28173.1\gamma\varepsilon^3\beta - 725.687\varepsilon^3\beta + \\
& 359.029\gamma^2\beta - 80140.8\gamma^5\varepsilon^2\beta + 207984\gamma^4\varepsilon^2\beta - 194044\gamma^3\varepsilon^2\beta + 77603.6\gamma^2\varepsilon^2\beta - 11638\gamma\varepsilon^2\beta + \\
& 303.278\varepsilon^2\beta - 53.6432\gamma\beta + 10235\gamma^5\varepsilon\beta - 26458.1\gamma^4\varepsilon\beta + 24556.3\gamma^3\varepsilon\beta - 9768.58\gamma^2\varepsilon\beta + 1464\gamma\varepsilon\beta - \\
& 38.4288\varepsilon\beta + 1.34397\beta + 6.35481\gamma^5 - 15.0286\gamma^4 + 16.6978\gamma^3 - 2229.81\gamma^5\varepsilon^3 + 5477.01\gamma^4\varepsilon^3 - \\
& 7416.09\gamma^3\varepsilon^3 + 6937.38\gamma^2\varepsilon^3 - 3070.83\gamma\varepsilon^3 + 0.31498\varepsilon^3 - 12.837\gamma^2 + 994.045\gamma^5\varepsilon^2 - 2421.96\gamma^4\varepsilon^2 + \\
& 3128.67\gamma^3\varepsilon^2 - 2809.02\gamma^2\varepsilon^2 + 1239.6\gamma\varepsilon^2 - 0.0393504\varepsilon^2 + 5.81263\gamma - 141.791\gamma^5\varepsilon + \\
& 341.78\gamma^4\varepsilon - 414.132\gamma^3\varepsilon + 349.545\gamma^2\varepsilon - 153.761\gamma\varepsilon + 0.0937655\varepsilon.
\end{aligned}
\tag{38}
$$

The error $\mathcal{E}$ according to Formula (38) is numerically estimated, as illustrated in Table 3.

Table 3. The error $\mathcal{E}$ according to Formula (38).

ε	β	γ	$\mathcal{E}(\varepsilon,\beta,\gamma)$	ε	β	γ	$\mathcal{E}(\varepsilon,\beta,\gamma)$	ε	β	γ	$\mathcal{E}(\varepsilon,\beta,\gamma)$
0.02	0.6	0.2	0.00680076	0.08	0.7	0.2	0.00671226	0.12	0.7	0.2	0.00712994
0.03	0.6	0.2	0.00623318	0.08	0.9	0.3	0.00885137	0.12	0.8	0.2	0.00872512
0.03	0.9	0.3	0.00971491	0.08	1.0	0.3	0.00892256	0.12	1.0	0.3	0.00912936
0.04	0.6	0.2	0.00623761	0.09	0.6	0.2	0.00808938	0.13	0.7	0.2	0.00791204
0.04	0.9	0.3	0.00899421	0.09	0.7	0.2	0.00666215	0.13	0.8	0.2	0.00839803
0.05	0.6	0.2	0.00638026	0.09	0.8	0.2	0.00972188	0.13	1.0	0.3	0.00865544
0.05	0.7	0.2	0.00934977	0.09	0.9	0.3	0.00921603	0.14	0.7	0.2	0.00867321
0.05	0.9	0.3	0.00892461	0.09	1.0	0.3	0.00857979	0.14	0.8	0.2	0.00801654
0.06	0.6	0.2	0.00650476	0.1	0.6	0.2	0.008758	0.14	1.0	0.3	0.00972908
0.06	0.7	0.2	0.00789632	0.1	0.7	0.2	0.00664228	0.15	0.7	0.2	0.00934578

Table 3. *Cont.*

ε	β	γ	$\mathcal{E}(\varepsilon,\beta,\gamma)$	ε	β	γ	$\mathcal{E}(\varepsilon,\beta,\gamma)$	ε	β	γ	$\mathcal{E}(\varepsilon,\beta,\gamma)$
0.06	0.9	0.3	0.00893119	0.1	0.8	0.2	0.00920778	0.15	0.8	0.2	0.00749827
0.06	1.0	0.3	0.00956233	0.1	0.9	0.3	0.00964525	0.16	0.7	0.2	0.00990592
0.07	0.6	0.2	0.00695091	0.1	1.0	0.3	0.00896661	0.16	0.8	0.2	0.00831075
0.07	0.7	0.2	0.0072119	0.11	0.6	0.2	0.0094164	0.17	0.8	0.2	0.00923937
0.07	0.9	0.3	0.00904325	0.11	0.7	0.2	0.00674026	0.17	0.9	0.2	0.00965559
0.07	1.0	0.3	0.00919536	0.11	0.8	0.2	0.00899557	0.18	0.9	0.2	0.00856151
0.08	0.6	0.2	0.00746038	0.11	1.0	0.3	0.00902475	0.19	0.9	0.2	0.00943644

4. Conclusions

In this investigation, some approximate analytical solutions to both the conserved complex Duffing oscillator (CDO) and the unconserved/damped CDO were derived using the ansatz and averaging methods. The most important main results that we obtained are summarized in the following brief points:

- The exact solution to the conserved/undamped CDO was derived in terms of the Jacobi elliptic function using the ansatz method.
- Concerning the damped CDO, both the ansatz and averaging methods were implemented to analyze this problem. To apply the ansatz method, the damped CDO was reduced to two decoupled standard damped Duffing oscillators, and their solutions were obtained in terms of trigonometric functions.
- The averaging method was applied directly to analyze and solve the damped CDO without decoupling, and two approximations were obtained in terms of trigonometric functions.
- There are restrictions on the values of the physical parameters when applying the approximations that were derived using the ansatz method. In contrast, for the averaging method, the obtained approximations can be discussed using arbitrary values for the physical parameters.
- All the obtained approximations were compared with the RK4 numerical approximations to ensure that the obtained approximations were accurate. Moreover, the global maximum error of all the obtained approximations as compared to RK4 numerical approximations was estimated. There was complete agreement between the obtained results and the RK4 numerical approximations, which enhances the accuracy and efficiency of the used methods.
- Furthermore, some formulas for the residual errors to the obtained approximations using averaging method were derived and discussed using different values for the physical parameters.

Future work: There are many evolution equations that are used to model many nonlinear waves in a plasma physics, such as a nonlinear Schrödinger's equation [23–25]

$$i\partial_t\varphi + \frac{P}{2}\partial_{x,x}\varphi + Q|\varphi|^2\varphi = 0. \tag{39}$$

- This equation and some linked equations can be reduced to a complex Duffing-type oscillator using the transformation $\zeta = x + \lambda t$ and $\varphi = v + iw$ to study the dynamics of nonlinear oscillations in different types of plasmas

$$\begin{cases} \ddot{v}(\zeta) = \frac{2\lambda}{P}\dot{w}(\zeta) - \frac{2Q}{P}vw^2 - \frac{2Q}{P}v^3, \\ \ddot{w}(\zeta) = -\frac{2\lambda}{P}\dot{v}(\zeta) - \frac{2Q}{P}v^2w - \frac{2Q}{P}w^3. \end{cases} \tag{40}$$

- Problem (40) will be discussed in detail in future work using the suggested method. Therefore, we expect that this study will help many researchers interested in studying nonlinear phenomena in a plasma physics. Furthermore, the suggested methods can be applied to analyze and study the dynamical systems to many evolution equations [25–27].

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