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On Linear Codes over Finite Singleton Local Rings

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Abstract: The study of linear codes over local rings, particularly non-chain rings, imposes difficulties that differ from those encountered in codes over chain rings, and this stems from the fact that local non-chain rings are not principal ideal rings. In this paper, we present and successfully establish a new approach for linear codes of any finite length over local rings that are not necessarily chains. The main focus of this study is to produce generating characters, MacWilliams identities and generator matrices for codes over singleton local Frobenius rings of order 32. To do so, we first start by characterizing all singleton local rings of order 32 up to isomorphism. These rings happen to have strong connections to linear binary codes and $\mathbb{Z}_4$ codes, which play a significant role in coding theory.

Keywords: MacWiliams relations; Frobenius rings; coding over rings; generating character; local rings

MSC: 94B05; 94B60; 16L30; 16P20

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1. Introduction

All rings considered in this article are finite commutative and have an identity. A ring R is defined to be local if it has a unique maximal ideal, denoted $J(R)$ or simply J, called the Jacobson radical. Furthermore, it is well established that all finite commutative Frobenius rings can be decomposed as a product of local Frobenius rings. Thus, studying codes over local rings is essential, as most structural coding theory findings have been transferred to such rings. The idea that the class of Frobenius rings is the appropriate class to describe codes is well known, and this is largely due to the fulfilment of both MacWilliams theorems. For more details on the theory of rings, we refer to [1–4] and the references therein.

Linear codes of length N over R correspond to subsets of R^N which are R-submodules of R. When N is divisible by the characteristic of the residue field R/J, then these codes are called repeated-root codes; otherwise, they are simple codes. Using Gray maps, linear codes over fields were related to those over chain rings. While codes over chain rings have been extensively investigated, codes over local non-chain rings have not gained as much attention. The main reason for this disparity is because chain rings are principle ideal rings (PIRs), and as PIRs are characterized as direct sums of chain rings, many conclusions on chain rings may also be applied to PIRs. However, to fully determine codes over Frobenius rings, it is essential to consider local rings which are not chains, even though this imposes challenges, since the rings are not PIRs. We recommend references [5–15] to readers.

This paper mainly concentrates on determining fundamental coding results over local Frobenius rings, with a specific focus on rings of order 32, to clarify the significance of general results. The study of singleton local rings was accomplished in [2], and their relevance in coding theory was demonstrated through connections to linear binary and $\mathbb{Z}_4$ codes, see [16]. We proceed to investigate, in this article, two crucial tools in coding theory: MacWilliams relations and generator matrices. In [17], the authors discussed these tools over local Frobenius rings with small order, i.e., 16, based on the classification of local

Frobenius rings with order 16 provided in [18]. Thus, we aim to concentrate on rings of order 32 and utilize them as examples. First, we present a constructive approach to finding a generating character χ associated with any singleton local Frobenius ring with invariants p, n, r, t. Given such an χ, determining the MacWilliams relations when working with rings of order 32 becomes straightforward.

On the other hand, generator matrices are highly advantageous for linear codes because they not only generate the code but also make it easier to compute the code size. Chain rings have a well-established standard form that satisfies this purpose, but this however cannot be said for codes with alphabets of local (non-chain) rings. In this regard, we introduce a natural generalization to local non-chain Frobenius rings whose orders are 32. We also demonstrate, through several numerical examples, why such a generator matrix does not necessarily result in determining the code size.

Following the preliminary definitions and results presented in Section 2, the classification of singleton local Frobenius rings with invariants p, n, r, t is described in Section 3, with special attention to giving the full details of characterizing rings of order 32. The method for finding, in general, generating characters for singleton Frobenius rings is given in Section 4. Additionally, specific generating characters are determined for all singleton local Frobenius rings of order 32. Subsequently, the matrix associated with the weight enumerator is obtained. Section 5 focuses on the results concerning generator matrices for linear codes over such rings of order 32.

2. Preliminaries

Throughout this section, we introduce some notations and basic facts which will be utilized later in our discussion. From now on, suppose that R is a finite commutative singleton local ring with identity, and J denotes its Jacobson radical. We will rely on the following well-established results from the theory of finite rings and coding theory (see [2,3,6,8,15,19]).

First, we define Jacobson radical J of R as the maximal ideal of R. The order of R, is $|R| = p^{mr}$, where p is a prime number, and that of J is $p^{(m-1)r}$ under the condition $J^m = 0$. The additive order of 1 in R (characteristic) is of the form p^n, such that $1 \leq n \leq m$ and $R/J \cong GF(p^r) = F$. Moreover, R has a coefficient subring S of the form $GR(p^n, r)$, known as the maximal Galois subring of R. It has been shown that there exists π in J such that

$$R = S + S\pi, \;\; J = pS + \pi S. \tag{1}$$

A chain ring is a ring for which its J is principal. When $m = n$, then R is commutative, and J is generated by the element p. Moreover, R can be constructed over $\mathbb{Z}_{p^n}$ as a Galois extension,

$$R = \mathbb{Z}_{p^n}[a] \cong \mathbb{Z}_{p^n}[x]/(g(x)),$$

where a has a multiplicative order of $p^r - 1$ and $g(x)$ is a monic basic polynomial (irreducible modulo p) of degree r over $\mathbb{Z}_{p^n}$. Elements of R can be uniquely expressed (p-adic expression) as a sum of terms involving $\alpha_i \in \Gamma(r) = (a) \cup \{0\}$,

$$\gamma = \alpha_1 + p\alpha_1 + p^2\alpha_2 + \cdots + p^{n-1}\alpha_{n-2}. \tag{2}$$

Suppose t is the additive order of π; $p^t\pi = 0$. We characterize the integers p, n, r and t as the invarints of R. The group of units of R, $U(R)$, is decomposed as

$$U(R) = (a) \times H, \tag{3}$$

where $H = 1 + J$ is called the one group.

We define the socle of R, $soc(R)$, as the sum of all minimal ideals of R. As the rings under consideration are commutative rings, then the socle coincides with the annihilator of J. In the literature, there are several equivalent definitions of Frobenius rings. However, in our discussion, we will focus on a specific definition that is most relevant to our subsequent analysis.

Definition 1 ([4]). *We call R a Frobenius ring if $soc(R) \cong R/J$, considered as F-vector spaces.*

Finite Frobenius rings have a very simple charaterization due to the work in [1]. A character χ of $(R,+)$ is an element of $\mathrm{Hom}_{\mathbb{Z}}(R,\mathbb{C}^*)$, the character group of $(R,+)$. We call χ a generating character if kerχ contains no left ideals of R which are not trivial.

Theorem 1 (Wood [1]). *A finite ring R is a Frobenius ring if and only if it has a generating character $\chi \in Hom_{\mathbb{Z}}(R,\mathbb{C}^*)$.*

Corollary 1 ([1]). *A finite ring R is a Frobenius ring if and only if it has a unique minimal ideal.*

Theorem 2 (Honold [4]). *A finite ring R is a Frobenius ring if and only if its $soc(R)$ is cyclic.*

A subset of R^N is called a code C of length N over R, and if C is a submodule, then it is called a linear code. Furthermore, we can incorporate the inner-product in R^N, and thus we can define the dual code $C^\perp$ of C as

$$C^\perp = \{\mathbf{u} : \mathbf{c} \cdot \mathbf{u} = 0, \mathbf{c} \in C\}, \tag{4}$$

where $\cdot$ denotes an inner product in R^N.

All symbols and notations mentioned above will be maintained throughout the manuscript.

3. Singleton Local Frobenius Rings

From now on, R will denote a finite commutative local ring with a singleton basis and invariants p,n,r,t. Moreover, let $g(x)$ always be defined as

$$g(x) = x^2 - p^d\beta h - p^e\beta_1 h_1 x, \tag{5}$$

where $\beta,\beta_1 \in \Gamma^*(r)$ and $h,h_1 \in 1+pS$. By the results of [2], R is structured as

$$R \cong S[x]/(g(x), p^t x). \tag{6}$$

For the purpose of simplicity, we need to agree on the following notations

$$\begin{cases} \pi^2 = p^d\beta h + p^e\beta_1 h_1 \pi; \\ \beta,\beta_1 \in \Gamma^*(r) \text{ and } h,h_1 \in 1+pS; \\ m \leq 2n; \\ t = m-n,\ n-t \leq d \leq n \text{ and } 1 \leq e \leq t; \\ 2 \leq l \leq m; J^l = 0 \text{ and } J^{l-1} \neq 0. \end{cases} \tag{$*$}$$

The following theorem establishes a powerful tool in characterizing singelton local Frobenius rings based just on their invariants p,n,r,t,d,e. When $t=1$, the case is trivial, so we assume, in the theorem, that $t>1$.

Theorem 3. *If R is a singleton local ring, then R is Frobenius if and only if $t=n$ or $(t,d) = (n-1,1)$.*

Proof. Suppose that R is Frobenius, then $soc(R)$ is the minimal ideal which is unique. Furthermore, $soc(R)$ is cyclic by Theorem 2, and thus we can write $soc(R) = (\theta)$. In this case, we have $\theta p = \theta\pi = 0$. As $\theta = s_0 + s_1\pi$, it is clear that $s_0 \in p^{n-1}S$ and $s_1 \in p^{t-1}S$, and hence

$$\theta = p^{n-1}u + p^{t-1}v\pi, \tag{7}$$

where $u,v \in \Gamma(r)$ but they are not both equal to 0. This means that $\theta \in (p^{n-1}, p^{t-1}\pi)$. But since $t-1 \leqslant n-1$, then $soc(R) \subseteq (p^{t-1})$. Therefore, $soc(R) = p^{n-1}R$. To finish the proof, we consider two cases. If $u=0$, then $soc(R) = (p^{t-1}\pi)$ and since $p^{n-1}\pi$ annihilates J, then $p^{n-1}\pi \in soc(R)$, which means that $n=t$. On the other hand, suppose that $u \neq 0$.

Since $\theta\pi = 0$, by Equation (7), $p^{n-1}\pi = 0$ and $p^{t-1}\pi^2 = 0$, implying that $p^{d+t-1}\beta h = 0$. Therefore, $t = n - 1$ and $d = 1$. For the converse, assume that $t = n$. As $p^{n-1}\pi \neq 0$ and $p^{n-1}\pi J = 0$, then clearly $soc(R) = (p^{n-1}\pi)$. This means R is Frobenius according to Theorem 2. The case when $t = n - 1$ and $d = 1$ will lead to $soc(R) = (p^{n-1})$ and again, by the same reasoning, R is Frobenius. □

Corollary 2. *Suppose that R is a chain ring, then R is Frobenius. In particular, if $n = 1$, R is Frobenius.*

Proof. The result follows from Theorem 3 since $t = n$ or $t = n - 1$ and $d = 1$. □

Remark 1. *For any singleton local Frobenius ring R with invariants p, n, r and t, then*

$$soc(R) = \begin{cases} (p^{n-1}\pi), & \text{if } n = t, \\ (p^{n-1}), & \text{if } t = n - 1, d = 1. \end{cases}$$

The following proposition is useful for the next section.

Proposition 1. *Let I be any non-zero ideal of a singleton local Frobenius ring R. Then, $soc(R)$ is contained in I.*

Proof. Assume that I is an ideal of R. If I is minimal, then $I = soc(R)$ because $soc(R)$ is the unique minimal ideal of R by Corollary 1. Now, suppose I is not minimal ideal, then I contains an ideal which is minimal, and thus contains $soc(R)$. □

Remark 2. *The number of singleton local rings (up to isomorphism) with $\pi^2 = p^d\beta$ and invariants p, n, r, t, d is*

$$N(p,n,r,t,d) = \begin{cases} 1, & \text{if } p = 2; \\ 2, & \text{if } p \neq 2. \end{cases}$$

Full Characterization of Singleton Local Frobenius and Non-Frobenius Rings of Order 32

The following theorem plays a crucial role in our subsequent discussion as it provides a comprehensive classification of all local rings with a singleton basis of order 32.

Theorem 4. *Suppose R is a singleton local ring of order 32. Then, R is isomorphic to a unique ring among those listed in Table 1.*

Table 1. Classification of all singleton local Frobenius and non-Fobenius rings of order 32.

Frobenius Rings		
Chain Rings	**Non-Chain Rings**	**Non-Frobenius Rings**
$\mathbb{Z}_{2^3}[x]/(x^2-2,4x)$ $\mathbb{Z}_{2^3}[x]/(x^2+2,4x)$ $\mathbb{Z}_{2^3}[x]/(x^2-2-2x,4x)$	$\mathbb{Z}_{2^4}[x]/(x^2-8,2x)$	$\mathbb{Z}_{2^3}[x]/(x^2-4-2x,4x)$ $\mathbb{Z}_{2^3}[x]/(x^2-4,4x)$ $\mathbb{Z}_{2^3}[x]/(x^2,4x)$ $\mathbb{Z}_{2^4}[x]/(x^2,2x)$ $\mathbb{Z}_{2^3}[x]/(x^2-2x,4x)$

Proof. Since $|R| = 32$, we have two possible cases: either $r = 5$ and $m = 1$, or $r = 1$ and $m = 5$. However, the first case does not result in a singleton ring. Therefore, we consider the case where $m = 5$ and n can be either 3 or 4.
Case a: Let us assume $n = 3$, which implies $t = 2$. In this case, e can take values of 1 or 2, and d can take values of 1, 2, or 3.

Case a1. Considering the sub-case where $d = 1$, then $g(x) = x^2 - 2\beta h - 2^e \beta_1 x$. Hence,

$$\begin{cases} R_1 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 2h, 4x), \\ R_2 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 2h - 2x, 4x). \end{cases}$$

As $h \in 1 + pS = \{1, 3, 5, 7\}$, then $1 = 5, 3 = 7 \bmod 4$. Then, the correspondence ϕ between $\mathbb{Z}_{2^3}[x]/(x^2 - 2, 4x)$ and $\mathbb{Z}_{2^3}[x]/(x^2 - 2(5), 4x)$ defined by $\phi(s_1 + s_2\pi) = s_1 + s_2\theta$ is an isomorphism. In addition, the same can be imposed on the rings $\mathbb{Z}_{2^3}[x]/(x^2 - 2(3), 4x)$ and $\mathbb{Z}_{2^3}[x]/(x^2 - 2(7), 4x)$. This concludes that there are two rings (up to isomorphism) of type R_1 which are of the form $\mathbb{Z}_{2^3}[x]/(x^2 - 2, 4x)$ and $\mathbb{Z}_{2^3}[x]/(x^2 + 2, 4x)$. Now, we classify rings of type R_2. Because $d = e$, then the last term of $g(x) = x^2 - 2h - 2x$ determines the classes of such rings; i.e., there is only one class that represents these rings, which is $\mathbb{Z}_{2^3}[x]/(x^2 - 2 - 2x, 4x)$.
Case a2. Now consider the option where d takes the values of 2 or 3. In this case, the construction and properties of such rings can be further explored and analyzed.

$$R_1 \cong \mathbb{Z}_{2^3}[x]/(x^2, 4x) \text{ if } d = 3, e = 2,$$

$$\begin{cases} R_2 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 4 - 2x, 4x), \\ R_3 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 4, 4x), \\ R_4 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 2x, 4x). \end{cases}$$

Based on Remark 2, there is exactly one ring of the form R_3. With respect to rings of type R_1, there exists a unique class up to isomorphism. Furthermore, there is only one class of each singleton local ring R_2 and R_4 by the same discussion as the previous case.
Case b. Let us consider the case where $n = 4$, which imposes $t = 1$. In this case, we have $e = 1$ and d can take values of 3 or 4. Therefore, we list all such rings as

$$\begin{cases} R_1 \cong \mathbb{Z}_{2^4}[x]/(x^2, 2x), & \text{if } d = 4 \\ R_2 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 8, 2x), & \text{if } d = 3. \end{cases}$$

The class of R_1 rings consists of one element, and additionally, there is only one ring of type R_2 in light of Remark 2.

We finally employ Theorem 3 to classify all singleton local rings of order 32. They are divided into two categories: chain and non-chain rings. □

Example 1. *The ring $\mathbb{Z}_{2^3}[x]/(x^2 - 2x, 4x)$ is not Frobenius because $d = 3, t = 2$ which does not satisfy the condition of Theorem 3. Note that $soc(R) = (4, 2x)$ which is not cyclic; $|\ soc(R)\ | \neq |\ F\ |$. This ring is not a chain ring and is not the only non-Frobenius singleton local ring, as shown in Table 1.*

Remark 3. *For non-chain and Frobenius rings, $l = m - 1 = 4$. On the other hand, for non-Frobenius rings, if we denote $l_i = l(J(R_i))$, then the index of nilpotency of J is R_i. Thus, simple calculations will lead to $l_1 = 3$, $l_2 = 3$, $l_3 = 3$, $l_4 = 4$ and $l_5 = 3$, where R_i runs through all rings in the third column.*

Remark 4. *By utilizing the results in our prior publication [2], we successfully classified singleton local rings of order 32. It is worthy to highlight the original contribution of distinguishing between Frobenius and non-Frobenius singleton local rings, as this distinction carries substantial significance in the construction of generating characters discussed in the subsequent section. Furthermore, this new approach has the potential to be extended for the purpose of studying codes over rings with higher orders.*

4. Generating Characters and MacWilliams Identities

Let R be a singleton local Frobenius ring with invariants p, n, r, t and associated polynomial $g(x)$. The following theorem describes an approach to construct a generating character for any singleton local Frobenius ring. In Table 2, we list the resulting generating characters for singleton local Frobenius rings of order 32.

Table 2. χ for singleton Frobenius local rings of order 32.

Ring	Additive Structure	Generating Character
$\mathbb{Z}_{2^4}[x]/(x^2-8, 2x)$	$\mathbb{Z}_{2^4} \times \mathbb{Z}_2$	$\chi(a+bx) = \zeta^a(-1)^b$
$\mathbb{Z}_{2^3}[x]/(x^2-2, 4x)$	$\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}$	$\chi(a+bx) = \gamma^a i^b$
$\mathbb{Z}_{2^3}[x]/(x^2+2, 4x)$	$\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}$	$\chi(a+bx) = \gamma^a i^b$
$\mathbb{Z}_{2^3}[x]/(x^2+2-2x, 4x)$	$\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}$	$\chi(a+bx) = \gamma^a i^b$

Theorem 5. *Suppose that R is a singleton local Frobenius ring with invariants p, n, r, t. Then, there exists an integer $q \geq 1$ such that*

$$\chi(\omega) = \gamma_1^{a_1}\gamma_2^{a_2}\ldots\gamma_q^{a_q}, \tag{8}$$

is a generating character of R, where γ_i is a p^{n_i}-root of unity and $a_i \leq mr$ for each $1 \leq i \leq q$.

Proof. Since R has a characteristic number p^n, there are $n = n_1, n_2, \ldots, n_q$ that satisfy $n_1 \geq n_2 \geq \cdots \geq n_q$ and so are additive groups,

$$R \cong \mathbb{Z}_{p^{n_1}} \times \mathbb{Z}_{p^{n_2}} \times \cdots \times \mathbb{Z}_{p^{n_q}}. \tag{9}$$

This means that there are $u_1, u_2, \ldots u_q$, generators satisfying the condition $o(u_i) = p^{n_i}$ for $1 \leq i \leq q$. Thus, every element ω of R is factorized uniquely (in an additive sense) as

$$\omega = a_1u_1 + a_2u_2 + \cdots + a_qu_q,$$

where $a_i \in \mathbb{Z}_{p^{n_i}}$. As R is Frobenius, then by Proposition 1, any nontrivial ideal encompasses $\mathrm{soc}(R)$. Therefore, in order for χ to be a generating character, it is enough to illustrate that a character χ is nontrivial on $\mathrm{soc}(R)$. Let us define the following map $\chi : R \to \mathbb{C}^*$ by

$$\chi(a_1u_1 + a_2u_2 + \cdots + a_qu_q) = \gamma_1^{a_1}\gamma_2^{a_2}\ldots\gamma_q^{a_q}, \tag{10}$$

where γ_i is a p^{n_i}th root of unity in $\mathbb{C}$ for each i. One can see that χ is in $\mathrm{Hom}(R, \mathbb{C}^*)$; that is, it is a character. Suppose that the image of $soc(R)$ under χ is not 1,; then, χ will be a generating character for R. On the contrary, assume that $\chi(soc(R)) = \{1\}$. Suppose that $\omega \in soc(R)$. As $\mathrm{soc}(R)$ is a cyclic of order p^r, then in the additive structure, we can identify $\mathrm{soc}(R)$ by $\mathbb{Z}_{p^r}$. From group theory, there exists a unique i, $1 \leq i \leq q$ such that $\mathbb{Z}_{p^r}$ is a subgroup of $\mathbb{Z}_{p^{n_i}}$. This imposes $\chi(a_iu_i) = \gamma^{a_i} \neq 1$ for all $a_i < p^r$, and hence χ is a non-trivial character on $soc(R)$. Therefore, χ is a generating character. □

The following table presents generating characters for singleton Frobenius local rings of order 32. In the table, γ and ζ are 2^3th root and 2^4th root of unity, respectively.

MacWilliams Relations

Now, we completely determine MacWilliams identities of various versions for singleton local Frobenius rings with p, n, r, t. In fact, these relations can hold for a broader class of finite rings, namely the class of all Frobenius rings. These identities play a vital role in coding theory as they establish a crucial link between the weight enumerator of a code and its dual.

Let C be a code over R. The complete weight enumerator is known as

$$CWE(\mathbf{c}) = \prod_i a_i^{n_i(\mathbf{c})}, \tag{11}$$

$$CWE(C) = \sum_{c \in C} CWE(\mathbf{c}). \tag{12}$$

where $n_i(\mathbf{c})$ means the number of occurrences of a_i in $\mathbf{c}$.

Suppose that A is a $p^{mr} \times p^{mr}$ matrix with $a_{ij} = \chi(a_i a_j)$. Then,

$$CWE_C(x_{a_1}, \dots, x_{a_{p^{mr}}}) = \frac{1}{\mid C^{\perp} \mid} CWE_{C^{\perp}}(A \cdot x_{a_1}, \dots, x_{a_{p^{mr}}}). \tag{13}$$

The Hamming weight enumerator is defined as

$$HW_C(a, b) = \sum_{c \in C} a^{N - wt(\mathbf{c})} b^{wt(\mathbf{c})}, \tag{14}$$

where $wt(\mathbf{c}) = \mid \{i : c_i \neq 0\} \mid$. First, note that $W_H(a, b) = W_C(a, b, b, \dots, b)$. Now, we introduce the MacWilliam identity for the Hamming weight enumerator as

$$HW_C(a, b) = \frac{1}{\mid C^{\perp} \mid} HW_{C^{\perp}}(a + (p^{mr} - 1)b, a - b). \tag{15}$$

We define $\sim$ on R as $a \sim b$ if and only if $a = \omega b$, where $\omega \in U(R)$. It can be easily justified that $\sim$ is an equivalence relation. Suppose that $\hat{b}_1, \dots, \hat{b}_q$ are its equivalence classes, and the symmetrized weight enumerator is hence defined by

$$SWE_C(x_{\hat{b}_1}, \dots, x_{\hat{b}_q}) = \sum_{\mathbf{c} \in C} \prod_i x_{\hat{b}_i}{}^{n'_i(\mathbf{c})}, \tag{16}$$

where $n'_i(\mathbf{c})$ is the number of occurrences of elements of $\hat{b}_i$ in the codeword $\mathbf{c}$. Now, assume that

$$b_{ij} = \sum_{a \in \hat{b}_j} \chi(a_i a).$$

Denote the matrix $(b_{ij})_{q \times q}$ by B. Then, we define the MacWiliams identity for the SWE (symmetrized weight enumerator) of a linear code C as

$$SWE_C(x_{\hat{b}_1}, \dots, x_{\hat{b}_q}) = \frac{1}{\mid C^{\perp} \mid} SWE_{C^{\perp}}(B \cdot (x_{\hat{b}_1}, \dots, x_{\hat{b}_q})). \tag{17}$$

It is easy to obtain the matrix A in (13) once we have χ. However, computing B in Equation (17) requires more computational steps, as we need to determine $\hat{b}_i$. This process involves more effort but is vital for constructing B. The following theorem gives us a detailed scheme of building B for a more general case.

Theorem 6. *If R is a singleton local Frobenius ring with $n = 1$, then $R \cong \mathbb{F}_{p^r}[x]/(x^2)$ and B is a matrix of 3×3 of the form*

$$B = \begin{pmatrix} 1 & (p^r - 1)p^r & p^r - 1 \\ 1 & 0 & \delta \\ 1 & -p^r & p^r - 1 \end{pmatrix}$$

Proof. The additive structure of $\mathbb{F}_{p^r}$ is $\underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r \text{ times}}$. Then, the generating character χ is defined on $\mathbb{F}_{p^r}$ as

$$\chi(a_1 u_1 + a_2 u_2 + \cdots + a_r u_r) = (\gamma)^{a_1 + a_2 + \cdots + a_r},$$

where γ is the $p-$root of unity in $\mathbb{C}$. The equivalence classes are $\hat{b}_1 = \{0\}$, $\hat{b}_2 = U(R)$ and $\hat{b_3} = J(R) \setminus \{0\}$. Therefore, it is clear that $b_{1j} = 1$ and $b_{j1} = \mid \hat{b}_j \mid$. Also, $b_{33} = p^r - 1$

and $b_{23} = \delta$, which is 0 or -1 according to whether p is odd or even, respectively. Finally, $b_{32} = -p^r$. $\square$

Remark 5. *Theorem 6 can be generalized to a broader class of rings, namely $R = \mathbb{F}_{p^r}[x]/(x^k)$, $k \geq 3$. In this case, the associated matrix B will have dimensions of $(k+1) \times (k+1)$.*

Example 2. *If R is with invariants $(p, n, r, t) = (2, 1, 2, 1)$, then the order of R is 2^4, and*

$$B = \begin{pmatrix} 1 & 12 & 3 \\ 1 & 0 & -1 \\ 1 & -4 & 3 \end{pmatrix}$$

Next, we proceed to illustrate these computations through practical demonstration of the steps involved for examples of singleton local Frobenius rings of order 32. Our attention will be focused on understanding the equivalence classes under $\sim$ and then constructing B. Table 3 presents the associated B of each ring, and also provides all equivalent classes.

From now onwards, the order of R is 32. First, we investigate B of chain rings.

Table 3. MacWilliams SWE matrices for singleton local Frobenius rings of order 32.

Ring	Associated Matrix	Equivalence Classes
$\mathbb{Z}_{2^4}[x]/(x^2-8, 2x)$	B_2	$\{0\}, U(R), (2) \setminus soc(R), (x) \setminus soc(R), (x+2) \setminus soc(R), (x+4) \setminus soc(R), soc(R) \setminus \{0\}$
$\mathbb{Z}_{2^3}[x]/(x^2-2, 4x)$	B_1	$\{0\}, U(R), J \setminus J^2, J^2 \setminus J^3, J^3 \setminus J^4, soc(R) \setminus \{0\}$
$\mathbb{Z}_{2^3}[x]/(x^2+2, 4x)$	B_1	$\{0\}, U(R), J \setminus J^2, J^2 \setminus J^3, J^3 \setminus J^4, soc(R) \setminus \{0\}$
$\mathbb{Z}_{2^3}[x]/(x^2+2-2x, 4x)$	B_1	$\{0\}, U(R), J \setminus J^2, J^2 \setminus J^3, J^3 \setminus J^4, soc(R) \setminus \{0\}$

Example 3. *Suppose that $R = \mathbb{Z}_{2^3}[x]/(x^2-2, 4x)$. Let us assume the elements of R have the following order:*

$$\begin{aligned} R = \{&0,1,2,3,4,5,6,7,x,2x,3x,1+x,1+2x, \\ &1+3x,2+x,2+2x,2+3x,3+x,3+2x,3+3x,4+x,4+2x, \\ &4+3x,5+x,5+2x,5+3x,6+x,6+2x,6+3x,7+x,7+2x,7+3x\}. \end{aligned}$$

Now, define for $a, b \in R$, $a_{ij} = \chi(ab)$, where i and j are the indexes of a and b, respectively. Thus, set $A = (a_{ij})$ of size 32×32 over $\mathbb{C}$. Note that this matrix is easy to calculate but it is too large. Next, we compute the matrix B which needs extensive computations. We need to obtain $\hat{b}_i$ on R which are listed as

$$\begin{cases} \hat{b}_1 = \{0\}, \\ \hat{b}_2 = U(R) = \{1,3,5,7,1+x,1+2x,1+3x,3+x, \\ 3+2x,3+3x,5+x,5+2x,5+3x,7+x, \\ 7+2x,7+3x\}, \\ \hat{b}_3 = J \setminus J^2 = \{x,3x,x+2,x+4,x+6,3x+2,3x+4,3x+6\}, \\ \hat{b}_4 = J^2 \setminus J^3 = \{2,6,2+2x,6+2x\}, \\ \hat{b}_5 = J^3 \setminus J^4 = \{2x,2x+4\}, \\ \hat{b}_6 = soc(R) \setminus \{0\} = J^4 \setminus \{0\} = \{4\}. \end{cases}$$

To demonstrate the computations, we introduce a few cases, noting that $\chi(a) = -\chi(a+2x)$ and $\chi(a+x) = -\chi(a+3x)$,

$$b_{21} = \chi(1(0)) = 1, \qquad b_{12} = \sum_{b_i \in U(R)} \chi(0(b_i)) = 16, b_{1j} = |\hat{b}_j|, j = 3,4,5,6$$

$$b_{i1} = 1, i = 1,2,\dots,6 \qquad b_{22} = \sum_{b_i \in \hat{b}_2} \chi(1(b_i)) = 0, b_{i2} = 0, i = 3,4,6$$

$$b_{23} = \sum_{b_i \in \hat{b}_3} \chi(1(b_i)) = 0 \qquad b_{2j} = 0, j = 4,5, b_{26} = -1.$$

Therefore,

$$B_1 = \begin{pmatrix} 1 & 16 & 8 & 4 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -4 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -16 & 8 & 4 & 2 & 1 \\ 1 & 0 & 0 & 4 & 0 & 1 \end{pmatrix}$$

Remark 6. *If α is a unit, then $\alpha\hat{b}_i = \hat{b}_i$, where $i = 1,2,3,\dots,q$. Thus, if the index of α is i, then $b_{ij} = b_{1j}$ for all j.*

Remark 7. *From Example 3, all chain rings have an equivalent matrix B.*

The following example will involve the case when R is a non-chain ring. There are only unique rings of this type, which are $R = \mathbb{Z}_{2^4}[x]/(x^2 - 8, 2x)$. In this ring, $J = (x, 2)$, which is of order 16 and its index of nilpotency is $l = 4$, and finally $soc(R) = (p^{n-1}\pi)$.

Example 4. *Consider the non-chain ring $R = \mathbb{Z}_{2^4}[x]/(x^2 - 8, 2x)$ which has $(p, n, r, t) = (2, 4, 1, 1)$ as invariants. Assume the order of elements of R as follows: if $i, j \in \mathbb{Z}_{2^4}$, then i is before j if $i < j$ as integers, and $i + x$ before $j + x$ if i precedes j. Thus, the equivalence classes are*

$$\begin{cases} \hat{b_1} = \{0\}, \hat{b}_2 = U(R) = \{i, i + x : \text{ where } i \text{ is odd as integer}\}, \\ \hat{b_3} = (2) \setminus soc(R) = \{2, 4, 6, 10, 12, 14\}, \\ \hat{b_4} = (x) \setminus soc(R) = \{x, x + 8\}, \\ \hat{b_5} = (x + 2) \setminus soc(R) = \{x + 2, x + 6, 10 + x, 14 + x\} \\ \hat{b_6} = (x + 4) \setminus soc(R) = \{x + 4, 12 + x\} \\ \hat{b_7} = soc(R) \setminus \{0\} = J^3 \setminus \{0\} = \{8\}. \end{cases}$$

Thus, using the associated generating character in Table 2 and after appropriate calculations as in Example 3,

$$B_2 = \begin{pmatrix} 1 & 16 & 6 & 2 & 4 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 2 & 4 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & -2 & 2 & 4 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & -2 & 2 & 4 & 2 & 1 \end{pmatrix}$$

Note that $\zeta^2 + \zeta^6 + \zeta^{10} + \zeta^{14} = 0$ and $\zeta^4 + \zeta^{12} = 0$.

5. Generator Matrices

In coding theory, one of the essential techniques is creating a generator matrix in standard form for a linear code over finite rings. In this section, we determine generator matrices for linear codes over singleton local Frobenuis rings with invariants $2, n, r, t$ and $n + t = 5$. Compared to codes over chain rings, constructing a generator matrix is more challenging when studying codes over singleton local Frobenius rings that are non-chains. Although a basic set of generators can still be found, such a generator matrix may not provide simple information on the code size or amount of codewords. This is not the case with codes across chain rings, where a generator matrix may be used to easily compute the code size.

Let C be a linear code over R_i, $i = 2, 3, 4$. Then, C has a generator matrix with standard form [11],

$$G = \begin{pmatrix} I_{e_0} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ 0 & \pi I_{e_1} & A_{23} & A_{24} & A_{25} & A_{26} \\ 0 & 0 & \pi^2 I_{e_2} & A_{34} & A_{35} & A_{36} \\ 0 & 0 & 0 & \pi^3 I_{e_3} & A_{45} & A_{46} \\ 0 & 0 & 0 & 0 & \pi^4 I_{e_4} & A_{56} \end{pmatrix}$$

where I_e denotes the $e \times e$ identity matrix. A linear code which has a generator matrix of this form is called a code of type $\{e_0, e_1, e_2, e_3, e_4\}$, and therefore its size is equal to

$$\mid C \mid = p^{\sum_{i=0}^{4} (5-i)e_i}. \tag{18}$$

Definition 2. *We call the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_e$ modular independent if no nontrivial linear combination of the vectors exists with coefficients from J that equals the zero vector. A generator matrix G over the ring R is considered as such if the rows of G are modularly independent and they generate the code C.*

The remainder of this section deals with singleton local Frobenius rings of order 32,

$$R = \mathbb{Z}_{2^4}[x]/(x^2 - 8, 2x). \tag{19}$$

Figure 1 below shows the ideal lattice of R. As we know that $\mid J \mid = 16$, $\mid (2) \mid = \mid (\pi) \mid = \mid (\pi + 2) \mid = 8$, $\mid (\pi + 4) \mid = \mid (4) \mid = 4$ and $\mid (8) = soc(R) \mid = 2$. Therefore, our aim, in this section, is to create a set of modularly independent elements which serve as the rows of a generator matrix for a given code. The subsequent theorem provides a full representation of the structure of a generator matrix.

Theorem 7. *Suppose C is a linear code over $R = \mathbb{Z}_{2^4}[x]/(x^2 - 8, 2x)$ with arbitrary length N. Then, any generator matrix for C is equivalent to*

$$G = \left(\begin{array}{c|c|c|c|c|c|c|c|c} I_{e_0} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} & M_{17} & M_{18} & M_{19} \\ \hline 0 & 2I_{e_1} & M_{23} & M_{24} & M_{25} & M_{26} & M_{27} & M_{28} & M_{29} \\ 0 & \pi I_{e_1} & & & & & & & \\ \hline 0 & 0 & \pi I_{e_2} & 0 & 0 & 0 & M_{37} & & \\ 0 & 0 & 0 & (\pi+4)I_{e_3} & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 2I_{e_4} & 0 & 0 & M_{48} & M_{49} \\ 0 & 0 & 0 & 0 & 0 & (2+\pi)I_{e_5} & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 4I_{e_6} & & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8I_{e_7} & M_{99} \end{array}\right)$$

where M_{ij} are matrices over R.

Proof. Suppose that G is a matrix whose rows generate C as an R-module. We move all columns that have a unit in them to the left of G. By performing row reduction on those columns, we get a matrix of the form

$$G = \begin{pmatrix} I_{e_0} & * \\ 0 & A \end{pmatrix}$$

Now, all elements of A are not units. We again move all columns that have elements in $J = (2, \pi)$ to the left, and impose the main row operations to transfer the matrix to a form of the following.

$$G = \begin{pmatrix} I_{e_0} & * & * \\ 0 & 2 & * \\ 0 & \pi & * \\ 0 & 0 & A_1 \end{pmatrix}$$

We proceed with this algorithm, ensuring that the matrix A_1 is constructed by arranging columns with elements that form a pair $(2, \pi)$, and we repeat this process until the matrix is in the desired form.

$$\left(\begin{array}{c|c|c} I_{e_0} & * & * \\ \hline 0 & 2I_{e_1} & * \\ \cline{1-2} 0 & \pi I_{e_1} & \\ \hline 0 & 0 & A_2 \end{array}\right)$$

where the entries of columns of the matrix A_2 are elements of only one ideal of the ideals (2), (π), $(2 + \pi)$ and $(4 + \pi)$. So, now we proceed with the matrix A_2. In order to create a unique representation of the matrix, we select a specific ordering for the four ideals: (π), $(4 + \pi)$, (2), and $(2 + \pi)$. This chosen order will be used consistently when constructing the matrix. Let u be a unit of R; we first continue with columns whose entries are of the form πu, then with columns of $(\pi + 4)u$, and next with columns that have elements of the form $(2)u$. Finally, we deal with columns of the form $(\pi + 2)u$. In each step, we perform row reduction in the usual manner. Note that the ideal (4) is contained in both (2) and $(\pi + 2)$, and thus we repeat the same procedure with this ideal, since all remaining entries of the columns will be from the ideal (4).

$$\left(\begin{array}{c|c|c|c|c|c} \pi I_{e_2} & 0 & 0 & 0 & * & \\ \cline{1-4} 0 & (\pi+4)I_{e_3} & 0 & 0 & & \\ \cline{1-5} 0 & 0 & 2I_{e_4} & 0 & 0 & * \\ \cline{1-5} 0 & 0 & 0 & (2+\pi)I_{e_5} & 0 & \\ \cline{1-5} 0 & 0 & 0 & 0 & 4I_{e_6} & \\ \hline 0 & 0 & 0 & 0 & 0 & A_3 \end{array}\right)$$

Finally, all elements of A_3 are from the ideal generated by 8. By eliminating any rows containing only zeros and performing a final round of row reduction, we ultimately obtain a matrix that precisely matches the desired form we were aiming for. $\square$

Proposition 2. *Suppose that* $\boldsymbol{u}$ *is a vector in* R^N. *Let* $M = (\boldsymbol{u})$ *be a cyclic* R*-submodule of* R^N. *Then,* $\mid M \mid \in \{32, 16, 8, 4, 2, 0\}$.

Proof. Assume that I is an ideal generated by coordinates of the vector $\mathbf{u}$. Let T be the set of all annihilators of $\mathbf{u}$ in R, which is an ideal of R. Thus,

$$\mid M \mid = \frac{\mid R \mid}{\mid T \mid} = \mid I \mid .$$

According to Figure 1, we have six possibilities for the order of $I \in \{32, 16, 8, 4, 2, 0\}$. The result follows. $\square$

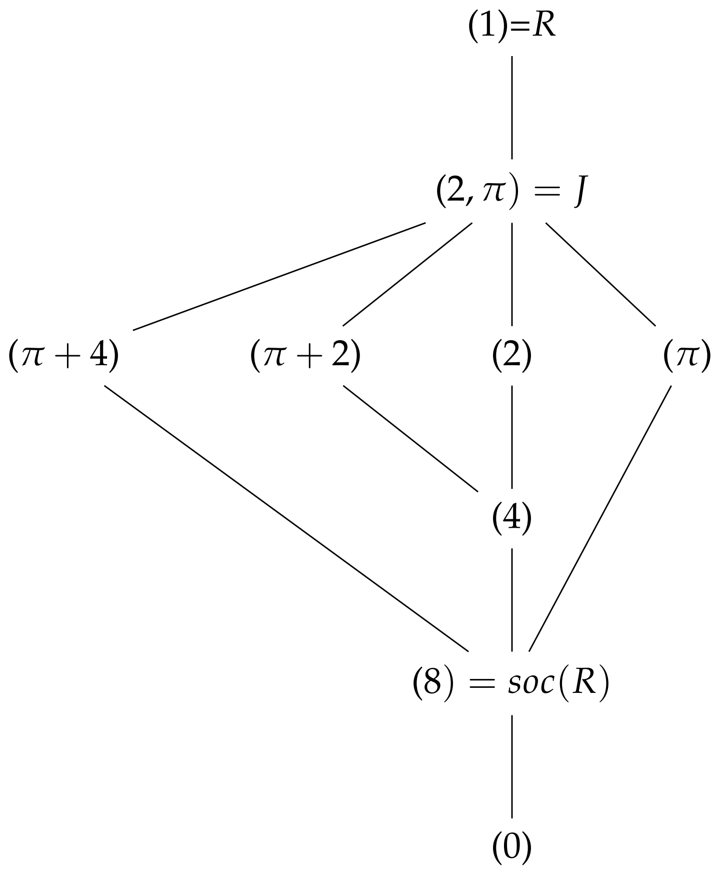

Figure 1. Ideals lattice of $\mathbb{Z}_{2^4}[x]/(x^2 - 8, 2x)$.

Theorem 8. *Suppose that $M = (\boldsymbol{u}, \boldsymbol{v})$ are R-submodules such that their coordinates contain no units. Then,*

$$\mid M \mid \in \{256, 128, 64, 32, 16, 8, 4\}.$$

Proof. By Proposition 2, we have $\mid M \mid \leq 256$. Since $\mid (8) \mid = 2$, then $4 \leq \mid M \mid$. $\square$

Example 5. *To obtain a linear code over $R = \mathbb{Z}_{2^4}[x]/(x^2 - 8, 2x)$ of order 16, let $N = 1$ and $C = (2, \pi)$. Therefore, $\mid C \mid = 16$. If we want a code of order 32, we put $C = (\boldsymbol{u}, \boldsymbol{v})$ such that $N = 2$ and $\boldsymbol{u} = (2, \pi)$, $\boldsymbol{v} = (\pi, 2)$. Thus, $\mid C \mid = 32$. Consider $N = 4$, $\boldsymbol{u} = (2, 0, \pi, 2)$ and $\boldsymbol{v} = (\pi, 2, 0, 0)$. Hence, $\mid C \mid = 256$. Moreover, C is a decomposable module, i.e.,*

$$C \cong (\boldsymbol{u}) \oplus (\boldsymbol{v}).$$

The last example illustrates that it is possible for a code over a singleton local Frobenius (non-chain) ring to not have a minimal set of generators in standard form, making it difficult to determine the code size. In other words, it illustrates the distinction between codes over local non-chain rings and codes over chain rings.

Example 6. *Assume C is a linear code over $\mathbb{Z}_{2^4}[x]/(x^2 - 8, 2x)$ with a generator matrix of the form*

$$G = \begin{pmatrix} 2 & \pi \\ \pi & 0 \\ 0 & 2 \end{pmatrix}$$

If M_1 is the R-submodule generated by the first and second row of G, and M_2 is the R-submodule generated by the third row of G, then $M_1 \cap M_2$ is always non-trivial. This means C is an indecomposable module.

6. Conclusions

In conclusion, we have successfully classified all singleton local Frobenius rings (up to isomorphism) with respect to fixed invariants and determined the MacWilliams relations and generator matrices for linear codes of arbitrary length over these rings. While MacWilliams relations and generator matrices are well known and significant for codes over chain rings, such a case may not be reachable for codes over local non-chain rings. The challenge lies in the fact that local non-chain rings are not principal ideal rings, which complicates the determination of a minimal set of generators and the enumeration of the code size. This limitation suggests that alternative approaches or techniques are needed to handle codes over local non-chain rings effectively.

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